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# COMPRESSED RESOLVENTS OF SELFADJOINT CONTRACTIVE EXIT SPACE EXTENSIONS AND HOLOMORPHIC OPERATOR-VALUED FUNCTIONS ASSOCIATED WITH THEM

YU. M. ARLINSKIĬ AND S. HASSI

*Dedicated to Yuri Makarovich Berezansky on the occasion of his 90th birthday*

Abstract. Contractive selfadjoint extensions of a Hermitian contraction $B$ in a Hilbert space $\mathfrak{H}$ with an exit in some larger Hilbert space $\mathfrak{H}\oplus\mathcal{H}$ are investigated. This leads to a new geometric approach for characterizing analytic properties of holomorphic operator-valued functions of Kreĭn-Ovcharenko type, a class of functions whose study has been recently initiated by the authors. Compressed resolvents of such exit space extensions are also investigated leading to some new connections to transfer functions of passive discrete-time systems and related classes of holomorphic operator-valued functions.

## 1. Introduction

Let $S$ be a closed symmetric, possibly nondensely defined, linear operator in a (complex separable) Hilbert space $\mathfrak{H}$. As is well known, the operator $S$ admits selfadjoint extensions possibly in a larger Hilbert space $\widetilde{\mathfrak{H}}=\mathfrak{H}\oplus\mathcal{H}$ [1], [36]. Let $\widetilde{A}$ be such an extension. Then there are two compressed resolvents $P_{\mathfrak{H}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{H}$ and $P_{\mathcal{H}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathcal{H}$. As is well known, the function $P_{\mathfrak{H}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{H}$ is called *generalized resolvent* of $S$. First results related to descriptions/parameterizations of canonical and generalized resolvents of densely defined closed symmetric operator with equal and finite deficiency indices, and their applications to the moment and interpolation problems were obtained by M. A. Naĭmark [37, 38] and M. G. Kreĭn [27, 28, 30]. Kreĭn's approach has been further developed in M. G. Kreĭn and H. Langer [31, 32], where densely defined symmetric operators in a Pontryagin space setting were considered. A. V. Shtraus in [40] suggested another approach for the investigation and parametrization of all generalized resolvents of an arbitrary symmetric, not necessary densely defined, operator. The Shtraus representation [40] for $P_{\mathfrak{H}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{H}$ takes the form

$$P_{\mathfrak{H}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{H}=(\mathcal{A}(\lambda)-\lambda I)^{-1},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}_+,$$

where $\mathcal{A}(\lambda)$ is a holomorphic family of quasi-selfadjoint extensions of $S$ ($S\subset\mathcal{A}(\lambda)\subset S^*$), $A(\lambda)$ is maximal dissipative for $\operatorname{Im}\lambda<0$, and maximal anti-dissipative for $\operatorname{Im}\lambda>0$. A recent survey on Shtraus approach, its developments, and corresponding references can be found in [45]. Extensions of symmetric linear relations and their generalized resolvents have been studied in [17, 18, 22, 34]. Furthermore, M. G. Kreĭn and I. E. Ovcharenko [33] and H. Langer and B. Textorius [35] obtained descriptions of all generalized resolvents of selfadjoint contractive extensions and contractive extensions of dual pair of contractions.

The main objective in this paper is to study compressed resolvents $P_{\mathfrak{H}}(z\widetilde{B}-I)^{-1}\upharpoonright\mathfrak{H}$ and $P_{\mathcal{H}}(z\widetilde{B}-I)^{-1}\upharpoonright\mathcal{H}$ of selfadjoint contractive extensions (*sc*-extensions) $\widetilde{B}$ (with exit

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in some larger complex separable Hilbert space $\mathfrak{H}\oplus\mathcal{H}$) of a nondensely defined Hermitian contraction $B$ in $\mathfrak{H}$ and investigate the interplay that occurs in certain associated analytic operator functions. This investigation is motivated by some further applications which involve boundary triplets, boundary relations, and the corresponding Weyl functions and Weyl families; cf. [19, 20, 21]. In this paper some new connections between compressed resolvents and transfer functions of corresponding passive selfadjoint discrete-time systems are established; see Theorems 4.1, 4.3 with a further consequence established in Theorem 4.4. There are also a couple of other new properties that complement some well-known results established in [29, 33] and are related to the shorted operators and selfadjoint contractive extensions; see Theorems 3.2 and 3.3. These results lead to a new construction of special pairs of $sc$-extensions of $B$ without exit by means of $sc$-extensions with exit with certain prescribed geometric properties. The main result in this connection is established in Theorem 3.6. The interest in studying such special pairs of $sc$-extensions of $S$ possessing certain specific geometric properties comes from the fact that they play a central role in characterizing analytic properties of Kreĭn-Ovcharenko type holomorphic operator functions which originally appeared in [33] and whose systematic study was initiated in [10].

## 2. Preliminaries

2.1. **Linear fractional transformation of sectorial operators and linear relations.** On the set of all linear relations (l.r.) in a Hilbert space $\mathfrak{H}$ define the linear fractional transformation (the Cayley transform)

$$\mathcal{C}(\mathbf{S})=\mathbf{T}=\{\langle x+x',x-x'\rangle:\langle x,x'\rangle\in\mathbf{S}\}\,. \tag{2.1}$$

Clearly, $\mathcal{C}(\mathcal{C}(\mathbf{S}))=\mathbf{S}$. Let $\mathbf{S}$ be an accretive l.r. in $\mathfrak{H}$, i.e., $\mathrm{Re}\,(x',x)\geq 0$ for all $\langle x,x'\rangle\in\mathbf{S}$; see [26, 39]. Then it follows from the identity

$$\|x+x'\|^2-\|x-x'\|^2=4\mathrm{Re}\,(x',x)$$

that $\mathbf{T}$ is the graph of a contraction $T$ in $\mathfrak{H}$, $||T||\leq 1$, and $\mathrm{dom}\,T=\mathrm{dom}\,\mathbf{T}$ is a subspace in $\mathfrak{H}$. Conversely, if $T$ is a contraction in $\mathfrak{H}$ defined on a subspace $\mathrm{dom}\,T\subseteq\mathfrak{H}$, then

$$\mathbf{S}=\{\langle(I+T)h,(I-T)h\rangle\,,\ h\in\mathrm{dom}\,T\}$$

is an accretive l.r. in $\mathfrak{H}$. The transformation $\mathcal{C}$ can be rewritten in operator form as follows

$$\mathcal{C}(\mathbf{S})=T=-I+2(I+\mathbf{S})^{-1},\quad \mathbf{S}=(I-T)(I+T)^{-1}=-I+2(I+T)^{-1}.$$

The following properties are clear from the above formulas:

- $\mathbf{S}$ is the graph of an accretive operator if and only if $\ker(I_H+T)=\{0\}$,
- $\mathbf{S}$ is $m$-accretive if and only if $\mathrm{dom}\,T=H$,
- $\mathbf{S}$ is nonnegative selfadjoint relation if and only if $T$ is a selfadjoint contraction.

In the sequel we will denote by $\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ the set of all linear bounded operators acting from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and by $\mathbf{L}(\mathfrak{H})$ the Banach algebra $\mathbf{L}(\mathfrak{H},\mathfrak{H})$.

Recall that for a contraction $T\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ the nonnegative square root $D_T=(I-T^*T)^{1/2}$ is called the defect operator of $T$ and $\mathfrak{D}_T$ (the so-called defect subspace) denotes the closure of the range $\mathrm{ran}\,D_T$. For the defect operators one has the well-known commutation relation $TD_T=D_{T^*}T$. Since

$$\begin{bmatrix}T & D_{T^*}\end{bmatrix}\begin{bmatrix}T & D_{T^*}\end{bmatrix}^*=TT^*+D_{T^*}^2=I_2,$$

one has $\mathrm{ran}\,T+\mathrm{ran}\,D_{T^*}=\mathfrak{H}_2$. In general this sum is not direct: one has

$$\mathrm{ran}\,T\cap\mathrm{ran}\,D_{T^*}=\mathrm{ran}\,TD_T=\mathrm{ran}\,D_{T^*}T, \tag{2.2}$$

as can be checked directly. It is also easily seen that

$$T(\ker D_T) = \ker D_{T^*}, \quad T^*(\ker D_{T^*}) = \ker D_T. \tag{2.3}$$

Hence, $\ker D_T = \{0\}$ if and only if $\ker D_{T^*} = \{0\}$.

**Definition 2.1.** [5]. *Let $\alpha \in (0, \pi/2)$ and let $A$ be a linear operator in the Hilbert space $H$ defined on a subspace* $\operatorname{dom} A$. *If*

$$||A \sin\alpha \pm i \cos\alpha I_H|| \le 1, \tag{2.4}$$

*then in the case* $\operatorname{dom} A = H$ *we say that $A$ belongs to the class $C_H(\alpha)$, and in the case* $\operatorname{dom} A \ne H$ *we say that $A$ is $C_H(\alpha)$-suboperator.*

The condition (2.4) is equivalent to

$$2|\mathrm{Im}\,(Af, f)| \le \tan\alpha(||f||^2 - ||Af||^2), \quad f \in \operatorname{dom} A. \tag{2.5}$$

Therefore, $C_H(\alpha)$-suboperator is a contraction. Due to (2.5) it is natural to consider Hermitian (selfadjoint) contractions in $H$ as $C_H(0)$-suboperators (operators of the class $C_H(0)$, respectively). In view of (2.5) one can write

$$C_H(0) = \bigcap_{\alpha \in (0, \pi/2)} C_H(\alpha).$$

Analogously, the convex hull $C(\alpha) = \{z \in \mathbb{C} : |z \sin\alpha \pm i \cos\alpha| < 1\}$ in the complex plane is denoted by $C(\alpha)$. If $\alpha = 0$, then the above intersection equals $C(0) = [-1, 1]$. Notice that the linear fractional transformation (2.1) establishes a one-to-one correspondence between $\alpha$-sectorial ($m - \alpha$-sectorial) l.r. (as defined in [26, 39]) in $H$ and $C_H(\alpha)$-suboperators (operators of the class $C_H(\alpha)$, respectively). In addition, $T \in C_H(\alpha)$ if and only if the operator $(I - T^*)(I + T)$ is a sectorial operator with the vertex at the origin and the semiangle $\alpha$; see [6]. Denote

$$\widetilde{C}_H := \bigcup_{\alpha \in [0, \pi/2)} C_H(\alpha).$$

Properties of operators of the class $\widetilde{C}_H$ were studied in [5, 6]. In [5] it was proved that if $T \in \widetilde{C}_H$, then

1. $\operatorname{ran}(D_{T^n}) = \operatorname{ran}(D_{T^{*n}}) = D_{T_R}$ for all natural numbers $n$, where $T_R = (T + T^*)/2$ is the real part of $T$,
2. the subspace $\mathfrak{D}_T$ reduces the operator $T$, and, moreover, the operator $T \restriction \ker(D_T)$ is a selfadjoint and unitary, and $T \restriction \mathfrak{D}_T$ is a completely nonunitary contraction of the class $C_{00}$ [44], i.e., $\lim\limits_{n\to\infty} T^n f = \lim\limits_{n\to\infty} T^{*n} f = 0$ for all $f \in \mathfrak{D}_T$.

Let $T \in \widetilde{C}_H$. Then, clearly, the operators $I_H \pm T$ are m-sectorial (bounded) operators. It follows that $I_H + T = (I_H + T_R)^{1/2}(I + iG)(I_H + T_R)^{1/2}$, where $T_R = (T + T^*)/2$ is the real part of $T$, $G$ is a bounded selfadjoint operator in the subspace $\overline{\operatorname{ran}}\,(I_H + T_R)^{1/2}$, and $I$ is the identity operator in $\overline{\operatorname{ran}}\,(I_H + T_R)^{1/2}$. Let

$$\mathbf{M} = -I + 2(I_H + T)^{-1} = \{\{(I_H + T)f, (I_H - T)f\}, \ f \in H\}.$$

Then $\mathbf{M}$ is m-sectorial linear relation, $\operatorname{dom}\mathbf{M} = \operatorname{ran}(I_H + T)$. The closed sectorial form $\mathbf{M}[u, v]$ generated by $\mathbf{M}$ can be described now explicitly.

**Proposition 2.2.** *The closed sectorial form associated with m-sectorial linear relation $\mathbf{M}$ is given by*

$$\mathbf{M}[u, v] = -(u, v) + 2\left((I + iG)^{-1}(I_H + T_R)^{-1/2}u, (I_H + T_R)^{-1/2}v\right) \tag{2.6}$$

*for all $u, v \in \mathcal{D}[\mathbf{M}] = \operatorname{ran}(I_H + T_R)^{1/2}$.*

*Proof.* Let $g = (I_H + T)f$, $g' = (I_H - T)f$. Then $\{g, g'\} \in \mathbf{M}$. With $u = g$ one gets

$$\begin{aligned}(\mathbf{M}u, u) &= (g', g) = ((I_H - T)f, (I_H + T)f)\\ &= -((I_H + T)f, (I_H + T)f) + 2(f, (I_H + T)f)\\ &= -||u||^2 + 2((I_H + T)^{-1}u, u)\\ &= -||u||^2 + 2((I_H + T_R)^{-1/2}(I + iG)^{-1}(I_H + T_R)^{-1/2}u, u)\\ &= -||u||^2 + 2((I + iG)^{-1}(I_H + T_R)^{-1/2}u, (I_H + T_R)^{-1/2}u).\end{aligned}$$

It follows that the righthand side of (2.6) coincides with $\mathbf{M}[u, v]$ for $u, v \in \operatorname{dom} \mathbf{M}$.

