

## THE MULTI-DIMENSIONAL TRUNCATED MOMENT PROBLEM: MAXIMAL MASSES

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*To Professor Ju. M. Berezanskii, on the occasion of his ninetieth birthday*

ABSTRACT. Given a subset  $\mathcal{K}$  of  $\mathbb{R}^d$  and a linear functional  $L$  on the polynomials  $\mathbb{R}_{2n}^d[\underline{x}]$  in  $d$  variables and of degree at most  $2n$  the truncated  $\mathcal{K}$ -moment problem asks when there is a positive Borel measure  $\mu$  supported by  $\mathcal{K}$  such that  $L(p) = \int p d\mu$  for  $p \in \mathbb{R}_{2n}^d[\underline{x}]$ . For compact sets  $\mathcal{K}$  we investigate the maximal mass of all representing measures at a given point of  $\mathcal{K}$ . Various characterizations of this quantity and related properties are developed and a close link to zeros of positive polynomials is established.

### 1. INTRODUCTION

The one-dimensional full moment problem ([1], see e.g. [3, Chapter VII] or [22, Chapter 16]) and the one-dimensional truncated moment problem [14] are well-developed classical fields. In contrast a theory of the multi-dimensional truncated moment problem is still at the beginning. Since the appearance of the work by R. Curto and L. Fialkow [8], [9] this problem is an active area of research, see e.g. [10], [12], [15], [16], [5], [13].

Let  $\mathcal{K}$  be a closed subset of  $\mathbb{R}^d$  and  $n \in \mathbb{N}$ . Given a finite real multi-sequence  $s_\alpha$ , where  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| \leq 2n$ , the *multi-dimensional truncated  $\mathcal{K}$ -moment problem* asks if there is a positive Borel measure  $\mu$  supported by  $\mathcal{K}$  such that

$$(1) \quad s_\alpha = \int x^\alpha d\mu \quad \text{for } \alpha \in \mathbb{N}_0^d, \quad |\alpha| \leq 2n.$$

If  $L$  denotes the corresponding Riesz functional  $L$  on the polynomials  $\mathbb{R}_{2n}^d[\underline{x}]$  of degree at most  $2n$  defined by  $L(x^\alpha) = s_\alpha$ , then equation (1) is equivalent to

$$L(p) = \int p d\mu \quad \text{for } p \in \mathbb{R}_{2n}^d[\underline{x}].$$

The main new contribution of the present paper is a study of the maximal mass  $\rho_L(t)$  at a point  $t \in \mathcal{K}$  among all representing measures for the functional  $L$ .

Probably the first study of the truncated multi-dimensional moment problem was in the unpublished thesis of John Matzke [17]. Several results from this thesis occur in the present paper, while others have been later rederived in the literature. In any case we establish a number of additional results and give new proofs.

In Section 2 we review and develop basic notions and some known results on the truncated  $\mathcal{K}$ -moment problem which are needed later. Sections 3–7 are devoted to a detailed study of maximal mass representations of atomic measures. In Section 3 a dual characterization of the quantity  $\rho_L(t)$  is given. Section 4 deals with atomic solutions of the moment problem. Several notions of representing measures (weakly maximal mass, maximal mass, and strongly maximal mass measures) and related properties of points

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are defined and investigated. A close interplay between maximal masses and properties of zeros of positive polynomials on  $\mathcal{K}$  is established. A number of examples are discussed in Section 5. In Section 6 a general method for constructing weakly maximal mass representing measures is proposed. In Section 7 we use Robinson’s ternary sextic [20] to develop an example of a weakly maximal mass measure which is not maximal mass.

Let us fix some notation. Let  $d \in \mathbb{N}$  and  $n \in \mathbb{N}$ . We denote by  $\mathbb{R}_n^d[\underline{x}]$  the real vector space of polynomials  $p \in \mathbb{R}[x_1, \dots, x_d]$  such that  $\deg(p) \leq n$ . Clearly,

$$\dim \mathbb{R}_n^d[\underline{x}] = \binom{n+d}{d}.$$

Let  $\mathcal{Z}(p) = \{x \in \mathbb{R}^d : p(x) = 0\}$  denote the real zero set of a polynomial  $p$ . For a subset  $\mathcal{K}$  of  $\mathbb{R}^d$  we define

$$\text{Pos}(\mathcal{K})_{2n} = \{p \in \mathbb{R}_{2n}^d[\underline{x}] : p(x) \geq 0 \text{ for } x \in \mathcal{K}\}.$$

Throughout this paper  $L$  denotes a nonzero linear functional on  $\mathbb{R}_{2n}^d[\underline{x}]$  and  $\mathcal{K}$  is a closed subset of  $\mathbb{R}^d$ .

2. THE TRUNCATED  $\mathcal{K}$ -MOMENT PROBLEM

For a positive Borel measure  $\mu$  on  $\mathbb{R}^d$  such that  $\mathbb{R}_{2n}^d[\underline{x}] \subseteq L^1(\mu)$  we denote by  $L_\mu$  the linear functional on  $\mathbb{R}_{2n}^d[\underline{x}]$  defined by

$$L_\mu(p) = \int p d\mu, \quad p \in \mathbb{R}_{2n}^d[\underline{x}].$$

**Definition 1.** A linear functional  $L$  on  $\mathbb{R}_{2n}^d[\underline{x}]$  is called a truncated  $\mathcal{K}$ -moment functional if there exists a positive regular Borel measure  $\mu$  on  $\mathbb{R}^d$  supported by  $\mathcal{K}$  such that  $L = L_\mu$ , that is,  $p \in L^1(\mu)$  and

$$(2) \quad L(p) = \int p(x) d\mu(x) \quad \text{for } p \in \mathbb{R}_{2n}^d[\underline{x}].$$

The set of such measures  $\mu$  is denoted by  $\mathcal{M}_{L,\mathcal{K}}$ .

We say that a functional  $L$  resp. a measure  $\mu \in \mathcal{M}_{L,\mathcal{K}}$  is  $\mathcal{K}$ -determinate if the set  $\mathcal{M}_{L,\mathcal{K}}$  is a singleton. In the special case  $\mathcal{K} = \mathbb{R}^d$  a  $\mathcal{K}$ -determinate functional or measure is called determinate.

The following simple fact is used in the proof of the next proposition. We use the standard multi-index notation  $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$  for  $\alpha \in \mathbb{N}_0^d$  and abbreviate by  $\sum_n^2$  the cone in  $\mathbb{R}_{2n}^d[\underline{x}]$  of all finite sums of squares  $\sum_i f_i^2$ , where  $f_i \in \mathbb{R}_n^d[\underline{x}]$ .

**Lemma 2.**  $\mathbb{R}_{2n}^d[\underline{x}] = \sum_n^2 - \sum_n^2$ .

*Proof.* One inclusion is trivial. For the other it suffices to show that  $x^\alpha \in \sum_n^2 - \sum_n^2$  for  $|\alpha| \leq 2n$ . To prove this we write  $x^\alpha$  as  $x^\alpha = x^{2\beta} x_{i_1} \dots x_{i_r}$  with  $i_1, \dots, i_r \in \{1, \dots, d\}$  and  $r \leq d$ . We write  $r = 2s$  if  $r$  is even and  $r = 2s - 1$  if  $r$  is odd. Since  $|\alpha| = 2|\beta| + r \leq 2n$ , we then have  $|\beta| + s \leq n$ . Using the identities

$$\begin{aligned} x_{i_{2j-1}} x_{i_{2j}} &= \frac{1}{4} [(x_{i_{2j-1}} + x_{i_{2j}})^2 - (x_{i_{2j-1}} - x_{i_{2j}})^2], \\ x_{i_{2s-1}} &= \frac{1}{4} [(x_{i_{2s-1}} + 1)^2 - (x_{i_{2s-1}} - 1)^2] \end{aligned}$$

we conclude that  $x_{i_1} \dots x_{i_r} \in \sum_s^2 - \sum_s^2$ . Therefore, since  $|\beta| + s \leq n$  and  $x^\alpha = x^{2\beta} x_{i_1} \dots x_{i_r}$ , it follows that  $x^\alpha \in \sum_n^2 - \sum_n^2$ .  $\square$

Let us recall the notion of an adapted space in the sense of Choquet [6]. Let  $X$  be a locally compact topological Hausdorff space. Let  $C(X; \mathbb{R})$  denote the continuous real-valued functions on  $X$  and  $C_c(X; \mathbb{R})$  the functions of  $C(X; \mathbb{R})$  with compact support. A linear subspace  $E$  of  $C(X; \mathbb{R})$  is called an *adapted space* if the following conditions are satisfied:

- (i)  $E = E_+ - E_+$ , where  $E_+ := \{f \in E : f(x) \geq 0 \text{ for } x \in X\}$ .
- (ii) For each  $x \in X$  there exists  $f_x \in E_+$  such that  $f_x(x) > 0$ .
- (iii) For  $f \in E_+$  there exists  $g \in E_+$  such that for  $\varepsilon > 0$  there is  $h_\varepsilon \in C_c(X; \mathbb{R})$  satisfying  $f(x) \leq \varepsilon g(x) + h_\varepsilon(x)$ ,  $x \in X$ .

The following proposition is crucial for our treatment. Assertion (i) follows from the paper [23] by V. Tchakaloff. We give another approach of this result.

**Proposition 3.** *Suppose that  $\mathcal{K}$  is a compact subset of  $\mathbb{R}^d$ .*

(i) *The linear functional  $L$  is a truncated  $\mathcal{K}$ -moment functional if and only if*

$$L(p) \geq 0 \quad \text{for all } p \in \text{Pos}(\mathcal{K})_{2n}.$$

(ii) *If  $L$  is a truncated  $\mathcal{K}$ -moment functional, then the set  $\mathcal{M}_{L, \mathcal{K}}$  is compact in the weak topology.*

*Proof.* Set  $\mathcal{K} = X$ . The linear subspace  $E := \mathbb{R}_{2n}^d[\underline{x}]$  of  $C(X; \mathbb{R})$  is an adapted space. Indeed, since  $\sum_n^2 \subseteq E_+$ , condition (i) follows from Lemma 2. Condition (ii) holds by setting  $f_x = 1$  and condition (iii) is obviously satisfied with  $f = g$  and  $h_\varepsilon = f$ , because  $\mathcal{K}$  is compact. Therefore, both assertions follow from Choquet’s theory of adapted spaces [6], see e.g. [4, Section 2.2]. □

For  $t \in \mathbb{R}^d$  let  $\delta_t$  denote the delta measure at  $t$ , that is,  $\delta_t(M) = 1$  if  $t \in M$  and  $\delta_t(M) = 0$  if  $t \notin M$ . Clearly,  $l_t := L_{\delta_t}$  is the evaluation functional at  $t$ , that is,

$$l_t(p) = p(t), \quad p \in \mathbb{R}_{2n}^d[\underline{x}].$$

A measure  $\mu$  is called *k-atomic* if there are  $k$  pairwise different points  $t_1, \dots, t_k \in \mathbb{R}^d$  and positive numbers  $m_1, \dots, m_k$  such that

$$\mu = \sum_{j=1}^k m_j \delta_{t_j}.$$

The points  $t_i$  are called *atoms* and the numbers  $m_i$  *masses* of  $\mu$ .

