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TANNAKA–KREIN DUALITY FOR COMPACT QUANTUM GROUP COACTIONS (SURVEY)

LEONID VAINERMAN

Dedicated to Professor Yurij M. Berezansky on the occasion of his 90th anniversary

Abstract. The last decade saw an appearance of a series of papers containing a very interesting development of the Tannaka-Krein duality for compact quantum group coactions on C^*-algebras. The present survey is intended to present the main ideas and constructions underlying this development.

1. Introduction

According to the Pontryagin's duality for abelian locally compact groups, such a group can be reconstructed if its dual (i.e., the group of its continuous characters) is known. Extending this result to non-commutative groups, T. Tannaka [30] showed that a compact group G can be reconstructed if the set $\mathrm{Rep}(G)$ of its continuous finite dimensional representations is known. In 1949, M. G. Krein [13] gave an abstract description of $\mathrm{Rep}(G)$.

Later on, mainly due to works by A. Grothendieck, P. Deligne, and N. Saavedra Rivano, these results referred to as "Tannaka-Krein duality for compact groups" were formulated in the language of symmetric monoidal (or tensor) categories and extended to affine algebraic groups.

After the discovery of quantum groups by V. G. Drinfeld and M. Jimbo, there was a number of papers on the Tannaka-Krein duality for this new class of objects in a purely algebraic context – see [12] and the references therein. Here the representation categories were not symmetric, in general.

Motivated by superselection principles in quantum field theory, S. Doplicher and J. E. Roberts introduced the notion of a C^*-tensor category with conjugates [8] and proved that any such a category with permutation symmetry is equivalent to the representation category of a unique compact group. In the setting of *compact quantum groups* (CQG) [40], [42], the S. L. Woronowicz's Tannaka-Krein duality [43] claims that any C^*-tensor category with conjugates and a unitary tensor functor to the category $\mathcal{H}_f$ of finite dimensional Hilbert spaces (fiber functor), is equivalent to the category $\mathrm{Rep}(G)$ of unitary finite dimensional representations of a unique CQG G with the canonical fiber functor sending any representation to the Hilbert space where it acts. This can be viewed as a reconstruction of a regular coaction of G via its coproduct on the C^*-algebra $C(G)$ "of all continuous functions on G".

As for a general continuous coaction α of a CQG G on a unital C^*-algebra B, the problem is to find a structure on the category $\mathrm{Rep}(G)$ which would replace the above mentioned canonical fiber functor in order to be able to reconstruct B and α from this structure.

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The results by A. Wassermann [37], [38] and M. B. Landstad [15] concerning usual group actions on C^*-algebras mean that there is a bijection between ergodic actions of full multiplicity of a compact group G and arbitrary fiber functors $\mathrm{Rep}(G) \to \mathcal{H}_f$. In particular, A. Wassermann [39] proved that any erogdic action of $SU(2)$ is obtained by induction from a projective representation of its closed subgroup. Thus, there are countably many isomorphism classes of such actions. The Hopf algebraic version of the above mentioned general result is due to K.-H. Ulbrich [34] and P. Schauenburg [28].

The general theory of ergodic coactions of CQG's on C^*-algebras has been initiated by F. Boca [5] and Landstad [16]. P. Podleś [25] showed that there is a continuum of non-isomorphic ergodic coactions of the famous compact quantum group $SU_q(2)(|q| < 1)$ [41] on the coideal C^*-subalgebras of $C(G)$ called "quantum 2-spheres" which do not necessarily correspond to quantum subgroups of $SU_q(2)$. More generally, R. Tomatsu [32] classified all the coideal C^*-subalgebras of $C(SU_q(2))$.

The bijection between unitary fiber functors of $\mathrm{Rep}(G)$ and ergodic coactions of full multiplicity of a CQG G has been established by J. Bichon, A. De Rijdt, and S. Vaes [4]; using this, they classified all such coactions for universal orthogonal and unitary quantum groups and constructed, for $|q|$ small enough, a family of such coactions of $SU_q(2)$ which are not coideals.

It is clear that in order to extend the categorical duality to wider classes of coactions, one has to replace fiber functors by some more general structure. In particular, C. Pinzari and J. E. Roberts [24] introduced *spectral* functors $F : \mathrm{Rep}(G) \to \mathcal{H}_f$ which are not tensor, but only *quasi-tensor* which means that, for any $U, V \in \mathrm{Rep}(G)$, $F(U) \otimes F(V)$ is isometric to a subspace of $F(U \otimes V)$ and not to the whole space. They showed that spectral functors exactly correspond to all ergodic coactions of G.

Later on, A. De Rijdt and N. Vander Vennet [9] obtained similar result for non-ergodic coactions, but in other terms. Finally, S. Neshveyev [18] introduced *weak tensor functors* and established the categorical duality for arbitrary CQG coactions. Note that this new notion in ergodic case is equivalent to the one of a quasi-tensor functor and that in both [24], [18] the reconstruction procedure follows the lines of [43].

There is an alternative formulation of the Tannaka-Krein duality for CQG coactions motivated by reconstruction theorems for finite semisimple tensor (or fusion) categories [11]. In particular, the notion of a *module category* [23] has been adopted by K. De Commer and M. Yamashita to C^*-tensor category context [6]. They showed that there is a one-to-one correspondence between ergodic coactions of a CQG G and semisimple irreducible module categories over $\mathrm{Rep}(G)$ with a simple generator, which enabled them to classify all ergodic coactions of $SU_q(2)$ in terms of weighted graphs.

This approach was extended by S. Neshveyev [18] to general coactions, he also showed that the two versions of the duality are equivalent. Finally, S. Neshveyev and M. Yamashita [21] proved that the module category in question can be defined by a unitary tensor functor if and only if G coacts on a *Yetter-Drinfeld C^*-algebra*.

In this short survey, we briefly discuss the main ideas and constructions contained in the cited papers, especially, in the recent ones because they give the most clear and general picture of the categorical duality for CQG coactions. The survey is organized as follows. Section 2 contains necessary definitions, results and notations concerning CQG's, their representation categories, and also abstract C^*-tensor categories and unitary tensor functors, our main reference is [19]. Here we freely use without special explanations the standard language of the tensor category theory [17]. Next we remind the reader of the relations between continuous and algebraic CQG coactions on C^*-algebras – see [5], [32] and [6] for ergodic and [18] for nonergodic case.

Section 3 is devoted to the presentation of the categorical duality. First, we explain the construction of the functor from the category of G-coactions to the category of pairs

$(\mathcal{M}, M)$, where $\mathcal{M}$ is a Rep(G)-module category and M its generator. The discussion is based on the ideas from [6] and uses essentially G-equivariant Hilbert modules introduced earlier in K-theory [1]. Conversely, such a pair $(\mathcal{M}, M)$ gives rise to an algebraic G-coaction on a unital $*$-algebra $\mathcal{B}$ – see [6], [18], and then to a continuous G-coaction. These two functors define an equivalence of the categories in question, i.e., the Tannaka-Krein duality for G-coactions. Then we present an alternative construction in terms of weak tensor functors introduced in [18], which are, in the ergodic case, equivalent to spectral functors studied earlier in [24]. If a C^*-algebra carries coactions of both, CQG and its dual, compatible in a special way (Yetter-Drinfeld algebra), the Tannaka-Krein duality can be formulated in terms of a unitary tensor functor. The discussion of this situation is based on [21].

Section 4 presents two applications of the Tannaka-Krein duality. The first of them is a characterisation of subalgebras of $C(G)$ invariant with respect to coactions of a CQG G and its dual as coideals of quotient type, which was obtained in [21]. Next we briefly explain, in which way the categorical duality was used in [7] in order to reduce the classification of ergodic $SU_q(2)$-coactions to the classification of weighted oriented graphs. The discussion of the last problem is beyond the scope of this survey, so we only mention some of the nice results obtained in this way.

We call a triple (B, G, α), where α is a continuous coaction of a CQG G on a unital C^*-algebra B, a $G-C^*$-algebra, this terminology is different from the one used in some other papers, for example, in [6], [7], and [32]. We consider only coactions on C^*-algebras and do not discuss their von Neumann algebraic counterparts.

All Hilbert spaces and C^*-algebras are supposed to be separable.

Notations. For an object X of a category $\mathcal{C}$, we write $X \in \mathcal{C}$. The set of morphisms between $X, Y \in \mathcal{C}$ is denoted by $\mathcal{C}(X, Y)$ except for $\mathcal{C} = \text{Rep}(G)$, where we write Mor$(X, Y)$. If $\mathcal{E}$ is a right Hilbert module over a C^*-algebra A, $\mathcal{L}(\mathcal{E})$ is the space of all bounded adjointable A-linear operators acting on $\mathcal{E}$ and $\mathcal{K}(\mathcal{E})$ is its closed ideal of "compact" operators [14]. In the context of C^*-algebras, $\otimes$ means the minimal C^*-tensor product. The algebraic tensor product is denoted by $\otimes_{alg}$. $M(A)$ is the multiplier algebra of a C^*-algebra A and $[B]$ is the norm closure of a vector subspace $B \subset A$. $\mathcal{K}(H)$ is the C^*-algebra of all compact operators on a Hilbert space H, $B(H, K)$ is the space of all bounded operators between Hilbert spaces H and K. $\omega_{\xi,\eta}$ is the linear functional on $B(H) := B(H, H)$ defined by $\omega_{\xi,\eta}(A) =< A\eta, \xi >$.

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2. Preliminaries

2.1. **Representation category of a CQG. 1.** A *compact quantum group* G [40]–[43] is defined by a unital C^*-algebra $C(G)$ equipped with a *coproduct*, i.e., a unital C^*-monomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ satisfying the coassociativity $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ and the cancellation properties: $[(1 \otimes C(G))\Delta(C(G))] = C(G) \otimes C(G) = [(C(G) \otimes 1)\Delta(C(G))]$. There is a unique state h on $C(G)$ satisfying $(h \otimes id)\Delta(a) = h(a)1$ (and/or $(id \otimes h)\Delta(a) = h(a)1$), for all $a \in C(G)$, called the *Haar state.* If h is faithful, G is called reduced.

