Methods of Functional Analysis and Topology
Vol. 21 (2015), no. 3, pp. 246–255

ON FOURIER ALGEBRA OF A LOCALLY COMPACT HYPERGROUP

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The paper is dedicated to the 90th birthday anniversary of Yu. M. Berezansky

Abstract. We give sufficient conditions for the Fourier and the Fourier-Stieltjes spaces of a locally compact hypergroup to be Banach algebras.

1. Introduction

The Fourier and the Fourier-Stieltjes algebras for a general locally compact group have been considered in [1]. Following a similar procedure for a locally compact hypergroup, one can introduce Banach spaces that are not Banach algebras in general. This has been done in [2], where the author has also considered the Fourier spaces of some commutative hypergroups, proving that the Fourier spaces are Banach algebras with respect to a certain norm. In [3], the author has introduced a special class of hypergroups, he calls ultraspherical, that also possess the property of the Fourier space being a Banach algebra.

In this paper, we give a sufficient condition for the Fourier space to be a Banach algebra in the case of a general locally compact hypergroup. The double coset hypergroup of a locally compact group satisfies this condition.

2. Preliminary

2.1. **Main notations and definitions.** Let Q be a Hausdorff locally compact topological space. The set of all compact subsets of Q is denoted by $\mathcal{K}$.

The linear space of complex-valued continuous functions on Q is denoted by $\mathscr{C}(Q)$, the subspace of $\mathscr{C}(Q)$ of bounded functions (resp., functions approaching zero at infinity) is denoted by $\mathscr{C}_b(Q)$ (resp., $\mathscr{C}_0(Q)$). The space $\mathscr{C}_b(Q)$ is endowed with the norm $\|f\|_\infty = \sup_{t\in Q}|f(t)|$. By $\mathscr{K}(Q)$, we denote the linear subspace of $\mathscr{C}_0(Q)$ of functions with compact supports. By 1_Q, we denote the constant function, $1_Q(s) = 1$ for all $s \in Q$.

A measure is understood as a complex Radon measure [4] on Q. The linear space of complex Radon measures, over the field $\mathbf{C}$ of complex numbers, is denoted by $\mathscr{M}(Q)$. For a measure μ, its norm is $\|\mu\|_1 = \sup_{f\in\mathscr{K}(Q),\ \|f\|_\infty\le 1}|\mu(f)|$. The subspace of $\mathscr{M}(Q)$ of bounded (resp., compactly supported) measures is denoted by $\mathscr{M}_b(Q)$ (resp., $\mathscr{M}_c(Q)$). The subset of $\mathscr{M}(Q)$ of nonnegative (resp., probability) measures is denoted by $\mathscr{M}_+(Q)$ (resp., $\mathscr{M}_p(Q)$). For a measure $\mu \in \mathscr{M}_+(Q)$, its support is denoted by $S(\mu)$. The set of measures μ such that $S(\mu)$ is compact is denoted by $\mathscr{M}_c(Q)$. If $\mu \in \mathscr{M}_+(Q) \cap \mathscr{M}_b(Q)$, then $\|\mu\|_1 = \mu(1_Q)$. The Dirac measure at a point $s \in Q$ is denoted by ε_s. The integral of $f \in \mathscr{K}(F)$ with respect to a measure $\mu \in \mathscr{M}$ is denoted by $\mu(f) = \langle f, \mu \rangle = \int_F \langle f, \varepsilon_t \rangle \, d\mu(t) = \int_F f(t)\, d\mu(t)$.

2000 *Mathematics Subject Classification.* Primary: 20N20; Secondary: 22D15, 22D35.

Key words and phrases. Fourier algebra, Fourier-Stieltjes algebra, DJS-hypergroup, locally compact hypergroup, dual algebras, Pontryagin duality.

A (locally compact) hypergroup is a locally compact Hausdorff topological space Q such that $\mathscr{M}_b(Q)$ is endowed with a multiplication, called composition and denoted by $*$, satisfying the following conditions [5]:

(H_1) $(\mathscr{M}_b(Q), *)$ is an algebra over $\mathbf{C}$.

(H_2) For all $s, t \in Q$, $\varepsilon_s * \varepsilon_t \in \mathscr{M}_p(Q)$ and $S(\varepsilon_s * \varepsilon_t)$ is compact.