Let $H_0 = \overline{\operatorname{ran}}\,(I_H + T)$. Then $\operatorname{ran}\,(I_H + T_R)^{1/2}$ is dense in $H_0$. Denote

$$\tau[u, v] = -||u||^2 + 2\left((I + iG)^{-1}(I_H + T_R)^{-1/2}u, (I_H + T_R)^{-1/2}v\right)$$

with $u, v \in \operatorname{ran}\,(I_H + T_R)^{1/2}$. Clearly, the form $\tau$ is closed and sectorial (with a vertex at the point $-1$ at least). Let $u = (I_H + T_R)^{1/2}h$, $h \in H_0$, and choose a sequence $\{h_n\} \subset H_0$ such that $\lim\limits_{n\to\infty}(I_H + T_R)^{1/2}h_n = (I + iG)^{-1}h \in H_0$. Then $\varphi_n = (I_H + T)h_n \in \operatorname{dom} \mathbf{M}$ and $\lim\limits_{n\to\infty} \varphi_n = (I + T_R)^{1/2}h = u$. Moreover,

$$\begin{aligned}&\tau[u - \varphi_n]\\ &\quad= -||u - \varphi_n||^2 + 2\left((I + iG)^{-1}(I_H + T_R)^{-1/2}(u - \varphi_n), (I_H + T_R)^{-1/2}(u - \varphi_n)\right)\\ &\quad= -||u - \varphi_n||^2 + 2\left((I + iG)^{-1}h - (I_H + T_R)^{1/2}h_n, h - (I + iG)(I_H + T_R)^{1/2}h_n)\right).\end{aligned}$$

Hence $\lim\limits_{n\to\infty} \tau[u - \varphi_n] = 0$. This shows that the form $\tau$ is the closure of the form $(\mathbf{M}\cdot, \cdot)$ and this completes the proof. □

2.2. **Passive discrete-time systems and their transfer functions.** Let $\mathfrak{M}, \mathfrak{N}$, and $\mathfrak{H}$ be separable Hilbert spaces. A linear system $\tau = \left\{\begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\right\}$ with bounded linear operators $A$, $B$, $C$, $D$ of the form

$$\left\{\begin{array}{l} \sigma_k = Ch_k + D\xi_k, \\ h_{k+1} = Ah_k + B\xi_k \end{array}\right., \quad k \in \mathbb{N}_0,$$

where $\{\xi_k\} \subset \mathfrak{M}$, $\{\sigma_k\} \subset \mathfrak{N}$, $\{h_k\} \subset \mathfrak{H}$ is called a *discrete time-invariant system.* The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathfrak{H}$ is called the state space. Associated with $\tau$ is the block operator

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{array} \to \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{array}.$$

If $U$ is contractive, then the corresponding discrete-time system is said to be *passive* [16]. If $U$ is unitary, then the system is called conservative. The *transfer function*

$$\Theta_\tau(\lambda) := D + zC(I_{\mathfrak{H}} - zA)^{-1}B, \quad z \in \mathbb{D},$$

of a passive system $\tau$ belongs to the *Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ [16]. Recall that the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is the set of all holomorphic and contractive $\mathbf{L}(\mathfrak{M}, \mathfrak{N})$-valued functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Define the following subsets of the complex plane

$$\begin{aligned}&\Pi_+(\alpha) := \{z \in \mathbb{C} : |z \sin\alpha + i\cos\alpha| < 1\},\ \Pi_-(\alpha) := \{z \in \mathbb{C} : |z \sin\alpha - i\cos\alpha| < 1\},\\ &\Pi(\alpha) := \Pi_+(\alpha) \cup \Pi_-(\alpha).\end{aligned}$$

Then, in particular $\Pi(0) = \mathbb{C} \setminus ((-\infty, 1] \cup [1, +\infty))$ and $C(\alpha) = \Pi_+(\alpha) \cap \Pi_-(\alpha)$.

**Theorem 2.3.** [6]. *Suppose that* $\mathfrak{N} = \mathfrak{M}$ *and that the operator*

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{array} \to \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{array}$$

*belongs to class* $C_{\mathfrak{N}\oplus\mathfrak{H}}(\alpha)$ *for some* $\alpha \in [0, \pi/2)$. *Then the function* $\Theta_\tau$ *possesses the following properties:*

(1) $\Theta_\tau$ *is holomorphic in* $\Pi(\alpha)$*;*
(2) *there exist strong non-tangential limits* $\Theta_\tau(\pm 1)$ *and* $\Theta_\tau(\pm 1) \in C_{\mathfrak{N}}(\alpha)$*;*
(3) *the implications*

$$\begin{array}{l} z \in \Pi_+(\alpha) \Longrightarrow ||\Theta_\tau(z) \sin\alpha + i \cos\alpha \, I_{\mathfrak{N}}|| \le 1, \\ z \in \Pi_-(\alpha) \Longrightarrow ||\Theta_\tau(z) \sin\alpha - i \cos\alpha \, I_{\mathfrak{N}}|| \le 1 \end{array}$$

*are valid. Therefore,* $z \in C(\beta) \Longrightarrow \Theta_\tau(z) \in C_{\mathfrak{N}}(\beta)$ *for each* $\beta \in [\alpha, \pi/2)$.

A particular case is *self-adjoint passive system*, i.e., the case when $\alpha = 0 \iff$ the operator $U = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ is a self-adjoint contraction in $\mathfrak{N} \oplus \mathfrak{H}$.

A more general class of passive systems is formed by *passive quasi-selfadjoint systems* (*pqs*-systems for short). The passive system

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{N}, \mathfrak{N}, \mathfrak{H} \right\}$$

is called a *pqs*-system if the operator $U = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ is a *quasi-selfadjoint contraction* (*qsc*-operator for short), i.e., $U$ is a contraction and $\operatorname{ran}(U - U^*) \subset \mathfrak{N} \times \{0\}$, cf. [11]. This last condition alone is equivalent to $A = A^*$ and $C = B^*$; for contractivity of $U$ see Theorem 2.4 below. If $\tau$ is a *pqs*-system, then the transfer function of $\tau$ takes the form

$$\Theta_\tau(z) = W(z) + D,$$

where the function $W(z)$ belongs to the class $\mathbf{N}(\mathfrak{N})$ of Herglotz-Nevanlinna functions and it is defined on the domain $\operatorname{Ext}\{(-\infty, -1] \cup [1, \infty)\}$. The class $\mathbf{S}^{qs}(\mathfrak{N})$ is the class of all transfer functions of *pqs*-systems $\tau = \{U; \mathfrak{N}, \mathfrak{N}, \mathfrak{H}\}$. A complete description of the class $\mathbf{S}^{qs}(\mathfrak{N})$ is given in [13]. Denote by $\mathbf{S}^s(\mathfrak{N})$ the subset of Herglotz-Nevanlinna functions from the class of $\mathbf{S}^{qs}(\mathfrak{N})$. Clearly,

$$\Theta(z) \in \mathbf{S}^s(\mathfrak{N}) \iff \left\{ \begin{array}{l} \Theta(z) \in \mathbf{S}^{qs}(\mathfrak{N}), \\ \Theta(0) = \Theta^*(0) \end{array} \right. .$$

The following equivalent statements for $\mathbf{L}(\mathfrak{N})$-valued Herglotz-Nevanlinna function $\Theta$, holomorphic in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, can be derived with the aid of the integral representation of $\Theta$; see also [33, Theorem 4.2]:

(1) $\Theta \in \mathbf{S}^s(\mathfrak{N})$;
(2) $\Theta(x)$ is selfadjoint contraction for each $x \in (-1, 1)$;
(3) $\Theta$ is the transfer function of a passive selfadjoint discrete-time system

$$\tau = \left\{ \begin{bmatrix} D & B \\ B^* & A \end{bmatrix}; \mathfrak{N}, \mathfrak{N}, \mathfrak{H} \right\}.$$

2.3. **The Schur-Frobenius formula for the resolvent.** Let

$$\mathcal{U} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{array}$$

be a bounded block operator. Then an applications of the Schur-Frobenius formula gives the following formula for the resolvent $R_{\mathcal{U}}(\lambda) = (\mathcal{U} - \lambda I)^{-1}$ of $\mathcal{U}$:

$$R_{\mathcal{U}}(\lambda)=\begin{bmatrix} -V^{-1}(\lambda) & V^{-1}(\lambda)CR_A(\lambda) \\ R_A(\lambda)BV^{-1}(\lambda) & R_A(\lambda)\left(I_{\mathcal{H}} - BV^{-1}(\lambda)CR_A(\lambda)\right) \end{bmatrix}, \ \lambda\in\rho(\mathcal{U})\cap\rho(A), \tag{2.7}$$

where

$$V(\lambda) := \lambda I_{\mathfrak{M}} - D + CR_A(\lambda)B, \quad \lambda \in \rho(A). \tag{2.8}$$

Moreover, $\lambda \in \rho(\mathcal{U}) \cap \rho(A) \iff V^{-1}(\lambda) \in \mathbf{L}(\mathfrak{M})$. In particular, (2.7) and (2.8) imply

$$(P_{\mathfrak{M}} R_U(\lambda)\upharpoonright \mathfrak{M})^{-1} = D - CR_A(\lambda)B - \lambda I_{\mathfrak{M}}. \tag{2.9}$$

2.4. **Kreĭn shorted operators.** For every bounded nonnegative operator $\mathcal{S}$ in the Hilbert space $\mathcal{H}$ and every subspace $\mathcal{K} \subset \mathcal{H}$ M.G. Kreĭn [29] defined the operator $\mathcal{S}_{\mathcal{K}}$ by the relation

$$\mathcal{S}_{\mathcal{K}} = \max\left\{ \mathcal{Z} \in \mathbf{L}(\mathcal{H}) : 0 \le \mathcal{Z} \le \mathcal{S}, \operatorname{ran}\mathcal{Z} \subseteq \mathcal{K} \right\}.$$

An equivalent description is

$$(\mathcal{S}_{\mathcal{K}} f, f) = \inf_{\varphi\in\mathcal{K}^\perp} \left\{(\mathcal{S}(f+\varphi), f+\varphi)\right\}, \quad f \in \mathcal{H}, \tag{2.10}$$

where $\mathcal{K}^\perp := \mathcal{H} \ominus \mathcal{K}$. The properties of $\mathcal{S}_{\mathcal{K}}$, have been studied by M.G. Kreĭn and by other authors (see [7] and references therein): in [2, 4] $\mathcal{S}_{\mathcal{K}}$ is called a *shorted operator*. The following representation of $\mathcal{S}_{\mathcal{K}}$ was also established in [29]:

$$\mathcal{S}_{\mathcal{K}} = \mathcal{S}^{1/2} P_\Omega \mathcal{S}^{1/2},$$

where $P_\Omega$ is the orthogonal projection in $\mathcal{H}$ onto $\Omega = \{ f \in \overline{\operatorname{ran}}\,\mathcal{S} : \mathcal{S}^{1/2} f \in \mathcal{K} \} = \overline{\operatorname{ran}}\,\mathcal{S} \ominus \mathcal{S}^{1/2}\mathcal{K}^\perp$. Moreover, it was shown in [29] that

$$\operatorname{ran}\mathcal{S}_{\mathcal{K}}^{1/2} = \operatorname{ran}(\mathcal{S}^{1/2}P_\Omega) = \mathcal{K} \cap \operatorname{ran}\mathcal{S}^{1/2}. \tag{2.11}$$

Hence,

$$\mathcal{S}_{\mathcal{K}} = 0 \iff \operatorname{ran}\mathcal{S}^{1/2} \cap \mathcal{K} = \{0\}. \tag{2.12}$$

As a bounded selfadjoint operator $\mathcal{S}$ admits the block operator representation

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \mathcal{S}_{12}^* & \mathcal{S}_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^\perp \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^\perp \end{matrix}.$$

It is well known (see [25, 33, 42]) that the operator $\mathcal{S}$ is nonnegative if and only if

$$\mathcal{S}_{22} \ge 0,\ \operatorname{ran}\mathcal{S}_{12}^* \subset \operatorname{ran}\mathcal{S}_{22}^{1/2},\ \mathcal{S}_{11} \ge \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right)^* \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right)$$

and the operator $\mathcal{S}_{\mathcal{K}}$ can be expressed in the block operator form

$$\mathcal{S}_{\mathcal{K}} = \begin{bmatrix} \mathcal{S}_{11} - \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right)^* \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.13}$$

where $\mathcal{S}_{22}^{-1/2}$ is the Moore-Penrose pseudo-inverse of $\mathcal{S}_{22}$. If $\mathcal{S}_{22}^{-1} \in \mathbf{L}(\mathcal{K}^\perp)$ then

$$\mathcal{S}_{\mathcal{K}} = \begin{bmatrix} \mathcal{S}_{11} - \mathcal{S}_{12}\mathcal{S}_{22}^{-1}\mathcal{S}_{12}^* & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mathcal{S}_{11} - \mathcal{S}_{12}\mathcal{S}_{22}^{-1}\mathcal{S}_{12}^*$ is called a *Schur complement* of $\mathcal{S}$. From (2.13) it follows that

$$\mathcal{S}_{\mathcal{K}} = 0 \iff \operatorname{ran}\mathcal{S}_{12}^* \subset \operatorname{ran}\mathcal{S}_{22}^{1/2} \quad \text{and} \quad \mathcal{S}_{11} = \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right)^* \left(\mathcal{S}_{22}^{-1/2}\mathcal{S}_{12}^*\right).$$

2.5. **Selfadjoint and quasi-selfadjoint contractive extensions of a nondensely defined Hermitian contraction.** Let $B$ be a closed nondensely defined Hermitian contraction in the Hilbert space $\mathfrak{H}$. Denote

$$\mathfrak{H}_0 := \operatorname{dom} B, \quad \mathfrak{N} := \mathfrak{H} \ominus \mathfrak{H}_0.$$

A description of all selfadjoint contractive extensions ($sc$-extensions [33]) of $B$ in $\mathfrak{H}$ was given by M. G. Kreĭn [29]. In fact, he showed that all $sc$-extensions of $B$ form an operator interval $[B_\mu, B_M]$, where the extensions $B_\mu$ and $B_M$ can be characterized by

$$(I + B_\mu)_{\mathfrak{N}} = 0, \quad (I - B_M)_{\mathfrak{N}} = 0 \tag{2.14}$$

respectively. The operator $B$ admits a unique $sc$-extension if and only if

$$\sup_{\varphi \in \operatorname{dom} B} \frac{|(B\varphi, h)|^2}{||\varphi||^2 - ||B\varphi||^2} = \infty$$

for all $h \in \mathfrak{N} \setminus \{0\}$.

The operator interval $[B_\mu, B_M]$ can be described as follows (cf. [29, 33]):

$$\widehat{B} = (B_M + B_\mu)/2 + (B_M - B_\mu)^{1/2} Y (B_M - B_\mu)^{1/2}/2, \tag{2.15}$$

where $Y = Y^*$ is a contraction in the subspace $\overline{\operatorname{ran}}\,(B_M - B_\mu) \subseteq \mathfrak{N}$. It follows from (2.14), for instance, that for every $sc$-extension $\widehat{B}$ of $B$ the following identities hold:

$$(I - \widehat{B})_{\mathfrak{N}} = B_M - \widehat{B}, \quad (I + \widehat{B})_{\mathfrak{N}} = \widehat{B} - B_\mu, \tag{2.16}$$

cf. [29]. Hence, according to (2.11)

$$\begin{aligned} \operatorname{ran}\,(I - \widehat{B})^{1/2} \cap \mathfrak{N} &= \operatorname{ran}\,(B_M - \widehat{B})^{1/2}, \\ \operatorname{ran}\,(I + \widehat{B})^{1/2} \cap \mathfrak{N} &= \operatorname{ran}\,(\widehat{B} - B_\mu)^{1/2}. \end{aligned}$$

Let $P_{\mathfrak{H}_0}$ and $P_{\mathfrak{N}}$ be the orthogonal projections in $\mathfrak{H}$ onto $\mathfrak{H}_0$ and $\mathfrak{N}$, respectively. Then the operator $B_0 = P_{\mathfrak{H}_0} B$ is contractive and self-adjoint in the subspace $\mathfrak{H}_0$. Let $D_{B_0} = (I - B_0^2)^{1/2}$ be the defect operator determined by $B_0$. The operator $B_{21} = P_{\mathfrak{N}} B$ is also contractive. Moreover, it follows from $B^* B \le I$ that $B_{21}^* B_{21} \le D_{B_0}^2$. Therefore, the identity

$$K_0 D_{B_0} f = P_{\mathfrak{N}} B f, \quad f \in \operatorname{dom} B = \mathfrak{H}_0,$$

defines a contractive operator $K_0$ from $\mathfrak{D}_{B_0} := \overline{\operatorname{ran}}\,(D_{B_0})$ into $\mathfrak{N}$, cf. [23, 24]. This gives the following decomposition for the Hermitian contraction $B$

$$B = B_0 + K_0 D_{B_0} = \begin{bmatrix} B_0 \\ K_0 D_{B_0} \end{bmatrix} : \mathfrak{H}_0 \to \mathfrak{H}. \tag{2.17}$$

An extension $\widehat{B}$ of $B$ in $\mathfrak{H}$ is called *quasi-selfadjoint* if also $\widehat{B}^*$ is an extension of $B$ and $\widehat{B}$ is said to be a *quasi-selfadjoint contractive* extension of $B$ ($qsc$-extension for short) if $\operatorname{dom} \widehat{B} = \mathfrak{H}$, $||\widehat{B}|| \le 1$, and $\ker(\widehat{B} - \widehat{B}^*) \supseteq \operatorname{dom} B = \mathfrak{H}_0$; cf. [14, 15].