The next proposition shows that truncated  $\mathcal{K}$ -moment functionals have always atomic solutions and gives a justification for the study of atomic solutions.

**Proposition 4.** *Suppose that the linear functional  $L$  on  $\mathbb{R}_{2n}^d[\underline{x}]$  is a truncated  $\mathcal{K}$ -moment functional. Then there exists a  $k$ -atomic measure  $\mu \in \mathcal{M}_{L, \mathcal{K}}$ , where  $k \leq \binom{d+2n}{2n}$ , with all atoms in  $\mathcal{K}$ , that is, there are pairwise different points  $t_1, \dots, t_k \in \mathcal{K}$  and positive numbers  $m_1, \dots, m_k$  such that  $\mu = \sum_{j=1}^k m_j \delta_{t_j}$  and*

$$L(p) = \int p(x) d\mu(x) \equiv \sum_{j=1}^k m_j p(t_j), \quad p \in \mathbb{R}_{2n}^d[\underline{x}].$$

Proposition 4 was first proved by H. Richter [19, p. 151]. It was shown almost simultaneously for compact sets by V. Tchakaloff [23, Theorem 2], and independently and slightly later for closed sets by W. W. Rogosinski [21, p. 4]. It seems that both references [19] and [21] have been overlooked in the literature, since several versions for closed sets have been published later.

Also we will use the following well-known fact.

**Proposition 5.** *Let  $\mu \in \mathcal{M}_{L,\mathcal{K}}$ . Suppose that  $p \in \text{Pos}(\mathcal{K})_{2n}$  and  $L(p) = 0$ . Then we have  $\text{supp } \mu \subseteq \mathcal{Z}(p) \cap \mathcal{K}$ .*

*Proof.* Suppose that  $t \in \mathcal{K} \setminus \mathcal{Z}(p)$ . Then there exist a ball  $U$  around  $t$  and a number  $\varepsilon > 0$  such that  $p(x) \geq \varepsilon$  on  $U$ . Then

$$0 = L(p) = \int_{\mathcal{K}} p \, d\mu \geq \int_{U \cap \mathcal{K}} p \, d\mu \geq \varepsilon \mu(U \cap \mathcal{K}) \geq 0.$$

Therefore,  $\mu(U \cap \mathcal{K}) = 0$ , that is,  $t$  is not in the support of  $\mu$ . This proves that  $\text{supp } \mu \subseteq \mathcal{Z}(p) \cap \mathcal{K}$ .  $\square$

At the end of this Section we briefly discuss the truncated moment problem for the real projective space  $\mathbb{P}^d(\mathbb{R})$ . Recall that the points of  $\mathbb{P}^d(\mathbb{R})$  are equivalence classes of  $(d + 1)$ -tuples  $(t_0, \dots, t_d) \neq 0$  of real numbers by the equivalence relation

$$(3) \quad (t_0, \dots, t_d) \sim (t'_0, \dots, t'_d) \quad \text{if} \quad (t_0, \dots, t_d) = \lambda(t'_0, \dots, t'_d)$$

for some nonzero real  $\lambda$ . Thus,  $\mathbb{P}^d(\mathbb{R}) = (\mathbb{R}^{d+1} \setminus \{0\}) / \sim$ . The map

$$\varphi : (t_1, \dots, t_d) \rightarrow (1, t_1, \dots, t_d)$$

is an injection of  $\mathbb{R}^d$  into  $\mathbb{P}^d(\mathbb{R})$ . We identify  $t \in \mathbb{R}^d$  with its image  $\varphi(t)$  in  $\mathbb{P}^d(\mathbb{R})$ . In this manner  $\mathbb{R}^d$  becomes a subset of  $\mathbb{P}^d(\mathbb{R})$ .

Let  $\mathbb{R}_{2n}^d[\underline{x}_0]$  denote the homogeneous polynomials from  $\mathbb{R}[x_0, x_1, \dots, x_d]$  of degree  $2n$ . Recall that  $\mathbb{R}_{2n}^d[\underline{x}]$  are the polynomials from  $\mathbb{R}[x_1, \dots, x_d]$  of degree at most  $2n$ . It is not difficult to verify that the map

$$\phi : p(x_0, \dots, x_d) \rightarrow \check{p}(x_1, \dots, x_d) := p(1, x_1, \dots, x_d)$$

is a bijection of the vector spaces  $\mathbb{R}_{2n}^d[\underline{x}_0]$  and  $\mathbb{R}_{2n}^d[\underline{x}]$  with inverse given by

$$(4) \quad \phi^{-1} : q(x_1, \dots, x_d) \rightarrow q_h(x_0, \dots, x_d) := x_0^{2n} q\left(\frac{x_1}{x_0}, \dots, \frac{x_d}{x_0}\right).$$

It is well-known that  $\mathbb{P}^d(\mathbb{R})$  is a compact topological Hausdorff space and that  $\mathbb{R}^d$  is dense in  $\mathbb{P}^d(\mathbb{R})$ . Each homogeneous polynomial  $q \in \mathbb{R}_d[x_0, \dots, x_d]_{2n}$  can be considered as a continuous function, denoted by  $\hat{q}$ , on the projective space by

$$(5) \quad \hat{q}(t) := \frac{q(t_0, \dots, t_d)}{(t_0^2 + \dots + t_d^2)^n}, \quad t = (t_0, \dots, t_d) \in \mathbb{P}^d(\mathbb{R}).$$

(The fraction in (5) is invariant under the relation (3) and continuous on  $\mathbb{R}^{d+1} \setminus \{0\}$ .)

For a subset  $\mathcal{K}$  of  $\mathbb{P}^d(\mathbb{R})$  we define

$$\text{Pos}(\mathcal{K})_{2n} = \{p \in \mathbb{R}_{2n}^d[\underline{x}_0] : \hat{p}(x) \geq 0 \quad \text{for} \quad x \in \mathcal{K}\}.$$

If we replace  $\mathbb{R}_{2n}^d[\underline{x}]$  by the homogeneous polynomials  $\mathbb{R}_{2n}^d[\underline{x}_0]$ , for each closed subset  $\mathcal{K}$  of  $\mathbb{P}^d(\mathbb{R})$  the truncated  $\mathcal{K}$ -moment problem can be defined in a similar manner as for  $\mathbb{R}^d$ . The notions and results of this paper remain valid in this setting. We shall use this projective truncated moment problem on  $\mathbb{P}^d(\mathbb{R})$  in Section 7.

### 3. MAXIMAL MASSES

Suppose that  $L$  is a truncated  $\mathcal{K}$ -moment functional on  $\mathbb{R}_{2n}^d[\underline{x}]$ .

For any  $t_0 \in \mathcal{K}$  we define two nonnegative numbers  $\rho_L(t_0)$  and  $\pi_L(t_0)$  by

$$(6) \quad \rho_L(t_0) = \sup \{ \mu(\{t_0\}); \mu \in \mathcal{M}_{L,\mathcal{K}} \},$$

$$(7) \quad \pi_L(t_0) = \inf \left\{ \frac{L(p)}{p(t_0)}; p \in \text{Pos}(\mathcal{K})_{2n} \right\}$$

$$(8) \quad = \inf \{ L(p); p \in \text{Pos}(\mathcal{K})_{2n}, p(t_0) = 1 \}.$$

where we set  $\frac{c}{0} := +\infty$  for  $c \geq 0$ . The equality in (8) is obvious.

The number  $\rho_L(t_0)$  is the supremum of all point masses of solutions of the truncated  $\mathcal{K}$ -moment problem for  $L$ . Note that  $\rho_L(t_0) \leq L(1) < \infty$ , since

$$\mu(\{t_0\}) \leq \int 1 d\mu = L(1) \quad \text{for } \mu \in \mathcal{M}_{L,\mathcal{K}}.$$

*Remark.* Determining the number  $\pi_L(t_0)$  in (8) is a *linear conic optimization problem* with linear functional  $L$ , cone  $C := \text{Pos}(\mathcal{K})_{2n}$  in the vector space  $\mathbb{R}_{2n}^d[\underline{x}]$ , and constraint  $p(t_0) = 1$ . From the duality theory of conic optimization (see e.g. [2]) it is known that the infimum in (8) is attained if  $L$  is an interior point of the dual cone  $C^*$ . A more important result of this kind is Theorem 24 below.

The following Propositions 6, 11, and 12 are taken from [17].

**Proposition 6.** *Let  $t_0 \in \mathcal{K}$ . Then we have  $\pi_L(t_0) \geq \rho_L(t_0)$ . If  $\mathcal{K}$  is compact, the supremum in equation (6) is a maximum and*

$$(9) \quad \rho_L(t_0) = \pi_L(t_0).$$

*Proof.* Let  $\nu \in \mathcal{M}_{L,\mathcal{K}}$ . For any  $p \in \text{Pos}(\mathcal{K})_{2n}$  we have

$$L(p) = \int p d\nu \geq \nu(\{t_0\})p(t_0)$$

and hence  $\frac{L(p)}{p(t_0)} \geq \nu(\{t_0\})$ . Taking the infimum over  $p \in \text{Pos}(\mathcal{K})_{2n}$  and the supremum over  $\nu \in \mathcal{M}_{L,\mathcal{K}}$ , we conclude that

$$(10) \quad \pi_L(t_0) \geq \rho_L(t_0).$$

Suppose now that  $\mathcal{K}$  is compact. Let  $(\mu_i)_{i \in J}$  be a net from  $\mathcal{M}_{L,\mathcal{K}}$  which converges weakly to some measure  $\mu \in \mathcal{M}_{L,\mathcal{K}}$ . Then  $\lim_i \mu_i(\{t_0\}) \leq \mu(\{t_0\})$  by the portmanteau theorem [4, p. 46]. That is, the map  $\mu \rightarrow \mu(\{t_0\})$  is an upper continuous functions on the compact set  $\mathcal{M}_{L,\mathcal{K}}$  (by Proposition 3(ii)) with respect to the weak topology. Since such a function on a compact set has always a maximum, the supremum in (6) is attained.