(H_3) The mapping $(s, t) \mapsto \varepsilon_s * \varepsilon_t$ of $Q \times Q$ into $\mathscr{M}_p(Q)$ is continuous with respect to the weak topology $\sigma(\mathscr{M}_p(Q), \mathscr{C}_0(Q))$ on $\mathscr{M}_p(Q)$.

(H_4) The mapping $(s, t) \mapsto S(\varepsilon_s * \varepsilon_t)$ of $Q \times Q$ into $\mathcal{K}$ is continuous with respect to the Michael topology on $\mathcal{K}$.

(H_5) There exists a (necessarily unique) element $e \in Q$ such that $\varepsilon_e * \varepsilon_s = \varepsilon_s * \varepsilon_e = \varepsilon_s$ for all $s \in Q$.

(H_6) There exists a (necessarily unique) homeomorphism $s \mapsto \check{s}$ of Q into Q such that $\check{\check{s}} = s$ and $(\varepsilon_s * \varepsilon_t)\check{} = \varepsilon_{\check{t}} * \varepsilon_{\check{s}}$, where $\check{\mu}$ denotes the image of the measure μ with respect to homeomorphism $s \mapsto \check{s}$, i.e. $\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle$, where $\check{f}(s) = f(\check{s})$.

(H_7) For $s, t \in Q$, $e \in S(\varepsilon_s * \varepsilon_t)$ if and only if $s = \check{t}$.

For a measure $\mu \in \mathscr{M}(Q)$ and $h \in \mathscr{C}(Q)$, the measure $h\mu$ is defined by $\langle f, h\mu \rangle = \langle fh, \mu \rangle$ for $f \in \mathscr{K}$; it is clear that $h\mu \in \mathscr{M}_c(Q)$ for $h \in \mathscr{K}(Q)$.

Everywhere in the sequel, we assume that the hypergroup possesses a left invariant measure, denoted by m, which means that

$$\varepsilon_s * m = m$$

for all $s \in Q$.

For $\mu \in \mathscr{M}_b(Q)$, denote by $\mu^\star$ the bounded measure defined by $\mu^\star(f) = \overline{\check{\mu}(\overline{f})}$ for $f \in \mathscr{K}(Q)$. It follows from the axiom (H_6) of a hypergroup that * is an involution on the algebra $(\mathscr{M}_b(Q), *)$. It is well known [5] that $(\mathscr{M}_b(Q), *, ^\star)$ is an involutive Banach algebra.

For C^*-algebras A and B, the tensor product $A \otimes B$ is the completion of the algebraic tensor product $A \odot B$ with respect to the min-C^*-norm on $A \odot B$,

$$\|\sum_{i=1}^n a_i \otimes b_i\|_{\min} = \sup_{\pi_A \in \Sigma_A, \pi_B \in \Sigma_B} \Big\|\sum_{i=1}^n \pi_A(a_i) \otimes \pi_B(b_i)\Big\|,$$

where Σ_A (resp., Σ_B) is the set of all representations of A (resp., B) [6].

For a C^*-algebra A, the C^*-algebra of multipliers of A is denoted by $M(A)$, see [7] for details.

2.2. **Fourier–Stieltjes and Fourier spaces.**

Definition 1. Let Q be a locally compact hypergroup. Let $\mathscr{H}$ be a Hilbert space, $B(\mathscr{H})$ the C^*-algebra of all linear bounded operators on $\mathscr{H}$, and $\pi\colon \mathscr{M}_b(Q) \to B(\mathscr{H})$ a linear map. Then the pair $(\mathscr{H}, \pi)$ is called a *representation* of $\mathscr{M}_b(Q)$ if π is an involutive homomorphism of the involutive Banach algebra $(\mathscr{M}_b(Q), *, ^\star)$ into $B(\mathscr{H})$.

A left invariant measure m on Q gives rise to an inner product $\big(\cdot \mid \cdot\big)$ and norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathscr{K}(Q)$ as usual,

$$\big(f \mid g\big) = \int_Q f(t)\overline{g}(t)\, dm(t),$$

$$\|f\|_1 = \big(|f| \mid 1_Q\big), \qquad \|f\|_2 = \big(f \mid f\big)^{1/2}.$$

The completions of $\mathscr{K}(Q)$ with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are denoted by $L_1(Q)$ and $L_2(Q)$, respectively.