For a proof of the following result and some history behind the well-known formula therein; see [8, Theorem 9.2.3], [12, Theorem 4.1], [25, Corollary 3.5].

**Theorem 2.4.** *Let $B$ be a Hermitian contraction in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{N}$ with $\operatorname{dom} B = \mathfrak{H}_0$ and decompose $B$ as in (2.17). Then the formula*

$$\widehat{B} = \begin{bmatrix} B_0 & D_{B_0} K_0^* \\ K_0 D_{B_0} & -K_0 B_0 K_0^* + D_{K_0^*} X D_{K_0^*} \end{bmatrix} : \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \end{matrix} \to \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \end{matrix} \tag{2.18}$$

*gives a one-to-one correspondence between all qsc-extensions $\widehat{B}$ of the Hermitian contraction $B = B_0 + K_0 D_{B_0}$ and all contractions $X$ in the subspace $\mathfrak{D}_{K_0^*} := \overline{\operatorname{ran}}\,(D_{K_0^*}) \subseteq \mathfrak{N}$. Furthermore, the following statements hold:*

(i) *$B$ has a unique sc-extension if and only if $K_0^*$ is an isometry ($\mathfrak{D}_{K_0^*} = \{0\}$);*

(ii) *if* $\mathfrak{D}_{K_0^*} \neq \{0\}$, *then the following equivalences hold*

$$\ker D_{K_0^*} = \{0\} \iff \ker(B_M - B_\mu) = \mathfrak{H}_0;$$

(iii) *if* $\mathfrak{D}_{K_0^*} \neq \{0\}$, *then the following equivalences hold*

$$\operatorname{ran} D_{K_0^*} = \mathfrak{N} \iff \operatorname{ran}(B_M - B_\mu) = \mathfrak{N}.$$

*Moreover,* $\widehat{B} \in C_{\mathfrak{H}}(\alpha)$, $\alpha \in [0, \pi/2)$, *if and only if* $X \in C_{\mathfrak{D}_{K_0^*}}(\alpha)$.

From (2.18) it follows that

$$B_\mu = \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0B_0K_0^* - D^2_{K_0^*} \end{bmatrix}, \quad B_M = \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0B_0K_0^* + D^2_{K_0^*} \end{bmatrix} \tag{2.19}$$

with $X = -I\restriction \mathfrak{D}_{K_0^*}$ and $X = I\restriction \mathfrak{D}_{K_0^*}$, respectively. From (2.19) it is seen that

$$\frac{B_\mu + B_M}{2} = \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0B_0K_0^* \end{bmatrix}, \quad \frac{B_M - B_\mu}{2} = \begin{bmatrix} 0 & 0 \\ 0 & D^2_{K_0^*} \end{bmatrix}.$$

Finally, we mention the following implications

$$\begin{array}{ll} X \in \mathbf{L}(\mathfrak{D}_{K_0^*}), & ||X\sin\alpha + i\cos\alpha|| \le 1 \Longrightarrow ||\widehat{B}\sin\alpha + i\cos\alpha|| \le 1, \\ X \in \mathbf{L}(\mathfrak{D}_{K_0^*}), & ||X\sin\alpha - i\cos\alpha|| \le 1 \Longrightarrow ||\widehat{B}\sin\alpha - i\cos\alpha|| \le 1, \end{array} \tag{2.20}$$

where $\widehat{B}$ is given by (2.18).

**Remark 2.5.** *Let $X$ be a selfadjoint contraction in the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$. From Theorem 2.4 one can derive the following two block representations for $X$:*

$$\begin{aligned} X &= \begin{bmatrix} X_{11} & D_{X_{11}}L^* \\ LD_{X_{11}} & -LX_{11}L^* + D_{L^*}YD_{L^*} \end{bmatrix} \\ &= \begin{bmatrix} -UX_{22}U^* + D_{U^*}VD_{U^*} & UD_{X_{22}} \\ D_{X_{22}}U^* & X_{22} \end{bmatrix} : \begin{array}{c} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{array} \to \begin{array}{c} \mathcal{H}_1 \\ \oplus \\ \mathcal{H}_2 \end{array}, \end{aligned}$$

*where $L \in \mathbf{L}(\mathfrak{D}_{X_{11}}, \mathcal{H}_2)$ and $U \in \mathbf{L}(\mathfrak{D}_{X_{22}}, \mathcal{H}_1)$ are contractions and $Y \in \mathbf{L}(\mathfrak{D}_{L^*})$ and $V \in \mathbf{L}(\mathfrak{D}_{U^*})$ are selfadjoint contractions. From* (2.14), (2.16), *and* (2.19) *we get*

$$\begin{aligned} &(I + X)_{\mathcal{H}_2} = D_{L^*}(I + Y)D_{L^*}P_{\mathcal{H}_2}, \quad (I - X)_{\mathcal{H}_2} = D_{L^*}(I - Y)D_{L^*}P_{\mathcal{H}_2}, \\ &(I + X)_{\mathcal{H}_1} = D_{U^*}(I + V)D_{U^*}P_{\mathcal{H}_1}, \quad (I - X)_{\mathcal{H}_1} = D_{U^*}(I - V)D_{U^*}P_{\mathcal{H}_1}, \\ &(I - X)_{\mathcal{H}_1} = (I + X)_{\mathcal{H}_1} = 0 \iff UU^* = I_{\mathcal{H}_1} \Longrightarrow X = \begin{bmatrix} -UX_{11}U^* & UD_{X_{22}} \\ D_{X_{22}}U^* & X_{22} \end{bmatrix}, \\ &(I - X)_{\mathcal{H}_2} = (I + X)_{\mathcal{H}_2} = 0 \iff LL^* = I_{\mathcal{H}_2} \Longrightarrow X = \begin{bmatrix} X_{11} & D_{X_{11}}L^* \\ LD_{X_{11}} & -LX_{11}L^* \end{bmatrix}. \end{aligned}$$

*In addition, for the defect operators the following identities hold (cf.* [12, Theorem 4.1]*):*

$$\begin{aligned} \left\| D_X \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|^2 &= \|D_L\left(D_{X_{11}}h_1 - X_{11}L^*h_2\right) - L^*YD_{L^*}h_2\|^2 + \|D_YD_{L^*}h_2\|^2 \\ &= \|D_U\left(D_{X_{22}}h_2 - X_{22}U^*h_1\right) - U^*VD_{U^*}h_1\|^2 + \|D_VD_{U^*}h_1\|^2. \end{aligned}$$

2.6. **Special pairs of selfadjoint contractive extensions and corresponding $Q$-functions.** The so-called $Q_\mu$ and $Q_M$-functions of a Hermitian contraction $B$ of the form

$$\begin{aligned} Q_\mu(\xi) &= \left(I_{\mathfrak{N}} + (B_M - B_\mu)^{1/2}\left(B_\mu - \xi I_{\mathfrak{H}}\right)^{-1}(B_M - B_\mu)^{1/2}\right)\restriction \mathfrak{N}, \\ Q_M(\xi) &= \left(-I_{\mathfrak{N}} + (B_M - B_\mu)^{1/2}\left(B_M - \xi I_{\mathfrak{H}}\right)^{-1}(B_M - B_\mu)^{1/2}\right)\restriction \mathfrak{N}, \quad \xi \in \mathbb{C} \setminus [-1, 1], \end{aligned}$$

were introduced and studied in [33]. These functions belong to the Herglotz-Nevanlinna class and they are connected to each other via

$$Q_\mu(\xi)Q_M(\xi) = -I_{\mathfrak{N}}, \quad \xi \in \mathbb{C} \setminus [-1,1].$$

They possess the following further properties:

$$s-\lim_{\xi\to\infty} Q_\mu(\xi) = I, \quad \lim_{\xi\uparrow -1}(Q_\mu(\xi)h,h) = +\infty \forall h \in \mathfrak{N}\setminus\{0\}, \quad s-\lim_{\xi\downarrow 1} Q_\mu(\xi) = 0,$$
$$s-\lim_{\xi\to\infty} Q_M(\xi) = -I_{\mathfrak{N}}, \quad \lim_{\xi\downarrow 1}(Q_M(\xi)h,h) = -\infty \forall h \in \mathfrak{N}\setminus\{0\}, \quad s-\lim_{\xi\uparrow -1} Q_M(\xi) = 0.$$

The following resolvent formula has been established in [33].

**Theorem 2.6.** *Let $C = B_M - B_\mu$. The formula*

$$\widetilde{R}_\xi = (B_\mu - \xi I)^{-1} - (B_\mu - \xi I)^{-1} C^{1/2} K(\xi)\left(I + (Q_\mu(\xi) - I)K(\xi)\right)^{-1} C^{1/2}(B_\mu - \xi I)^{-1}$$

*gives a bijective correspondence between the generalized resolvents $\widetilde{R}_\xi = P_{\mathfrak{H}}(\widetilde{B} - \xi I)^{-1}\upharpoonright\mathfrak{H}$ of sc-extensions $\widetilde{B}$ of $B$ with exit and the $\mathbf{L}(\mathfrak{N})$-valued operator functions $K(\xi)$ holomorphic on* $\operatorname{Ext}[-1,1]$ *and possessing the following two further properties:*

*1) $-K(\xi)$ is a Herglotz-Nevanlinna function,*

*2) $K(\xi)$ is a nonnegative selfadjoint contraction for every $\xi \in \mathbb{R}\setminus[-1,1]$.*

*Here canonical resolvents correspond to constant functions $K(\xi) = K$ and vice versa.*

A further study of functions of Kreĭn-Ovcharenko type was initiated in [10]. Given an arbitrary pair $\{\widehat{B}_0, \widehat{B}_1\}$ of *sc*-extensions of $B$ in $\mathfrak{H}$ satisfying the condition $\widehat{B}_0 \le \widehat{B}_1$, define a pair of Herglotz-Nevanlinna functions via

$$\widehat{Q}_0(\xi) = \left[(\widehat{B}_1 - \widehat{B}_0)^{1/2}(\widehat{B}_0 - \xi I)^{-1}(\widehat{B}_1 - \widehat{B}_0)^{1/2} + I\right]\upharpoonright\mathfrak{N}, \tag{2.21}$$

$$\widehat{Q}_1(\xi) = \left[(\widehat{B}_1 - \widehat{B}_0)^{1/2}(\widehat{B}_1 - \xi I)^{-1}(\widehat{B}_1 - \widehat{B}_0)^{1/2} - I\right]\upharpoonright\mathfrak{N}, \quad \xi \in \operatorname{Ext}[-1,1]. \tag{2.22}$$

It is easy to verify that $\widehat{Q}_0(\xi)\widehat{Q}_1(\xi) = \widehat{Q}_1(\xi)\widehat{Q}_0(\xi) = -I_{\mathfrak{N}}$, $\xi \in \operatorname{Ext}[-1,1]$. Now proceed by introducing the classes of Kreĭn-Ovcharenko type Herglotz-Nevanlinna functions.

**Definition 2.7.** [10]. *Let $\mathfrak{N}$ be a Hilbert space. An $\mathbf{L}(\mathfrak{N})$-valued function $\widehat{Q}(\xi)$ is said to belong to the subclass $\mathfrak{S}_\mu(\mathfrak{N})$ (respectively, $\mathfrak{S}_M(\mathfrak{N})$) of Herglotz-Nevanlinna operator functions if it is holomorphic on* $\operatorname{Ext}[-1,1]$ *and, in addition, has the following properties:*

1) $s-\lim_{\xi\to\infty}\widehat{Q}(\xi) = I$ *(respectively, $s-\lim_{\xi\to\infty}\widehat{Q}(\xi) = -I$);*

2) $\lim_{\xi\uparrow -1}(\widehat{Q}(\xi)h,h) = +\infty$ *for all $h \in \mathfrak{N}\setminus\{0\}$ (respectively, $s-\lim_{\xi\uparrow -1}\widehat{Q}(\xi) = 0$);*

3) $s-\lim_{\xi\downarrow 1}\widehat{Q}(\xi) = 0$ *(respectively, $\lim_{\xi\downarrow 1}(\widehat{Q}(\xi)h,h) = -\infty$ for all $h \in \mathfrak{N}\setminus\{0\}$).*

The function $Q_\mu$ belongs $\mathfrak{S}_\mu(\mathfrak{N})$ while $Q_M$ is of the class $\mathfrak{S}_M(\mathfrak{N})$. It is stated in [33] that if the function $\widehat{Q}$ belongs to $\mathfrak{S}_\mu(\mathfrak{N})$ (respectively, $\widehat{Q} \in \mathfrak{S}_M(\mathfrak{N})$), then it is a $Q_\mu$-function (respectively, $Q_M$-function) of some nondensely defined Hermitian contraction $B$. However, it is shown in [10] that this statements is true only when $\dim\mathfrak{N} < \infty$.

**Theorem 2.8.** [10]. *Assume that $\widehat{Q} \in \mathfrak{S}_\mu(\mathfrak{N})$ ($\widehat{Q} \in \mathfrak{S}_M(\mathfrak{N})$). Then there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{N}$ as a subspace, a Hermitian contraction $B$ in $\mathfrak{H}$ defined on* $\operatorname{dom} B = \mathfrak{H}\ominus\mathfrak{N}$, *and a pair $\{\widehat{B}_0, \widehat{B}_1\}$ of sc-extensions of $B$, satisfying $\widehat{B}_0 \le \widehat{B}_1$,* $\ker(\widehat{B}_1 - \widehat{B}_0) = \operatorname{dom} B$, *such that $\widehat{Q}(\xi)$ admits the representation in the form* (2.21) *(in the form* (2.22)*, respectively). Moreover, the pair $\{\widehat{B}_0, \widehat{B}_1\}$ possesses the following properties:*

$$\operatorname{ran}(\widehat{B}_1 - \widehat{B}_0)^{1/2} \cap \operatorname{ran}(\widehat{B}_0 - B_\mu)^{1/2} = \operatorname{ran}(\widehat{B}_1 - \widehat{B}_0)^{1/2} \cap \operatorname{ran}(B_M - \widehat{B}_1)^{1/2} = \{0\}, \tag{2.23}$$

*If* $\dim\mathfrak{N} < \infty$, *then necessarily $\widehat{B}_0 = B_\mu$ and $\widehat{B}_1 = B_M$.*

In particular, in the case that $\dim\mathfrak{N}=\infty$ [10] (see also [9]) contains a construction of pairs $\{\widehat{B}_0,\widehat{B}_1\}$ of *sc*-extensions which differ from $\{B_\mu,B_M\}$ and satisfy the conditions in (2.23): in other words, the corresponding $Q$-functions given by (2.21) and (2.22) belong to $\mathfrak{S}_\mu(\mathfrak{N})$ and $\mathfrak{S}_M(\mathfrak{N})$, respectively, but they do not coincide with the $Q_\mu$- and $Q_M$-functions of $B$.