Define a linear functional  $L_0 := L - \pi_L(t_0)l_{t_0}$  on  $\mathbb{R}_{2n}^d[\underline{x}]$ , that is,

$$L_0(f) = L(f) - \pi_L(t_0)f(t_0), \quad f \in \mathbb{R}_{2n}^d[\underline{x}].$$

From (7) it follows at once that  $L_0(p) \geq 0$  for  $p \in \text{Pos}(\mathcal{K})_{2n}$ . Therefore, by Proposition 3(i),  $L_0$  is a truncated  $\mathcal{K}$ -moment functional, that is, there exists a measure  $\mu_0 \in \mathcal{M}_{L_0,\mathcal{K}}$ . Then

$$L(f) = \pi_L(t_0)f(t_0) + L_0(f) = \pi_L(t_0)f(t_0) + \int f d\mu_0, \quad f \in \mathbb{R}_{2n}^d[\underline{x}].$$

This shows that the measure  $\mu := \pi_L(t_0)\delta_{t_0} + \mu_0$  belongs to  $\mathcal{M}_{L,\mathcal{K}}$ . Therefore,  $\rho_L(t_0) \geq \mu(\{t_0\}) \geq \pi_L(t_0)$ . Combined with the converse inequality (10) proved above we obtain  $\rho_L(t_0) = \pi_L(t_0)$ . This proves (9).  $\square$

Now we introduce two basic objects associated with  $L$  and  $\mathcal{K}$ .

**Definition 7.** *For a closed subset  $\mathcal{K}$  of  $\mathbb{R}^d$  we define*

$$\begin{aligned} \mathcal{N}_+(L, \mathcal{K}) &= \{p \in \text{Pos}(\mathcal{K})_{2n} : L(p) = 0\}, \\ \mathcal{V}_+(L, \mathcal{K}) &= \{t \in \mathbb{R}^d : p(t) = 0 \text{ for } p \in \mathcal{N}_+(L, \mathcal{K})\}. \end{aligned}$$

Note that it may happen that  $\mathcal{N}_+(L, \mathcal{K}) = \{0\}$ ; in this case  $\mathcal{V}_+(L, \mathcal{K}) = \mathbb{R}^d$ .

Clearly,  $\mathcal{V}_+(L, \mathcal{K})$  is a real algebraic variety. The importance of this variety for the truncated moment problem stems from the following result.

**Proposition 8.** *If  $\mathcal{K}$  is a compact subset of  $\mathbb{R}^d$  and  $L$  is a truncated  $\mathcal{K}$ -moment functional on  $\mathbb{R}_{2n}^d[\underline{x}]$ , then  $\mathcal{V}_+(L, \mathcal{K}) \cap \mathcal{K}$  is the smallest subset of  $\mathbb{R}^d$  which contains the supports of all measures  $\mu \in \mathcal{M}_{L, \mathcal{K}}$ .*

The proof of Proposition 8 is deferred to Section 6 after Theorem 23.

The following two objects are often used in the literature:

$$\begin{aligned} \mathcal{N}(L) &:= \{p \in \mathbb{R}_{2n}^d[\underline{x}] : L(p^2) = 0\}, \\ \mathcal{V}(L) &= \{t \in \mathbb{R}^d : p(t) = 0 \text{ for } p \in \mathcal{N}(L)\}. \end{aligned}$$

Obviously, if  $p \in \mathcal{N}(L)$ , then  $p^2 \in \mathcal{N}_+(L, \mathcal{K})$ . Hence  $\mathcal{V}_+(L, \mathcal{K}) \subseteq \mathcal{V}(L)$ .

If  $L$  is a positive functional (that is,  $L(p^2) \geq 0$  for all  $p \in \mathbb{R}_n^d[\underline{x}]$ ), it follows easily from the Cauchy-Schwarz inequality that  $\mathcal{N}(L)$  is a vector space. Further, there is a natural bijection of  $\mathcal{N}(L)$  on the kernel of the Hankel matrix  $\mathcal{H}_n(L) = (L(x^{i+j}))_{i,j=0}^n$  associated with  $L$ .

The real algebraic variety  $\mathcal{V}_+(L)$  is of fundamental importance for the truncated moment problem. The following example illustrates the difference between both varieties  $\mathcal{V}_+(L)$  and  $\mathcal{V}(L)$ . We shall use the projective versions of both sets.

**Example 9.** *Set  $d = 2, n = 3$ , and  $\mathcal{K} = \mathbb{P}^2(\mathbb{R})$ . Let  $R$  denote the homogeneous Robinson polynomial [20] used in Section 7 below. As noted therein,  $\hat{R} \in \text{Pos}(\mathcal{K})_6$  and  $R$  has precisely 10 projective zeros  $t_1, \dots, t_{10} \in \mathbb{P}^2(\mathbb{R})$ . Putting  $\mu = \sum_{i=1}^{10} \delta_{t_i}$  and  $L := L_\mu = \sum_{i=1}^{10} l_{t_i}$ , we clearly have  $\hat{R} \in \mathcal{N}_+(L, \mathcal{K})$  and  $\{t_1, \dots, t_{10}\} \subseteq \mathcal{V}_+(L, \mathcal{K})$ . (We have even equality, but we do not need this here.) Since there is no cubic that vanishes at all ten points  $t_i$  (see e.g. [7]), we have  $\mathcal{N}(L) = \{0\}$  and  $\mathcal{V}(L) = \mathbb{P}^2(\mathbb{R})$ .*

The next example deals with the one-dimensional case.

**Example 10.** *Consider a  $k$ -atomic measure  $\mu = \sum_{j=1}^k m_j \delta_{t_j}$  on  $\mathbb{R}$  and set  $L = L_\mu$ .*

*Let  $\mathcal{K} = \mathbb{R}$ . Since each polynomial of  $\text{Pos}(\mathbb{R})_{2n}$  is a sum of squares from  $\mathbb{R}_n[x]$ ,  $\mathcal{N}(L) = \mathcal{N}_+(L, \mathcal{K})$  and hence  $\mathcal{V}(L) = \mathcal{V}_+(L, \mathcal{K})$ . It is easily verified that  $\mathcal{N}(L) \neq \{0\}$  if and only if  $k \leq n$ ; in this case  $L$  is determinate and  $\mathcal{V}(L) = \{t_1, \dots, t_k\}$ .*

*Now let  $\mathcal{K} = [-1, 1]$ ,  $k = n + 1$ . Assume that  $t_1, \dots, t_k \in [-1, 1]$ . Then  $\mathcal{N}(L) = \{0\}$  as just noted. If at least  $n$  atoms  $t_j$  are in  $(-1, 1)$ , then  $\mathcal{N}_+(L, \mathcal{K}) = \{0\}$ . But if both end points  $-1$  and  $1$  are atoms of  $\mu$ , then  $\mathcal{N}_+(L, \mathcal{K}) \neq \{0\}$ .*

**Proposition 11.** *Suppose that  $\mathcal{K}$  is compact and let  $t_0 \in \mathcal{K}$ .*

- (i) *If  $\rho_L(t_0) > 0$ , then  $t_0 \in \mathcal{V}_+(L, \mathcal{K})$ .*
- (ii) *If the infimum in (7), or equivalently in (8), is a minimum, then  $t_0 \in \mathcal{V}_+(L, \mathcal{K})$  if and only if  $\rho_L(t_0) > 0$ .*
- (iii) *Suppose that  $\mathcal{N}_+(L, \mathcal{K}) = \{0\}$ . Then  $t \in \mathcal{V}_+(L, \mathcal{K})$  and  $\rho_L(t) > 0$  for all  $t \in \mathcal{K}$ .*

*Proof.* (i): Suppose that  $\rho_L(t_0) > 0$  and assume to the contrary that  $t_0 \notin \mathcal{V}_+(L, \mathcal{K})$ . Then there exists a  $p_0 \in \mathcal{N}_+(L)$  such that  $p_0(t_0) \neq 0$ . Upon scaling we can assume that  $p_0(t_0) = 1$ . Since  $L(p_0) = 0$ , we then have  $\pi_L(t_0) = 0$  by (8) and hence  $\rho_L(t_0) = 0$  by (9) which is the desired contradiction.

(ii): By (i) it suffices to prove that  $t_0 \in \mathcal{V}_+(L, \mathcal{K})$  implies  $\rho_L(t_0) > 0$ . Let  $t_0 \in \mathcal{V}_+(L, \mathcal{K})$  and assume that the infimum in (8) is a minimum. Let  $p_0$  be a polynomial for which this infimum is attained. If  $L(p_0)$  would be zero, then  $p_0 \in \mathcal{N}_+(L, \mathcal{K})$  and hence  $p_0(t_0) = 0$ , since  $t_0 \in \mathcal{V}_+(L, \mathcal{K})$ . But  $p_0(t_0) = 1$  by (8). Hence  $L(p_0) > 0$  and therefore  $L(p_0) = \pi_L(t_0) = \rho_L(t_0) > 0$  again by (9).

(iii) follows by combining (i) and (ii). □

4. ATOMIC SOLUTIONS

Let  $\mu \in \mathcal{M}_{L,\mathcal{K}}$ . Then  $\mathbb{R}_{2n}^d[\underline{x}]$  is contained in  $L_{\mathbb{R}}^1(\mu)$ . Let  $J_\mu$  denote this embedding of  $\mathbb{R}_{2n}^d[\underline{x}]$  into  $L_{\mathbb{R}}^1(\mu)$ . The next proposition is a simple consequence of a well-known fact of Douglas [11].

**Proposition 12.** *For any measure  $\mu \in \mathcal{M}_{L,\mathcal{K}}$  the following are equivalent:*

- (i)  $\mu$  is an extreme point of the set  $\mathcal{M}_{L,\mathcal{K}}$ .
- (ii)  $J(\mathbb{R}_{2n}^d[\underline{x}]) = L_{\mathbb{R}}^1(\mu)$ .
- (iii)  $\mu$  is atomic, that is,  $\mu = \sum_{l=1}^k m_l \delta_{t_l}$  with  $t_1, \dots, t_k \in \mathcal{K}$ , and there are polynomials  $p_1, \dots, p_k \in \mathbb{R}_{2n}^d[\underline{x}]$  such that  $p_j(t_l) = \delta_{jl}$  for  $j, l = 1, \dots, k$ .