A *unitary representation* of G on a Hilbert space H_U is a unitary $U \in M(\mathcal{K}(H_U) \otimes C(G))$ such that $(id \otimes \Delta)(U) = U_{12}U_{13}$ (we use the standard leg notation). In particular, 1 is the *trivial representation* of G on $\mathbb{C}$, and the unitary W defined by $W^*(a \otimes b) := \Delta(b)(a \otimes 1)$ $(a, b \in C(G))$ is the *regular representation* of G acting on the GNS Hilbert space $L^2(G, h)$. We have

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \quad \text{and} \quad \Delta(a) = W^*(1 \otimes a)W,$$

for all $a \in C(G)$. The tensor product of unitary representations U and V acting on Hilbert spaces H_U and H_V, respectively, is defined by $U \oplus V := U_{13}V_{23}$. Let $\mathrm{Mor}(U, V)$ be the set of intertwiners between U and V:

$$\mathrm{Mor}(U, V) := \{T \in B(H_U, H_V) | V(T \otimes 1) = (T \otimes 1)U\}.$$

We say that U and V are equivalent (resp., unitarily equivalent) if $\mathrm{Mor}(U, V)$ contains an invertible element (resp., a unitary). In general, $U \oplus V$ and $V \oplus U$ are not equivalent. By definition, $\dim(U) := \dim(H_U)$. U is called *irreducible* if $\mathrm{Mor}(U, U) = \mathbb{C}id$.

The *regular $*$-subalgebra* $\mathbb{C}[G] \subset C(G)$ is defined as the linear span of *matrix coefficients* $u_{\xi,\eta} := (\omega_{\xi,\eta} \otimes id)(U)$ $(\xi, \eta \in H_U)$, where U runs through the finite dimensional unitary representations of G. Denote by $\hat{G}$ the (countable) set of equivalence classes of irreducible unitary representations of G (all of them are finite dimensional), and choose a representative U^x of $x \in \hat{G}$ acting on a Hilbert space H_x. As a linear space,

$$\mathbb{C}[G] = \oplus_{x \in \hat{G}} \overline{H_x} \otimes H_x \cong \oplus_{x \in \hat{G}} B(H_x). \tag{1}$$

It is known that $\mathbb{C}[G]$ is a Hopf $*$-algebra with coproduct Δ, antipode S and counit ε, it is norm dense in $C(G)$, and the state h is faithful on $\mathbb{C}[G]$. Every finite dimensional representation U defines a $*$-representation π_U of the algebraic dual of $\mathbb{C}[G]$ acting on H_U: $\mu \mapsto (id \otimes \mu)(U)$.

We also denote by $\mathbb{C}[G]_U$ the linear span of matrix coefficients of U (in particular, $\mathbb{C}[G]_x := \mathbb{C}[G]_{U^x}$), and by $\mathbf{1}$ the class of the trivial representation.

The C^*-algebra $C^r(G)$ generated by $C(G)$ in the GNS-representation defined by h, is called *the reduced form* of G. The universal C^*-algebraic envelope $C^f(G)$ of $\mathbb{C}[G]$ is called *the universal form* of G.

The C^*-algebra of the dual (discrete) quantum group denoted by $c_0(\hat{G})$ is isomorphic to $c_0 - \oplus_{x \in \hat{G}} B(H_x)$ and can be viewed as part of the algebraic dual of $\mathbb{C}[G]$ with respect to the duality

$$< u^x_{i,j}, e^y_{k,l} >= \delta_{x,y}\delta_{i,k}\delta_{j,l}, \quad \forall x, y \in \hat{G},$$

where $u^x_{i,j}$ are matrix coefficients of U^x and $e^y_{k,l}$ are matrix units in $B(H_y)$ with respect to some orthogonal basis in H_y.

We say that a CQG G *is of Kac type*, if either the antipode S of $\mathbb{C}[G]$ satisfies $S^2 = id$ or h is a tracial state. In this case, it is a *G. I. Kac algebra* in the sense of [10].

2. From an abstract point of view, the category $\mathrm{Rep}(G)$ of unitary finite dimensional representations of G with the above mentioned tensor product and morphisms is a semi-simple rigid tensor C^*-category in the sense of the following definitions (for the details see [19]).

Definition 2.1. A C^*-category $\mathcal{D}$ is a $\mathbb{C}$-linear category whose morphism spaces $\mathcal{D}(X, Y)$ are Banach spaces such that $\|S \circ T\| \leqslant \|S\|\|T\|$ ($\circ$ is a composition of morphisms), admitting antilinear contravariant "conjugation" $* : \mathcal{D}(X, Y) \to \mathcal{D}(Y, X) : T \mapsto T^*$, and satisfying the C^*-condition $\|T^*T\| = \|T\|^2$ for any morphism T. A linear functor between two C^*-categories preserving the $*$-operation is called a unitary C^*-functor. An object $X \in \mathcal{D}$ is called *simple* if $\mathrm{End}(X) := \mathcal{D}(X, X) \simeq \mathbb{C}$. $\mathcal{D}$ is called *semisimple* if it admits finite direct sums and if any its object is isomorphic to a finite direct sum of simples.

Definition 2.2. A strict C^*-tensor category $\mathcal{C} = (\mathcal{D}, \otimes, \mathbf{1})$ is a C^*-category $\mathcal{D}$ with a bilinear C^*-functor $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ and an object $\mathbf{1} \in \mathcal{D}$, which is also a strict tensor category in the sense of [17]. We say that an object $X \in \mathcal{D}$ admits a conjugate or dual if there exists a triple $(\overline{X}, R_X, \overline{R}_X)$, where $\overline{X} \in \mathcal{C}$ and $R_X : \mathbf{1} \to \overline{X} \otimes X$, $\overline{R}_X : \mathbf{1} \to X \otimes \overline{X}$ are morphisms satisfying the *conjugate equations*:

$$(\overline{R}^*_X \otimes id_X)(id_X \otimes R_X) = id_X, \quad (R^*_X \otimes id_{\overline{X}})(id_{\overline{X}} \otimes \overline{R}_X) = id_{\overline{X}}.$$

The category $\mathcal{C}$ is called *rigid* if any its object admits a conjugate.

Remark 2.3. (see [19], 2.2). A rigid C^*-tensor category $\mathcal{C}$ is automatically semisimple and $\mathrm{End}(X)$ is finite dimensional, for all $X \in \mathcal{C}$. If X is a simple object of such a category, then the number $\dim_q(X) = \|R_X\|\|R_{\overline{X}}\|$ is independent on the solution of the conjugate equations and is called the *quantum (or intrinsic) dimension* of X. In general, if $X = \oplus_i X_i$ with X_i simples, put $\dim_q(X) := \Sigma_i \dim_q(X_i)$.

Example 2.4. 1. The category $\mathcal{H}$ of Hilbert spaces with $\mathcal{H}(H,K) := B(H,K)$, $\mathbf{1} = \mathbb{C}$ and usual tensor product, is a C^*-tensor category. Its maximal rigid subcategory $\mathcal{H}_f$ is formed by finite dimensional Hilbert spaces. If $H \in \mathcal{H}_f$, its complex conjugate space $\overline{H}$ gives a dual,

$$R_H^* : \overline{H} \otimes H \mapsto \mathbb{C} : \overline{\zeta} \otimes \eta \mapsto <\eta, \zeta>, \quad \overline{R}_H^* : H \otimes \overline{H} \mapsto \mathbb{C} : \zeta \otimes \overline{\eta} \mapsto <\zeta, \eta> .$$

Here $<\zeta, \eta>$ is the scalar product in H. Then $\dim_q(H) = \dim H$, $\forall H \in \mathcal{H}_f$.

2. $\mathrm{Rep}(G)$ with intertwiners as morphisms, and $\oplus$ as a tensor product, is a rigid C^*-tensor category. If $U \in \mathrm{Rep}(G)$, its conjugate is

$$\overline{U} := (j(\pi_U(\rho)^{1/2}) \otimes 1)(j \otimes id)(U^*)(j(\pi_U(\rho)^{-1/2}) \otimes 1) \in B(\overline{H}_U) \otimes C(G),$$

$$R_U(\mathbf{1}) := \Sigma_i(\overline{\zeta}_i \otimes \pi_U(\rho^{-1/2})(\zeta_i)), \quad \overline{R}_U(\mathbf{1}) := \Sigma_i(\pi_U(\rho^{1/2})(\zeta_i) \otimes \overline{\zeta}_i), \tag{2}$$

where $\rho = f_1$ is the Woronowicz character [40], $j : B(H_U) \to B(\overline{H}_U)$ is the canonical $*$-anti-isomorphism defined by $j(T)\overline{\zeta} = \overline{T^*\zeta}$ and $\{\zeta_i\}_i$ is an orthonormal basis in H_U (see [19]). Note that $\dim_q(U) \geqslant \dim(H_U)$.

3. A *correspondence over* a C^*-algebra A is a right Hilbert A-module $\mathcal{E}$ (see [14]) with a $*$-homomorphism $\varphi : A \to \mathcal{L}(\mathcal{E})$ such that $\varphi(A)(\mathcal{E})$ is dense in $\mathcal{E}$. A-correspondences form a C^*-tensor category $\mathrm{Corr}(A)$ with tensor product $\otimes_A$ and adjointable A-bilinear maps as morphisms [18].

Definition 2.5. A unitary tensor functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between C^*-tensor categories is a $*$-functor $\mathcal{C}_1 \to \mathcal{C}_2$ which is also a linear tensor functor whose natural transformations $F(X) \otimes F(Y) \to F(X \otimes Y)$ and $\mathbf{1}_{\mathcal{C}_2} \to F(\mathbf{1}_{\mathcal{C}_1})$ are unitary. It is called a tensor C^*-equivalence if F is an equivalence.

