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ON COMPLEX PERTURBATIONS OF INFINITE BAND SCHRÖDINGER OPERATORS

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Dedicated to Yu. M. Berezanskii on occasion of his 90th birthday

Abstract. Let $H_0 = -\frac{d^2}{dx^2} + V_0$ be an infinite band Schrödinger operator on $L^2(\mathbb{R})$ with a real-valued potential $V_0 \in L^\infty(\mathbb{R})$. We study its complex perturbation $H = H_0 + V$, defined in the form sense, and obtain the Lieb–Thirring type inequalities for the rate of convergence of the discrete spectrum of H to the joint essential spectrum. The assumptions on V vary depending on the sign of $\operatorname{Re} V$.

Introduction

Different characteristics of the distribution of the discrete spectrum for non-self-adjoint perturbations of model differential self-adjoint operators, e.g., a Laplacian on $\mathbb{R}^d$, a discrete Laplacian on $\mathbb{Z}^d$, etc., were studied in a number of papers (see Frank–Laptev–Lieb–Seiringer [4], Borichev–Golinskii–Kupin [1], Demuth–Hansmann–Katriel [3]). This paper focuses on complex perturbations of one dimensional Schrödinger operators with infinite band spectrum and certain behavior of the lengths of its gaps (the case of finite band Schrödinger operators was studied in [5]).

So, consider a real-valued measurable function V_0 on $\mathbb{R}$ and denote by M_{V_0} a maximal multiplication operator by V_0. The standing assumption is that the Schrödinger operator

$$H_0 = -\Delta + M_{V_0}, \quad \Delta := \frac{d^2}{dx^2}, \tag{0.1}$$

is self-adjoint, $H_0^* = H_0$, and its spectrum $\sigma(H_0)$ is an infinite band, i.e.,

$$\sigma(H_0) = \sigma_{ess}(H_0) = I = \bigcup_{k=1}^{\infty} [a_k, b_k], \quad a_k \to +\infty. \tag{0.2}$$

We say that the gaps are relatively bounded if

$$r = r(I) := \sup_k \frac{r_k}{b_k} < \infty, \quad r_k := a_{k+1} - b_k \tag{0.3}$$

is the length of k' gap in (0.2). A typical example here is the Hill operator with a periodic potential (see [10, Section XIII.16]). It is well known (see [9]) that $r_k \to 0$ as $k \to \infty$ for potentials V_0 from L^2 on a period, so (0.3) obviously holds for such potentials.

Furthermore, consider the form sum

$$H = H_0 + M_V, \tag{0.4}$$

where V is a complex-valued potential. If M_V is a relatively compact perturbation of H_0, that is, $\mathrm{dom(M_V)} \supset \mathrm{dom(H_0)}$, and $M_V(H_0 - z)^{-1}$ is a compact operator for

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$z \in \rho(H_0)$, then, by the celebrated theorem of Weyl (see, e.g., [8, Section IV.5.6]), $\sigma_{ess}(H) = \sigma_{ess}(H_0)$ and

$$\sigma(H) = I \,\dot{\cup}\, \sigma_d(H)$$

(disjoint union), the discrete spectrum $\sigma_d(H)$ of H, i.e., the set of isolated eigenvalues of finite algebraic multiplicity, can accumulate only on I. The main goal of the paper is to obtain certain quantitative bounds for the rate of this accumulation.

The assumption on the background V_0 looks as follows:

$$V_0 \ge 0, \quad V_0 \in L^\infty(\mathbb{R}). \tag{0.5}$$

As for the perturbation V the conditions will vary depending on the sign of $\operatorname{Re} V$. Precisely, for general V's we assume that

$$V \in L^p(\mathbb{R}), \quad p \ge 2, \tag{0.6}$$

and for accretive perturbations with $\operatorname{Re} V \ge 0$ we put

$$V \in L^p(\mathbb{R}), \quad p > 1. \tag{0.7}$$

Under assumptions (0.5)–(0.7) H is a well-defined, closed and m-sectorial operator, and there is a number $\omega_1 \le 0$ such that

$$\sigma(H) \subset \overline{N(H)} \subset \{z : \operatorname{Re} z \ge \omega_1\}, \tag{0.8}$$

where $N(H) = \{(Hf, f) : f \in \operatorname{dom}(H),\ \|f\|_2 \le 1\}$ is the numerical range of H (see, e.g., [8, Chapter VI]). Moreover, H appears to be a relatively compact (even $\mathcal{S}_p$) perturbation of H_0, $\mathcal{S}_p$ being the Schatten–von Neumann class of compact operators.