To finish this section the following simple observation is mentioned: if $V$ is an isometry in $\mathfrak{N}$ and $\widehat{B}_0\le\widehat{B}_1$ are *sc*-extensions, then the operator-valued functions

$$\begin{aligned}\widehat{Q}_0(\xi)&:=\left(I_{\mathfrak{N}}+V(\widehat{B}_1-\widehat{B}_0)^{1/2}\left(\widehat{B}_0-\xi I_{\mathfrak{H}}\right)^{-1}(\widehat{B}_1-\widehat{B}_0)^{1/2}V^*\right)\restriction\mathfrak{N},\\ \widehat{Q}_1(\xi)&:=\left(-I_{\mathfrak{N}}+V(\widehat{B}_1-\widehat{B}_0)^{1/2}\left(\widehat{B}_1-\xi I_{\mathfrak{H}}\right)^{-1}(\widehat{B}_1-\widehat{B}_0)^{1/2}V^*\right)\restriction\mathfrak{N},\ \xi\in\operatorname{Ext}[-1,1],\end{aligned}$$

belong to the Herglotz-Nevanlinna class and $\widehat{Q}_1^{-1}(\xi)=-\widehat{Q}_0(\xi)$, $\xi\in\operatorname{Ext}[-1,1]$.

**Remark 2.9.** *If $F$ and $G$ are bounded nonnegative selfadjoint operators, then the parallel sum $F:G$ can be defined* [3], [24]. *The conditions $F:G=0$ and $\operatorname{ran}F^{1/2}\cap\operatorname{ran}G^{1/2}=\{0\}$ are equivalent.*

## 3. Selfadjoint contractive extensions of nondensely defined Hermitian contractions with exit

Let $B$ be a nondensely defined Hermitian contraction in the Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be an auxiliary Hilbert space. If $B$ is given by (2.17), then all *qsc*-extensions of $B$ in the extended Hilbert space $\mathfrak{H}\oplus\mathcal{H}$ can be described as follows. Let

$$\widehat{\mathcal{H}}=\mathfrak{N}\oplus\mathcal{H},$$

let $j_{\widehat{\mathcal{H}}}$ be the canonical embedding operator $\mathfrak{N}\to\widehat{\mathcal{H}}$, and define $\widehat{K}_0=j_{\widehat{\mathcal{H}}}K_0$. Then

$$D_{\widehat{K}_0^*}=(I_{\widehat{\mathcal{H}}}-\widehat{K}_0\widehat{K}_0^*)^{1/2}=\begin{bmatrix}D_{K_0^*}&0\\0&I_{\mathcal{H}}\end{bmatrix}:\begin{array}{c}\mathfrak{N}\\\oplus\\\mathcal{H}\end{array}\to\begin{array}{c}\mathfrak{N}\\\oplus\\\mathcal{H}\end{array}.$$

Clearly, $\mathfrak{D}_{\widehat{K}_0^*}=\mathfrak{D}_{K_0^*}\oplus\mathcal{H}\subset\widehat{\mathcal{H}}$. In what follows we identify $B$ with its image in $\mathfrak{H}_0\oplus\widehat{\mathcal{H}}$. By Theorem 2.4 a *qsc*-extension $\widetilde{B}$ of $B$ in $\mathfrak{H}\oplus\mathcal{H}$ with respect to the decomposition $\mathfrak{H}\oplus\mathcal{H}=\mathfrak{H}_0\oplus\widehat{\mathcal{H}}$ takes the block form

$$\widetilde{B}=\widetilde{B}_X=\begin{bmatrix}B_0&D_{B_0}\widehat{K}_0^*\\\widehat{K}_0D_{B_0}&-\widehat{K}_0B_0\widehat{K}_0^*+D_{\widehat{K}_0^*}XD_{\widehat{K}_0^*}\end{bmatrix}:\begin{array}{c}\mathfrak{H}_0\\\oplus\\\widehat{\mathcal{H}}\end{array}\to\begin{array}{c}\mathfrak{H}_0\\\oplus\\\widehat{\mathcal{H}}\end{array},$$

where $X:\mathfrak{D}_{\widehat{K}_0^*}\to\mathfrak{D}_{\widehat{K}_0^*}$ is a contraction. Let

$$X=\begin{bmatrix}X_{11}&X_{12}\\X_{21}&X_{22}\end{bmatrix}:\begin{array}{c}\mathfrak{D}_{K_0^*}\\\oplus\\\mathcal{H}\end{array}\to\begin{array}{c}\mathfrak{D}_{K_0^*}\\\oplus\\\mathcal{H}\end{array}\tag{3.1}$$

be the block representation of the operator $X$. Then

$$\widetilde{B}=\begin{bmatrix}B_0&D_{B_0}K_0^*&0\\K_0D_{B_0}&-K_0B_0K_0^*+D_{K_0^*}X_{11}D_{K_0^*}&D_{K_0^*}X_{12}\\0&X_{21}D_{K_0^*}&X_{22}\end{bmatrix}:\begin{array}{c}\mathfrak{H}_0\\\oplus\\\mathfrak{N}\\\oplus\\\mathcal{H}\end{array}\to\begin{array}{c}\mathfrak{H}_0\\\oplus\\\mathfrak{N}\\\oplus\\\mathcal{H}\end{array}.\tag{3.2}$$

Let $\mathcal{K}$ be a Hilbert space. Associate with any selfadjoint contraction

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix}$$

two further selfadjoint contractions in $\mathcal{K}$ via

$$\begin{aligned} \widehat{Z}_0 &:= ((I+X)_{\mathcal{K}} - I) \restriction \mathcal{K} = X_{11} - \left((I+X_{22})^{(-1/2)} X_{12}^*\right)^* (I+X_{22})^{(-1/2)} X_{12}^*, \\ \widehat{Z}_1 &:= (I - (I-X)_{\mathcal{K}}) \restriction \mathcal{K} = X_{11} + \left((I-X_{22})^{(-1/2)} X_{12}^*\right)^* (I-X_{22})^{(-1/2)} X_{12}^*. \end{aligned} \tag{3.3}$$

By Remark 2.5 selfadjoint contractions $X$ in $\mathcal{K} \oplus \mathcal{H}$ are of the form

$$X = \begin{bmatrix} -UX_{22}U^* + D_{U^*}VD_{U^*} & UD_{X_{22}} \\ D_{X_{22}}U^* & X_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix}, \tag{3.4}$$

where $X_{22} \in \mathbf{L}(\mathcal{H})$, $U \in \mathbf{L}(\mathfrak{D}_{X_{22}}, \mathcal{K})$, $V \in \mathbf{L}(\mathfrak{D}_{U^*})$ are contractions, and $X_{22}$ and $V$ are selfadjoint. Then from (3.3) and (3.4) one obtains

$$\widehat{Z}_0 = D_{U^*}VD_{U^*} - UU^*, \quad \widehat{Z}_1 = D_{U^*}VD_{U^*} + UU^*.$$

Hence,

$$UU^* = \frac{1}{2}(\widehat{Z}_1 - \widehat{Z}_0), \quad D_{U^*}VD_{U^*} = \frac{1}{2}(\widehat{Z}_1 + \widehat{Z}_0). \tag{3.5}$$

Then clearly $\widehat{Z}_0 \le \widehat{Z}_1$ and, moreover,

$$\ker(\widehat{Z}_1 - \widehat{Z}_0) = \{0\} \iff \ker X_{12}^* = \{0\}. \tag{3.6}$$

With $\mathcal{K} = \mathfrak{D}_{K_0^*}$ as above, $\widehat{Z}_0$ and $\widehat{Z}_1$ determine two *sc*-extensions $\widehat{B}_0$ and $\widehat{B}_1$ of $B$ in $\mathfrak{H}$:

$$\widehat{B}_0 := \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}\widehat{Z}_0D_{K_0^*} \end{bmatrix} = B_\mu + D_{K_0^*}(I+\widehat{Z}_0)D_{K_0^*}, \tag{3.7}$$

$$\widehat{B}_1 := \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}\widehat{Z}_1D_{K_0^*} \end{bmatrix} = B_M - D_{K_0^*}(I-\widehat{Z}_1)D_{K_0^*}. \tag{3.8}$$

From definitions and Remark 2.5 we get

$$(I+\widehat{B}_0)_{\mathfrak{N}} = D_{K_0^*}P_{\mathfrak{N}}(I+X)_{\mathfrak{D}_{K_0^*}}D_{K_0^*}P_{\mathfrak{N}}, \quad (I-\widehat{B}_1)_{\mathfrak{N}} = D_{K_0^*}P_{\mathfrak{N}}(I-X)_{\mathfrak{D}_{K_0^*}}D_{K_0^*}P_{\mathfrak{N}}.$$

**Proposition 3.1.** *Let $\widehat{Z}_0$ and $\widehat{Z}_1$ be two selfadjoint contractions in a Hilbert space $\mathcal{K}$, such that $\widehat{Z}_0 \le \widehat{Z}_1$. If the Hilbert space $\mathcal{H}$ satisfies $\dim \mathcal{H} \ge \dim \overline{\operatorname{ran}}\,(\widehat{Z}_1 - \widehat{Z}_0)$, then all selfadjoint contractions $X$ in $\mathcal{K} \oplus \mathcal{H}$ possessing the properties $((I+X)_{\mathcal{K}} - I) \restriction \mathcal{K} = \widehat{Z}_0$ and $(I - (I-X)_{\mathcal{K}}) \restriction \mathcal{K} = \widehat{Z}_1$ are given by the formula*

$$X = \begin{bmatrix} \frac{\widehat{Z}_1+\widehat{Z}_0}{2} - \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2} \mathcal{V}^* X_{22} \mathcal{V} \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2} & \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2} \mathcal{V}^* D_{X_{22}} \\ D_{X_{22}} \mathcal{V} \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2} & X_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix},$$

*where $X_{22}$ is an arbitrary selfadjoint contraction in $\mathcal{H}$ and $\mathcal{V}$ is an arbitrary isometry from $\overline{\operatorname{ran}}\,(\widehat{Z}_1 - \widehat{Z}_0)$ into $\mathfrak{D}_{X_{22}}$. In particular, if $\widehat{Z}_0 = -I_{\mathcal{K}}$ and $\widehat{Z}_1 = I_{\mathcal{K}}$, then*

$$X = \begin{bmatrix} -\mathcal{V}^* X_{22} \mathcal{V} & \mathcal{V}^* D_{X_{22}} \\ D_{X_{22}} \mathcal{V} & X_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix},$$

*where $\mathcal{V}$ is an arbitrary isometry from $\mathcal{K}$ into $\mathfrak{D}_{X_{22}}$.*

*Proof.* Conclusions in the proposition follow from relations (3.3), (3.4), and (3.5). □

The next result clarifies the definitions of $\widehat{B}_0$ and $\widehat{B}_1$ in (3.7), (3.8) by establishing an exit space version for the identities in (2.16).

**Theorem 3.2.** *Assume that* $\mathfrak{D}_{K_0^*} = \mathfrak{N}$, *let* $X = (X_{ij})_{i,j=1}^2$ *be a selfadjoint contraction in* $\mathfrak{N} \oplus \mathcal{H}$ *as in* (3.1), *and let*

$$\widetilde{B}_X = \begin{bmatrix} B_0 & D_{B_0}\widehat{K}_0^* \\ \widehat{K}_0 D_{B_0} & -\widehat{K}_0 B_0 \widehat{K}_0^* + D_{\widehat{K}_0^*} X D_{\widehat{K}_0^*} \end{bmatrix} : \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{matrix}.$$

*Then* $\widehat{B}_0$ *and* $\widehat{B}_1$ *defined in* (3.7) *and* (3.8) *satisfy the relations*

$$\widehat{B}_0 = B_\mu + \left(I + \widetilde{B}_X\right)_{\mathfrak{N}} \upharpoonright \mathfrak{H}, \quad \widehat{B}_1 = B_M - \left(I - \widetilde{B}_X\right)_{\mathfrak{N}} \upharpoonright \mathfrak{H}. \tag{3.9}$$

*Proof.* Let $\widetilde{B}_\mu := B_\mu P_{\mathfrak{H}} \oplus (-P_{\mathcal{H}})$, $\widetilde{B}_M := B_M P_{\mathfrak{H}} \oplus P_{\mathcal{H}}$. Then it follows from (2.19) that

$$\widetilde{B}_X = \widetilde{B}_\mu + D_{\widehat{K}_0^*}(I + X) D_{\widehat{K}_0^*} = \widetilde{B}_M - D_{\widehat{K}_0^*}(I - X) D_{\widehat{K}_0^*}.$$

Moreover, using (2.10) and (2.14) it is seen that for all $f \in \mathfrak{H} \oplus \mathcal{H}$

$$\left(\left(I + \widetilde{B}_X\right)_{\mathfrak{N}} f, f\right) = \inf_{\substack{f_0 \in \mathfrak{H}_0 \\ h \in \mathcal{H}}} \left(\left(I + \widetilde{B}_X\right)(f + f_0 + h), f + f_0 + h\right)$$

$$= \inf_{f_0 \in \mathfrak{H}_0} \left((I + B_\mu)(f + f_0), f + f_0\right) + \inf_{h \in \mathcal{H}} \left((I + X) D_{\widehat{K}_0^*}(f + h), D_{\widehat{K}_0^*}(f + h)\right)$$

$$= \inf_{h \in \mathcal{H}} \left((I + X) D_{\widehat{K}_0^*}(f + h), D_{\widehat{K}_0^*}(f + h)\right) = \left((I + X)_{\mathfrak{N}} D_{K_0^*} P_{\mathfrak{N}} f, D_{K_0^*} P_{\mathfrak{N}} f\right).$$

In view of (3.3) $(I + X)_{\mathfrak{N}} = I + \widehat{Z}_0$ which combined with the identity (3.7) leads to

$$D_{K_0^*}(I + \widehat{Z}_0) D_{K_0^*} = \widehat{B}_0 - B_\mu.$$

This proves the first identity in (3.9). The second identity in (3.9) is proved similarly. □

It is also useful to describe shortenings of $I \pm \widetilde{B}_X$ to the exit space $\mathcal{H}$.