CQG's G and G' are said to be *monoidally equivalent* if $\mathrm{Rep}(G) \cong \mathrm{Rep}(G')$.

Example 2.6. 1. Let G be a CQG and $F : \mathrm{Rep}(G) \to \mathcal{H}_f$ the *forgetful* functor $f : U \mapsto H_U$ (H_U is the underlying Hilbert space of U) acting as the identity on morphisms. The above transformations are identities and F is a faithful unitary tensor functor.

The Woronowicz's Tannaka-Krein duality claims that any rigid C^*-tensor category admitting a unitary tensor functor to $\mathcal{H}_f$ (called a *fiber functor*), is equivalent to the category $\mathrm{Rep}(G)$ of some CQG G. The corresponding Hopf $*$-algebra $\mathbb{C}[G]$ is unique up to isomorphism – see [19], Theorem 2.3.2.

2. By a *closed quantum subgroup* of a CQG G we mean a CQG H together with a surjective homomorphism $P_H : \mathbb{C}[G] \to \mathbb{C}[H]$ of Hopf $*$-algebras (this definition is slightly less restrictive than the one in [26]). Then the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is a unitary tensor functor.

Any CQG G has a unique *maximal Kac quantum subgroup*, i.e., a CQG of Kac type K and a surjective homomorphism P_K as above, and if (H, P_H) is any other closed quantum subgroup of Kac type of G, then P_H factors through P_K [29]. Indeed, the ideal $I \subset \mathbb{C}[G]$ generated by the elements $a - S^2(a)$, for all $a \in \mathbb{C}[G]$, is clearly a Hopf $*$-ideal. Hence the quotient $\mathbb{C}[K] := \mathbb{C}[G]/I$ is a unital Hopf $*$-algebra with involutive antipode, so it defines a closed quantum subgroup K of Kac type of G. Clearly, K is maximal in the above sense, and $\mathbb{C}[K]$ is called *the canonical Kac quotient* of $\mathbb{C}[G]$.

Definition 2.7. [19]. The fusion ring of a rigid tensor C^*-category $\mathcal{C}$ is the universal ring $R(\mathcal{C})$ generated by the equivalence classes of its objects with operations of the direct sum and of the tensor product.

Let I be the set of the equivalence classes of simple objects of $\mathcal{C}$. Choose a representative U^α in every class, then the *fusion rule* is the decomposition

$$U^\zeta \otimes U^\eta = \oplus_\alpha \; N^\alpha_{\zeta,\eta} U^\alpha, \quad \forall \zeta, \eta \in I,$$

where the multiplicities $N^\alpha_{\zeta,\eta} \in \mathbb{Z}^+$, only finitely many nonzero. The conjugation $\alpha \mapsto \overline{\alpha}$ is a bijection of I which extends to a $\mathbb{Z}$-linear anti-multiplicative involution of $R(\mathcal{C})$. We also have the Frobenius reciprocity

$$N^\alpha_{\zeta,\eta} = N^\eta_{\overline{\zeta},\alpha} = N^\zeta_{\alpha,\overline{\eta}}, \quad \forall \zeta, \eta, \alpha \in I,$$

which follows from the Frobenius reciprocity for morphisms in $\mathcal{C}$ [19], Theorem 2.2. The map $\dim_q : I \to [1, \infty[$ is such that $d_q(\zeta) = d_q(\overline{\zeta})$ and extends to a $\mathbb{Z}$-linear multiplicative map $R(\mathcal{C}) \to \mathbb{R}$.

Remark 2.8. A fusion ring $R(\mathcal{C})$ is an example of a *hypercomplex system with discrete basis* in the sense of Yu. M. Berezanskij and A. A. Kaljuzhnyi [2] (also called a hypergroup) with the structural constants

$$C^\alpha_{\zeta,\eta} = \mathcal{N}^\alpha_{\zeta,\eta} \frac{d(\alpha)}{d(\zeta)d(\eta)}.$$

2.2. $G - C^*$**-algebras.**

Definition 2.9. A unital left $G - C^*$-algebra is a triple (B, G, α), where B is a unital C^*-algebra and $\alpha : B \to C(G) \otimes B$ is a left continuous coaction of a CQG G on B, i.e., an injective unital $*$-homomorphism such that $(id \otimes \alpha)\alpha = (\Delta \otimes id)\alpha$ and $[(C(G) \otimes 1)\alpha(B)] = C(G) \otimes B$. The C^*-subalgebra $B^\alpha := \{b \in B | \alpha(b) = 1 \otimes b\}$ is called the *fixed point subalgebra* of B.

An *algebraic coaction* of a CQG G on a unital $*$-algebra $\mathcal{B}$ is a Hopf $*$-algebra coaction $\alpha : \mathcal{B} \to \mathbb{C}[G] \otimes_{alg} \mathcal{B}$ such that $(\varepsilon \otimes_{alg} id) \circ \alpha = id$, the fixed point subalgebra $\mathcal{B}^\alpha := \{b \in \mathcal{B} | \alpha(b) = 1 \otimes b\}$ is a unital C^*-algebra, and the map $E = (h \otimes id) \circ \alpha : \mathcal{B} \to \mathcal{B}^\alpha$ is completely positive. So, $\mathcal{B}$ is a right pre-Hilbert $\mathcal{B}^\alpha$-module with inner product $< a, b >= E(a^* b)$. In this case, we call $(\mathcal{B}, G, \alpha)$ a unital left $G - *$-algebra.

Right $G - C^*$-algebras and $G - *$-algebras are defined similarly.

For any $U \in \mathrm{Rep}(G)$, H_U is a left $\mathbb{C}[G]$-comodule via $\delta_U : \xi_i \mapsto \Sigma_k u^*_{i,k} \otimes \xi_k$, where $\{\xi_i\}$ is an orthonormal basis in H_U. Given a unital $G - C^*$-algebra B, the *spectral subspace* $\mathcal{B}_U$ of B is the linear span of the images of all comodule maps $S : H_U \to B$. Put $\mathcal{B}_x = \mathcal{B}_{U^x}$, so $\mathcal{B}_{\mathbf{1}} = B^\alpha$ and $\alpha(\mathcal{B}_x) \subset \mathbb{C}[G]_x \otimes_{alg} \mathcal{B}_x$ with $\mathbb{C}[G]_x$ spanned by matrix coefficients of U^x. It was proved essentially in [5], [26] that $B_x B_y \subset \mathrm{span}\{B_z\}$, $\forall\, x, y \in \hat{G}$, where z are the classes of direct summands U^z of $U^x \oplus U^y$ and that $\mathcal{B} := \oplus_{x \in \hat{G}} \mathcal{B}_x$ is a unital $*$-algebra $\mathcal{B}$ norm dense in B. It is also characterized either as $\mathcal{B} := \{b \in B | \alpha(b) \in \mathbb{C}[G] \otimes_{alg} B\}$ or, equivalently, as $\mathcal{B} := \{(h \otimes id)[(a \otimes 1)\alpha(b)] |\; \forall\; a \in \mathbb{C}[G],\; b \in B\}$. The conditional expectation E as above is faithful on $\mathcal{B}$ and clearly $E(\mathcal{B}) = E(B)$. Thus, $\mathcal{B}$ is a unital $G - *$-algebra called the *regular subalgebra* of B.

Conversely, see [6], Proposition 4.4:

Proposition 2.10. *Given an algebraic coaction α of a CQG G on a unital $*$-algebra $\mathcal{B}$, there is a unique C^*-completion B of $\mathcal{B}$ to which α extends as a coaction of the reduced form of $C(G)$ with the same fixed point subalgebra.*

Indeed, for any $b \in \mathcal{B}$, there are $U \in Rep(G)$, an intertwiner $S : H_U \to B$, and $\xi \in H_U$ such that $b = S\xi$, so there are $b_i \in \mathcal{B}$ and $z_i \in \mathbb{C}$ such that $b = \Sigma_i z_i b_i$ and

$\alpha(b_i) = \Sigma_k u^*_{i,k} \otimes b_k$. Then one shows that $\Sigma_i b_i^* b_i \in B^\alpha$ which implies that all b_i and b can be faithfully represented by left multiplication operators on the right pre-Hilbert $\mathcal{B}^\alpha$-module $\mathcal{B}$, which defines a C^*-completion $B = [\mathcal{B}]$. Further, one shows that the map V defined on $\mathbb{C}[G] \otimes \mathcal{B}$ by $V(a \otimes b) := \alpha(b)(a \otimes 1)$ extends to a unitary on a Hilbert $\mathcal{B}^\alpha$-module $L^2(G,h) \otimes B$, and that $\alpha(b) = V(1 \otimes b)V^*$, so α extends to a continuous coaction of G on B. Finally, if $\tilde{B}$ is any C^*-completion of $\mathcal{B}$ for which the assertion of the proposition holds, then the map E as above is faithful because h is faithful on $C(G)$. Then $\tilde{B}$ is presented faithfully on the Hilbert $\mathcal{B}^\alpha$-module B, from where the unicity follows.

A coaction α is said to be universal (resp., reduced) if B is the universal enveloping C^*-algebra of $\mathcal{B}$ (resp., E is faithful on B). In particular, if $B = C(G)$ and $\alpha = \Delta$, we have $\mathcal{B} = \mathbb{C}[G]$) which leads to the usual notions of universal and reduced forms of G.

2.3. **The ergodic case.** A $G - C^*$-algebra (B, G, α) is called *ergodic* if the coaction α is ergodic, i.e., if $B^\alpha = \mathbb{C}1_B$. In this case, the state ω on B defined by $\omega(b)1_B = E(b)$, for all $b \in B$, is the unique α-invariant state (i.e., such that $(id \otimes \omega)\alpha(b) = \omega(b)1, \ \forall b \in B$), the spectral subspaces are finite dimensional and orthogonal for both scalar products $\omega(b^*c)$ and $\omega(cb^*)$ $(b, c \in B)$ [5]. An ergodic coaction is reduced if and only if the above ω is faithful on B.