Denote by $\mathrm{d}(z, E)$ the Euclidean distance from a point $z \in \mathbb{C}$ to a set $E \subset \mathbb{R}$.

Theorem 0.1. *Let H_0 be the Schrödinger operator* (0.1) *with V_0 satisfying* (0.5)*. Assume that H_0 is an infinite band operator with the spectrum*

$$\sigma(H_0) = \sigma_{ess}(H_0) = \bigcup_{k=1}^{\infty} [a_k, b_k], \quad 0 \le a_1 < b_1 < a_2 < b_2 < \ldots, \quad a_n \to +\infty,$$

and the lengths of gaps are relatively bounded (0.3)*. Then for the perturbation H* (0.4) *with V* (0.6) *and for each $\omega < \omega_1$* (0.8) *the following Lieb–Thirring type inequality*

$$\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z, I)}{(|z-\omega| + |\omega|)^{2p}} \le \frac{C(p, I)\, \|V\|_p^p}{(\omega_1 - \omega)^p |\omega|^{p-1/2}} \left(1 + \frac{\|V_0\|_\infty}{a_1 + |\omega|}\right)^p, \tag{0.9}$$

holds, where the positive constant $C(p, I)$ depends on p and $I = \sigma(H_0)$.

Remark 0.2. If we take $\omega < \omega_1 - 1$, bound (0.9) can be simplified. Indeed, now $\omega_1 - \omega > 1$,

$$|z - \omega| < |\omega|(1 + |z|), \quad 1 < a_1 + |\omega| < |\omega|(1 + a_1),$$

and so

$$\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z, I)}{(1 + |z|)^{2p}} \le C(p, I) |\omega|^{p+1/2} (1 + \|V_0\|_\infty)^p \, \|V\|_p^p. \tag{0.10}$$

There is an elementary way to specify ω and eliminate it from the final expression. The price we pay is an additional factor in the right hand side.

Theorem 0.3. *Under assumptions* (0.5), (0.6),

$$\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z, I)}{(1 + |z|)^{2p}} \le C(p, I)(1 + \|V_0\|_\infty)^p \, (1 + \|V\|_p)^{p \frac{2p+1}{2p-1}} \, \|V\|_p^p, \tag{0.11}$$

where the positive constant $C(p, I)$ depends on p and $I = \sigma(H_0)$.

Denote $\mathbb{D}_+ = \{|z| < 1\}$, $\mathbb{D}_- = \{|z| \ge 1\}$.

Theorem 0.4. *Let* $\operatorname{Re} V \ge 0$. *Under assumptions* (0.5), (0.7) *the following Lieb–Thirring type inequality holds for each* $0 < \varepsilon < 1$:

$$\sum_{z\in\sigma_d(H)\cap\mathbb{D}_+} \frac{\mathrm{d}^p(z,I)}{|z|^{1/2-\varepsilon}} + \sum_{z\in\sigma_d(H)\cap\mathbb{D}_-} \frac{\mathrm{d}^p(z,I)}{|z|^{1/2+\varepsilon}} \le C(p,I,\varepsilon)\|V\|_p^p. \tag{0.12}$$

Remark 0.5. The only reason we restricted ourselves to the case of one dimensional Schrödinger operator H_0 as a background is that the class of multidimensional Schrödinger operators with spectra (0.2) is not well understood. Our technique works for any dimension $d \ge 1$, and the corresponding problem will be elaborated elsewhere.

1. Distortion for linear fractional transformations

The main analytic tool in the proof of Theorem 0.1 is the following distortion lemma for linear fractional transformations of the form

$$\lambda_\omega(z) := \frac{1}{z-\omega}, \quad \omega\in\mathbb{R}. \tag{1.1}$$

The argument here is quite elementary (though, rather lengthy).