**Theorem 3.3.** *Let* $X = (X_{ij})_{i,j=1}^2$ *be a selfadjoint contraction in* $\mathfrak{D}_{K_0^*} \oplus \mathcal{H}$ *as in* (3.1) *and let*

$$\widetilde{B}_X = \begin{bmatrix} B_0 & D_{B_0}\widehat{K}_0^* \\ \widehat{K}_0 D_{B_0} & -\widehat{K}_0 B_0 \widehat{K}_0^* + D_{\widehat{K}_0^*} X D_{\widehat{K}_0^*} \end{bmatrix} : \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \widehat{\mathcal{H}} \end{matrix} \to \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \widehat{\mathcal{H}} \end{matrix}.$$

*Then*

$$(I \pm \widetilde{B}_X)_{\mathcal{H}} \upharpoonright \mathcal{H} = (I \pm X)_{\mathcal{H}} \upharpoonright \mathcal{H}. \tag{3.10}$$

*Proof.* Rewrite $\widetilde{B}_X$ as in (3.2)

$$\widetilde{B}_X = \begin{bmatrix} B_0 & D_{B_0} K_0^* & 0 \\ K_0 D_{B_0} & -K_0 B_0 K_0^* + D_{K_0^*} X_{11} D_{K_0^*} & D_{K_0^*} X_{12} \\ 0 & X_{12}^* D_{K_0^*} & X_{22} \end{bmatrix} : \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \\ \oplus \\ \mathcal{H} \end{matrix}.$$

Let $\mathcal{X}$ be the Hermitian contraction determined by the first column of $X$,

$$\mathcal{X} = \begin{bmatrix} X_{11} \\ X_{12}^* \end{bmatrix} : \mathfrak{D}_{K_0^*} \to \begin{matrix} \mathfrak{D}_{K_0^*} \\ \oplus \\ \mathcal{H} \end{matrix}.$$

Then one can consider $X$ as an $sc$-extension of $\mathcal{X}$. Analogously, define the Hermitian contraction $\mathcal{B}_{\mathcal{X}}$ by

$$\mathcal{B}_{\mathcal{X}} = \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0 B_0 K_0^* + D_{K_0^*} X_{11} D_{K_0^*} \\ 0 & X_{12}^* D_{K_0^*} \end{bmatrix} : \begin{array}{c} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \end{array} \to \begin{array}{c} \mathfrak{H}_0 \\ \oplus \\ \mathfrak{N} \\ \oplus \\ \mathcal{H} \end{array}.$$

Now we consider $sc$-extensions of $\mathcal{X}$ in the Hilbert space $\widehat{\mathcal{H}} = \mathfrak{D}_{K_0^*} \oplus \mathcal{H}$ and $sc$-extensions of $\mathcal{B}_{\mathcal{X}}$ in the Hilbert space $\mathfrak{H} \oplus \mathcal{H} = \mathfrak{H}_0 \oplus \mathfrak{N} \oplus \mathcal{H}$. It is evident that

$$\widetilde{B}_X \supset \mathcal{B}_{\mathcal{X}} \iff X \supset \mathcal{X}.$$

All $sc$-extensions of $\mathcal{X}$ form the operator interval $[(\mathcal{X})_\mu, (\mathcal{X})_M]$. On the other hand, the form of $\widetilde{B}_X$ shows that

$$X_1 \le X_2 \iff \widetilde{B}_{X_1} \le \widetilde{B}_{X_2}.$$

Hence,

$$X \in [(\mathcal{X})_\mu, (\mathcal{X})_M] \Rightarrow \widetilde{B}_X \in [\widetilde{B}_{(\mathcal{X})_\mu}, \widetilde{B}_{(\mathcal{X})_M}].$$

On the other hand, every $sc$-extension $\widetilde{B}$ of $B$ in $\mathfrak{H} \oplus \mathcal{H}$ is of the form $\widetilde{B}_X$, where $X$ is a selfadjoint contraction in $\mathfrak{D}_{K_0^*} \oplus \mathcal{H}$; see (3.1), (3.2). It follows that if $\widetilde{B}$ is an $sc$-extension of $B_{\mathcal{X}}$, then $\widetilde{B}$ is also an $sc$-extension of $B\,(\subset B_{\mathcal{X}})$, i.e., $\widetilde{B} = \widetilde{B}_X$, where $X$ is the $sc$-extension of $\mathcal{X}$. Hence,

$$\widetilde{B}_X \in [(\mathcal{B}_{\mathcal{X}})_\mu, (\mathcal{B}_{\mathcal{X}})_M] \Rightarrow X \in [(\mathcal{X})_\mu, (\mathcal{X})_M].$$

One concludes that

$$(\mathcal{B}_{\mathcal{X}})_\mu = \widetilde{B}_{(\mathcal{X})_\mu}, \quad (\mathcal{B}_{\mathcal{X}})_M = \widetilde{B}_{(\mathcal{X})_M}. \tag{3.11}$$

Since for all $X_1, X_2 \in [(\mathcal{X})_\mu, (\mathcal{X})_M]$ one has

$$\left(\widetilde{B}_{X_1} - \widetilde{B}_{X_2}\right) \restriction \mathcal{H} = (X_1 - X_2) \restriction \mathcal{H},$$

the equalities (2.16) applied to $I \pm \widetilde{B}_X$ and $(I \pm X)$ yield (3.10) in view of (3.11). □

**Corollary 3.4.** *The following statements are equivalent:*

(i) $(I + \widetilde{B}_X)_{\mathcal{H}} = 0$ *and* $(I - \widetilde{B}_X)_{\mathcal{H}} = 0$*;*
(ii) $(I + X)_{\mathcal{H}} = 0$ *and* $(I - X)_{\mathcal{H}} = 0$*;*
(iii) $\widetilde{B}_X$ *is a unique sc-extension of Hermitian contraction* $\mathcal{B}_{\mathcal{X}}$*;*
(iv) $X$ *is a unique sc-extension of Hermitian contraction* $\mathcal{X}$*.*

Theorem 3.3 and Corollary 3.4 have important implications on the contractions $\widehat{Z}_0$, $\widehat{Z}_1$, therefore, also on the $sc$-extensions $\widehat{B}_0$, $\widehat{B}_1$ of $B$ in the original Hilbert space $\mathfrak{H}$.

**Theorem 3.5.** *Let*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} : \begin{array}{c} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{array} \to \begin{array}{c} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{array}$$

*be a selfadjoint contraction. Suppose that*

$$(I - X)_{\mathcal{H}} = (I + X)_{\mathcal{H}} = 0 \tag{3.12}$$

*and*

$$||X_{22}|| < 1. \tag{3.13}$$

*Let the selfadjoint contractions* $\widehat{Z}_0$ *and* $\widehat{Z}_1$ *in* $\mathcal{K}$ *be defined by* (3.3)*. Then*

$$\begin{aligned}&\operatorname{ran}(\widehat{Z}_1-\widehat{Z}_0)^{1/2}\cap\operatorname{ran}(I+\widehat{Z}_0)^{1/2}=\{0\},\\&\operatorname{ran}(\widehat{Z}_1-\widehat{Z}_0)^{1/2}\cap\operatorname{ran}(I-\widehat{Z}_1)^{1/2}=\{0\}.\end{aligned}\tag{3.14}$$

*Proof.* By (3.3) we have

$$I_\mathcal{K}+\widehat{Z}_0=(I+X)_\mathcal{K}\restriction\mathcal{K},\quad I_\mathcal{K}-\widehat{Z}_1=(I-X)_\mathcal{K}\restriction\mathcal{K}.$$

Due to the assumption (3.12), the operator $X$ takes the form

$$X=\begin{bmatrix}X_{11}&D_{X_{11}}L^*\\LD_{X_{11}}&-LX_{11}L^*\end{bmatrix}:\begin{matrix}\mathcal{K}\\\oplus\\\mathcal{H}\end{matrix}\to\begin{matrix}\mathcal{K}\\\oplus\\\mathcal{H}\end{matrix},$$

where $LL^*=I_\mathcal{H}$; see Remark 2.5. On the other hand, $X_{12}=D_{X_{11}}L^*=UD_{X_{22}}$ for a contraction $U\in\mathbf{L}(\mathfrak{D}_{X_{22}},\mathcal{K})$. From the assumption (3.13) it follows that $D_{X_{22}}$ has a bounded inverse. Hence $U=D_{X_{11}}L^*D_{X_{22}}^{-1}$ and

$$\operatorname{ran}U=D_{X_{11}}\operatorname{ran}L^*.$$

Furthermore, since $\widehat{Z}_1-\widehat{Z}_0=2UU^*$, see (3.5), one obtains

$$\operatorname{ran}(\widehat{Z}_1-\widehat{Z}_0)^{1/2}=\operatorname{ran}U=D_{X_{11}}\operatorname{ran}L^*.$$

On the other hand, from the formula for $X$ above it is clear that

$$I\pm X=\begin{bmatrix}(I\pm X_{11})^{1/2}\\L(I\mp X_{11})^{1/2}\end{bmatrix}\begin{bmatrix}(I\pm X_{11})^{1/2}\\L(I\mp X_{11})^{1/2}\end{bmatrix}^*.$$

This gives a description of $\operatorname{ran}(I\pm X)^{1/2}$ and now an application of (2.11) leads to

$$\begin{aligned}\operatorname{ran}(I+\widehat{Z}_0)^{1/2}&=(I+X_{11})^{1/2}(I-X_{11})^{-1/2}\ker L,\\\operatorname{ran}(I-\widehat{Z}_1)^{1/2}&=(I-X_{11})^{1/2}(I+X_{11})^{-1/2}\ker L.\end{aligned}$$

Since $\operatorname{ran}L^*\perp\ker L$, one concludes that

$$\begin{aligned}&(I-X_{11})^{1/2}\operatorname{ran}L^*\cap(I-X_{11})^{-1/2}\ker L=\{0\},\\&(I+X_{11})^{1/2}\operatorname{ran}L^*\cap(I+X_{11})^{-1/2}\ker L=\{0\}.\end{aligned}$$

This implies the equalities (3.14). □

Observe that if $B$ is a Hermitian contraction in $\mathfrak{H}$, if $\widehat{Z}_0$ and $\widehat{Z}_1$ are selfadjoint contractions in $\mathfrak{N}(=\mathfrak{H}\ominus\operatorname{dom}B)$ satisfying (3.14), and if the *sc*-extensions $\widehat{B}_0$ and $\widehat{B}_1$ of $B$ are given by

$$\widehat{B}_j=\begin{bmatrix}B_0&D_{B_0}K_0^*\\K_0D_{B_0}&-K_0B_0K_0^*+D_{K_0^*}\widehat{Z}_jD_{K_0^*}\end{bmatrix},\quad j=0,1,$$

then the pair $\{\widehat{B}_0,\widehat{B}_1\}$ possesses the properties in (2.23). If $\mathfrak{D}_{K_0^*}=\mathfrak{N}$ and $\ker(\widehat{Z}_1-\widehat{Z}_0)=\{0\}$, then $\ker(\widehat{B}_1-\widehat{B}_0)=\operatorname{dom}B$. We also note that if

$$X=\begin{bmatrix}0&\mathcal{V}^*\\\mathcal{V}&0\end{bmatrix}:\begin{matrix}\mathcal{K}\\\oplus\\\mathcal{H}\end{matrix}\to\begin{matrix}\mathcal{K}\\\oplus\\\mathcal{H}\end{matrix},$$

where $\mathcal{V}$ is an isometry from $\mathcal{K}$ into $\mathcal{H}$, then $(I\pm X)_\mathcal{K}=0$ and $\widehat{Z}_0=-I_\mathcal{K}$, $\widehat{Z}_1=I_\mathcal{K}$. On the other hand, $(I\pm X)_\mathcal{H}=I-\mathcal{V}\mathcal{V}^*$ and hence $(I\pm X)_\mathcal{H}=0$ if and only if $\mathcal{V}$ is unitary, i.e., $\operatorname{ran}\mathcal{V}=\mathcal{H}$. Therefore, it is possible that (3.13) and (3.14) are satisfied, while (3.12) fails to hold.

The next result completes the role of exit space extensions in the study of pairs $\{\widehat{B}_0,\widehat{B}_1\}$ of *sc*-extensions of $B$ in the original Hilbert space $\mathfrak{H}$ whose $Q$-functions belong to the classes $\mathfrak{S}_\mu(\mathfrak{N})$ and $\mathfrak{S}_M(\mathfrak{N})$; see Definition 2.7 and Theorem 2.8.

**Theorem 3.6.** *1) Let* $\dim \mathcal{K} = \dim \mathcal{H} = \infty$. *Then there exists a selfadjoint contractive block operator*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix}$$

*satisfying the conditions* (3.12), (3.13), *and the additional conditions*

$$\ker X_{12}^* = \{0\}, \tag{3.15}$$

$$\widehat{Z}_0 \neq -I_{\mathcal{K}}, \quad \widehat{Z}_1 \neq I_{\mathcal{K}}, \quad \ker(\widehat{Z}_1 - \widehat{Z}_0) = \{0\},$$

*where* $\widehat{Z}_0$ *and* $\widehat{Z}_1$ *are as in* (3.3), *i.e.,* $\widehat{Z}_0 = ((I+X)_{\mathcal{K}} - I)\restriction \mathcal{K}$, $\widehat{Z}_1 = (I - (I-X)_{\mathcal{K}})\restriction \mathcal{K}$.

*2) Let* $\dim \mathcal{K} = \infty$ *and suppose that* $\widehat{Z}_0$ *and* $\widehat{Z}_1$, $\widehat{Z}_0 \leq \widehat{Z}_1$, *are two selfadjoint contractions in* $\mathcal{K}$ *which satisfy the conditions* (3.14) *and the condition*

$$\ker(\widehat{Z}_1 - \widehat{Z}_0) = \{0\}. \tag{3.16}$$

*Then there exists a selfadjoint contractive block operator* $X$ *in the Hilbert space* $\mathcal{K} \oplus \mathcal{H}$, $\dim \mathcal{H} = \dim \mathcal{K}$, *such that*

$$(I \pm X)_{\mathcal{H}} = 0, \quad ||X_{22}|| < 1$$

*and* $\widehat{Z}_0 = ((I+X)_{\mathcal{K}} - I)\restriction \mathcal{K}$, $\widehat{Z}_1 = (I - (I-X)_{\mathcal{K}})\restriction \mathcal{K}$.

*Proof.* 1) We give a construction of a required $X$ in two steps.

**Step 1.** In $\mathcal{K}$ choose an infinite dimensional subspace $\Omega_0$ with an infinite dimensional orthogonal complement $\mathfrak{M}_0 = \mathcal{K} \ominus \Omega_0$. In this step we construct a special selfadjoint contraction $X_{11}$ in $\mathcal{K} = \Omega_0 \oplus \mathfrak{M}_0$.