Example 2.11. Let G be a CQG and H its closed quantum subgroup (see Example 2.6, 2). Then the norm closure $C(G/H)$ of the $*$-algebra $\mathbb{C}[G/H] := \{x \in \mathbb{C}[G] | (id \otimes P)\Delta(x) = x \otimes 1\}$ is called an algebra of continuous functions on the quantum homogeneous space G/H. The triple $(C(G/H), G, \Delta|_B)$ is an ergodic $G - C^*$-algebra. Indeed, $C(G/H)^\Delta$ is the same for both, continuous and algebraic coactions, so it suffices to consider $b \in \mathbb{C}[G/H]^\Delta$ for which we have $(id \otimes P)\Delta(b) = b \otimes 1$ and $\Delta(b) = 1 \otimes b$, so $b \in \mathbb{C}1$.

However, there is an example of a commutative ergodic $G - C^*$-algebra, where α is not a quotient action as above [36].

Let (B, G, α) be an ergodic $G - C^*$-algebra and $x \in \hat{G}$. We call $\dim(B_x)$ the *multiplicity of x in α*. It is known [5] that $\dim(B_x) \leqslant \dim_q(x)$.

3. Categorical duality

3.1. Tannaka-Krein reconstruction in terms of Rep(G)-module categories.

Definition 3.1. [6]. Let $\mathcal{C}$ be a C^*-tensor category with unit object $\mathbf{1}$. A C^*-category $\mathcal{M}$ is called a right $\mathcal{C}$-module C^*-category if there is a bilinear $*$-functor $\boxtimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ with natural unitary transformations $M \boxtimes (X \otimes Y) \to (M \boxtimes X) \boxtimes Y$ and $M \boxtimes \mathbf{1} \to M$ $(X, Y \in \mathcal{C}, M \in \mathcal{M})$ making $\mathcal{M}$ a right module category over $\mathcal{C}$ viewed as tensor category, i.e., making some pentagonal and triangular diagrams commutative – see [23]. We say that $\mathcal{M}$ is strict (resp., indecomposable) if these natural transformations are identities (resp., if, for all non-zero $M, N \in \mathcal{M}$, there is $X \in \mathcal{C}$ such that $\mathcal{M}(M \boxtimes X, N) \neq 0$).

We say that an object $M \in \mathcal{M}$ generates $\mathcal{M}$ if any object of $\mathcal{M}$ is isomorphic to a subobject of $M \boxtimes X$ for some $X \in \mathcal{C}$. $\mathcal{M}$ is said to be semisimple if the underlying C^*-category is semisimple.

We will always consider C^*-categories closed with respect to subobjects, i.e., such that for any object M and any projection $p \in \mathrm{End}(M)$, there are an object N and isometry $v \in \mathcal{M}(N, M)$ satisfying $p = vv^*$ (if necessary, one can complete given C^*-category with respect to subobjects).

One naturally defines a morphism $F : \mathcal{M}_1 \to \mathcal{M}_2$ between two $\mathcal{C}$-module C^*-categories as a morphism of the underlying C^*-categories equipped with a unitary natural equivalence $F(M \boxtimes X) \to F(M) \boxtimes X, \ \forall\, X \in \mathcal{C}, \ M \in \mathcal{M}$ satisfying some coherence conditions (see [6], 2.17).

Example 3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a unitary tensor functor. Then $\mathcal{M} = \mathcal{D}$ has a structure of a $\mathcal{C}$-module C^*-category with $M \boxtimes X := M \otimes F(X)$, for all $X \in \mathcal{C}, M \in \mathcal{D}$. For particular examples of this kind see Example 2.6.

Definition 3.3. [1]. Let (G, α, B) be a unital $G - C^*$-algebra . A G-equivariant Hilbert B-module is a right Hilbert B-module $\mathcal{E}$ with a coaction $\alpha_{\mathcal{E}} : \mathcal{E} \to C(G) \otimes \mathcal{E}$ ($C(G) \otimes \mathcal{E}$ is considered as a right Hilbert $C(G) \otimes B$-module), such that $[(C(G) \otimes 1)\alpha_{\mathcal{E}}(\mathcal{E})] = C(G) \otimes \mathcal{E}$, and

$$1.\ \alpha_{\mathcal{E}}(\zeta \cdot b) = \alpha_{\mathcal{E}}(\zeta)\alpha(b), \quad \forall\, b \in B, \quad \zeta \in \mathcal{E}.$$

$$2.\ < \alpha_{\mathcal{E}}(\zeta), \alpha_{\mathcal{E}}(\eta) >_{C(G)\otimes B} = \alpha(< \zeta, \eta >_B), \quad \forall\, \zeta, \eta \in \mathcal{E}.$$

$\mathcal{E}$ is called *irreducible* if $\dim(\mathcal{L}_G(\mathcal{E})) = 1$, where

$$\mathcal{L}_G(\mathcal{E}) := \{T \in \mathcal{L}(\mathcal{E}) | \alpha_{\mathcal{E}}(T\zeta) = (1 \otimes T)\alpha_{\mathcal{E}}(\zeta), \quad \forall\, \zeta \in \mathcal{E}\}.$$

Example 3.4. Let $\mathcal{M}_B$ be the category of finitely generated G-equivariant right Hilbert B-modules, its morphisms are G-equivariant maps of Hilbert B-modules (they are automatically adjointable since Hilbert B-modules are finitely generated). If $\mathcal{E} \in \mathcal{M}_B$ and $U \in \mathrm{Rep}(G)$, construct a new object $\mathcal{E} \boxtimes U \in \mathcal{M}_B$ by taking the right Hilbert B-module $H_U \otimes \mathcal{E}$ with the coaction

$$\tilde{\alpha} : H_U \otimes \mathcal{E} \to C(G) \otimes (H_U \otimes \mathcal{E}) : \tilde{\alpha}(\zeta \otimes \eta) = U_{21}^*(\zeta \otimes \alpha_{\mathcal{E}}(\eta))_{213}.$$

One can check that $(\mathcal{E} \boxtimes U) \boxtimes V \cong \mathcal{E} \boxtimes (U \oplus V)$ and that this operation is natural in both U and $\mathcal{E}$, so it defines a strict right $\mathrm{Rep}(G)$-module C^*-category structure on $\mathcal{M}_B$. If α is ergodic, $\mathcal{M}_B$ is semisimple and indecomposable [6], Proposition 3.11. The equivariant version of Kasparov's stabilisation theorem (see [35], 3.2 or [20], Lemma 3.2) shows that B is its generator.

The next theorem has been proved in ergodic case in [6], and then in general case in [18]:

Theorem 3.5. *Let G be a reduced CQG. Then the following two categories are equivalent:*

(i) The category $G - \mathrm{Alg}$ *of unital* $G - C^*$*-algebras with unital* G*-equivariant* $*$*-homomorphisms as morphisms.*

(ii) The category $\mathrm{Rep}(G) - \mathrm{Mod}$ *of pairs* $(\mathcal{M}, M)$*, where* $\mathcal{M}$ *is a right* $\mathrm{Rep}(G)$*-module* C^**-category and* M *is its generator, with equivalence classes of unitary* $\mathrm{Rep}(G)$*-module functors respecting the generators as morphisms.*

The coactions in (i) are ergodic if and only if the $\mathrm{Rep}(G)$*-module* C^**-categories in (ii) are semisimple, indecomposable, and with simple generators.*

The idea of the proof is as follows (for the details see [6], [18]).

1. The construction of the functor $\mathcal{T} : G - \mathrm{Alg} \to \mathrm{Rep}(G) - \mathrm{Mod}$ on the level of objects was explained in Example 3.4: given $B \in G - \mathrm{Alg}$, construct the category $\mathcal{M}_B$ with generator B. Now, if $f : B_0 \to B_1$ is a morphism in $G - \mathrm{Alg}$, then the morphism $\mathcal{T}(f) : \mathcal{M}_{B_0} \to \mathcal{M}_{B_1}$ is given by $\mathcal{E} \mapsto \mathcal{E} \otimes_{B_0} B_1$.

2. Next, in order to define the functor $\mathcal{S} : \mathrm{Rep}(G) - \mathrm{Mod} \to G - \mathrm{Alg}(G)$, we have to construct a unital $G - C^*$-algebra B from a given pair $(\mathcal{M}, M)$. According to Proposition 2.10, it suffices to construct a unital $G - *$-algebra $\mathcal{B}$. Looking at the construction (1) of $\mathbb{C}[G]$, considering $\mathcal{H}_f$ as a $\mathrm{Rep}(G)$-module category with generator $\mathbb{C}$ via the forgetful fiber functor $U \mapsto H_U$, $\forall U \in \mathrm{Rep}(G)$ (see Example 3.2), and identifying H_U with $\mathcal{H}_f(\mathbb{C}, H_U)$, it is natural to define $\mathcal{B}$ as a vector space by

$$\mathcal{B} = \oplus_{x \in \hat{G}} (\overline{H}_x \otimes \mathcal{M}(M, M \boxtimes U^x)).$$

The next reasoning follows [18], [21]. In order to simplify notations, it is convenient to replace $\mathcal{M}$ by an equivalent $\mathrm{Rep}(G)$-module C^*-category $\mathcal{N}$ as follows. Identify M with $\mathbf{1} \in \mathrm{Rep}(G)$, $M \boxtimes U$ with $\mathbf{1} \otimes U = U$, and define $\mathcal{N}(U, V) := \mathcal{M}(M \boxtimes U, M \boxtimes V)$, $\forall U, V \in$