Lemma 1.1. *Let*

$$I = I_z = \bigcup_{k=1}^{\infty} [a_k, b_k], \quad 0 \le a_1 < b_1 < a_2 < b_2 < \dots, \quad a_n \to +\infty, \tag{1.2}$$

and let $\lambda_\omega(I) = I_\lambda$ *be its image under the linear fractional transformation* (1.1),

$$\lambda_\omega(I) = I_\lambda = \bigcup_{k=1}^{\infty} [\beta_k(\omega), \alpha_k(\omega)], \quad \beta_k(\omega) = \frac{1}{b_k-\omega}, \quad \alpha_k(\omega) = \frac{1}{a_k-\omega}.$$

Then for $\omega < a_1$ *the following bounds hold:*
for $\operatorname{Re} z < a_1$ *or* $\operatorname{Re} z \in I$,

$$\frac{\mathrm{d}(\lambda_\omega(z),\lambda_\omega(I))}{\mathrm{d}(z,I_z)} > \frac{1}{3|z-\omega|(|z-\omega|+a_1-\omega)}; \tag{1.3}$$

for $b_k < \operatorname{Re} z < a_{k+1}$, $k = 1,2,\dots$,

$$\frac{\mathrm{d}(\lambda_\omega(z),\lambda_\omega(I))}{\mathrm{d}(z,I_z)} \ge \frac{1}{2|z-\omega|^2}\left(1+\frac{a_{k+1}-b_k}{b_k-\omega}\right)^{-1}. \tag{1.4}$$

Moreover, if $\omega \le 0$ *and the gaps are relatively bounded* (0.3), *then the unique bound is valid*

$$\frac{\mathrm{d}(\lambda_\omega(z),\lambda_\omega(I))}{\mathrm{d}(z,I_z)} \ge \frac{1}{5(1+r(I))}\,\frac{1}{|z-\omega|(|z-\omega|+a_1-\omega)}, \quad z\in\mathbb{C}\backslash I. \tag{1.5}$$

Proof. With no loss of generality we can assume that $a_1 > 0$.

We begin with the case $\omega = 0$ and put $\lambda_0 = \lambda = z^{-1}$. If $z = x+iy$ and $x = \operatorname{Re} z \le 0$, then $\operatorname{Re}\lambda = x|z|^{-2} \le 0$ and so

$$\frac{\mathrm{d}(\lambda,I_\lambda)}{\mathrm{d}(z,I_z)} = \frac{|\lambda|}{|z-a_1|} = \frac{1}{|z||z-a_1|} \ge \frac{1}{|z|(|z|+a_1)}. \tag{1.6}$$

Similarly, if $x \in I_z$, then $x \ge a_1$ and

$$0 < \operatorname{Re}\lambda = \frac{x}{|z|^2} \le \frac{1}{x} \le a_1^{-1} = \alpha_1, \quad \mathrm{d}(\lambda,[0,\alpha_1]) = |\operatorname{Im}\lambda| = \frac{|y|}{|z|^2}.$$

Since now $\mathrm{d}(z,I_z) = |y|$, we have

$$\frac{\mathrm{d}(\lambda,I_\lambda)}{\mathrm{d}(z,I_z)} \ge \frac{\mathrm{d}(\lambda,[0,\alpha_1])}{\mathrm{d}(z,I_z)} = \frac{1}{|z|^2} > \frac{1}{|z|(|z|+a_1)}. \tag{1.7}$$

Consider now the case when $x = \operatorname{Re} z \notin I_z$. Fix x in k's gap,

$$b_k < x < a_{k+1}, \quad k = k(x) = 0, 1, \dots \tag{1.8}$$

(we put $b_0 = 0$ and treat (b_0, a_1) as a number zero gap). Then

$$\mathrm{d}(z, I_z) = \min(|z - b_k|, |z - a_{k+1}|), \quad k = 1, 2, \dots, \quad \mathrm{d}(z, I_z) = |z - a_1|, \quad k = 0.$$

Define two sets of positive numbers

$$u_j = u_j(x), \quad v_j = v_j(x), \quad j = k+1, k+2, \dots$$

by the equalities

$$\operatorname{Re}(\lambda(x + iu_j)) = \frac{x}{x^2 + u_j^2} = \alpha_j, \quad \operatorname{Re}(\lambda(x + iv_j)) = \frac{x}{x^2 + v_j^2} = \beta_j,$$

or, equivalently,

$$u_j(x) = \sqrt{x(a_j - x)}, \quad v_j(x) = \sqrt{x(b_j - x)}.$$

We also put $v_k = 0$, so

$$0 = v_k < u_{k+1} < v_{k+1} < u_{k+2} < v_{k+2} < \dots, \quad u_n, v_n \to \infty, \quad n \to \infty.$$

While the point z traverses the line $x + iy$, $y \in \mathbb{R}$, its image $\lambda(z)$ describes a circle with diameter $[0, 1/x]$. We distinguish the following two cases.