Let $\mathcal{A}$ be a selfadjoint operator in $\Omega_0$ such that $||\mathcal{A}|| < 1$. Then choose a contraction $\mathcal{M} \in \mathbf{L}(\Omega_0, \mathfrak{M}_0)$ such that $\ker D_{\mathcal{M}^*} = \{0\}$ and $\operatorname{ran} D_{\mathcal{M}^*} \neq \mathfrak{M}_0$, i.e., $||\mathcal{M}f|| < ||f||$ ($\Leftrightarrow$ $\ker D_{\mathcal{M}} = \{0\}$) for all $f \in \Omega_0 \setminus \{0\}$, while $||\mathcal{M}|| = 1$; cf. (2.3). Moreover, let $\mathfrak{L}_0$ be a subspace in $\mathfrak{M}_0$ such that

$$\mathfrak{L}_0 \cap \operatorname{ran} D_{\mathcal{M}^*} = \{0\} \text{ and } \mathfrak{L}_0^{\perp} \cap \operatorname{ran} D_{\mathcal{M}^*} = \{0\} \tag{3.17}$$

cf. [41]. Next define the selfadjoint and unitary operator $J_0$ in $\mathfrak{M}_0$ by

$$J_0 = 2P_{\mathfrak{L}_0} - I_{\mathfrak{M}_0}. \tag{3.18}$$

Due to (3.17) $J_0$ satisfies

$$J_0 \operatorname{ran} D_{\mathcal{M}^*} \cap \operatorname{ran} D_{\mathcal{M}^*} = \{0\}. \tag{3.19}$$

Now, introduce

$$X_{11} = \begin{bmatrix} \mathcal{A} & D_{\mathcal{A}}\mathcal{M}^* \\ \mathcal{M}D_{\mathcal{A}} & -\mathcal{M}\mathcal{A}\mathcal{M}^* + D_{\mathcal{M}^*}J_0 D_{\mathcal{M}^*} \end{bmatrix} : \begin{matrix} \Omega_0 \\ \oplus \\ \mathfrak{M}_0 \end{matrix} \to \begin{matrix} \Omega_0 \\ \oplus \\ \mathfrak{M}_0 \end{matrix}.$$

We claim that $X_{11}$ satisfies the equalities

$$\ker D_{X_{11}} = \{0\} \tag{3.20}$$

and

$$\operatorname{ran} D_{X_{11}} \cap \mathfrak{M}_0 = \{0\}. \tag{3.21}$$

Since $J_0$ in (3.18) is unitary, $D_{J_0} = 0$ and hence Remark 2.5 shows that for all $\vec{a} = \begin{bmatrix} h \\ g \end{bmatrix}$

$$\|D_{X_{11}}\vec{a}\|^2 = \|D_{\mathcal{M}}(D_{\mathcal{A}}h - \mathcal{A}\mathcal{M}^*g) - \mathcal{M}^* J_0 D_{\mathcal{M}^*} g\|^2. \tag{3.22}$$

Hence, if $\|D_{X_{11}}\vec{a}\|^2 = 0$ then it follows from (2.2) that there exists $x \in \mathfrak{M}_0$ such that $D_{\mathcal{A}}h - \mathcal{A}\mathcal{M}^*g = \mathcal{M}^*x$ and $J_0 D_{\mathcal{M}^*}g - D_{\mathcal{M}^*}x \in \ker \mathcal{M}^* \subset \operatorname{ran} D_{\mathcal{M}^*}$. Now (3.19) gives $J_0 D_{\mathcal{M}^*}g = 0$ and, hence, $g = 0$ and $h = 0$, since also $\ker D_A = 0$. So (3.20) holds true.

On the other hand, by applying (2.10) to (3.22) it is seen that $\left(D^2_{X_{11}}\right)_{\mathfrak{M}_0} = 0$, and hence (3.21) is obtained from (2.12). Furthermore, an application of Remark 2.5 shows that when $\mathfrak{L}_0 \neq \{0\} \neq \mathfrak{L}_0^\perp$ then, equivalently,

$$(I + X_{11})_{\mathfrak{M}_0} \neq 0, \quad (I - X_{11})_{\mathfrak{M}_0} \neq 0. \tag{3.23}$$

**Step 2.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, let $L^*$ be an isometry from $\mathcal{H}$ into $\mathcal{K}$ such that $\operatorname{ran} L^* = \Omega_0$, and define

$$X = \begin{bmatrix} X_{11} & D_{X_{11}}L^* \\ LD_{X_{11}} & -LX_{11}L^* \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix}.$$

It follows from $\operatorname{ran} L^* = \Omega_0$ that $X_{22} = -LP_{\Omega_0}\mathcal{A}L^*$ and thus $||X_{22}|| < 1$ by the choice of $\mathcal{A}$. Since $\ker L = \mathfrak{M}_0$, the equalities (3.20), (3.21) yield $\ker LD_{X_{11}} = \{0\}$. Therefore,

$$(I \pm X)_{\mathcal{H}} = 0, \quad ||X_{22}|| < 1, \quad \ker X_{12}^* = \{0\}, \quad \ker X_{12} = \{0\},$$

where the first equality holds by Remark 2.5. By applying Theorem 3.5 one concludes that $\widehat{Z}_0$ and $\widehat{Z}_1$ have properties (3.14). Moreover, from (3.23) it follows that $\widehat{Z}_0 \neq -I_{\mathcal{K}}$, $\widehat{Z}_1 \neq I_{\mathcal{K}}$. Thus, relations in (3.12), (3.13), (3.15) are valid for $X$. Finally, the condition $\ker(\widehat{Z}_1 - \widehat{Z}_0) = \{0\}$ is obtained from (3.15) and (3.6).

2) Define $X$ by

$$X = \begin{bmatrix} \frac{\widehat{Z}_1+\widehat{Z}_0}{2} & \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2}\mathcal{V}^* \\ \mathcal{V}\left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2} & 0 \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{H} \end{matrix},$$

where $\mathcal{V} : \mathcal{H} \to \mathcal{K}$ is unitary. Clearly, $X$ is a selfadjoint contraction in $\mathcal{H} \oplus \mathcal{K}$; cf. Theorem 2.4, Proposition 3.1. Next observe that

$$I + X = \begin{bmatrix} (I + \widetilde{Z}_0)^{1/2} & \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2}\mathcal{V}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} (I + \widetilde{Z}_0)^{1/2} & \left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2}\mathcal{V}^* \\ 0 & I \end{bmatrix}^*$$

and

$$I - X = \begin{bmatrix} (I - \widetilde{Z}_0)^{1/2} & -\left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2}\mathcal{V}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} (I - \widetilde{Z}_0)^{1/2} & -\left(\frac{\widehat{Z}_1-\widehat{Z}_0}{2}\right)^{1/2}\mathcal{V}^* \\ 0 & I \end{bmatrix}^*.$$

These two formulas give descriptions for $\operatorname{ran}(I + X)^{1/2}$ and $\operatorname{ran}(I - X)^{1/2}$, respectively. Now using the assumptions (3.14) and (3.16) one concludes that

$$\operatorname{ran}(I \pm X)^{1/2} \cap (\{0\} \oplus \mathcal{H}) = \{0\}.$$

According to (2.11) this means that $(I \pm X)_{\mathcal{H}} = 0$.

Finally, the equalities $\widehat{Z}_0 = ((I + X)_{\mathcal{K}} - I) \upharpoonright \mathcal{K}$ and $\widehat{Z}_1 = (I - (I - X)_{\mathcal{K}}) \upharpoonright \mathcal{K}$ are clear from Proposition 3.1. □

In particular, Theorem 3.6 contains an improvement of Theorem 2.8: given any Hermitian contraction $B$ in $\mathfrak{H}$ with $\dim \mathfrak{N} = \dim(\mathfrak{H} \ominus \operatorname{dom} B) = \infty$ it enables to construct pairs $\{\widehat{B}_0, \widehat{B}_1\}$ of *sc*-extensions of $B$ in $\mathfrak{H}$, which differ from the pair $\{\widehat{B}_\mu, \widehat{B}_M\}$ and satisfy the conditions (2.23), directly from one exit space extension $\widetilde{B}_X$ of $B$ via the formulas (3.2)–(3.8). Furthermore, all the key properties of $\{\widehat{B}_0, \widehat{B}_1\}$ are expressed in simple terms and the choice of appropriate parameters $X$ is specified explicitly.

## 4. Compressed resolvents

Let $B$ be Hermitian contraction in $\mathfrak{H}$ and let $\widetilde{B}$ be a $qsc$-extension of $B$ in the Hilbert space $\mathfrak{H} \oplus \mathcal{H}$. Recall that then $\widetilde{B} = \widetilde{B}_X$ can be rewritten in the form (3.2) for some contractive block operator $X$ of the form (3.1). To formulate the next result it is useful to associate with $X$ the operator function

$$\Phi_X(z) = X_{11} + zX_{12}(I - zX_{22})^{-1}X_{21}, \quad |z| < 1. \tag{4.1}$$

If, in addition, $X_{22}$ is selfadjoint, then $\Phi_X(z)$ admits a holomorphic continuation to all points $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Observe, that $\Phi_X(z)$ can be also interpreted as the transfer function of the passive system

$$\sigma = \left\{ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}; \mathfrak{D}_{K_0^*}, \mathfrak{D}_{K_0^*}, \mathcal{H} \right\},$$

see Section 2.2. In particular, $\Phi_X(z)$ is contractive on the unit disk $\mathbb{D}$.

**Theorem 4.1.** *Let $B$ be Hermitian contraction in $\mathfrak{H}$, let $\widetilde{B} = \widetilde{B}_X$ be a qsc-extension of $B$ in $\mathfrak{H} \oplus \mathcal{H}$ rewritten in the form (3.2) with $X$ given by (3.1), and let $\Phi_X(z)$ be as in (4.1). Then*

$$P_{\mathfrak{H}}(z\widetilde{B} - I)^{-1} \upharpoonright \mathfrak{H} = \left(z\widehat{B}_X(z) - I_{\mathfrak{H}}\right)^{-1}, \quad |z| < 1, \tag{4.2}$$

*where*

$$\begin{aligned} \widehat{B}_X(z) &= \frac{1}{2}(B_\mu + B_M) + \frac{1}{2}(B_M - B_\mu)^{1/2}\Phi_X(z)(B_M - B_\mu)^{1/2} \\ &= \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0 B_0 K_0^* + D_{K_0^*}\Phi_X(z)D_{K_0^*} \end{bmatrix}. \end{aligned} \tag{4.3}$$

*With $z$ fixed, the operator $\widehat{B}_X(z)$ is a qsc-extension of $B$ in the Hilbert space $\mathfrak{H}$.*

*Furthermore, if $\widetilde{B} \in C_{\mathfrak{H} \oplus \mathcal{H}}(\alpha)$, then $\Phi_X(z)$ and $\widehat{B}_X(z)$ can be defined for $z \in \Pi(\alpha)$ and*

(1) *the implications*

$$\begin{cases} z \in \Pi_+(\beta), \ z \neq \pm 1, \\ \beta \in [\alpha, \pi/2) \end{cases} \Longrightarrow ||\widehat{B}_X(z)\sin\beta + i\cos\beta|| \leq 1,$$
$$\begin{cases} z \in \Pi_-(\beta), \ z \neq \pm 1, \\ \beta \in [\alpha, \pi/2) \end{cases} \Longrightarrow ||\widehat{B}_X(z)\sin\beta - i\cos\beta|| \leq 1$$

*are valid, therefore*

$$z \in C(\beta) \Longrightarrow B_X(z) \in C_{\mathfrak{H}}(\beta);$$

(2) *there exist strong limits*

$$\Phi_X(\pm 1) \in C_{\mathfrak{D}_{K_0^*}}(\alpha), \quad \widehat{B}_X(\pm 1) \in C_{\mathfrak{H}}(\alpha).$$

*In particular, for $\alpha = 0$ the operator functions $\Phi_X(z)$ and $\widehat{B}_X(z)$ are defined for $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ and $\Phi_X(\pm 1)$ are selfadjoint contractions given by*

$$\Phi_X(-1) = \widehat{Z}_0, \quad \Phi_X(1) = \widehat{Z}_1, \tag{4.4}$$

*where $\widehat{Z}_0$ and $\widehat{Z}_1$ are as in (3.3), and the sc-extensions $\widehat{B}_X(-1)$ and $\widehat{B}_X(+1)$ of $B$ in $\mathfrak{H}$ coincide with $\widehat{B}_0$ and $\widehat{B}_1$ in (3.7) and (3.8), respectively.*

*Proof.* Since $||\Phi(z)|| \le 1$, the operator $\widehat{B}_X(z)$ in (4.3) is a *qsc*-extension of $B$ for each $z$, $|z|<1$; see Theorem 2.4. Using (2.9) and (3.2) we get for $|\lambda|>1$

$$\begin{aligned}\left(P_{\mathfrak{H}}(\widetilde{B}-\lambda)^{-1}\upharpoonright\mathfrak{H}\right)^{-1} &= \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0 D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}X_{11}D_{K_0^*}\end{bmatrix} - \lambda I_{\mathfrak{H}} \\ &- \begin{bmatrix} 0 \\ D_{K_0^*}X_{12}\end{bmatrix}(X_{22}-\lambda)^{-1}\begin{bmatrix}0 & X_{21}D_{K_0^*}\end{bmatrix} \\ &= \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}\left(X_{11}-X_{12}(X_{22}-\lambda)^{-1}X_{21}\right)D_{K_0^*}\end{bmatrix} - \lambda I_{\mathfrak{H}}.\end{aligned}$$

Consequently, with $|z|<1$ this leads to

$$\begin{aligned}\left(P_{\mathfrak{H}}(z\widetilde{B}-I)^{-1}\upharpoonright\mathfrak{H}\right)^{-1} &= z\begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}\Phi_X(z)D_{K_0^*}\end{bmatrix} - I_{\mathfrak{H}} \\ &= z\widehat{B}_X(z) - I\end{aligned}$$

and this proves (4.2).

Suppose that $\widetilde{B} \in C_{\mathfrak{H}\oplus\mathcal{H}}(\alpha)$. Then $X \in C_{\mathfrak{D}_{K_0^*}\oplus\mathcal{H}}(\alpha)$ by Theorem 2.4 and this implies that $X \in C_{\mathfrak{D}_{K_0^*}\oplus\mathcal{H}}(\beta)$ for $\beta \in [\alpha,\pi/2)$. Theorem 2.3 combined with (2.20) shows that

$$\begin{cases} z \in \Pi_+(\beta), \\ z \ne \pm 1, \\ \beta \in [\alpha,\pi/2) \end{cases} \Longrightarrow ||\Phi_X(z)\sin\beta + i\cos\beta|| \le 1 \Longrightarrow ||\widehat{B}_X(z)\sin\beta + i\cos\beta|| \le 1,$$

$$\begin{cases} z \in \Pi_-(\beta), \\ z \ne \pm 1, \\ \beta \in [\alpha,\pi/2) \end{cases} \Longrightarrow ||\Phi_X(z)\sin\beta - i\cos\beta|| \le 1 \Longrightarrow ||\widehat{B}_X(z)\sin\beta - i\cos\beta|| \le 1.$$

Moreover, according to Theorem 2.3 the strong limit values $\Phi_X(\pm1)$ exist and in view of (4.3) $\widehat{B}_X(\pm1)$ exist, too, and they satisfy the inclusions in (2). For $\alpha=0$ the equalities in (4.4) can be obtained directly from the formulas in (3.3) and (4.1). Finally, by comparing (3.7), (3.8) and (4.3) one concludes that $\widehat{B}_X(-1)=\widehat{B}_0$ and $\widehat{B}_X(+1)=\widehat{B}_1$. □

Let $X$ and $\widetilde{B}_X$ be given by (3.1) and (3.2). Define the operator $\widetilde{C}$ in $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{N}$ by

$$\widetilde{C} = \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}X_{11}D_{K_0^*}\end{bmatrix} : \begin{matrix}\mathfrak{H}_0 \\ \oplus \\ \mathfrak{N}\end{matrix} \to \begin{matrix}\mathfrak{H}_0 \\ \oplus \\ \mathfrak{N}\end{matrix}, \tag{4.5}$$

and let $\widetilde{M}:\mathcal{H}\to\mathfrak{H}$ and its adjoint $\widetilde{M}^*:\mathfrak{H}\to\mathcal{H}$ be given by

$$\widetilde{M} = \begin{bmatrix} 0 \\ D_{K_0^*}X_{12}\end{bmatrix} : \mathcal{H} \to \begin{matrix}\mathfrak{H}_0 \\ \oplus \\ \mathfrak{N}\end{matrix}, \quad \widetilde{M}^* = \begin{bmatrix}0 & X_{12}^*D_{K_0^*}\end{bmatrix} : \begin{matrix}\mathfrak{H}_0 \\ \oplus \\ \mathfrak{N}\end{matrix} \to \mathcal{H}.$$

Let the operator $\widetilde{B}=\widetilde{B}_X$ be given by (3.2). We rewrite it in the form

$$\widetilde{B}_X = \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M}^* & X_{22}\end{bmatrix} : \begin{matrix}\mathfrak{H} \\ \oplus \\ \mathcal{H}\end{matrix} \to \begin{matrix}\mathfrak{H} \\ \oplus \\ \mathcal{H}\end{matrix}.$$

Consider a passive $pqs$-system $\mathcal{T}_X = \left\{\widetilde{B}_X; \mathfrak{H}, \mathfrak{H}, \mathcal{H}\right\}$ with the state space $\mathcal{H}$ and the input-output space $\mathfrak{H}$; see Subsection 2.2. The transfer function of $\mathcal{T}_X$ is given by

$$\begin{aligned}&\widetilde{C} + z\widetilde{M}(I - zX_{22})^{-1}\widetilde{M}^* \\ &= \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}X_{11}D_{K_0^*}\end{bmatrix} + z\begin{bmatrix} 0 \\ D_{K_0^*}X_{12}\end{bmatrix}(I - zX_{22})^{-1}\begin{bmatrix}0 & X_{12}^*D_{K_0^*}\end{bmatrix} \\ &= \begin{bmatrix} B_0 & D_{B_0}K_0^* \\ K_0D_{B_0} & -K_0B_0K_0^* + D_{K_0^*}\Phi_X(z)D_{K_0^*}\end{bmatrix}.\end{aligned}$$

Comparing this with (4.3) it is seen that the transfer function of $\mathcal{T}_X$ is in fact $\widehat{B}_X(z)$.