*Proof.* By a result of Douglas [11],  $\mu$  is an extreme point of  $\mathcal{M}_{L,\mathcal{K}}$  if and only if the image of  $\mathbb{R}_{2n}^d[\underline{x}]$  is dense in  $L_{\mathbb{R}}^1(\mu)$ . Since  $\mathbb{R}_{2n}^d[\underline{x}]$  is finite dimensional, the latter holds if and only if  $J_\mu(\mathbb{R}_{2n}^d[\underline{x}]) = L_{\mathbb{R}}^1(\mu)$ . This proves that (i) and (ii) are equivalent. Then  $J_\mu(\mathbb{R}_{2n}^d[\underline{x}]) = L_{\mathbb{R}}^1(\mu)$  is finite dimensional, so that  $\mu$  is atomic, say  $\mu = \sum_{l=1}^k m_l \delta_{t_l}$  with  $t_1, \dots, t_k \in \mathcal{K}$ . Clearly,  $J_\mu(\mathbb{R}_{2n}^d[\underline{x}]) = L_{\mathbb{R}}^1(\mu)$  if and only if the characteristic function  $\chi_{t_j}$  of each point  $\{t_j\}$ ,  $j = 1, \dots, k$ , is in the image  $J_\mu(\mathbb{R}_{2n}^d[\underline{x}])$ , that is,  $\chi_{t_j}$  is equal to  $J_\mu(p_j)$  for some polynomial  $p_j \in \mathbb{R}_{2n}^d[\underline{x}]$ , or equivalently,  $p_j(t_l) = \delta_{jl}$ , where  $j, l = 1, \dots, k$ . This proves the equivalence of (ii) and (iii).  $\square$

**Proposition 13.** *Let  $\mu = \sum_{l=1}^k m_l \delta_{t_l} \in \mathcal{M}_{L,\mathcal{K}}$  be  $k$ -atomic with all atoms in  $\mathcal{K}$  and let  $j \in \{1, \dots, k\}$ .*

- (i) *The number  $m_j$  is the maximal mass  $\rho_L(t_j)$  if and only there is a sequence  $(p_r)_{r \in \mathbb{N}}$  of polynomials  $p_r \in \text{Pos}(\mathcal{K})_{2n}$  such that  $p_r(t_j) = 1$  and  $\lim_r p_r(t_l) = 0$  for all  $l \neq j$ .*
- (ii) *If there exists a polynomial  $p \in \text{Pos}(\mathcal{K})_{2n}$  such that  $p(t_j) = 1$  and  $p(t_l) = 0$  for  $l \neq j$ , then  $m_j = \rho_L(t_j)$ .*
- (iii) *If  $m_j = \rho_L(t_j)$  for all  $j = 1, \dots, k$ , then  $k \leq \binom{d+2n}{2n} = \dim \mathbb{R}_{2n}^d[\underline{x}]$ .*

*Proof.* (i): From the definition of  $\rho_L$  it is obvious that  $m_j \leq \rho_L(t_j)$ . Suppose that there exists a sequence  $(p_r)$  of polynomials as above. Since  $p_r(t_j) = 1$ , we have  $\rho_L(t_j) = \pi_L(t_j) \leq L(p_r)$  by (7). Combined with

$$\lim_r L(p_r) = \sum_{l=1}^k m_l (\lim_r p_r(t_l)) = \sum_{l=1}^k m_l \delta_{jl} = m_j$$

we conclude that  $\rho_L(t_j) \leq m_j$  by (7). Thus,  $m_j = \rho_L(t_j)$ .

Conversely, suppose that  $m_j = \rho_L(t_j) = \pi_L(t_j)$ . By the definition of  $\pi_L(t_j)$ , there is a sequence  $(p_r)$  of polynomials  $p_r \in \text{Pos}(\mathcal{K})_{2n}$  such that  $p_r(t_j) = 1$  and  $m_j = \lim_r L(p_r)$ . Since  $p_r(x_l) \leq m_l^{-1} L(p_r)$  and the sequence  $((L(p_r))_{r \in \mathbb{N}}$  converges, the sequence  $(p_r(t_l))_{r \in \mathbb{N}}$  is bounded. By passing to some subsequence if necessary we can assume that  $a_l := \lim_r p_r(t_l)$  exists for all  $l$ . Since  $p_r \in \text{Pos}(\mathcal{K})_{2n}$ , we have  $a_l \geq 0$ . Then

$$m_j = \lim_r L(p_r) = m_j + \sum_{l=1, l \neq j}^k m_l a_l.$$

Since  $m_l > 0$  for all  $l$ , this implies that  $a_l = \lim_r p_r(t_l) = 0$ .

(ii) follows at once from (i).

(iii): Since  $m_j = \rho_L(t_j)$ , there is a sequence  $(p_{jr})_{r \in \mathbb{N}}$  of polynomials as in (i). We define a linear functional  $F_j$  on  $\mathbb{R}_{2n}^d[\underline{x}]$  by  $F_j = l_{t_j}$ . To prove that  $k \leq \dim \mathbb{R}_{2n}^d[\underline{x}]$  it suffices to show that the functionals  $F_1, \dots, F_k$  are linearly independent. Assume that there are numbers  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  such that  $0 = \sum_j \lambda_j F_j(f)$  for  $f \in \mathbb{R}_{2n}^d[\underline{x}]$ . Setting

$f = p_{jr}$  and passing to the limit  $r \rightarrow \infty$ , we obtain

$$0 = \lim_r \sum_{l=1}^k \lambda_l F_l(p_{jr}) = \sum_{l=1}^k \lambda_l (\lim_r p_{jr}(t_l)) = \sum_{l=1}^k \lambda_l \delta_{jl} = \lambda_j. \quad \square$$

Let  $t_0, \dots, t_k \in \mathbb{R}^d$  be pairwise different points and let  $n \in \mathbb{N}$ . Consider the following three properties:

(SP) $_{2n}$  (*Separation property*)

There are polynomials  $p_0, \dots, p_k \in \mathbb{R}_{2n}^d[\underline{x}]$  such that  $p_j(t_l) = \delta_{jl}, j, l = 0, \dots, k$ .

Suppose in addition that  $t_0, \dots, t_k \in \mathcal{K}$ .

(PSP) $_{2n, \mathcal{K}}$  (*Positive separation property*)

There are polynomials  $q_0, \dots, q_k \in \text{Pos}(\mathcal{K})_{2n}$  such that  $q_j(t_l) = \delta_{jl}, j, l = 0, \dots, k$ .

(APSP) $_{2n, \mathcal{K}}$  (*Asymptotic positive separation property*)

There are sequences  $(q_{jr})_{r \in \mathbb{N}}, j = 1, \dots, k$ , of polynomials  $q_{jr} \in \text{Pos}(\mathcal{K})_{2n}$  such that  $q_{jr}(t_j) = 1$  for  $r \in \mathbb{N}$  and  $\lim_r q_{jr}(t_l) = 0$  for  $j \neq l$  and  $j, l = 0, \dots, k$ .

Note that the two last properties depend on the fixed closed subset  $\mathcal{K}$  of  $\mathbb{R}^d$ . In the important special case  $\mathcal{K} = \mathbb{R}^d$  we write (APSP) $_{2n}$  and (PSP) $_{2n}$ , respectively. Obviously, (PSP) $_{2n, \mathcal{K}}$  implies (APSP) $_{2n, \mathcal{K}}$  and (SP) $_{2n}$ .

These properties are closely related to the following concepts.

**Definition 14.** A  $k$ -atomic measure  $\mu = \sum_{l=1}^k m_l \delta_{t_l} \in \mathcal{M}_{L, \mathcal{K}}$  with all atoms in  $\mathcal{K}$  is called

- maximal mass for  $L$  if  $m_j = \rho_L(t_j)$  for all  $j = 1, \dots, k$ .
- weakly maximal mass for  $L$  if  $m_{j+1} = \rho_{L_j}(t_{j+1})$  for  $j = 0, \dots, k-1$ , where  $L_j$  is the functional defined by  $L_j(f) = \sum_{l=j+1}^k m_l f(t_l)$  and  $L_0 := L$ .
- strongly maximal mass for  $L$  if there are polynomials  $p_j \in \text{Pos}(\mathcal{K})_{2n}$  such that  $p_j(t_l) = \delta_{jl}$  for  $j, l = 1, \dots, k$ .

We briefly discuss these concepts. Suppose that  $\mu = \sum_{l=1}^k m_l \delta_{t_l} \in \mathcal{M}_{L, \mathcal{K}}$  and all atoms of  $\mu$  are in  $\mathcal{K}$ . Let  $L_j$  be the  $\mathcal{K}$ -truncated moment functional defined by  $L_j(f) = \sum_{l=j+1}^k m_l f(t_l)$ . Note that  $L_0 := L$ .

That  $\mu$  is maximal mass means that the masses  $m_j$  of all atoms  $t_j$  coincide with the maximal masses  $\rho_L(t_j)$ . By Proposition 13(ii), if  $\mu$  is strongly maximal mass, then  $\mu$  is maximal mass. Further,  $\mu$  is an weakly maximal measure if and only if  $m_1$  is the maximal mass  $\rho_L(t_1)$ ,  $m_2$  is the maximal mass  $\rho_{L_1}(t_2)$  etc. It is easily checked that if  $\mu$  is maximal mass, it is also weakly maximal mass. The converse is not true as shown in Section 7 below. The reason for introducing the notion of weakly maximal mass measures is that such measures can be constructed easily by a general and simple procedure. This will be developed in the next Section.

The next proposition reformulates the preceding results in terms of these properties.

**Proposition 15.** Suppose that  $\mu = \sum_{l=0}^k m_l \delta_{t_l} \in \mathcal{M}_{L, \mathcal{K}}$  with all atoms in  $\mathcal{K}$ .

- (i)  $\mu$  is an extreme point of  $\mathcal{M}_{L, \mathcal{K}}$  if and only if  $t_0, \dots, t_k$  satisfy (SP) $_{2n}$ .
- (ii)  $\mu$  is maximal mass, that is,  $m_j = \rho_L(t_j)$  for  $j = 0, \dots, k$ , if and only if (APSP) $_{2n, \mathcal{K}}$  holds for  $t_0, \dots, t_k$ .
- (iii)  $\mu$  is strongly maximal mass, that is, for all atoms  $t_j$  the infimum in the definition (7) of  $\rho_L(t_j)$  is a minimum, if and only if (PSP) $_{2n, \mathcal{K}}$  holds for  $t_0, \dots, t_k$ .