Rep(G). We have also to complete $\mathcal{N}$ with respect to subobjects, so that $\mathcal{N}$ can be considered as the completion of Rep(G) with larger morphism spaces than in Rep(G). In particular, the generator $\mathbf{1}$ is not necessarily simple in $\mathcal{N}$, and the definition of $\mathcal{B}$ gives:

$$\mathcal{B} = \oplus_{x\in\hat{G}}(\overline{H}_x \otimes \mathcal{N}(\mathbf{1}, U^x)). \tag{3}$$

Now, it is convenient to introduce a much larger auxiliary vector space

$$\tilde{\mathcal{B}} = \oplus_{U\in\mathrm{Rep}(G)}(\overline{H}_U \otimes \mathcal{N}(\mathbf{1}, U)), \tag{4}$$

which is an associative unital algebra with respect to the product defined by

$$(\overline{\zeta}\otimes T)(\overline{\eta}\otimes S) = (\overline{\zeta}\otimes\overline{\eta})\otimes(T\otimes id)S, \quad \forall\zeta \in H_U, \quad \eta \in H_V, \quad T \in \mathcal{N}(\mathbf{1}, U), \quad S \in \mathcal{N}(\mathbf{1}, V).$$

Note that $(T \otimes id)S \in \mathcal{N}(\mathbf{1}, U \oplus V)$. Define now an antilinear map $\bullet : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$:

$$(\overline{\zeta} \otimes T)^{\bullet} := \overline{(id \otimes \overline{\zeta})R_U(\mathbf{1})} \otimes (T^* \otimes id)\overline{R}_U,$$

where R_U and $\overline{R}_U$ were defined in (2). Note that $(T^* \otimes id)\overline{R}_U \in \mathcal{N}(\mathbf{1}, \overline{U})$.

Then, for any $U \in$ Rep(G), choose isometries $w_x : H_x \to H_U$ defining the decomposition of U into irreducibles and define the projection $p : \tilde{\mathcal{B}} \to \mathcal{B}$ by

$$p(\overline{\zeta} \otimes T) = \Sigma_x(\overline{w_x^*\zeta} \otimes w_x^*T), \quad \forall\overline{\zeta} \in \overline{H}_U, \quad T \in \mathcal{N}(\mathbf{1}, U), \tag{5}$$

which is independent of the choice of w_x. One checks that $\mathcal{B}$ is a unital $*$-algebra with the product $p(b)p(c) := p(bc)$ and the involution $p(b)^* := p(b^{\bullet})$, for all $b, c \in \tilde{\mathcal{B}}$. It is also a $\mathbb{C}[G]$-comodule via the map $\alpha : \mathcal{B} \to \mathbb{C}[G] \otimes \mathcal{B}$ defined as follows. If $\{\xi_i\}$ is an orthonormal basis in H_U ($U \in$ Rep(G)), put

$$\alpha(p(\overline{\xi_i} \otimes T)) := \Sigma_j(u_{i,j} \otimes p(\overline{\xi_j} \otimes T)), \quad \text{where} \quad u_{i,j} = (\omega_{\xi_i,\xi_j} \otimes id)U.$$

One shows that α is an algebraic coaction with $\mathcal{B}^{\alpha} = \mathrm{End}_{\mathcal{N}}(\mathbf{1}) = \mathrm{End}_{\mathcal{M}}(M)$ that gives rise to a continuous G-coaction. Clearly, this construction allows to get a morphism in $G -$ Alg from a given morphism in Rep(G) $-$ Mod.

3. **Equivalence.** In order to explain the equivalence of the Rep(G)-module categories $\mathcal{N}$ and $\mathcal{M}_B$, replace the last one by an equivalent Rep(G)-module category $\mathcal{N}_B$, which is a completion of Rep(G) with respect to subobjects, with generator $\mathbf{1}$. Now it suffices to explain why $\mathcal{N}_B(U, V) \cong \mathcal{N}(U, V)$, for all $U, V \in$ Rep(G). One can check, as in [18], that the first of them consists of elements $S \in B(H_U, H_V) \otimes B$, such that $V_{12}^*(id\otimes\alpha)(S)U_{12} = S_{13}$, and if $T \in \mathcal{N}(U, V)$, the needed isomorphism can be given explicitly by:

$$S = \Sigma_{i,j}\theta_{\zeta_j,\eta_i}\pi_U(\rho^{-1/2}) \otimes p(\overline{(\zeta_j \otimes \eta_i)} \otimes (T \otimes id)\overline{R}_U),$$

where $\{\zeta_j\}$ and $\{\eta_i\}$ are orthonormal bases in H_V and H_U, respectively, and $\theta_{\zeta_j,\eta_i}(\eta) :=< \eta, \eta_i > \zeta_j$ is a rank 1 operator from H_U to H_V (see [21]).

Example 3.6. 1. The regular coaction Δ of G on $C(G)$ corresponds to the Rep(G)-module category $\mathcal{M} = \mathcal{H}_f$ with the generator $\mathbb{C}$ via the forgetful fiber functor $U \mapsto H_U$. Namely, identifying $\mathcal{M}(\mathbb{C}, H_U)$ with H_U, we see that the algebras $\tilde{\mathcal{B}}$ and $\mathcal{B}$ constructed from the pair $(\mathcal{H}_f, \mathbb{C})$ are isomorphic to $\widetilde{\mathbb{C}[G]} = \oplus_{U\in\mathrm{Rep}(G)}(\overline{H}_U \otimes H_U)$ and to $\mathbb{C}[G]$, respectively. Then the map $p : \tilde{\mathcal{B}} \to \mathcal{B}$ turns into the map $p_G : \widetilde{\mathbb{C}[G]} \to \mathbb{C}[G]$ sending $\overline{\zeta} \otimes \eta$ (where $\zeta, \eta \in H_U$) to the matrix coefficient $(\omega_{\eta,\zeta} \otimes id)U$ of U.

2. If G is a CQG and H its closed quantum subgroup H (see Example 2.6, 2), the $G-C^*$-algebra $C(G/H)$ corresponds to the category Rep(H) with generator $\mathbf{1}$ viewed as a Rep(G)-module category via the restriction tensor functor Rep(G) $\to$ Rep(H). Namely, identifying $\mathrm{Mor}_H(\mathbb{C}, H_U)$ with a subspace of H_U, we can view the corresponding algebra $\tilde{\mathcal{B}}$ as a subalgebra of $\widetilde{\mathbb{C}[G]}$. Then the above map p_G induces a G-equivariant isomorphism $\mathcal{B} \cong \mathbb{C}[G/H]$.

3.2. **Spectral functors.** Let (B, G, α) be a unital $G - C^*$-algebra and $U \in \mathrm{Rep}(G)$. Consider the tensor product comodule $H_U \otimes B$ and its subcomodule of G-invariant vectors

$$(H_U \otimes B)^G := \{X = \Sigma_i(\xi_i \otimes b_i) | \alpha(b_i) = \Sigma_j u_{i,j} \otimes b_j\},$$

where $\{\xi_i\}_i$ is an orthonormal basis in H_U and $u_{i,j}$ are the matrix coefficients of U with respect to this basis. The spectral subspaces can be recovered from $(H_U \otimes B)^G$ using the canonical surjective maps

$$\overline{H}_U \otimes (H_U \otimes B)^G \to B_{\overline{U}} : \overline{\xi} \otimes X \mapsto (\overline{\xi} \otimes id)(X),$$

which are isomorphisms for irreducible U. The space $(H_U \otimes B)^G$ is a B^α-bimodule and a right Hilbert B^α-module with inner product $< X, Y >= \Sigma_i b_i^* c_i$ for $X = \Sigma_i(\xi_i \otimes b_i)$ and $Y = \Sigma_i(\xi_i \otimes c_i)$ (note that the element $< X, Y >$ is in B^α and is independent of the choice of $\{\xi_i\}_i$ - we will write it as X^*Y). So, $(H_U \otimes B)^G \in \mathrm{Corr}(B^\alpha)$ (see Example 2.4, 3).

Remark also that for all $U, V \in \mathrm{Rep}(G)$, the map $X \otimes Y \mapsto X_{13}Y_{23}$ is an isometry from $(H_U \otimes B)^G \otimes_{B^\alpha} (H_V \otimes B)^G$ to $(H_{U \oplus V} \otimes B)^G$.

Definition 3.7. [18]. Given a $G - C^*$-algebra (B, G, α), the associated spectral functor is the unitary functor $F : \mathrm{Rep}(G) \to \mathrm{Corr}(B^\alpha)$ defined by

$$F(U) = (H_U \otimes B)^G, \quad \text{for all} \quad U \in \mathrm{Rep}(G),$$

$F(T) = T \otimes id$, for all morphisms, together with the B^α-bilinear isometries

$$F_2(U, V) : F(U) \otimes_{B^\alpha} F(V) \to F(U \oplus V) : \; X \otimes Y \mapsto X_{13}Y_{23}.$$

The spectral functor is not, in general, a tensor functor because the maps F_2 are not surjective. It was abstractly characterized in [18], Definition 2.1 as a *weak tensor functor* $F : \mathcal{C} \to \mathrm{Corr}(A)$ (where $\mathcal{C}$ is a strict C^*-tensor category and A is a C^*-algebra). The only difference between such F and a unitary tensor functor is that F_2 are not isomorphisms but just A-bilinear isometries satisfying some coherence condition.