Now consider the passive selfadjoint discrete-time system $\Sigma_X = \left\{\widetilde{B}_X; \mathcal{H}, \mathcal{H}, \mathfrak{H}\right\}$ with the state space $\mathfrak{H}$ and the input-output space $\mathcal{H}$. The transfer function $\Theta$ of the system $\Sigma_X$ is given by

$$\Theta(z) = X_{22} + z\widetilde{M}^*(I_{\mathfrak{H}} - z\widetilde{C})^{-1}\widetilde{M} = X_{22} + zX_{12}^*D_{K_0^*}P_{\mathfrak{N}}(I_{\mathfrak{H}} - z\widetilde{C})^{-1}D_{K_0^*}X_{12}. \tag{4.6}$$

The function $Q_{\widetilde{C}}(\lambda) = P_{\mathfrak{N}}(\widetilde{C} - \lambda I_{\mathfrak{H}})^{-1}\upharpoonright\mathfrak{N},\ \lambda \in \rho(\widetilde{C})$ is called the $Q$-function [11] of $\widetilde{C}$. Hence,

$$\Theta(z) = X_{22} - X_{12}^*D_{K_0^*}Q_{\widetilde{C}}(1/z)D_{K_0^*}X_{12}, \quad 1/z \in \rho(\widetilde{C}).$$

The transfer function $\Theta$ possesses the following properties (see Subsection 2.2):

(1) $\Theta$ belongs to Herglotz-Nevanlinna class for $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$,
(2) $\Theta$ is a contraction for $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
(3) $\Theta$ has strong limit values $\Theta(\pm 1)$,
(4) if $\beta \in [0, \pi/2)$, then

$$\begin{cases} |z\sin\beta + i\cos\beta| \le 1 \\ \quad z \ne \pm 1 \end{cases} \Longrightarrow \|\Theta(z)\sin\beta + i\cos\beta\, I_{\mathcal{H}}\|_{\mathcal{H}} \le 1, \tag{4.7}$$
$$\begin{cases} |z\sin\beta - i\cos\beta| \le 1 \\ \quad z \ne \pm 1 \end{cases} \Longrightarrow \|\Theta(z)\sin\beta - i\cos\beta\, I_{\mathcal{H}}\|_{\mathcal{H}} \le 1.$$

Furthermore, it follows from Theorem 3.3 and the formula (4.6) that

$$\begin{aligned}\Theta(-1) &= (I + \widetilde{B}_X)_{\mathcal{H}}\upharpoonright\mathcal{H} - I_{\mathcal{H}} = (I + X)_{\mathcal{H}}\upharpoonright\mathcal{H} - I_{\mathcal{H}}, \\ \Theta(1) &= I_{\mathcal{H}} - (I - \widetilde{B}_X)_{\mathcal{H}}\upharpoonright\mathcal{H} = I_{\mathcal{H}} - (I - X)_{\mathcal{H}}\upharpoonright\mathcal{H}.\end{aligned}$$

Using the Schur-Frobenius formula (2.7) one gets the following analog of Theorem 4.1.

**Corollary 4.2.** *The relation*

$$P_{\mathcal{H}}(z\widetilde{B}_X - I)^{-1}\upharpoonright\mathcal{H} = (z\Theta(z) - I)^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$$

*is valid.*

In the next theorem we show that a simple Hermitian contraction $B$ and its $sc$-extension $\widetilde{B}$ can be recovered up to the unitary equivalence by means of $\widehat{B}(z)$ or $\Theta(z)$.

**Theorem 4.3.** *1) Let $\mathfrak{H}$ be a Hilbert space and let the Herglotz-Nevanlinna function $\widehat{B}(z)$ be from the class $\mathbf{S}^s(\mathfrak{H})$. Then there exist a Hermitian contraction $B$ in $\mathfrak{H}$ and its sc-extension $\widetilde{B}$ in the Hilbert space $\mathfrak{H} \oplus \mathcal{H}$ such that*

$$P_{\mathfrak{H}}(z\widetilde{B} - I)^{-1}\upharpoonright\mathfrak{H} = (z\widehat{B}(z) - I)^{-1}.$$

*2) Let $\mathcal{H}$ be a Hilbert space and let the Herglotz-Nevanlinna function $\Theta(z)$ be from the class $\mathbf{S}^s(\mathcal{H})$. Then there exist a Hilbert space $\mathfrak{H}$, a simple Hermitian contraction $B$ in $\mathfrak{H}$ and its sc-extension $\widetilde{B}$ in the Hilbert space $\mathfrak{H} \oplus \mathcal{H}$ such that*

$$P_{\mathcal{H}}(z\widetilde{B} - I)^{-1}\upharpoonright\mathcal{H} = (z\Theta(z) - I)^{-1}.$$

*Proof.* 1) It is well known that the function $\widehat{B}(z)$ can be realized as the transfer function of a minimal passive selfadjoint system

$$\mathcal{T} = \left\{ \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M}^* & Y \end{bmatrix}, \mathfrak{H}, \mathfrak{H}, \mathcal{H} \right\}$$

with input-output space $\mathfrak{H}$ and the state space $\mathcal{H}$. Here the operator

$$\widetilde{B} = \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M}^* & Y \end{bmatrix} : \begin{array}{c} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{array} \to \begin{array}{c} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{array}$$

is a selfadjoint contraction, $\overline{\operatorname{span}} \left\{ Y^n \widetilde{M}^* \mathfrak{H} : \ n \in \mathbb{N}_0 \right\} = \mathcal{H}$, and

$$\widehat{B}(z) = \widetilde{C} + z\widetilde{M}(I - zY)^{-1}\widetilde{M}^*.$$

The minimal system $\mathcal{T}$ is determined by $\widehat{B}(z)$ uniquely up to unitary equivalence (see [13]). For the derivative $\widehat{B}'(0)$ one has $\widehat{B}'(0) = \widetilde{M}\widetilde{M}^*$. Now introduce

$$\mathfrak{H}_0 := \ker \widehat{B}'(0) = \ker \widetilde{M}^*, \quad B := \widetilde{C} \upharpoonright \mathfrak{H}_0.$$

Then $B$ is a Hermitian contraction, $\widetilde{C}$ is an *sc*-extension of $B$ in $\mathfrak{H}$, and $\widetilde{B}$ is an *sc*-extension of $B$ in $\mathfrak{H} \oplus \mathcal{H}$. Notice, that $B$ is nondensely defined precisely when

$$\mathfrak{H}_0 \neq \mathfrak{H} \iff \widetilde{M} \neq 0 \iff \mathcal{H} \neq \{0\}.$$

Therefore, one can write (cf. (2.15))

$$\widetilde{C} = \frac{1}{2}(B_M + B_\mu) + \frac{1}{2}(B_M - B_\mu)^{1/2} X_{11} (B_M - B_\mu)^{1/2}$$

and

$$\widetilde{B} = \begin{bmatrix} \dfrac{B_M + B_\mu}{2} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (B_M - B_\mu)^{1/2} & 0 \\ 0 & \sqrt{2} I \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & Y \end{bmatrix} \begin{bmatrix} (B_M - B_\mu)^{1/2} & 0 \\ 0 & \sqrt{2} I \end{bmatrix},$$

where $X_{11}$ is selfadjoint contraction in $\overline{\operatorname{ran}}\,(B_M - B_\mu)$ and $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & Y \end{bmatrix}$ is a selfadjoint contraction in $\overline{\operatorname{ran}}\,(B_M - B_\mu) \oplus \mathcal{H}$. Thus

$$\widetilde{B} = \begin{bmatrix} \frac{1}{2}(B_M + B_\mu) + \frac{1}{2}(B_M - B_\mu)^{1/2} X_{11} (B_M - B_\mu)^{1/2} & \frac{1}{\sqrt{2}}(B_M - B_\mu)^{1/2} X_{12} \\ \frac{1}{\sqrt{2}} X_{12}^* (B_M - B_\mu)^{1/2} & Y \end{bmatrix}.$$

Hence $\widetilde{M} = \frac{1}{\sqrt{2}}(B_M - B_\mu)^{1/2} X_{12}$, $\widetilde{M}^* = \frac{1}{\sqrt{2}} X_{12}^* (B_M - B_\mu)^{1/2}$, and

$$\begin{aligned} \widehat{B}(z) &= \widetilde{C} + \frac{1}{2}(B_M - B_\mu)^{1/2} z X_{12} (I - zY)^{-1} X_{12}^* (B_M - B_\mu)^{1/2} \\ &= \frac{1}{2}(B_M + B_\mu) + \frac{1}{2}(B_M - B_\mu)^{1/2} \left( X_{11} + z X_{12} (I - zY)^{-1} X_{12}^* \right) (B_M - B_\mu)^{1/2}. \end{aligned}$$

Therefore, $\widehat{B}(z)$ is of the form (4.3). Applying Theorem 4.1 and the formula (4.2) one gets the first statement of the theorem.

2) The function $\Theta$ can be realized as the transfer function of the minimal passive selfadjoint system

$$\Sigma = \left\{ \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M}^* & Y \end{bmatrix}, \mathcal{H}, \mathcal{H}, \mathfrak{H} \right\}$$

with input-output space $\mathcal{H}$ and the state space $\mathfrak{H}$. Again the operator

$$\widetilde{B} = \begin{bmatrix} \widetilde{C} & \widetilde{M} \\ \widetilde{M}^* & Y \end{bmatrix} : \begin{array}{c} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{array} \rightarrow \begin{array}{c} \mathfrak{H} \\ \oplus \\ \mathcal{H} \end{array}$$

is a selfadjoint contraction,

$$\overline{\operatorname{span}} \left\{ \widetilde{C}^n \widetilde{M} \mathcal{H},\ n \in \mathbb{N}_0 \right\} = \mathfrak{H}, \tag{4.8}$$

and

$$\Theta(z) = Y + z\widetilde{M}^*(I_{\mathfrak{H}} - z\widetilde{C})^{-1}\widetilde{M}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

The minimal system $\Sigma$ is determined by $\Theta$ uniquely up to unitary equivalence; see [13]. Define

$$\mathfrak{N} := \overline{\operatorname{ran}}\, \widetilde{M}, \quad \mathfrak{H}_0 := \mathfrak{H} \ominus \mathfrak{N} = \ker \widetilde{M}^*, \quad B := \widetilde{C} \upharpoonright \mathfrak{H}_0.$$

Then $B$ is a Hermitian contraction, $\operatorname{dom} B = \mathfrak{H}_0$, and $\widetilde{B}$ is an *sc*-extension of $B$. Moreover, (4.8) means that the operator $B$ is simple, i.e., it has no reducing subspace on which $B$ is selfadjoint. To complete the proof it remains to apply Corollary 4.2. □

The last part of this section is devoted to the study of the following linear fractional transformation of the transfer function $\Theta(z)$ of the form (4.6):

$$\begin{aligned} \mathcal{N}(\lambda) &= I_{\mathcal{H}} - 2\left(I_{\mathcal{H}} + \Theta\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1} \\ &= \left\{ \left\{ \left(I_{\mathcal{H}} + \Theta\left(\frac{1+\lambda}{1-\lambda}\right)\right) h, \left(\Theta\left(\frac{1+\lambda}{1-\lambda}\right) - I_{\mathcal{H}}\right) h \right\},\ h \in \mathcal{H} \right\}, \end{aligned} \tag{4.9}$$

where $\lambda \in \mathbb{C} \setminus [0, +\infty)$. From the properties in (4.7) it follows that for all $\beta \in (0, \pi/2)$

$$\arg\lambda \in [\pi - \beta, \pi + \beta] \implies |\operatorname{Im}(\mathcal{N}(\lambda)f, f)_{\mathcal{H}}| \le \tan\beta \operatorname{Re}(\mathcal{N}(\lambda)f, f)_{\mathcal{H}}, \quad f \in \operatorname{dom}\mathcal{N}(\lambda).$$

Hence, the linear relation $\mathcal{N}(\lambda)$ is $m$-sectorial for each $\operatorname{Re}\lambda < 0$ and, in particular, if $\lambda < 0$ then $\mathcal{N}(\lambda)$ is nonnegative and selfadjoint. In the next theorem the main analytic properties of $\mathcal{N}(\lambda)$ are established and an explicit representation for $\mathcal{N}(\lambda)$ is obtained. Using the terminology in [26] the result shows in particular that $\mathcal{N}(\lambda)$ forms a holomorphic family of the type (B) in the left open half-plane.