Case 1. Assume that λ lies over the "gaps for λ". For each $k = 0, 1, \dots$ there are two options for λ: the interior gaps

$$\operatorname{Re}\lambda \in (\alpha_{j+1}, \beta_j) \Longleftrightarrow v_j < |y| < u_{j+1}, \quad j = k+1, k+2, \dots, \tag{1.9}$$

and the rightmost gap

$$\operatorname{Re}\lambda \in (\alpha_{k+1}, 1/x) \Longleftrightarrow 0 < |y| < u_{k+1}. \tag{1.10}$$

For gaps (1.9) we have

$$\mathrm{d}(\lambda, I_\lambda) = \min(|\lambda - \alpha_{j+1}|, |\lambda - \beta_j|) = \frac{1}{|z|} \min\left(\frac{|z - a_{j+1}|}{a_{j+1}}, \frac{|z - b_j|}{b_j}\right). \tag{1.11}$$

Define an auxiliary function h on the right half-line,

$$h(t) = h(t, z) := \frac{|z - t|}{t} = \sqrt{\left(\frac{x}{t} - 1\right)^2 + y^2}, \quad t > 0.$$

Clearly, h is monotone increasing on $(x, +\infty)$ and decreasing on $(0, x)$ with the minimum $h(x) = |y|$. Hence (1.11) and (1.8) give

$$\begin{aligned} \mathrm{d}(\lambda, I_\lambda) = \frac{\min(h(a_{j+1}, z), h(b_j, z))}{|z|} &\geq \frac{h(b_j, z)}{|z|} \geq \frac{h(b_{k+1}, z)}{|z|} \\ &\geq \frac{h(a_{k+1}, z)}{|z|} = \frac{|z - a_{k+1}|}{a_{k+1}|z|}. \end{aligned}$$

Since by (1.8) $\mathrm{d}(z, I_z) \leq |z - a_{k+1}|$, we see that

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} \geq \frac{1}{a_{k+1}|z|}. \tag{1.12}$$

For gaps (1.10) let first $k \geq 1$. Then, as above in (1.11),

$$\mathrm{d}(\lambda, I_\lambda) = \frac{1}{|z|} \min\left(\frac{|z - a_{k+1}|}{a_{k+1}}, \frac{|z - b_k|}{b_k}\right),$$

but it is not clear now which term prevails. If $|z - a_{k+1}| \leq |z - b_k|$ then $\mathrm{d}(z, I_z) = |z - a_{k+1}|$ and

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} = \frac{1}{|z|} \min\left(\frac{1}{a_{k+1}}, \frac{|z - b_k|}{b_k|z - a_{k+1}|}\right) = \frac{1}{a_{k+1}|z|}.$$

Otherwise $|z-a_{k+1}|>|z-b_k|$ implies

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}=\frac{1}{|z|}\min\left(\frac{1}{b_k}, \frac{|z-a_{k+1}|}{a_{k+1}|z-b_k|}\right)\geq\frac{1}{a_{k+1}|z|}\,.$$

Next, for $k=0$ one has $0<x<a_1$, and in case (1.10),

$$\mathrm{d}(\lambda, I_\lambda)=|\lambda-\alpha_1|=\frac{|z-a_1|}{a_1|z|}\,,\quad \mathrm{d}(z, I_z)=|z-a_1|,$$

and so

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}=\frac{1}{a_1|z|}\,. \tag{1.13}$$

Finally, in the case of "gaps for λ" we come to the bound

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}\geq\frac{1}{a_{k+1}|z|}\,,\quad k=0,1,\dots\,. \tag{1.14}$$