*Proof.* Proposition 12 yields (i). (ii) follows by comparing Proposition 13(i) and Definition 14. (iii) follows from the proof of Proposition 13(ii).  $\square$

**Proposition 16.** Let  $p \in \text{Pos}(\mathcal{K})_{2n}$ . Suppose that  $\mathcal{Z}(p) = \{t_0, \dots, t_k\}$ , where  $t_0, \dots, t_k \in \mathcal{K}$  are pairwise different points, and  $t_0, \dots, t_k$  satisfy (SP) $_{2n}$ . Assume that  $I$  is a subset



of  $\{0, \dots, k\}$  such that  $\mu = \sum_{i \in I} m_i \delta_{t_i}$  is an  $|I|$ -atomic measure in  $\mathcal{M}_{L, \mathcal{K}}$ . Then  $\mu$  is  $\mathcal{K}$ -determinate and hence maximal mass.

*Proof.* Let  $\nu$  be another measure from  $\mathcal{M}_{L, \mathcal{K}}$ . Since  $p \in \text{Pos}(\mathcal{K})_{2n}$  and

$$\int p d\nu = L(p) = \int p d\mu = \sum_{i \in I} m_i p(t_i) = 0,$$

it follows from Proposition 5 that  $\text{supp } \nu \subseteq \mathcal{Z}(p) = \{t_0, \dots, t_k\}$ . Hence  $\nu$  is of the form  $\nu = \sum_{j=0}^k n_j \delta_{t_j}$  for some numbers  $n_j \geq 0$ .

Now fix  $i \in \{0, \dots, k\}$  and write  $m_i = 0$  if  $i \notin I$ . By assumption,  $t_0, \dots, t_k$  satisfy  $(SP)_{2n}$ , so there exists a polynomial  $q \in \mathbb{R}_{2n}^d[\underline{x}]$  such that  $q(t_j) = \delta_{ij}$  for  $j = 1, \dots, k$ . Then we have  $L(q) = \int q d\mu = m_i$  and  $L(q) = \int q d\nu = n_i$  which implies that  $m_i = n_i$ . This proves that  $\mu = \nu$ , that is,  $\mu$  is  $\mathcal{K}$ -determinate.  $\square$

*Remark.* If  $k \leq 2n$  in Proposition 16, then the points  $t_0, \dots, t_k$  satisfy  $(SP)_{2n}$  by Proposition 17(ii) below.

5. EXAMPLES AND DISCUSSION

Let  $t_0, \dots, t_k$  be pairwise different points of  $\mathbb{R}^d$ .

First we note that if  $(SP)_{2n}$  holds for  $t_0, \dots, t_k$ , it is easily seen that the polynomials  $p_0, \dots, p_k$  are linearly independent, so that  $k + 1 \leq \binom{2n+d}{2n} = \dim \mathbb{R}_{2n}^d[\underline{x}]$ .

Further, since  $t_j \neq t_l$ , there is a number  $i_{jl} \in \{1, \dots, d\}$  such that  $t_j^{i_{jl}} \neq t_l^{i_{jl}}$ . Define the interpolation polynomial

$$(11) \quad p_j(x) = \prod_{l=0, l \neq j}^k \frac{x^{j i_{jl}} - t_l^{i_{jl}}}{t_j^{i_{jl}} - t_l^{i_{jl}}}, \quad j = 0, \dots, k.$$

Then we have  $p_j \in \mathbb{R}^d[\underline{x}]$ ,  $k = \deg p_j$  and  $p_j(t_l) = \delta_{jl}$ ,  $j, l = 0, \dots, k$ . Therefore, if  $k \leq 2n$ , then  $p_j \in \mathbb{R}_{2n}^d[\underline{x}]$  and  $t_0, \dots, t_k$  satisfies  $(SP)_{2n}$ .

For  $k \leq n$ , it is obvious that  $p_j^2 \in \text{Pos}(\mathcal{K})_{2n}$  and  $p_j^2(t_l) = \delta_{jl}$ ,  $j, l = 0, \dots, k$ . Hence, if  $t_0, \dots, t_k \in \mathcal{K}$  and  $k \leq n$ , then  $t_0, \dots, t_k$  obey  $(PSP)_{2n, \mathcal{K}}$ . Instead of  $p_j^2$  we could have also taken the polynomials

$$(12) \quad q_j(x) = \prod_{l=0, l \neq j}^k \frac{\|x - t_l\|^2}{\|t_j - t_l\|^2}, \quad j = 0, \dots, k.$$

The following proposition summarizes the preceding discussion.

**Proposition 17.** *Suppose that  $\mu = \sum_{l=0}^k m_l \delta_{t_l}$  is  $k + 1$ -atomic.*

- (i) *If  $(SP)_{2n}$  is satisfied for  $t_0, \dots, t_k$ , then  $k + 1 \leq \binom{d+2n}{2n}$ .*
- (ii) *If  $k \leq 2n$ , then  $(SP)_{2n}$  holds for  $t_0, \dots, t_k$ .*
- (iii) *If  $k \leq n$  and  $t_0, \dots, t_k \in \mathcal{K}$ , then  $t_0, \dots, t_k$  obey  $(PSP)_{2n, \mathcal{K}}$ .*

We now develop a number of simple examples.

**Example 18.** Let  $d = 2$  and  $t_0 = (0, 0), t_1 = (1, 0), t_2 = (0, 1), t_3 = (1, 1)$ . Then the polynomials

$$p_0 = (1 - x_1)(1 - x_2), \quad p_1 = x_1(1 - x_1), \quad p_2 = x_1(1 - x_2), \quad p_3 = x_1 x_2$$

satisfy  $p_j(t_l) = \delta_{jl}$  for  $j, l = 0, \dots, 3$ , that is, the points  $t_0, \dots, t_3$  obey  $(SP)_2$ . Hence the 4-atomic measure  $\mu := \sum_{j=0}^3 \delta_{t_j}$  is an extreme point of the set  $\mathcal{M}_{L, \mu, \mathcal{K}}$  for any set  $\mathcal{K}$  containing  $t_0, \dots, t_3$ .

Let  $\mathcal{K}$  be a rectangle containing the points  $t_0, \dots, t_3$ . It is not difficult to verify that there is a 3-atomic measure  $\nu \in \mathcal{M}_{L, \mu, \mathcal{K}}$ . In particular,  $\mu$  is *not determinate*.

**Example 19.** (*Points on a line*)

Suppose that  $t_0, \dots, t_k$  are pairwise different points lying on a line in  $\mathbb{R}^d$ . Upon a linear transformation we can assume that this line is the  $x_1$ -axis, so that  $t_j = (t_{j1}, 0, \dots, 0)$  for  $j = 0, \dots, k$ . Fix  $j$  and suppose that  $p_j \in \mathbb{R}^d[x]$  satisfies  $p_j(t_l) = 0$  for all  $l \neq j$ . Then  $p_j(x_1, 0, \dots, 0)$  has at least  $k$  pairwise different zeros  $t_{l1}$ ,  $l \neq j$ . Hence  $\deg p_j \geq k$ . On the other hand, the polynomials

$$p_j(x) := \prod_{l=0, l \neq j}^k (x_1 - t_{l1})(t_{j1} - t_{l1})^{-1}$$

satisfy  $p_j(t_l) = \delta_{jl}$  for  $j, l = 0, \dots, k$  and  $\deg p_j = k$ . That is,  $(SP)_{2n}$  holds for  $t_0, \dots, t_k$  if and only if  $k \leq 2n$ .

Let  $[a, b]$  be an interval which contains all numbers  $t_{j1}$  and suppose that  $\mathcal{K}$  contains a neighbourhood of the set  $\{(y, 0, \dots, 0); y \in [a, b]\}$ . If  $p \in \text{Pos}(\mathcal{K})$  and  $p(y_0, 0, \dots, 0) = 0$  for some  $y_0 \in [a, b]$ , then  $y_0$  is a zero of even order. From this fact it follows easily that the points  $t_0, \dots, t_k$  obey  $(PSP)_{2n, \mathcal{K}}$  if and only if  $k \leq n$ . Therefore, by Proposition 15(iii), if  $k \leq n$ , each  $k+1$ -atomic measure  $\mu = \sum_{l=0}^k m_l \delta_{t_l} \in \mathcal{M}_{L, \mathcal{K}}$  is strongly maximal mass.

**Example 20.** Let  $d = n = 1$  and  $\mathcal{K} = [0, 1]$ . Set  $t_0 = 0, t_1 = \frac{1}{2}, t_2 = 1$  and define  $L(f) = \sum_{j=0}^2 f(t_j)$ ,  $f \in \mathbb{R}_2[x]$ . That is,  $\mu = \sum_{j=0}^2 m_j \delta_{t_j}$ , where  $m_0 = m_1 = m_2 = 1$ , is in  $\mathcal{M}_{L, \mathcal{K}}$ . One easily checks that the 2-atomic measure  $\mu_0 = \frac{6}{5} \delta_0 + \frac{9}{5} \delta_{\frac{1}{2}}$  is also in  $\mathcal{M}_{L, \mathcal{K}}$ , so that  $\rho_L(0) = \frac{6}{5} \neq m_0$  by the preceding example.

The next example gives a simple recipe for constructing sequences of points satisfying  $(PSP)_{2n}$ .

**Example 21.** (*Products of squares of polynomials of degree one*)

First we take points  $t_0, \dots, t_d$  of  $\mathbb{R}^d$  and assume that these points are not contained in a  $(d-1)$ -dimensional affine subspace. This assumption is equivalent to the requirement that for any (and then for all)  $i \in \{0, \dots, d\}$  the  $d$  vectors  $t_l - t_i$ ,  $l \neq i$ , do not lie in a  $(d-1)$ -dimensional linear subspace of  $\mathbb{R}^d$ .

Now fix  $j \in \{0, \dots, d\}$ . Take an index  $i \in \{0, \dots, d\}$  such that  $i \neq j$ . We choose a vector  $a_j \in \mathbb{R}^d$ ,  $a_j \neq 0$ , which is orthogonal to the  $d-1$  vectors  $t_l - t_i$ , where  $l = 0, \dots, d$ ,  $l \neq j, i$ . That is,  $a_j \cdot (t_l - t_i) = 0$ , where  $\cdot$  is the usual scalar product of  $\mathbb{R}^d$ . Further,  $a_j$  is not orthogonal to  $t_j - t_i$ , since otherwise the vectors  $t_l - t_i$ ,  $l \neq i$ , would be contained in a  $(d-1)$ -dimensional linear subspace. Upon scaling  $a_j$  we can assume that  $a_j \cdot (t_j - t_i) = 1$ . Putting

$$q_j(x) := (a_j \cdot x - a_j \cdot t_i)^2,$$

we have  $q_j(t_l) = \delta_{jl}$  for  $j, l = 0, \dots, d$  by construction. This means that the points  $t_0, \dots, t_d$  satisfy  $(PSP)_2$ .