Let us explain the relation between $\mathrm{Rep}(G)$-module C^*-categories and spectral functors. Let G be a reduced QCG G and $\mathcal{M}$ be a strict right $\mathrm{Rep}(G)$-module C^*-category with generator M. Put $A = \mathrm{End}(M)$ and consider the functor $F : \mathrm{Rep}(G) \to \mathrm{Corr}(A)$ defined by $F : U \to \mathcal{M}(M, M \otimes U)$ with the right and left A-module structures on $F(U)$ given by, respectively, composition of morphisms and by $aX = (a \otimes id)X$, and the inner product $< X, Y >= X^*Y$. The action of F on morphisms is defined by $F(T)X = (id \otimes T)X$, and $F_2 : F(U) \otimes_A F(V) \to F(U \otimes V)$ is given by $X \otimes Y \mapsto (X \otimes id)Y$. Then it was shown in [18], Theorem 2.2 that (B_F, G, α_F) is a unital $G - C^*$-algebra, where

$$B_F = \oplus_{x \in \hat{G}} \overline{H}_x \otimes F(U^x), \quad \alpha_F(\overline{\xi}_i \otimes X) = \Sigma_j (u_{i,j} \otimes \xi_i \otimes X).$$

Here $\{\xi_i\}_i$ is an orthonormal basis in H_x and $u_{i,j}$ are the corresponding matrix coefficients of U^x. Moreover, $A = B^\alpha$, $\mathcal{M} \cong \mathcal{M}_B$ with $M \cong B$, and the functor F is a spectral functor (there are canonical isomorphisms $(H_U \otimes B)^G \cong \mathcal{M}_B(B, B \otimes U)$ sending $\Sigma_i(\xi_i \otimes b_i)$ to the morphism $b \mapsto \Sigma_i(b_i b \otimes \xi_i)$). The module $B \otimes U$ is just $H_U \otimes B$ discussed above.

Remark 3.8. 1. The main part of the proof of [18], Theorem 2.2 is the construction of a $G - C^*$-algebra (B_F, G, α_F) from a given weak tensor functor F which can be viewed as another version of the Tannaka-Krein reconstruction. It follows the lines going back to [43].

2. In the ergodic case, the definition of a weak tensor functor is equivalent to the one of a *quasitensor functor* introduced earlier in [24]. An example of such a functor is the functor sending any $x \in \hat{G}$ to the corresponding spectral subspace – see [24], Definition 7.2 and Theorem 7.3. These results imply that isomorphism classes of ergodic coactions of monoidally equivalent CQG's are in canonical correspondence with each other.

In particular, when $F_2(U,V)$ are unitary, F is a unitary tensor functor and the corresponding ergodic coaction is said to be *of full quantum multiplicity.* This situation was studied systematically in [4] in other terms.

3.3. **Yetter-Drinfeld $G-C^*$-algebras** [21]**.** *Yetter-Drinfeld* (YD) C^*-algebras constitute a special class of $G-C^*$-algebras for which one can formulate the categorical duality in terms of a unitary tensor functor. Let (B,G,α) be a unital $G-C^*$-algebra equipped with a continuous right coaction $\beta : B \to M(B\otimes c_0(\hat{G}))$ of the dual discrete quantum group $\hat{G}$, where $c_0(\hat{G})$ is viewed as a subalgebra of the algebraic dual of $\mathbb{C}[G]$. Similarly to the definition of a YD algebra for a general locally compact quantum group G given in [22], Definition 3.1, we will say that (B,G,α,β) is a unital YD C^*-algebra, if

$$(\alpha\otimes id)\beta(b) = W_{31}(id\otimes\beta)\alpha(b)W_{31}^*,$$

where W is the fundamental unitary of G and $b\in B$. However, it is more convenient to formulate this condition, like in [21], in terms of the left *action* $\rhd : \mathbb{C}[G]\otimes_{alg} B \to B$ defined by $u\rhd b = (id\otimes u)\beta(b)$, for all $u\in\mathbb{C}[G], b\in B$. Then B is a left $\mathbb{C}[G]$-module unital $*$-algebra, i.e., for all $u\in\mathbb{C}[G], b,c\in B$:

$$u\rhd(bc) = (u_{(1)}\rhd b)(u_{(2)}\rhd c),\quad u\rhd b^* = (S(u)^*\rhd b)^*,\quad u\rhd 1 = \varepsilon(u)1. \tag{6}$$

Definition 3.9. We say that $(B,G,\alpha,\rhd)$ is a YD C^*-algebra if, for all $u\in\mathbb{C}[G]$ and b from the regular subalgebra $\mathcal{B}$:

$$\alpha(u\rhd b) = u_{(1)}b_{(1)}S(u_{(3)})\otimes(u_{(2)}\rhd b_{(2)}). \tag{7}$$

We used here the Sweedler's leg notations: $\Delta(u) = u_{(1)}\otimes u_{(2)}, \alpha(b) = b_{(1)}\otimes b_{(2)}$. A YD C^*-algebra is said to be braided-commutative if

$$ab = b_{(2)}(S^{-1}(b_{(1)})\rhd a),\quad \text{for all}\quad a,b\in\mathcal{B}. \tag{8}$$

The coaction β can be reconstructed from the action $\rhd$ as follows:

Proposition 3.10. ([21], Proposition 1.3). *Let (B,G,α) be a $G-C^*$-algebra, where G is a reduced CQG, such that $\mathcal{B}$ is a left $\mathbb{C}[G]$-module unital $*$-algebra and (7) holds. Then there is a unique continuous right coaction $\beta : B\to M(B\otimes c_0(\hat{G}))$ such that $u\rhd b := (id\otimes u)\beta(b)$, for all $u\in\mathbb{C}[G],\ b\in\mathcal{B}$.*

Indeed, if $U = \Sigma_{i,j}(e_{i,j}\otimes u_{i,j})\in \mathrm{Rep}(G)$ and $b\in\mathcal{B}$, the map $\beta_U(b) := \Sigma_{i,j}(u_{i,j}\rhd b)\otimes e_{i,j}$ defines a unital $*$-homomorphism $\mathcal{B}\to\mathcal{B}\otimes B(H_U)$ satisfying

$$U_{31}^*(\alpha\otimes id)\beta_U(b)U_{31} = (id\otimes\beta_U)\alpha(b).$$

Then the map $y\mapsto U_{31}^*(\alpha\otimes id)(y)U_{31}$ extends to a continuous left coaction of G on $[\beta_U(\mathcal{B})]\in B\otimes B(H_U)$. This implies that α viewed as an algebraic coaction of G on $\mathcal{B}$, extends to a continuous coaction of G on the completion of $\mathcal{B}$ with respect to the C^*-norm $\max\{\|\beta_U(\cdot)\|,\|\cdot\|\}$, but due to the unicity statement in Proposition 2.10, this completion equals to B.

Since $c_0(\hat{G})\cong c_0-\oplus_x B(H_x)$, the family of homomorphisms β_{U^x} $(x\in\hat{G})$ define a unital $*$-homomorphism $\beta : B\to M(B\otimes c_0(\hat{G})) = l^\infty-\oplus_x(B\otimes B(H_x))$ such that $(id\otimes u)\beta(b) = u\rhd b$ for all $u\in\mathbb{C}[G],\ b\in\mathcal{B}$. Finally, one checks that it is a continuous coaction, and the unicity is clear.

Remark 3.11. 1. One can regard a YD $G-C^*$-algebra as a $D(G)-C^*$-algebra for the *Drinfeld double* $D(G)$ of G – see [22], Proposition 3.2.

2. If $b\in\mathcal{B}^\alpha$, then (8) gives $ab = ba$, so $\mathcal{B}^\alpha\subset Z(\mathcal{B})$.

Example 3.12. Let G be a CQG and H its closed quantum subgroup (see Example 2.6, 2). Then $B = C(G/H)$ is a braided commutative YD $G-C^*$-algebra with $\alpha = \Delta|_B$ and $u\rhd b = u_{(1)}bS(u_{(2)})$ – the *adjoint action* of $\mathbb{C}[G]$.

Let us formulate the Tannaka-Krein duality for Yetter-Drinfeld $G - C^*$-algebras.

Theorem 3.13. ([21], Theorem 2.1). *Let G be a reduced compact quantum group. Then the following two categories are equivalent:*

(i) The category $YD_{brc}(G)$ of unital braided-commutative YD $G - C^*$-algebras with unital G- and $\hat{G}$-equivariant $*$-homomorphisms as morphisms.

(ii) The category $\mathrm{Tens}(\mathrm{Rep}(G))$ *of pairs $(\mathcal{C}, F)$, where $\mathcal{C}$ is a tensor C^*-category and $F : \mathrm{Rep}(G) \to \mathcal{C}$ is a unitary tensor functor such that $\mathcal{C}$ is generated by the image of F. Its morphisms $(\mathcal{C}, F) \to (\mathcal{C}', F')$ are equivalence classes of pairs $(\mathcal{F}, \eta)$, where $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a unitary tensor functor and $\eta : \mathcal{F}(F) \to F'$ is a natural unitary monoidal isomorphism of functors.*

Moreover, given a morphism $(\mathcal{F}, \eta) : (\mathcal{C}, F) \to (\mathcal{C}', F')$, the corresponding homomorphism of YD $G - C^*$-algebras is injective (resp., surjective) if and only if $\mathcal{F}$ is faithful (resp., full).

Here is the idea of the proof (for the corresponding calculations see [21]).

1. Functor $\tilde{\mathcal{T}} : YD_{brc}(G) \to \mathrm{Tens}(\mathrm{Rep}(G))$. Apply the functor $\mathcal{T}$ from the proof of Theorem 3.5 to $B \in YD_{brc}(G)$ and turn the corresponding category $\mathcal{M}_B$ into a tensor C^*-category by showing that any $M \in \mathcal{M}_B$ is an object of the tensor C^*-category $\mathrm{Corr}(B)$ – see Example 2.4, 3. As B is a generator of $\mathcal{M}_B$, it suffices to consider $M = H_U \otimes B$, $\forall\ U \in \mathrm{Rep}(G)$, and to construct a non-degenerate $*$-homomorphism $\varphi_U : B \to \mathrm{End}_B(H_U \otimes B) = B(H_U) \otimes B$. As $B \in YD_{brc}(G)$, one can check that the map

$$b \mapsto \varphi_U(b) := \Sigma_{i,j} e_{i,j} \otimes (u_{i,j} \rhd b),$$

where $b \in \mathcal{B}$, $U = \Sigma_{i,j} e_{i,j} \otimes u_{i,j}$ and $e_{i,j}$ are the matrix units in $B(H_U)$ related to an orthonormal basis $\{e_i\}$ in H_U, gives the needed $*$-homomorphism. Here $\varphi_U(b) \in \mathcal{L}(M)$ acts on any regular vector $\zeta \in M$ (i.e., such that $\alpha_M(\zeta) \in \mathbb{C}[G] \otimes_{alg} M$) as follows: $\varphi_U(b)\zeta = \zeta_{(2)}(S^{-1}(\zeta_{(1)}) \rhd b)$. We have also $\alpha_M(\varphi_U(b)\zeta) = (id \otimes \varphi_U(b))\alpha(b)\alpha_M(\zeta)$ – see [21], Lemma 2.2.