We denote $L_1(m) = \{fm : f \in L_1(Q)\}$. It is well known that $(L_1(m), *, ^\star)$ is a closed two-sided ideal of $(\mathscr{M}_b(Q), *, ^\star)$.

Identifying each $f \in L_1(Q)$ with the measure $fm \in L_1(m)$ yields an involutive Banach algebra structure on $L_1(Q)$, denoted by $(L_1(Q), *, ^\star)$, where

$$(f * g)(s) = \int_Q f(t)\langle\, g, \check{\varepsilon}_t * \varepsilon_t \,\rangle \, dm(t), \tag{1}$$

$$f^\star(s) = \Delta^{-1}(s)\overline{f}(\check{s}), \tag{2}$$

where $\Delta : Q \to \mathbf{R}$ is the modular function, $m * \varepsilon_s = \Delta(s)m$, see [5]. Also denote by

$$f^\dagger(s) = \overline{f(\check{s})}. \tag{3}$$

A function $f \in \mathscr{C}_b$ is called *positive definite* if $\langle\, f, \mu^\star * \mu \,\rangle \geq 0$ for all $\mu \in \mathscr{M}_c(Q)$. It is well known that the function $f * f^\dagger$, $f \in \mathscr{K}$, is positive definite, see [5].

Each representation $(\mathscr{H}, \pi)$ of $(\mathscr{M}_b(Q), *, ^\star)$, being restricted to $(L_1(m), *, ^\star)$, defines a representation of the involutive Banach algebra $(L_1(Q), *, ^\star)$, which we also denote by $(\mathscr{H}, \pi)$.

A representation $(\mathscr{H}, \pi)$ of $(\mathscr{M}_b(Q), *, ^\star)$ is called a representation of Q if it gives rise to a nondegenerate representation of $(L_1(m), *, ^\star)$, hence of $(L_1(Q), *, ^\star)$. The set of all representation of Q will be denoted by Σ.

The left regular representation $(L_2(Q), \lambda)$ of $(L_1(Q), *, ^\star)$ is defined by

$$\lambda(f).\,\xi = f * \xi$$

for $f \in L_1(Q)$ and $\xi \in L_2(Q)$.

Each representation $(\mathscr{H}, \pi)$ of Q induces a seminorm $\|\cdot\|_\pi$ on $(L_1(Q), *, ^\star)$, defined by $\|f\|_\pi = \|\pi(f)\|_{B(\mathscr{H})}$, $f \in L_1(Q)$, where $\|\cdot\|_{B(\mathscr{H})}$ denotes the operator norm in $B(\mathscr{H})$.

For a subset $\Sigma' \subset \Sigma$, we define a seminorm $\|\cdot\|_{\Sigma'}$ on $(L_1(Q), *, ^\star)$ by

$$\|f\|_{\Sigma'} = \sup_{\pi \in \Sigma'} \|f\|_\pi, \qquad f \in L_1(Q).$$

Definition 2. The enveloping C^*-algebra of $(L_1(Q), *, ^\star)$ with respect to the norm $\|\cdot\|_\Sigma$ (resp., $\|\cdot\|_\lambda$) will be called the *full* (resp., the *reduced*) *C^*-algebra* of the hypergroup Q. The full (resp., the reduced) C^*-algebra of Q will be denoted by $C^*(Q)$ (resp., $C^*_r(Q)$).

Definition 3. The Banach space dual to the full C^*-algebra $C^*(Q)$ is called the *Fourier-Stieltjes space* and will be denoted by $\mathscr{B}(Q)$.

The Banach space dual to the reduced C^*-algebra $C^*_r(Q)$ will be denoted by $\mathscr{B}_\lambda(Q)$.