The preceding construction can be extended to the case  $(PSP)_{2n}$  for  $n \in \mathbb{N}$ . Let us take  $1 + dn$  points  $t_0, \dots, t_{dn}$  of  $\mathbb{R}^d$  and assume the following: For each index  $j \in \{0, \dots, dn\}$  there is a decomposition of the remaining indices into a disjoint union of subsets  $N_{j1}, \dots, N_{jn}$  such that each set  $N_{jl}$  consists of  $d$  numbers and the points  $t_j$  and all  $t_i$  for  $i \in N_{jl}$  do not lie in a  $(d-1)$ -dimensional affine subspace of  $\mathbb{R}^d$ . Then the points  $t_0, \dots, t_{dn}$  obey  $(PSP)_{2n}$ . Therefore, if  $t_0, \dots, t_k \in \mathcal{K}$  and  $\mu = \sum_{l=0}^k m_l \delta_{t_l} \in \mathcal{M}_{L, \mathcal{K}}$ , then  $\mu$  is strongly maximal mass by Proposition 15(iii).

For notational simplicity we only construct a polynomial  $q_0 \in \text{Pos}(\mathbb{R}^d)_{2n}$  such that  $q_0(t_0) = 1$  and  $q_0(t_l) = 0$  for  $l = 1, \dots, dn$ . Interchanging indices we get the family  $q_0, \dots, q_{dn}$  of polynomials needed for  $(PSP)_{2n}$ . By assumption the set  $\{1, \dots, dn\}$  is a disjoint union of sets  $N_{01}, \dots, N_{0n}$  such that  $\{t_0, t_i; i \in N_{0l}\}$  is not contained in a  $(d-1)$ -dimensional affine subspace for each  $l = 1, \dots, n$ . Therefore, as shown above,

there is a polynomial  $p_l \in \text{Pos}(\mathbb{R}^d)_2$  such that  $p_l(t_0) = 1$  and  $p_l(t_i) = 0$  for  $i \in N_{0l}$ . Then  $q_0 := p_1 \dots p_n$  has the required properties.

6. CONSTRUCTING WEAKLY MAXIMAL MASS MEASURES

Let  $L$  be a truncated  $\mathcal{K}$ -moment functional on  $\mathbb{R}_{2n}^d[\underline{x}]$  such that  $L \neq 0$  and suppose that  $\mathcal{K}$  is **compact**. The following construction was proposed in [17].

By Proposition 4, there is an atomic measure in  $\mathcal{M}_{L,\mathcal{K}}$  such that all atoms are in  $\mathcal{K}$ . Hence  $\rho_L(t) > 0$  for some  $t \in \mathcal{K}$ . Set  $L_0 := L$ . We choose a point  $t_1 \in \mathcal{K}$  such that  $\rho_{L_0}(t_1) > 0$  and define  $L_1(f) = L(f) - \rho_{L_0}(t_1)f(t_1)$ ,  $f \in \mathbb{R}_{2n}^d[\underline{x}]$ . If  $L_1 = 0$ , we are done and set  $L_j = 0$  for all  $j \in \mathbb{N}, j \geq 2$ . Suppose that  $L_1 \neq 0$ . Since  $L_1(f) \geq 0$  for  $f \in \text{Pos}(\mathcal{K})_{2n}$  by definition and the set  $\mathcal{K}$  is compact,  $L_1$  is also a nonzero truncated  $\mathcal{K}$ -moment functional by Proposition 3(i) and we can apply the preceding construction with  $L$  replaced by  $L_1$ . Continuing this procedure, we obtain functionals  $L_m$  and points  $t_j \in \mathcal{K}$  such that  $\rho_{L_{j-1}}(t_j) > 0$  and

$$L_m(f) = L(f) - \sum_{j=1}^m \rho_{L_{j-1}}(t_j)f(t_j), \quad f \in \mathbb{R}_{2n}^d[\underline{x}].$$

**Lemma 22.** *There is a number  $k \in \{1, \dots, \dim \mathbb{R}_{2n}^d[\underline{x}]\}$  such that  $L_{k+1} = 0$ .*

*Proof.* We modify the reasoning used in the proof of Proposition 13(iii). Put  $m := \dim \mathbb{R}_{2n}^d[\underline{x}]$  and assume to the contrary that  $L_{m+1} \neq 0$ . We define linear functionals  $F_j, j = 1, \dots, m + 1$ , on  $\mathbb{R}_{2n}^d[\underline{x}]$  by  $F_j = l_{t_j}$ . Since  $m + 1 > \dim \mathbb{R}_{2n}^d[\underline{x}]$ , the functionals  $F_1, \dots, F_{m+1}$  must be linearly dependent. Hence there are numbers  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  such that at least one is nonzero and  $\sum_{j=1}^{m+1} \lambda_j F_j = 0$ . Let  $s$  be the smallest index for which  $\lambda_s \neq 0$ . By definition  $m_s := \rho_{L_{s-1}}(t_s)$ . Therefore, by Proposition 13(i), there is a sequence  $(p_r)_{r \in \mathbb{N}}$  of polynomials  $p_r \in \text{Pos}(\mathcal{K})_{2n}$  such that  $p_r(t_s) = 1$  and  $\lim_r p_r(t_l) = 0$  for  $l > s$ . Applying the functionals to  $p_r$  and passing to the limit  $r \rightarrow \infty$ , we obtain

$$0 = \lim_{r \rightarrow \infty} \sum_{j=1}^{m+1} \lambda_j F_j(p_r) = \sum_{j=s}^{m+1} \lambda_j \left( \lim_{r \rightarrow \infty} p_r(t_j) \right) = \sum_{j=s}^{m+1} \lambda_j \delta_{js} = \lambda_s,$$

This contradicts the choice of  $s$  and proves the assertion. □

Lemma 22 shows that this procedure terminates. More precisely, there is an index  $k$  such that  $1 \leq k \leq \dim \mathbb{R}_{2n}^d[\underline{x}]$  and  $L_{k+1} = 0$ . Then we have

$$(13) \quad L(f) = \sum_{l=1}^k \rho_{L_{l-1}}(t_l)f(t_l),$$

$$(14) \quad L_j(f) = \sum_{l=j+1}^k \rho_{L_{l-1}}(t_l)f(t_l),$$

for  $j = 1, \dots, k$  and  $f \in \mathbb{R}_{2n}^d[\underline{x}]$ . Equation (13) means that the  $k$ -atomic measure

$$\mu := \sum_{j=1}^k m_j \delta_{t_j}, \quad \text{where } m_j := \rho_{L_{j-1}}(t_j), \quad j = 1, \dots, k,$$

belongs to  $\mathcal{M}_{L,\mathcal{K}}$ . The functionals  $L_j$  in equation (14) are precisely the corresponding linear functionals  $L_j$  associated with  $L$  as in Definition 14. Therefore, from the preceding construction it is clear that the  $k$ -atomic measure  $\mu$  is *weakly maximal mass* for  $L$  according to Definition 14. Moreover,  $\mu$  can be chosen such that an arbitrary point  $t_1 \in \mathcal{K}$  satisfying  $\rho_L(t_1) > 0$  appears as an atom of  $\mu$ . In particular we have proved the following.

**Theorem 23.** *Suppose that  $\mathcal{K}$  is compact. Let  $t_1 \in \mathcal{K}$  be such that  $\rho_L(t_1) > 0$ . (In particular, then  $t_1 \in \mathcal{V}_+(L, \mathcal{K})$ .) Then for any truncated  $\mathcal{K}$ -moment functional  $L \neq 0$  the preceding construction leads to a  $k$ -atomic measure  $\mu = \sum_{j=1}^k m_j \delta_{t_i} \in \mathcal{M}_{L, \mathcal{K}}$ ,  $k \leq \dim \mathbb{R}_{2n}^d[\underline{x}]$ , such that all atoms of  $\mu$  are in  $\mathcal{K}$  and  $\mu$  is weakly maximal mass.*

*Proof of Proposition 8:*

Proposition 5 implies that  $\text{supp } \mu \subseteq \mathcal{V}_+(L, \mathcal{K}) \cap \mathcal{K}$  for any measure  $\mu \in \mathcal{M}_{L, \mathcal{K}}$ . If  $\mathcal{K}$  is compact, Theorem 23 applies and shows that each point of  $\mathcal{V}_+(L, \mathcal{K}) \cap \mathcal{K}$  is in the support of some measure  $\mu \in \mathcal{M}_{L, \mathcal{K}}$ .  $\square$

As an application of the preceding construction we derive the following result.

**Theorem 24.** *Let  $L$  be a truncated  $\mathcal{K}$ -moment functional and  $t_0 \in \mathcal{K}$ . Suppose that  $\mathcal{K}$  is compact and  $\rho_L(t_0) > 0$ . Then the infimum in (7) is a minimum.*

*Proof.* Since  $\mathcal{K}$  is compact and  $\rho_L(t_0) > 0$  by assumption, it follows from Theorem 23 that there exists a  $k$ -atomic measure  $\mu \in \mathcal{M}_{L, \mathcal{K}}$  such that  $t_0$  is an atom of  $\mu$ . Let  $p$  be the seminorm on the vector space  $\mathbb{R}_{2n}^d[\underline{x}]$  given by  $p(f) := \int |f| d\mu$ ,  $f \in \mathbb{R}_{2n}^d[\underline{x}]$ , and let  $E$  be the quotient space  $\mathbb{R}_{2n}^d[\underline{x}] / \ker(p)$  equipped with quotient norm defined by  $\tilde{p}(\tilde{f}) := p(f)$ , where  $\tilde{f}$  denotes the equivalence class  $\tilde{f} = f + \ker(p)$ . Further, let  $\tilde{C}$  be the cone of elements  $\tilde{f}$ ,  $f \in \text{Pos}(\mathcal{K})_{2n}$ . Since  $L(\ker(p)) = \{0\}$  by the definition of  $p$  and  $f(t_0) = 0$  for  $f \in \ker(p)$  because  $t_0$  is an atom of  $\mu$ , it follows that  $\tilde{L}(\tilde{f}) := L(f)$  and  $\tilde{L}_0(\tilde{f}) := f(t_0)$  are well-defined (!) linear functionals on  $E$ . By these definitions we have

$$(15) \quad \inf \{L(f) : f \in \text{Pos}(\mathcal{K})_{2n}, f(t_0) = 1\} = \inf \{\tilde{L}(\tilde{f}) : f \in \mathcal{N}_{t_0}\},$$

where  $\mathcal{N}_{t_0} := \{\tilde{f} \in \tilde{C} : \tilde{L}_0(\tilde{f}) = 1\}$ . In addition let us consider the set

$$\mathcal{M}_{t_0} = \{\tilde{f} \in \tilde{C} : \tilde{L}(\tilde{f}) \leq \tilde{L}(\tilde{1}), \tilde{L}_0(\tilde{f}) = 1\}.$$