Thus, M is a G-equivariant B-correpondence. Moreover, if $M, N \in \mathcal{M}_B$, then $M \otimes_B N \in \mathcal{M}_B$, and $\mathcal{C} = (\mathcal{M}_B, \otimes_B, B)$ is a full tensor C^*-subcategory of $\mathrm{Corr}(B)$ (see [21], Theorem 2.4). Note that $\mathcal{C}$ is generated by the image of the unitary tensor functor $F : Rep(G) \to \mathcal{M}_B :\ U \mapsto H_U \otimes B$.

If $f : B_0 \to B_1$ is a morphism in $YD_{brc}(G)$, the morphism $\mathcal{T}(f) : \mathcal{M}_{B_0} \to \mathcal{M}_{B_1}$ maps any M to $M \otimes_{B_0} B_1$ (see the proof of Theorem 3.5), and the tensor structure of $\mathcal{T}$ is described by the isomorphisms

$$(M \otimes_{B_0} B_1) \otimes_{B_1} (N \otimes_{B_0} B_1) \cong (M \otimes_{B_0} N) \otimes_{B_0} B_1, \quad \text{for all} \quad M, N \in \mathcal{M}_{B_0},$$

which are defined, for all $a, b \in \mathcal{B}_1$, and regular $\xi \in M, \zeta \in N$, as follows:

$$(\xi \otimes a) \otimes (\zeta \otimes b) \mapsto \xi \otimes \zeta_{(2)} \otimes (S^{-1}(\zeta_{(1)}) \rhd a)b.$$

Together with the obvious isomorphisms $\eta_U : (H_U \otimes B_0) \otimes_{B_0} B_1 \to H_U \otimes B_1$, we define a morphism $(\mathcal{M}_{B_0}, F_{B_0}) \to (\mathcal{M}_{B_1}, F_{B_1})$.

2. Functor $\tilde{\mathcal{S}} : \mathrm{Tens}(\mathrm{Rep}(G)) \to YD_{brc}(G)$. Conversely, given a pair $(\mathcal{C}, F)$ as above, $\mathcal{C}$ has a structure of a singly generated $\mathrm{Rep}(G)$-module C^*-category coming from F. As in the proof of Theorem 3.5, identify $\mathcal{C}$ with a completion $\mathcal{N}$ of $\mathrm{Rep}(G)$, and its generator with $\mathbf{1}$. The functor F is automatically faithful because the category $\mathrm{Rep}(G)$ is rigid (indeed, $F(\mathbf{1})$ is the direct summand of $F(U) \otimes \overline{F(U)}$, for any $U \in \mathrm{Rep}(G)$). So we can assume that it is just the embedding functor.

Now, define a $\mathbb{C}[G]$-module algebra structure on $\mathcal{B}$ (see (3)) using the algebra homomorphisms $p : \tilde{\mathcal{B}} \to \mathcal{B}$ from (5) and $p_G : \widetilde{\mathbb{C}[G]} \to \mathbb{C}[G]$ from Example 3.6. First, define a linear map $\tilde{\triangleright} : \widetilde{\mathbb{C}[G]} \otimes \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ by

$$(\overline{\xi} \otimes \zeta)\tilde{\triangleright}(\overline{\eta} \otimes T) = \overline{(\xi \otimes \eta \otimes \overline{\pi_U(\rho^{-1/2})(\zeta)})} \otimes (id \otimes T \otimes id)\overline{R}_U,$$

where $\xi, \zeta \in H_U, \eta \in H_V, U, V \in \mathrm{Rep}(G), T \in \mathcal{N}(\mathbf{1}, V), \overline{R}_U$ was introduced in (2). Note that $(id \otimes T \otimes id)\overline{R}_U \in \mathcal{N}(\mathbf{1}, U \oplus V \oplus \overline{U})$. Then define an action $\triangleright : \mathbb{C}[G] \otimes \mathcal{B} \to \mathcal{B}$ by $p_G(u) \triangleright p(b) := p(u\tilde{\triangleright}b)$, for all $u \in \widetilde{\mathbb{C}[G]}$ and $b \in \tilde{\mathcal{B}}$. Finally, one shows by direct but tedious calculations that the norm completion B of $\mathcal{B}$ is in $YD_{brc}(G)$.

Considering the category of pairs $(\mathcal{C}, F)$ as above, we may assume that the restriction of any morphism $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ on $\mathrm{Rep}(G)$ is the identical tensor functor, and that $\eta = id$. Then the construction of the algebras $\mathcal{B}$ and $\mathcal{B}'$ shows that the maps $\overline{H}_U \otimes \mathcal{C}(\mathbf{1}, U) \to \overline{H}_U \otimes \mathcal{C}'(\mathbf{1}, U)$, defined by $\overline{\xi} \otimes T \mapsto \overline{\xi} \otimes \mathcal{F}(T)$, give a unital $*$-homomorphism $\mathcal{B} \to \mathcal{B}'$ that respects the $\mathbb{C}[G]$-comodule and $\mathbb{C}[G]$-module structures. It extends to a homomorphism f of C^*-algebras by [6], Prop. 4.5. The above definition of morphisms in the category of pairs $(\mathcal{C}, F)$ shows that f depends only on the equivalence class of $(\mathcal{F}, \eta)$, and this gives the needed functor.

Furthermore, it is clear from the construction that the morphism $f : B \to B'$ defined by a morphism $(\mathcal{F}, \eta) : (\mathcal{C}, F) \to (\mathcal{C}', E')$ is injective (resp., surjective) if and only if the maps $\mathcal{C}(\mathbf{1}, U^x) \to \mathcal{C}'(\mathbf{1}, U^x), T \mapsto \mathcal{F}(T)$, are injective (resp., surjective), for all $x \in \hat{G}$. It follows from the Frobenius reciprocity for morphisms and the fact that the categories $\mathcal{C}$ and $\mathcal{C}'$ are generated by $\mathrm{Rep}(G)$, that f is injective (resp., surjective) if and only if $\mathcal{F}$ is faithful (resp., full).

Finally, it can be proved by direct calculations that the functors $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{S}}$ define an equivalence of the categories in question.

Example 3.14. If H is a closed quantum subgroup of a CQG G, then $B = C(G/H) \in YD_{brc}(G)$ with $\alpha = \Delta|_B$ and $u \triangleright b = u_{(1)} b S(u_{(2)})$ – see Example 3.12. It corresponds to the tensor C^*-category $\mathrm{Rep}(H)$ with generator $\mathbf{1}$ viewed as a $\mathrm{Rep}(G)$-module category via the restriction unitary tensor functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ – see Example 3.6. Indeed, identifying $\mathrm{Mor}_{\mathrm{Rep}(H)}(\mathbb{C}, H_U)$ with a subspace of H_U, we can view the corresponding algebra $\tilde{\mathcal{B}}$ as a subalgebra of $\widetilde{\mathbb{C}[G]} = \oplus_{U \in \mathrm{Rep}(G)}(\overline{H}_U \otimes H_U)$. Then the map $p_G : \widetilde{\mathbb{C}[G]} \to \mathbb{C}[G]$ induces a G-equivariant isomorphism $\mathcal{B} \cong \mathbb{C}[G/H]$.

Conversely, the action of $\mathbb{C}[G]$ on $\mathbb{C}[G/H]$ defined by the tensor functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is exactly the adjoint one. Indeed, it suffices to consider the trivial H corresponding to the functor $\mathrm{Rep}(G) \to \mathcal{H}_f$, while the general case corresponds to the intermediate functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$. Let $U, V \in \mathrm{Rep}(G)$, $\{\xi_i\} \in H_U$ and $\{\eta_k\} \in H_V$ be orthonormal bases, and $u_{i,j}$ (resp., $v_{k,l}$) their matrix coefficients. Then the definitions of the maps $\tilde{\triangleright}$ and $\triangleright$ imply

$$u_{i,j} \triangleright v_{k,l} = \Sigma_m u_{i,m} v_{k,l} S(u_{m,j})$$

(see [21], 3.1), which is exactly our claim.

For more examples of braided commutative YD $G - C^*$-algebras see [22].

4. Some applications

4.1. Invariant subalgebras of compact quantum groups.

Theorem 4.1. [21], [27], [29]. *Let G be a CQG. Then any unital left G- and right $\hat{G}$-invariant C^*-subalgebra B of $C(G)$ is of the form $C(G/H)$ for some closed quantum subgroup H of G.*

To prove this, note that B is a subobject of $C(G)$ in $YD_{brc}(G)$, and applying the functor $\tilde{\mathcal{T}}$ to this inclusion, we get a morphism in $\mathrm{Tens}(\mathrm{Rep}(G))$:

$$(\mathcal{C}_B, F_B) \to (\mathcal{C}_{C(G)}, F_{C(G)}) \cong (\mathcal{H}_f, \mathcal{F}),$$

where $\mathcal{F} : \mathrm{Rep}(G) \to \mathcal{H}_f$ is the fiber functor. So, there is a unitary fiber functor $\mathcal{E} : \mathcal{C}_B \to \mathcal{H}_f$, and by Woronowicz's Tannaka-Krein duality theorem, the pair $(\mathcal{C}_B, \mathcal{E})$ defines a CQG H. Then the functor F_B defines a functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ whose composition with $\mathcal{E}$ is $\mathcal{F}$. This factorization of $\mathcal{F}$ corresponds to the inclusion $C(G/H) \to C(G)$, so H is a closed quantum subgroup of G. As $\tilde{\mathcal{T}}$ is an equivalence of categories, we have $B = C(G/H)$.