A modified version of (1.14) will be convenient in the sequel. For $k\geq 1$ in view of $|z|\geq x>b_k$ we have

$$\frac{1}{a_{k+1}|z|}\geq\frac{b_k}{a_{k+1}|z|^2}$$

and so, for $k=1,2,\dots$,

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}\geq\frac{1}{|z|^2}\left(1+\frac{a_{k+1}-b_k}{b_k}\right)^{-1}=\frac{1}{|z|^2}\left(1+\frac{r_k}{b_k}\right)^{-1}\,, \tag{1.15}$$

$r_k=a_{k+1}-b_k$ is the length of k's gap. Similarly, for $k=0$ one has from (1.13) that

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}\geq\frac{1}{|z|(|z|+a_1)}\,. \tag{1.16}$$

Case 2. Assume that λ lies over the "bands for λ",

$$\operatorname{Re}\lambda\in[\beta_j, \alpha_j]\iff u_j\leq|y|\leq v_j,\quad j=k+1,k+2,\dots\,. \tag{1.17}$$

Now

$$\mathrm{d}(\lambda, I_\lambda)=|\operatorname{Im}\lambda|=\frac{|y|}{|z|^2}\,,$$

$$\begin{aligned}\mathrm{d}(z, I_z)&\leq|z-a_{k+1}|\leq|y|+a_{k+1}-x=|y|+\frac{u_{k+1}^2}{x}\leq|y|\left(1+\frac{u_{k+1}}{x}\right)\\&=|y|\left(1+\sqrt{\frac{a_{k+1}-x}{x}}\right),\end{aligned}$$

so that

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}\geq\frac{1}{|z|^2}\left(1+\sqrt{\frac{a_{k+1}}{x}-1}\right)^{-1}\,. \tag{1.18}$$

For $k\geq 1$ (interior gap for z) inequality (1.18) can be simplified in view of $x>b_k$,

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}\geq\frac{1}{|z|^2}\left(1+\sqrt{\frac{r_k}{b_k}}\right)^{-1}\,. \tag{1.19}$$

Let now $k=0$, i.e., $0<x=\operatorname{Re}z<a_1$. In our case $\mathrm{d}(z, I_z)=|z-a_1|$ and

$$|y|\geq u_1=\sqrt{x(a_1-x)}.$$

If $|y|\geq 2x$ then $|y|\geq\frac{2}{3}|z|$ and so

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)}=\frac{|y|}{|z|^2|z-a_1|}\geq\frac{2}{3}\frac{1}{|z|(|z|+a_1)}\,. \tag{1.20}$$

Otherwise, $|y| < 2x$ implies

$$2\sqrt{x} > \sqrt{a_1 - x}, \quad x > \frac{a_1}{5}.$$

It follows now from (1.18) with $k = 0$ that

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} \ge \frac{1}{3|z|^2} > \frac{1}{3}\frac{1}{|z|(|z| + a_1)}\,. \tag{1.21}$$

We can summarize the results obtained above in the following two bounds from below. A combination of (1.6), (1.7), (1.16), and (1.21) gives

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} > \frac{1}{3|z|(|z| + a_1)}\,, \quad \operatorname{Re} z < a_1 \quad \text{or} \quad \operatorname{Re} z \in I_z. \tag{1.22}$$

A combination of (1.15) and (1.19) provides

$$\begin{gathered}\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} \ge \frac{1}{\gamma_k |z|^2}\,, \quad \gamma_k = \max\left\{1 + \frac{r_k}{b_k},\ 1 + \sqrt{\frac{r_k}{b_k}}\right\}, \\ b_k < \operatorname{Re} z < a_{k+1}, \quad k = 1, 2, \dots .\end{gathered} \tag{1.23}$$

Since $\gamma_k < 2(1 + r_k/b_k)$, the latter can be written as

$$\frac{\mathrm{d}(\lambda, I_\lambda)}{\mathrm{d}(z, I_z)} \ge \frac{1}{2|z|^2}\left(1 + \frac{r_k}{b_k}\right)^{-1}, \quad b_k < \operatorname{Re} z < a_{k+1}, \quad k = 1, 2, \dots . \tag{1.24}$$