Since  $\tilde{1} \in \mathcal{N}_{t_0}$ , we have  $\inf \{\tilde{L}(\tilde{f}) : f \in \mathcal{N}_{t_0}\} = \inf \{\tilde{L}(\tilde{f}) : f \in \mathcal{M}_{t_0}\}$ . Obviously,  $\tilde{1} \in \mathcal{M}_{t_0}$ , so  $\mathcal{M}_{t_0}$  is not empty. For  $\tilde{f} \in \mathcal{M}_{t_0}$ , we can choose  $f \in \text{Pos}(\mathcal{K})_{2n}$ , so that

$$\tilde{p}(\tilde{f}) = p(f) = \int |f| d\mu = \int f d\mu = L(f) = \tilde{L}(\tilde{f}) \leq \tilde{L}(\tilde{1}) = L(1),$$

that is, the set  $\mathcal{M}_{t_0}$  is bounded in the normed space  $(E, \tilde{p})$ . Since each linear functional on the finite dimensional space  $(E, \tilde{p})$  is continuous,  $\mathcal{M}_{t_0}$  is closed and hence compact in  $(E, \tilde{p})$ . Therefore,  $\tilde{L}$  has a minimum on the compact set  $\mathcal{M}_{t_0}$  and hence on the set  $\mathcal{N}_{t_0}$ . By (15) this means that the infimum in (7) is attained.  $\square$

Combining Theorem 24 and Proposition 11(ii) yields the following.

**Corollary 25.** *Suppose that  $\mathcal{K}$  is compact. For  $t \in \mathcal{K}$ , we have  $\rho_L(t) > 0$  if and only if  $t \in \mathcal{V}_+(L, \mathcal{K})$ .*

**Example 26.** Let us consider Robinson's polynomial [20]

$$R(x_1, x_2) = x_1^6 + x_2^6 + 1 - x_1^4(x_2^2 + 1) - x_2^4(x_1^2 + 1) - x_1^2 - x_2^2 + 3x_1^2x_2^2.$$

It is well-known (see [7]) that  $R$  has precisely eight zeros in  $\mathbb{R}^2$  which are given by

$$\begin{aligned} \mathcal{Z}(R) = \{t_1 = (-1, -1), t_2 = (0, -1), t_3 = (1, -1), t_4 = (-1, 1), \\ t_5 = (0, 1), t_6 = (1, 1), t_7 = (-1, 0), t_8 = (1, 0)\}. \end{aligned}$$

It is straightforward to check that the polynomials

$$\begin{aligned} q_1 = (x_1 - 1)x_1x_2(x_2 - 1), \quad q_2 = (x_1^2 - 1)x_2(x_2 - 1), \quad q_3 = x_1(x_1 + 1)x_2(x_2 - 1), \\ q_4 = x_1(x_1 - 1)x_2(x_2 + 1), \quad q_5 = (x_1^2 - 1)x_2(x_2 + 1), \quad q_6 = x_1(x_1 + 1)x_2(x_2 + 1), \\ q_7 = (x_1 - 1)(x_2^2 - 1), \quad q_8 = (x_1 + 1)(x_2^2 - 1) \end{aligned}$$

of  $\mathbb{R}_6^2[x_1, x_2]$  satisfy  $q_j(t_l) = 0$  for  $j \neq l$  and  $q_j(t_j) \neq 0$ . This implies that the points  $t_1, \dots, t_8$  satisfy  $(SP)_6$ .

Let  $t_0 \in \mathbb{R}^2 \setminus \mathcal{Z}(p)$  and choose a compact subset  $\mathcal{K}$  which contains  $t_0, \dots, t_8$ . Define  $L(f) = \sum_{j=0}^8 f(t_j)$  for  $f \in \mathbb{R}_6^2[x_1, x_2]$ . Then  $\mu := \sum_{j=0}^8 \delta_{t_j} \in \mathcal{M}_{L, \mathcal{K}}$ . Since  $R(t_j) = \delta_{j0}$ , we have  $\rho_L(t_0) = 1$ . Then  $\mu_1 = \sum_{l=1}^8 \delta_{t_l}$  is a representing measure for the functional  $L_1 = L - l_{t_0}$ . Since the points  $t_1, \dots, t_8$  form the zero set of  $R$  and obey  $(SP)_6$ , it follows (by arguing as in the proof of Proposition 16) that  $L_1$  is determinate. Therefore,  $\mu_1 = \sum_{l=1}^8 \delta_{t_l}$  is maximal mass. Hence  $\mu = \sum_{j=0}^8 \delta_{t_j}$  is a weakly maximal mass representation of the functional  $L$  according to the preceding construction.

7. AN EXAMPLE

The example constructed below shows that the procedure from the preceding Section gives only weakly maximal mass measures but not maximal mass measures. (This example has been used in [17] in a different context.) In particular this proves that there exist weakly maximal mass measures that are not maximal mass. In order to keep the beauty of Robinson’s sextic and its zero set we prefer to work on the real projective plane  $\mathbb{P}^2(\mathbb{R})$  (see the discussion at the end of Section 2) and use the projective counter-parts of Proposition 13(iii) and Theorem 24.

We now consider the *homogeneous* Robinson polynomial [20]

$$R(x_0, x_1, x_2) = x_0^2 + x_1^6 + x_2^6 - x_0^4(x_1^2 + x_2^2) - x_1^4(x_0^2 + x_2^2) - x_2^4(x_0^2 + x_1^2) + 3x_0^2x_1^2x_2^2.$$

It is well-known that  $\hat{R}(x_0, x_1, x_2) \geq 0$  for all  $(x_0, x_1, x_2) \in \mathbb{P}^2(\mathbb{R})$  and that the projective zero set of  $R$  (see e.g. [7]) and so of  $\hat{R}$  consists of ten projective zeros

$$\mathcal{Z}(R) = \{t_1 = (1, -1, -1), t_2 = (1, 0, -1), t_3 = (1, 1, -1), t_4 = (1, -1, 1), t_5 = (1, 0, 1), t_6 = (1, 1, 1), t_7 = (1, -1, 0), t_8 = (1, 1, 0), t_9 = (0, 1, 1), t_{10} = (0, -1, 1)\}.$$

First we note that the points  $t_1, \dots, t_{10}$  satisfy  $(SP)_6$ . Indeed, the polynomials

$$\begin{aligned} q_1 &= x_0^2(x_0 - x_1)x_1x_2(x_0 - x_2), q_2 = x_0^2(x_0^2 - x_1^2)x_2(x_0 - x_2), \\ q_3 &= x_0^2x_1(x_0 + x_1)x_2(x_0 - x_2), q_4 = x_0^2x_1(x_0 - x_1)x_2(x_0 + x_2), \\ q_5 &= x_0^2(x_0^2 - x_1^2)x_2(x_0 + x_2), q_6 = x_0^2x_1(x_0 + x_1)x_2(x_0 + x_2), \\ q_7 &= x_0^2(x_0 - x_1)(x_0^2 - x_2^2), q_8 = x_0^2(x_0 + x_1)(x_0^2 - x_2^2), \\ q_9 &= (x_0^2 - x_1^2)(x_0^2 - x_2^2)(x_1 - x_2)^2, q_{10} = (x_0^2 - x_1^2)(x_0^2 - x_2^2)(x_1 + x_2)^2 \end{aligned}$$

belong to  $\mathbb{R}_6^2[\underline{x}_0]$  and satisfy  $\hat{q}_j(t_l) = 0$  for  $j \neq l$  and  $\hat{q}_j(t_j) \neq 0$ . This implies that the points  $t_1, \dots, t_{10}$  satisfy  $(SP)_6$ .

Set  $\mathcal{K} = \mathbb{P}^2(\mathbb{R})$  and  $g_0(t) := (t_0^2 + t_1^2 + t_2^2)^3$ . Then, by (5), we have  $f(t) = \hat{f}(t)g_0(t)$  for  $f \in \mathbb{R}_6^2[\underline{x}_0]$  and  $t \in \mathbb{R}^3$ . Now take a point  $t_0 \in \mathbb{R}^3$  such that  $R(t_0) \neq 0$ . Setting  $m_j = g_0(t_j)$  we define a linear functional  $L$  by

$$L(\hat{f}) = \sum_{j=0}^{10} m_j \hat{f}(t_j) \equiv \sum_{j=0}^{10} f(t_j), \quad f \in \mathbb{R}_6^2[\underline{x}_0].$$

Then  $p := g_0(t_0)R(t_0)^{-1}R$  satisfies  $\hat{p}(t_j) = \delta_{j0}$ , so  $\rho_L(t_0) = m_0$  by the projective version of Proposition 13(ii). Then  $\mu_1 = \sum_{j=1}^{10} m_j \delta_{t_j}$  is a representing measure for the linear functional  $L_1 = L - m_0 l_{t_0}$ . Since the points  $t_1, \dots, t_{10}$  form the zero set of  $R$  and obey  $(SP)_6$ , it follows (as in the proof of Proposition 16) that  $L_1$  is determinate. Hence  $\mu_1 = \sum_{j=1}^{10} m_j \delta_{t_j}$  is maximal mass, so that  $\mu = \sum_{j=0}^{10} m_j \delta_{t_j}$  is a **weakly maximal mass** representing measure on  $\mathbb{P}^2(\mathbb{R})$  for the functional  $L$ .