**Theorem 4.4.** *The domain $\mathfrak{L} := \mathcal{D}[\mathcal{N}(\lambda)]$ of the closed form $\mathcal{N}(\lambda)[\cdot,\cdot]$ associated with the family $\mathcal{N}(\lambda)$ in* (4.9) *does not depend on $\lambda$, $\operatorname{Re}\lambda < 0$, and the form $\mathcal{N}(\lambda)[h, g]$ admits the representation*

$$\begin{aligned} &\mathcal{N}(\lambda)[h, g] \\ &\quad = \left(\left(I_{\mathcal{H}} + V(\widehat{B}_1 - \widehat{B}_0)^{1/2}\left(\widehat{B}_0 - \frac{1-\lambda}{1+\lambda} I_{\mathfrak{H}}\right)^{-1}(\widehat{B}_1 - \widehat{B}_0)^{1/2}V^*\right)Yh, Yg\right), \\ &\qquad\qquad \operatorname{Re}\lambda < 0, \quad h, g \in \mathfrak{L} := \operatorname{ran}(I_{\mathcal{H}} + \Theta(0))^{1/2}, \end{aligned}$$

*where $\widehat{B}_0$ and $\widehat{B}_1$ are as defined in* (3.7) *and* (3.8),

$$Y = (I_{\mathcal{H}} - \Theta(0))^{1/2}(I_{\mathcal{H}} + \Theta(0))^{(-1/2)} : \mathfrak{L} \to \overline{\mathfrak{L}},$$

*and $V : \overline{\operatorname{ran}}\,(\widehat{B}_1 - \widehat{B}_0) \to \mathcal{H}$ is an isometry. Here $(I_{\mathcal{H}} + \Theta(0))^{(-1/2)}$ is the Moore-Penrose pseudo inverse.*

*Proof.* Since $\Theta$ is the transfer function of a passive selfadjoint discrete-time system, see (4.6), $||\Theta(z)|| \le 1$ for all $|z| < 1$ and $\Theta^*(z) = \Theta(\bar{z})$. Then the real part

$$\operatorname{Re}(\Theta(z)) = \frac{1}{2}(\Theta(z) + \Theta^*(z))$$

satisfies $I_{\mathcal{H}} \pm \operatorname{Re}(\Theta(z)) \ge 0$ for all $z \in \mathbb{D}$. Since $I_{\mathcal{H}} \pm \operatorname{Re}(\Theta(z))$ are harmonic functions, a result of Yu.L. Shmul'yan [43] yields the following invariance equalities

$$\operatorname{ran}(I_{\mathcal{H}} + \operatorname{Re}\Theta(z))^{1/2} = \operatorname{ran}(I_{\mathcal{H}} + \Theta(0))^{1/2},$$
$$\operatorname{ran}(I_{\mathcal{H}} - \operatorname{Re}\Theta(z)))^{1/2} = \operatorname{ran}(I_{\mathcal{H}} - \Theta(0))^{1/2}$$

for all $z \in \mathbb{D}$; observe that $\Theta(0) = \Theta(0)^*$. From Douglas Theorem [23] we get

$$(I_{\mathcal{H}} + \operatorname{Re}\Theta(z))^{1/2} = (I_{\mathcal{H}} + \Theta(0))^{1/2} F(z),$$

where $F^{-1}(z)$ is bounded for all $z \in \mathbb{D}$ in $\overline{\operatorname{ran}}(I_{\mathcal{H}} + \Theta(0))$. Since $\Theta(z) \in \widetilde{C}_{\mathcal{H}}$ for all $z \in \mathbb{D}$, the operators $I_{\mathcal{H}} + \Theta(z)$ are $m$-sectorial bounded operators. Therefore,

$$I_{\mathcal{H}} + \Theta(z) = (I_{\mathcal{H}} + \Theta_R(z))^{1/2}(I + iG(z))(I_{\mathcal{H}} + \Theta_R(z))^{1/2}, \quad z \in \mathbb{D},$$

where $G(z) = G^*(z)$ in the subspace $\overline{\operatorname{ran}}(I_{\mathcal{H}} + \Theta_R(0))$ and $I$ is the identity operator in $\overline{\operatorname{ran}}(I_{\mathcal{H}} + \Theta_R(0))$. Hence

$$I_{\mathcal{H}} + \Theta(z) = (I_{\mathcal{H}} + \Theta_R(0))^{1/2} F(z)(I + iG(z))F^*(z)(I_{\mathcal{H}} + \Theta_R(0))^{1/2}, \quad z \in \mathbb{D}.$$

In addition, the function $\Theta$ can be represented in the form (see [43])

$$\Theta(z) = \Theta(0) + D_{\Theta(0)}\Phi(z)D_{\Theta(0)}, \quad z \in \mathbb{D},$$

where $\Phi(z)$ is holomorphic in $\mathbb{D}$. Since $D_{\Theta(0)} = (I_{\mathcal{H}} + \Theta(0))^{1/2}(I_{\mathcal{H}} - \Theta(0))^{1/2}$ we obtain

$$\Theta(z) = \Theta(0) + (I_{\mathcal{H}} + \Theta(0))^{1/2}\Psi(z)(I_{\mathcal{H}} + \Theta(0))^{1/2}, \quad z \in \mathbb{D},$$

where

$$\Psi(z) = (I_{\mathcal{H}} - \Theta(0))^{1/2}\Phi(z)(I_{\mathcal{H}} - \Theta(0))^{1/2}.$$

On the other hand,

$$I_{\mathcal{H}} + \Theta(z) = (I_{\mathcal{H}} + \Theta(0))^{1/2}(I + \Psi(z))(I_{\mathcal{H}} + \Theta(0))^{1/2}. \tag{4.10}$$

Thus $I + \Psi(z) = F(z)(I + iG(z))F^*(z)$. It follows that $I + \Psi(z)$ has bounded inverse in $\overline{\operatorname{ran}}(I_{\mathcal{H}} + \Theta(0))^{1/2}$. Furthermore we use Proposition 2.2. For $\lambda$ with $\operatorname{Re}\lambda < 0$ we get

$$\mathcal{D}[\mathcal{N}(\lambda)] = \operatorname{ran}\left(I_{\mathcal{H}} + \operatorname{Re}\Theta\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{1/2} = \operatorname{ran}(I_{\mathcal{H}} + \Theta(0))^{1/2}, \quad \operatorname{Re}\lambda < 0.$$

Consequently, the domain $\mathcal{D}[\mathcal{N}(\lambda)]$ of the closed sectorial form $\mathcal{N}(\lambda)[\cdot,\cdot]$ is constant if $\operatorname{Re}\lambda < 0$. For $u \in \operatorname{ran}(I_{\mathcal{H}} + \Theta(0))^{1/2}$ if $\operatorname{Re}\lambda < 0$ and $\lambda = (z-1)(z+1)^{-1}$ we get

$$\begin{aligned} \mathcal{N}(\lambda)[u] &= -||u||^2 + 2((I+iG(z))^{-1}(I_{\mathcal{H}} + \operatorname{Re}\Theta(z))^{-1/2}u, (I_{\mathcal{H}} + \operatorname{Re}\Theta(z))^{-1/2}u) \\ &= -||u||^2 + 2((I+iG(z))^{-1}F^{-1}(z)(I_{\mathcal{H}} + \Theta(0))^{-1/2}u, F^{-1}(z)(I_{\mathcal{H}} + \Theta(0))^{-1/2}u) \\ &= -||u||^2 + 2((I+\Psi(z))^{-1}(I_{\mathcal{H}} + \Theta(0))^{-1/2}u, (I_{\mathcal{H}} + \Theta(0))^{-1/2}u). \end{aligned} \tag{4.11}$$

Therefore, $\mathcal{N}(\lambda)[u]$ is holomorphic in $\lambda$ in the left half-plane. Consequently, $\mathcal{N}(\lambda)$ forms a holomorphic family of type (B) in the left half-plane in the sense of [26].

Next the representation of the form $\mathcal{N}(\lambda)[\cdot,\cdot]$ is derived. Let $\widetilde{B}=\widetilde{B}_X$ be as in (3.2), let $\widehat{Z}_0$ and $\widehat{Z}_1$ be given by (3.3), and let $\widehat{B}_0$ and $\widehat{B}_1$ be given by (3.7) and (3.8), respectively. Then using the representation

$$X=\begin{bmatrix} X_{11} & UD_{X_{22}} \\ D_{X_{22}}U^* & X_{22}\end{bmatrix} : \begin{matrix}\mathfrak{N}\\ \oplus \\ \mathcal{H}\end{matrix} \to \begin{matrix}\mathfrak{N}\\ \oplus \\ \mathcal{H}\end{matrix},$$

where $U\in\mathbf{L}(\mathfrak{D}_{X_{22}},\mathfrak{N})$ is a contraction, see Remark 2.5, one can write

$$\widehat{Z}_0=X_{11}-U(I_{\mathfrak{D}_{X_{22}}}-X_{22})U^*,\quad \widehat{Z}_1=X_{11}+U(I_{\mathfrak{D}_{X_{22}}}+X_{22})U^*.$$

Moreover, $\widehat{B}_1-\widehat{B}_0=2D_{K_0^*}UU^*D_{K_0^*}P_{\mathfrak{N}}$ and if $\widetilde{C}$ is an in (4.5) then

$$\widetilde{C}-\widehat{B}_0=D_{K_0^*}(X_{11}-\widehat{Z}_0)D_{K_0^*}P_{\mathfrak{N}}=D_{K_0^*}U(I_{\mathfrak{D}_{X_{22}}}-X_{22})U^*D_{K_0^*}P_{\mathfrak{N}}.$$

Define

$$Q_{\widetilde{C}}(\xi)=P_{\mathfrak{N}}(\widetilde{C}-\xi I_{\mathfrak{H}})^{-1}{\upharpoonright}\mathfrak{N},\quad \xi\in\rho(\widetilde{C}).$$

Then it follows from (2.7) that

$$Q_{\widetilde{C}}(\xi)=Q_{\widehat{B}_0}(\xi)\left(I_{\mathfrak{N}}+(\widetilde{C}-\widehat{B}_0)Q_{\widehat{B}_0}(\xi)\right)^{-1},\quad \xi\in\mathbb{C}\setminus[-1,1],$$

cf. [11]. Furthermore, for $\xi\in\mathbb{C}\setminus[-1,1]$

$$\begin{aligned}
&X_{12}^*D_{K_0^*}Q_{\widetilde{C}}(\xi)D_{K_0^*}X_{12}=D_{X_{22}}U^*D_{K_0^*}Q_{\widetilde{C}}(\xi)D_{K_0^*}UD_{X_{22}}\\
&\quad=D_{X_{22}}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)\left(I_{\mathfrak{N}}+D_{K_0^*}U(I_{\mathfrak{D}_{X_{22}}}-X_{22})U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)\right)^{-1}D_{K_0^*}UD_{X_{22}}\\
&\quad=(I_{\mathcal{H}}+X_{22})^{1/2}\left(I_{\mathcal{H}}+(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UP_{\mathfrak{D}_{X_{22}}}(I_{\mathcal{H}}-X_{22})^{1/2}\right)^{-1}\\
&\quad\times(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UD_{X_{22}},
\end{aligned}$$

where $P_{\mathfrak{D}_{X_{22}}}$ is the orthogonal projection in $\mathcal{H}$ onto $\mathfrak{D}_{X_{22}}$ and the last identity follows from

$$\begin{aligned}
&\left(I_{\mathcal{H}}+(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UP_{\mathfrak{D}_{X_{22}}}(I_{\mathcal{H}}-X_{22})^{1/2}\right)\\
&\qquad\times(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)\\
&\qquad=(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)\left(I_{\mathfrak{N}}+D_{K_0^*}U(I_{\mathfrak{D}_{X_{22}}}-X_{22})U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)\right).
\end{aligned}$$

This yields, see (4.6),

$$\begin{aligned}
&I_{\mathcal{H}}+\Theta\left(1/\xi\right)=I_{\mathcal{H}}+X_{22}-X_{12}^*D_{K_0^*}Q_{\widetilde{C}}(\xi)D_{K_0^*}X_{12}\\
&\quad=(I_{\mathcal{H}}+X_{22})^{1/2}\left(I_{\mathcal{H}}+(I_{\mathcal{H}}-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UP_{\mathfrak{D}_{X_{22}}}(I_{\mathcal{H}}-X_{22})^{1/2}\right)^{-1}\\
&\quad\times(I_{\mathcal{H}}+X_{22})^{1/2}.
\end{aligned}$$

Since $\Theta(0)=X_{22}$, it follows from (4.10) that

$$\begin{aligned}
&I+\Psi\left(1/\xi\right)\\
&\quad=\left(I+(I-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UP_{\mathfrak{D}_{X_{22}}}(I-X_{22})^{1/2}\right)^{-1}{\upharpoonright}\overline{\operatorname{ran}}\,(I_{\mathcal{H}}+\Theta(0)),
\end{aligned}$$

where $I=I_{\overline{\operatorname{ran}}\,(I_{\mathcal{H}}+\Theta(0))}$. The equality $\widehat{B}_1-\widehat{B}_0=2D_{K_0^*}UU^*D_{K_0^*}P_{\mathfrak{N}}$ implies that

$$\sqrt{2}U^*D_{K_0^*}P_{\mathfrak{N}}=V(\widehat{B}_1-\widehat{B}_0)^{1/2},$$

holds for some isometry $V$ mapping $\overline{\operatorname{ran}}\,(\widehat{B}_1 - \widehat{B}_0)$ onto $\overline{\operatorname{ran}}\,U^*(\subseteq \mathfrak{D}_{X_{22}} \subseteq \mathcal{H})$. Hence,

$$\begin{aligned}(I+\Psi\,(1/\xi))^{-1} &= I + (I-X_{22})^{1/2}U^*D_{K_0^*}Q_{\widehat{B}_0}(\xi)D_{K_0^*}UP_{\mathfrak{D}_{X_{22}}}(I-X_{22})^{1/2}\\ &= I + \frac{1}{2}(I-X_{22})^{1/2}V(\widehat{B}_1-\widehat{B}_0)^{1/2}Q_{\widehat{B}_0}(\xi)(\widehat{B}_1-\widehat{B}_0)^{1/2}V^*(I-X_{22})^{1/2}\\ &= \frac{1}{2}(I+X_{22})\\ &+ \frac{1}{2}(I-X_{22})^{1/2}\left(I+V(\widehat{B}_1-\widehat{B}_0)^{1/2}Q_{\widehat{B}_0}(\xi)(\widehat{B}_1-\widehat{B}_0)^{1/2}V^*\right)(I-X_{22})^{1/2}.\end{aligned}$$

It remains to substitute this expression into the representation of $\mathcal{N}(\lambda)$ in (4.11) to conclude that for $h, g \in \operatorname{ran}(I+X_{22})^{1/2}$ and for $\operatorname{Re}\lambda < 0$ with $\xi = (1-\lambda)(1+\lambda)^{-1}$,

$$\mathcal{N}(\lambda)[h,g] = \left(\left(I+V(\widehat{B}_1-\widehat{B}_0)^{1/2}Q_{\widehat{B}_0}(\xi)(\widehat{B}_1-\widehat{B}_0)^{1/2}V^*\right)Yh, Yg\right),$$

where

$$Y = (I-X_{22})^{1/2}(I+X_{22})^{(-1/2)} = (I_{\mathcal{H}}-\Theta(0))^{1/2}(I_{\mathcal{H}}+\Theta(0))^{(-1/2)} : \mathfrak{L} \to \overline{\mathfrak{L}}.$$

This completes the proof. □

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Department of Mathematical Analysis, East Ukrainian National University; Department of Mathematics, Dragomanov National Pedagogical University, 9 Pirogova Str., Kyiv, 01601, Ukraine
*E-mail address*: yury.arlinskii@gmail.com

Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, 65101 Vaasa, Finland
*E-mail address*: sha@uwasa.fi

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