To work out the general case $\omega \ne 0$ and prove (1.3) and (1.4), it remains only to shift the variable and apply the results just obtained to the shifted sequence of bands

$$I_z(\omega) = \bigcup_{k \ge 1} [a_k - \omega, b_k - \omega].$$

The final statement follows from a simple observation that

$$\frac{r_k}{b_k - \omega} \le \frac{r_k}{b_k} \le r.$$

The proof is complete. □

2. Lieb–Thirring type inequalities

The key ingredient in the proof of our main statements is the following result of Hansmann [7, Theorem 1]. Let $A_0 = A_0^*$ be a bounded self-adjoint operator on the Hilbert space, and let A be a bounded operator with $A - A_0 \in \mathcal{S}_p$, $p > 1$. Then

$$\sum_{\lambda \in \sigma_d(A)} \mathrm{d}^p(\lambda, \sigma(A_0)) \le K_p \|A - A_0\|_{\mathcal{S}_p}^p, \tag{2.1}$$

K_p is an explicit (in a sense) constant, which depends only on p. We set

$$A_0(\omega) = R(\omega, H_0) = (H_0 - \omega)^{-1}, \quad A(\omega) = R(\omega, H) = (H - \omega)^{-1},$$

ω is defined above, and $\omega \in \rho(H_0) \cap \rho(H)$ in view of (0.2) and (0.8).

Let $\lambda = \lambda_\omega(z) = (z - \omega)^{-1}$. The Spectral Mapping Theorem implies that

$$\lambda \in \sigma_d(A(\omega)) \quad (\lambda \in \sigma(A_0(\omega))) \quad \Longleftrightarrow \quad z \in \sigma_d(H) \quad (z \in \sigma(H_0))\,.$$

Proof of Theorem 0.1. The second resolvent identity reads

$$R(z, H) - R(z, H_0) = -R(z, H) M_V R(z, H_0), \quad z \in \rho(H) \cap \rho(H_0).$$

We wish to show that this difference belongs to $\mathcal{S}_p$ and to obtain the bound for its $\mathcal{S}_p$-norm.

First, we have

$$\begin{aligned} W = W(z) &:= M_V R(z, H_0) = M_V(-\Delta - z)^{-1}(-\Delta - z)(H_0 - z)^{-1} \\ &= M_V(-\Delta - z)^{-1}\,(1 - M_{V_0}(H_0 - z)^{-1}), \end{aligned} \tag{2.2}$$

and so

$$\|W(z)\|_{\mathcal{S}_p} \le \|M_V\, R(z,-\Delta)\|_{\mathcal{S}_p} \|I - M_{V_0}\, R(z,H_0)\|, \quad z \in \rho(H)\cap\rho(H_0).$$

It is clear that

$$\|I - M_{V_0}\, R(z,H_0)\| \le 1 + \frac{\|V_0\|_\infty}{\mathrm{d}(z,I)} = 1 + \frac{\|V_0\|_\infty}{|a_1 - z|}, \quad \mathrm{Re}\, z < 0.$$

Next, write

$$M_V(-\Delta - z)^{-1} = V(x) g_z(-i\nabla), \quad g_z(x) = (x^2 - z)^{-1}, \quad x \in \mathbb{R}.$$

By [11, Theorem 4.1]

$$\|M_V(-\Delta - z)^{-1}\|_{\mathcal{S}_p} \le (2\pi)^{-1/p} \|V\|_p\, \|g_z\|_p, \quad p \ge 2.$$

Since $2|t-z|^2 \ge (t+|z|)^2$ for $t \ge 0$ and $\mathrm{Re}\, z < 0$, we have

$$\|g_z\|_p \le \sqrt{2} \|g_{-|z|}\|_p$$

and so

$$\begin{gathered} \|M_V(-\Delta - z)^{-1}\|_{\mathcal{S}_p} \le \frac{C_1}{|z|^{1-1/2p}}\, \|V\|_p, \\ C_1 = C_1(p) = \sqrt{2} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^p} \right\}^{1/p}. \end{gathered} \tag{2.3}$$

Thus,

$$\|W(z)\|_{\mathcal{S}_p} \le \frac{C_1(p) \|V\|_p}{|z|^{1-1/2p}} \left(1 + \frac{\|V_0\|_\infty}{|a_1 - z|} \right), \quad \mathrm{Re}\, z < 0. \tag{2.4}$$