Now we set  $t_0 = (1, 0, 0)$  and prove that  $\mu = \sum_{j=0}^{10} m_j \delta_{t_j}$  is not maximal mass. For this it suffices to show that  $\rho_L(t_5) > m_5$ , where  $t_5 = (1, 0, 1)$ . It is obvious that  $\rho_L(t_5) \geq m_5$ . Assume to the contrary that  $\rho_L(t_5) = m_5$ . Then, by the projective variant of Theorem 24, the infimum is attained in (7), so there exists a polynomial  $p \in \mathbb{R}_6^2[\underline{x}_0]$  such that  $\hat{p}(x_0, x_1, x_2) \geq 0$  on  $\mathbb{P}^2(\mathbb{R})$ ,  $\hat{p}(t_5) = 1$  and  $\hat{p}(t_l) = 0$  for  $l = 0, \dots, 10, l \neq 5$ . This implies that  $p(x_0, x_1, x_2) \geq 0$  on  $\mathbb{R}^3$ ,  $p(t_5) = m_5$  and  $p(t_l) = 0$  for  $l \neq 5$ . Put

$$q(x_0, x_1, x_2) := p(x_0, x_1, x_2) + p(x_0, -x_1, x_2) + p(-x_0, x_1, -x_2) + p(-x_0, -x_1, -x_2).$$

Since the polynomial  $q$  and the zero set of  $p$  are invariant under the mappings  $(x_0, x_1, x_2) \rightarrow (x_0, -x_1, x_2)$  and  $(x_0, x_1, x_2) \rightarrow (-x_0, x_1, -x_2)$ ,  $q$  is of the form  $q = q_0 + x_0 x_2 q_1$ , where  $q_0 \in \mathbb{R}_6^2[\underline{x}_0]$  and  $q_1 \in \mathbb{R}_4^2[\underline{x}_0]$  are even in each variable, and  $q$  has the zeros

$$t_0 = (1, 0, 0), t_2 = (1, 0, -1), t_3 = (1, 1, -1), t_6 = (1, 1, 1), t_8 = (1, 1, 0), t_9 = (0, 1, 1).$$

Further, we have  $q(1, 0, 1) = 4m_5$  and  $q(x_0, x_1, x_2) \geq 0$  on  $\mathbb{R}^3$ . The latter implies that  $q$  and all partial derivatives of  $q$  vanish at the zeros  $t_0, t_2, t_3, t_6, t_8, t_9$ . This leads to a number of linear equations which will determine  $q$  up to a constant factor. In fact, we prove that these equations imply that  $q = m_5 \tilde{q}$ , where is

$$(16) \quad \begin{aligned} \tilde{q} := & -x_1^2(x_1^2 - x_2^2)^2 + 2x_0x_2(x_1^2 - x_2^2)(x_1^2 - x_0^2) \\ & + x_0^2x_1^2(-x_0^2 + 2x_1^2 - x_2^2) + x_0^2x_2^2(x_0^2 - 2x_1^2 + x_2^2). \end{aligned}$$

Let us write

$$\begin{aligned} q_0 = & ax_0^6 + bx_1^6 + cx_2^6 + dx_0^4x_1^2 + ex_0^4x_2^2 + fx_1^4x_0^2 + gx_1^4x_2^2 + hx_2^4x_0^2 + kx_2^4x_1^2 + lx_0^2x_1^2x_2^2, \\ x_0x_2q_1 = & Ax_0^5x_2 + Bx_0^3x_1^2x_2 + Cx_0^3x_2^3 + Dx_0x_1^2x_2^3 + Ex_0x_1^4x_2 + Fx_0x_2^5. \end{aligned}$$

From  $q(0, 0, 1) = 0$  and  $\frac{\partial q}{\partial x_0}(0, 0, 1) = 0$  we obtain  $c = 0$  and  $F = 0$ . Since  $c = 0$ ,

$$q(0, 1, 1) = b + c + g + k = 0, \quad \text{and} \quad \frac{\partial q}{\partial x_2}(0, 1, 1) = 6c + 2g + 4k = 0,$$

we obtain  $b = k$  and  $g = -2k$ . That is, we have

$$(17) \quad cx_2^6 + bx_1^6 + gx_1^4x_2^2 + kx_2^4x_1^2 = kx_1^2(x_1^2 - x_2^2)^2.$$

Now  $\frac{\partial q}{\partial x_0}(0, 1, 1) = D + E + F = 0$  gives  $D = -E$ . Further,

$$q(1, 1, 1) - q(1, 1, -1) = 2(A + B + C + D + E + F) = 0,$$

so that  $A + B + C = 0$ ,

$$\frac{\partial q}{\partial x_0}(1, 1, 1) - \frac{\partial q}{\partial x_0}(1, 1, -1) = 2D + 2E + 6B + 6C + 10A = 0,$$

so that  $A = 0$  and  $B = -C$ , and

$$\frac{\partial q}{\partial x_1}(1, 1, 1) - \frac{\partial q}{\partial x_1}(1, 1, -1) = 4B + 4D + 8E = 0.$$

Since  $D + E = 0$ , we get  $B = -E$ . The preceding results imply that

$$(18) \quad x_0x_2q_1 = -Bx_0x_2(x_1^2 - x_2^2)(x_1^2 - x_0^2).$$

Further, we have

$$(19) \quad \begin{aligned} q(1, 1, 1) + q(1, 1, -1) &= 2(a + b + c + d + e + f + g + h + k + l) = 0, \\ q(1, 1, 0) &= a + b + d + f = 0. \end{aligned}$$

Since  $c = 0$ , this yields  $e + g + h + k + l = 0$ . Combined with

$$\frac{\partial q}{\partial x_2}(1, 1, 1) + \frac{\partial q}{\partial x_2}(1, 1, -1) = 12c + 8h + 8k + 4e + 4g + 4l = 0$$

and  $c = 0$ , the latter equality gives  $h = -k$  and  $e + g + l = 0$ . From the equations

$$(20) \quad \begin{aligned} \frac{\partial q}{\partial x_1}(1, 1, 0) &= 6b + 2d + 4f = 0, \\ \frac{\partial q}{\partial x_1}(1, 1, 1) + \frac{\partial q}{\partial x_1}(1, 1, -1) &= 12b + 8f + 8g + 4d + 4k + 4l = 0 \end{aligned}$$

it follows that  $2g + k + l = 0$ . Therefore,  $l = 3k$ , since  $g = -2k$  as shown above. Hence  $e + g + l = 0$  leads to  $e = -k$ . Since  $c = A = F = 0$  as noted above, we obtain

$$\frac{\partial q}{\partial x_0}(1, 0, -1) - 3q(1, 0, -1) = 3a + e - h = 0.$$

Since  $e = -k = h$ ,  $a = 0$ . Combining (19) and (20) by using that  $a = 0$  we obtain  $2b + f = 0$ , so that  $f = -2b = -2k$ . Therefore, by (19),  $d = -a - b - f = k$ . Thus

$$(21) \quad \begin{aligned} ax_0^6 + dx_0^4x_1^2 + ex_0^4x_2^2 + fx_1^4x_0^2 + hx_2^4x_0^2 + lx_0^2x_1^2x_2^2 \\ = -k[x_0^2x_1^2(-x_0^2 + 2x_1^2 - x_2^2) + x_0^2x_2^2(x_0^2 - 2x_1^2 + x_2^2)]. \end{aligned}$$

Moreover, the equality

$$0 = q(1, 0, -1) = a + c + e + h - C - F = -2k - C$$

yields  $C = -2k$ , so that  $B = 2k$ . Therefore it follows from (17), (18), and (21) that  $q = -k\tilde{q}$ . Since  $\tilde{q}(1, 0, 1) = 4$  by (16) and  $q(1, 0, 1) = 4m_5$  by assumption, we get  $k = -m_5$ , that is,  $q = m_5\tilde{q}$ .

By (16) we have  $q(0, x_1, 0) = m_5\tilde{q}(0, x_1, 0) = -m_5x_1^6$ . Since  $q \geq 0$  on  $\mathbb{R}^3$  by assumption, this is the desired contraction. This completes the proof of the fact that  $\rho_L(t_5) > m_5$ . Hence the representing measure  $\mu = \sum_{j=0}^{10} m_j\delta_{t_j}$  of the functional  $L$  is **not maximal mass**.

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#### REFERENCES

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh, 1965. (Russian edition: Goc. Izd. Fiz.-Mat. Lit., Moscow, 1961)
2. A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithm and Engineering Applications*, SIAM, Philadelphia, 2001.
3. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
4. C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups*, Graduate Texts in Mathematics, Vol. 100, Springer-Verlag, Berlin—Heidelberg—New York, 1984.
5. G. Blekherman and J. B. Lasserre, *The truncated  $K$ -moment problem for closure of open sets*, J. Funct. Anal. **263** (2012), 3604–3616; Corrigendum J. Funct. Anal. **265** (2013), 300–302.
6. G. Choquet, *Lectures on Analysis*, Vol. III, Benjamin, New York, 1969.
7. M. D. Choi, T. Y. Lam, and B. Reznick, *Real zeros of positive semidefinite forms. I*, Math. Z. **171** (1980), 1–26.
8. R. Curto and L. Fialkow, *Solution of the Truncated Complex Moment Problem for Flat Data*, Memoirs of the American Mathematical Society, Vol. 119, no. 568, Amer. Math. Soc., Providence, RI, 1996.
9. R. Curto and L. Fialkow, *Flat Extensions of Positive Moment Matrices: Recursively Generated Relations*, Memoirs of the American Mathematical Society, Vol. 136, no. 648, Amer. Math. Soc., Providence, RI, 1998.
10. R. Curto and L. Fialkow, *The truncated complex  $K$ -moment problem*, Trans. Amer. Math. Soc. **352** (2000), 2825–2855.
11. R. Douglas, *On extremal measures and subspace density*, Michigan Math. J. **11** (1964), 243–246.
12. L. Fialkow, *Truncated multivariable moment problems with finite variety*, J. Operator Theory **60** (2008), 343–377.
13. J. W. Helton and J. Nie, *A semidefinite approach for truncated  $K$ -moment problems*, Found. Comput. Math. **12** (2012), 851–881.

14. M. G. Krein and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs, Vol. 50, Amer. Math. Soc., Providence, RI, 1977. (Russian edition: Nauka, Moscow, 1973)
15. J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, London, 2009.
16. M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, M. Putinar, S. Sullivant (Eds.), *Emerging Applications of Algebraic Geometry*, Springer-Verlag, Berlin, 2009, pp. 157–270.
17. J. Matzke, *Mehrdimensionale Momentenprobleme und Positivitätskegel*, Dissertation, Universität Leipzig, 1992.
18. M. Marshall, *Positive Polynomials and Sums of Squares*, Math. Surveys Monogr., Vol. 146, Amer. Math. Soc., Providence, RI, 2008.
19. H. Richter, *Parameterfreie Abschätzung und Realisierung von Erwartungswerten*, *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* **3** (1957), 147–161.
20. R. M. Robinson, *Some definite polynomials which are not sums of squares of real polynomials*, *Selected Questions of Algebra and Logic*, Acad. Sci. USSR, 1973, pp. 264–282.
21. W. W. Rogosinski, *Moments of non-negative mass*, *Proc. Roy. Soc. London Ser. A* **245** (1958), 1–27.
22. K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Vol. 265, Springer-Verlag, Berlin, 2012.
23. V. Tchakaloff, *Formules de cubatures mecaniques a coefficients non negatifs*, *Bull. Sci. Math.* **81** (1957), 123–134.

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