As for the uniqueness, we have to show that the inclusion $C(G/H) \to C(G)$ determines the kernel of the restriction map $\mathbb{C}[G] \to \mathbb{C}[H]$. Here are the main arguments: (i) $\mathbb{C}[G/H]$ is spanned by matrix elements $u_{\xi,\zeta}$ of $U \in \mathrm{Rep}(G)$ with $\xi, \zeta \in H_U$ invariant with respect to H; (ii) from $u_{\xi,\zeta}$ and the duality morphisms in $\mathrm{Rep}(G)$, one can recover the spaces $\mathrm{Mor}_{\mathrm{Rep}(H)}(U, V)$ for all $U, V \in \mathrm{Rep}(G)$; (iii) a finite combination $\Sigma_i u_{\xi_i,\zeta_i}$ of matrix coefficients in $\mathbb{C}[G]$, with $\xi_i, \zeta_i \in H_U$, is in the kernel of the restriction map $\mathbb{C}[G] \to \mathbb{C}[H]$ if and only if $\Sigma_i \omega_{\xi_i,\zeta_i}$ vanishes on the commutant of $\mathrm{End}_{\mathrm{Rep}(H)}(H_U)$ in $B(H_U)$.

To extend this result, consider a forgetful fiber functor $\mathcal{F} : \mathrm{Rep}(G) \to \mathcal{H}_f$ and another unitary fiber functor $\mathcal{F}' : \mathrm{Rep}(G) \to \mathcal{H}_f$ which gives rise to a CQG G' monoidally equivalent to G. Denote by $B(\mathcal{F}, \mathcal{F}')$ the C^*-algebra constructed as above from the pair $(\mathcal{H}_f, \mathcal{F}')$. Its regular subalgebra is $\mathcal{B}(\mathcal{F}, \mathcal{F}') = \oplus_{x \in \hat{G}} (\overline{H}_x \otimes H'_x)$, from where one can see that $B(\mathcal{F}, \mathcal{F}')$ carries commuting coactions of both G and G'. This is exactly the *linking C^*-algebra* introduced in [4] (and in purely algebraic setting earlier in [28]), and its regular subalgebra is a *Hopf-Galois extension* of $\mathbb{C}$ over $\mathbb{C}[G]$ – see [3].

Now, using the same arguments as above, one shows that any unital left G- and right $\hat{G}$-invariant C^*-algebra of $B(\mathcal{F}, \mathcal{F}')$ is of the form $B(\mathcal{F}, \mathcal{F}')^{H'}$, where the last C^*-subalgebra of $B(\mathcal{F}, \mathcal{F}')$ is constructed by a closed quantum subgroup H' of G' exactly in the same way as $C(G/H) \subset C(G)$.

4.2. $SU_q(2)$ **ergodic coactions** [7]. A lot of papers devoted to the study of CQG-coactions were motivated by their classification in the extremely important special case of ergodic coactions of $G = SU_q(2)$ $(0 < |q| \leqslant 1)$. Recall the basic definition [41], [43].

Definition 4.2. The $*$-algebra $P(SU_q(2))$ of polynomials on the CQG $SU_q(2)$ is the universal unital $*$-algebra over $\mathbb{C}$ with generators u_{ij} $(i, j \in \{1, 2\})$, subject to the relations

$$\begin{pmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{pmatrix} = \begin{pmatrix} u_{22} & -q u_{21} \\ -q^{-1} u_{12} & u_{11} \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

It is a Hopf $*$-algebra with the coproduct

$$\Delta(u_{ij}) = u_{i1} \otimes u_{1j} + u_{i2} \otimes u_{2j}, \quad \text{for all} \quad i, j \in \{1, 2\}.$$

These generators and relations give rise to the CQG $SU_q(2)$ defined by the unital C^*-algebra $C(SU_q(2))$, which is the completion of $P(SU_q(2))$, and the extension of Δ to it. It is known that $\hat{G}$ is labelled by the set $J = \frac{1}{2}\mathbb{N} = \{0, 1/2, 1, 3/2, \ldots\}$. The fusion rule in $\mathrm{Rep}(SU_q(2))$ is as follows:

$$U_m \oplus U_n = U_{|m-n|} + U_{|m-n|+1} + \cdots + U_{m+n}, \quad \text{for all} \quad m, n \in \hat{G}.$$

Now, any semisimple C^*-category $\mathcal{M}$ whose set J of equivalence classes of simple objects is countable, is equivalent to the category of finite dimensional J-graded Hilbert spaces $\mathcal{H}_f^J$ (see [6], Lemma A.1.6). Then the category $\mathrm{End}_f(\mathcal{M})$ of its adjointable endofunctors is equivalent, as a rigid semisimple C^*-tensor category, to the category $\mathcal{E}_f^J$ of $J \times J$-graded Hilbert spaces $H = \oplus_{r,s \in J} H_{rs}$, where all H_{rs} are finite dimensional and only finitely many of them in each row and column are non-zero (see [6], Lemma A.3.3).

On the other hand, for any C^*-category $\mathcal{M}$, there is an equivalence between $\mathrm{Rep}(SU_q(2))$-module C^*-category structures on $\mathcal{M}$ and unitary tensor functors

$$\mathrm{Rep}(SU_q(2)) \to \mathrm{End}(\mathcal{M})$$

given by $U \mapsto \cdot \boxtimes U$ and $M \boxtimes U := F(U)(M)$, for all $U \in \mathrm{Rep}(SU_q(2)))$ and $M \in \mathcal{M}$. So that, there is an equivalence between $\mathrm{Rep}(SU_q(2))$-module C^*-category structures on C^*-categories $\mathcal{M}$ with countable J and unitary tensor functors $F : \mathrm{Rep}(SU_q(2)) \to \mathcal{E}_f^J$ which can not be decomposed as a direct sum $F_1 \oplus F_2$ with the F_i unitary tensor functors into J_i-bi-graded Hilbert spaces, $J = J_1 \cup J_2$ with J_1 and J_2 disjoint (see [6], Prop. A.4.2).

This result and Theorem 3.5 imply that the classification of ergodic unital $SU_q(2)-C^*$-algebras is the same as the classification of the above tensor $*$-functors $\mathrm{Rep}(SU_q(2)) \to \mathcal{E}_f^J$. The construction of unitary tensor functors from $\mathrm{Rep}(SU_q(2))$ to a strict C^*-tensor category $\mathcal{C}$ in terms of so called q-fundamental solutions in $\mathcal{C}$ was known earlier (see, for example, [33], [24]), it is summarized in [7], Theorem 1.4. For $\mathcal{C} = \mathcal{E}_f^J$, it is convenient to translate it into the language of weighted graphs, this was done also in [7].

Definition 4.3. An oriented graph Γ consists of two countable sets, $\Gamma^{(0)}$ of vertices and $\Gamma^{(1)}$ of (oriented) edges, and two maps, $s, t : \Gamma^{(1)} \to \Gamma^{(0)}$ called the source and the target map. A cost (or weight) on Γ is a function $W : \Gamma^{(1)} \to \mathbb{R}^+$. When v is a vertex, the source cost $W(v) \in [0, \infty]$ is the sum of the costs of all the edges leaving from v. We call Γ symmetric if there is an involution $e \to \overline{e}$ on the edge set interchanging source and target vertex of each edge. We call a cost on a symmetric graph balanced if one can choose an involution satisfying $W(e)W(\overline{e}) = 1$.

A fair and balanced T-cost on Γ (where $T \in \mathbb{R} - \{0\}$) is a balanced cost such that the source cost at any vertex is equal to $|T|$, and with an even number of loops at each vertex if $T > 0$. A graph with a fair and balanced T-cost is also called a fair and balanced T-graph.

Recall also that two ergodic unital $SU_q(2) - C^*$-algebras, A and B, are called *equivariantly Morita equivalent* if there is an irreducible equivariant Hilbert B-module $\mathcal{E}$ and an equivariant C^*-algebra isomorphism $A \to \mathcal{K}(\mathcal{E})$ [6], Definition 3.7. Now we are ready to formulate the following:

Theorem 4.4. ([7], Theorem 2.4). *For $0 < |q| \leqslant 1$, the ergodic unital $SU_q(2) - C^*$-algebras are classified, up to equivariant Morita equivalence, by connected fair and balanced $q + q^{-1}$-graphs.*

A classification of these graphs is a separate combinatorial problem which we do not discuss here but we want to mention some nice classification results for ergodic unital $SU_q(2) - C^*$-algebras obtained in this way. Namely, it was shown in [7], Propositions 4.1 and 4.2, and Theorem 4.3 that there is a number $0 < q_0 < 1$ such that for q satisfying $q_0 < |q| \leqslant 1$, all such algebras are coideals of $C(SU_q(2))$. In particular, for $q = 1$ this gives a new proof of A. Wassermann's classification of $SU(2)$-actions.

When $|q|$ is smaller, the picture is more complicated: in [4] a family of ergodic $SU_q(2)$-coactions of non-coideal type was constructed, their interpretation in terms of graphs is given in [7], Example 7.5.

It is also worth to mention that an ergodic unital $SU_q(2) - C^*$-algebra is necessarily nuclear (this follows from [5], Corollary 23), and it is of type I if and only if it is equivariantly Morita equivalent to a coideal [7], Proposition 7.2. As the property of being of type I is stable under the passage to a C^*-subalgebra and to a strong Morita equivalent C^*-algebra, the part "if" of the last statement is clear. The proof of the part "only if" is based on the analysis of the underlying graphs.

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Université de Caen, LMNO, Campus II, B.P. 5186, F-14032 Caen Cedex, France
E-mail address: leonid.vainerman@unicaen.fr

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