We put $z = \omega < \omega_1$. Relation (0.8) implies, in view of [8, Theorem V.3.2], that

$$\|R(\omega, H)\| \le \frac{1}{\mathrm{d}(\omega, \overline{N(H)})} \le \frac{1}{\omega_1 - \omega}, \tag{2.5}$$

and the combination of (2.4) and (2.5) leads to the following bound for each $\omega < \omega_1$:

$$\begin{aligned} \|R(\omega,H) - R(\omega,H_0)\|^p_{\mathcal{S}_p} &\le \|R(\omega,H)\|^p\, \|W(\omega)\|^p_{\mathcal{S}_p} \\ &\le \frac{C_2(p)\, \|V\|_p^p}{(\omega_1 - \omega)^p |\omega|^{p-1/2}} \left(1 + \frac{\|V_0\|_\infty}{a_1 + |\omega|} \right)^p. \end{aligned} \tag{2.6}$$

We go back to (2.1) with

$$A_0 = A_0(\omega) = R(\omega, H_0), \quad A = A(\omega) = R(\omega, H),$$

so by the Spectral Mapping Theorem, in the notation of Lemma 1.1, we have

$$\begin{aligned} \sum_{\lambda \in \sigma_d(A(\omega))} \mathrm{d}^p(\lambda, \sigma(A_0(\omega))) &= \sum_{z \in \sigma_d(H)} \mathrm{d}^p(\lambda_\omega(z), \lambda_\omega(I)) \\ &\le K_p\, \|R(\omega,H) - R(\omega,H_0)\|^p_{\mathcal{S}_p}. \end{aligned} \tag{2.7}$$

We apply Lemma 1.1 in the form (1.5) to obtain

$$\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z,I)}{|z-\omega|^p (|z-\omega| + a_1 + |\omega|)^p} \le \frac{C_3(p,I)\, \|V\|_p^p}{(\omega_1 - \omega)^p |\omega|^{p-1/2}} \left(1 + \frac{\|V_0\|_\infty}{a_1 + |\omega|} \right)^p,$$

and (0.9) follows. The proof is complete. □

The proof of Theorem 0.1 shows that bound (0.9) essentially depends on the parameter ω. Roughly speaking, it comes from a bound from below of $\inf \mathrm{Re}\, \sigma(H)$, and so it seems to be rather important to estimate this quantity in terms of V_0 and V only.

Proof of Theorem 0.3. Put

$$\Omega = \{\mathrm{Re}\, z < 0\} \bigcap \{|a_1 - z| > (1 + \|V_0\|_\infty)\} \bigcap \left\{ |z|^{1-1/2p} > 4C_1(p)(1 + \|V\|_p) \right\},$$

C_1 is defined in (2.3). We wish to show that $\Omega \subset \rho(H_0) \cap \rho(H)$. Indeed, by (2.4),

$$\|W(z)\|_\infty \le \|W(z)\|_{\mathcal{S}_p} \le \frac{\|V\|_p}{2(1+\|V\|_p)} < \frac{1}{2}, \quad z \in \Omega, \tag{2.8}$$

so $I + W(z)$ is invertible and $\|(I+W(z))^{-1}\| < 2$. An application of the identity $H - z = (1 + W(z))(H_0 - z)$ completes the proof of our claim.

Next, write the difference of the resolvents in another way,

$$R(z,H) - R(z,H_0) = -R(z,H_0)(1+W(z))^{-1}\, W(z),$$

to obtain for $z \in \Omega$ that

$$\begin{aligned}\|R(z,H) - R(z,H_0)\|_{\mathcal{S}_p} &\le \|R(z,H_0)\|\, \|(1+W(z))^{-1}\|\, \|W(z)\|_{\mathcal{S}_p} \\ &\le \frac{\|V\|_p}{|a_1 - z|(1+\|V\|_p)} \le \frac{\|V\|_p}{(1+\|V_0\|_\infty)(1+\|V\|_p)}.\end{aligned}$$

It is clear by the definition of Ω that if $t \in \Omega$, $t < 0$, then $\{\operatorname{Re} z \le t\} \subset \Omega$. Take $z = \omega' < 0$ so that

$$\frac{|\omega'|}{2} = \frac{a_1}{2} + 1 + \|V_0\|_\infty + (4C_1(1+\|V\|_p))^{1/(1-1/2p)}. \tag{2.9}$$

It is easy to check that $\{z : \operatorname{Re} z < \frac{\omega'}{2}\} \subset \Omega$, so, in particular, $\omega' \in \Omega$ and hence

$$\|R(\omega',H) - R(\omega',H_0)\|_{\mathcal{S}_p} \le \frac{\|V\|_p}{(1+\|V_0\|_\infty)(1+\|V\|_p)}.$$

Once again, (2.1) says

$$\sum_{\lambda \in \sigma_d(A(\omega'))} \mathrm{d}^p(\lambda, \sigma(A_0(\omega'))) \le K_p \left(\frac{\|V\|_p}{(1+\|V_0\|_\infty)(1+\|V\|_p)}\right)^p \tag{2.10}$$

for $p > 1$ and, using Lemma 1.1, we come to

$$\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z,I)}{|z-\omega'|^p(|z-\omega'| + a_1 + |\omega'|)^p} \le C_4(p,I) \left(\frac{\|V\|_p}{(1+\|V_0\|_\infty)(1+\|V\|_p)}\right)^p.$$

By the choice of ω' (2.9), we have $\operatorname{Re} z \ge \omega'/2$ for $z \in \sigma_d(H)$, and so

$$|z - \omega'| \ge \frac{|\omega'|}{2} > \frac{|a_1| + |\omega'|}{4},$$

and

$$\begin{aligned}|z-\omega'| + a_1 + |\omega'| &< 5|z-\omega'|, \\ |z-\omega'|(|z-\omega'| + a_1 + |\omega'|) &< 5|z-\omega'|^2.\end{aligned}$$

Next, $|z-\omega'| \le (1+|z|)(1+|\omega'|) \le 2|\omega'|(1+|z|)$ and hence

$$\begin{aligned}\sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z,I)}{(1+|z|)^{2p}} &\le C_5(p,I)|\omega'|^{2p} \left(\frac{\|V\|_p}{(1+\|V_0\|_\infty)(1+\|V\|_p)}\right)^p \\ &\le C_6(p,I)(1+\|V_0\|_\infty)^p(1+\|V\|_p)^{p\frac{2p+1}{2p-1}}\, \|V\|_p^p.\end{aligned}$$

The proof is complete. □

Proof of Theorem 0.4. For accretive perturbations we have $\sigma(H) \subset \{z : \operatorname{Re} z \ge 0\}$, so one can take $\omega_1 = 0$.

The lower bound for the difference of the resolvents is the same as above in Theorem 0.1. It is a consequence of the result of Hansmann and Lemma 1.1,

$$\|R(\omega,H) - R(\omega,H_0)\|_{\mathcal{S}_p}^p \ge \sum_{z \in \sigma_d(H)} \frac{\mathrm{d}^p(z,I)}{(|z-\omega| - \omega)^{2p}}, \quad p > 1.$$

As for the upper bound, we follow the line of reasoning from [6, Proof of Theorem 3.2], where such a bound was proved in the case $V_0 = 0$. As a matter of fact, the argument goes through under assumption (0.5) as well. At any rate, we have

$$\|R(-a,H) - R(-a,H_0)\|_{\mathcal{S}_p}^p \le \frac{C_5(p)}{a^{2p-1/2}}\,\|V\|_p^p\,, \quad a := -\omega > 0.$$

Since $\sqrt{2}|z+a| \ge |z| + a$ for $\operatorname{Re} z > 0$ and $a > 0$, we come to the following inequality:

$$\sum_{z\in\sigma_d(H)} \frac{\mathrm{d}^p(z,I)}{(|z|+a)^{2p}} \le \frac{C_6(p)}{a^{2p-1/2}}\,\|V\|_p^p\,, \quad a > 0. \tag{2.11}$$

As in [6, Proof of Theorem 3.3], we multiply (2.11) through by $(1+a)^{2\varepsilon}$, $\varepsilon > 0$, to obtain

$$\sum_{z\in\sigma_d(H)} \frac{\mathrm{d}^p(z,I)\,a^{2p-3/2+\varepsilon}}{(|z|+a)^{2p}(1+a)^{2\varepsilon}} \le \frac{C_6(p)\|V\|_p^p}{a^{1-\varepsilon}(1+a)^{2\varepsilon}}, \quad a > 0,$$

and then integrate the latter inequality with respect to $a \in (0,\infty)$. The proof is complete. □

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