It is known, see [2], that for each $\alpha \in \mathscr{B}(Q)$ (resp., $\alpha \in \mathscr{B}_\lambda(Q)$) there is a representation $(\mathscr{H}, \pi)$ of Q (resp., weakly contained in $(L_2(Q), \lambda)$) and two vectors $\xi, \eta \in \mathscr{H}$ such that the function $a \in \mathscr{C}_b(Q)$ given by

$$a(s) = \big(\, \pi(\varepsilon_s).\,\xi \;\big|\; \eta \,\big)_{\mathscr{H}}, \qquad s \in Q, \tag{4}$$

defines the functional α, namely,

$$\alpha(f) = \int_Q a(s) f(s)\, dm(s), \qquad f \in L_1(Q), \tag{5}$$

and

$$\|\alpha\| = \sup_{f \in L_1(Q),\, \|f\|_{\Sigma'} = 1} \Big| \int_Q a(s) f(s)\, dm(s) \Big| = \|\xi\|_{\mathscr{H}} \, \|\eta\|_{\mathscr{H}}, \tag{6}$$

where $\Sigma' = \Sigma$ (resp., $\Sigma' = \lambda$).

Henceforth, we identify $\mathscr{B}(Q)$ (resp., $\mathscr{B}_\lambda(Q)$) with a linear space of functions $a \in \mathscr{C}_b(Q)$ given by (4), where $(\mathscr{H}, \pi)$ is a representation of Q (resp., a representation weakly contained in $(L_2(Q), \lambda)$), and endow this space with the norm

$$\|a\|^\circ = \sup_{\mu \in L_1(m),\, \|\mu\|_{\Sigma'} = 1} |\mu(a)|, \tag{7}$$

where $\Sigma' = \Sigma$ (resp, $\Sigma' = \lambda$).

Definition 4. The closure of the subspace spanned by the elements $f * f^\dagger$, $f \in \mathscr{K}(Q)$, is in $\mathscr{B}_\lambda(Q)$ is called the *Fourier space* of Q and is denoted by $\mathscr{A}(Q)$.

3. Main results

Definition 5. For an involutive algebra $(B, *, ^\star)$ and a C^*-algebra $(A, \cdot, ^*)$, a linear map $\varphi\colon B \to A$ will be called *positive*, if $\varphi(b^\star * b)$ is a nonnegative element of the C^*-algebra A for any $b \in B$, and is called *completely positive*, if the linear map $\mathrm{id} \otimes \varphi\colon M_n(\mathbf{C}) \otimes B \to M_n(\mathbf{C}) \otimes A$ is positive for all $n \in \mathbf{N}$, where $M_n(\mathbf{C})$ denotes the C^*-algebra of complex $(n \times n)$-matrices.

It follows from [6] that a map $\varphi\colon B \to A$ is completely positive if and only if, for any $n \in \mathbf{N}$, $b_i \in B$ and $a_i \in A$, $i = 1, \ldots, n$, we have

$$\sum_{i,j=1}^{n} a_j^* \varphi(b_j^\star * b_i) a_i \geq 0. \tag{8}$$

For a hypergroup Q, consider the product hypergroup $Q \times Q$ [5], and let $\delta\colon \mathscr{M}_b(Q) \to \mathscr{M}_b(Q \times Q)$ denote a linear extension of the map defined by

$$\delta(\varepsilon_s) = \varepsilon_s \otimes \varepsilon_s, \qquad s \in Q, \tag{9}$$

that is, for $\mu \in \mathscr{M}_b(Q)$ and $F \in \mathscr{K}(Q \times Q)$,

$$\langle \delta(\mu), F \rangle = \int_Q F(s,s)\, d\mu(s). \tag{10}$$

Let $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. For $\tilde{\mu} \in \mathscr{M}_b(Q \times Q)$, we define $(\pi_1 \otimes \pi_2)(\tilde{\mu}) \in B(\mathscr{H}_1 \otimes \mathscr{H}_2)$ by

$$(\pi_1 \otimes \pi_2)(\tilde{\mu}) = \int \pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_t)\, d\tilde{\mu}(s,t).$$

Proposition 1. *Let G be a locally compact group, H a compact subgroup of G, and $Q = H \setminus G/H$. Denote by λ_Q the left regular representation of $L_1(m)$ on $L_2(Q)$, and let δ be defined by (9). Then the linear map*

$$(\lambda_Q \otimes \lambda_Q) \circ \delta\colon L_1(m) \to B(L_2(Q) \otimes L_2(Q))$$

is completely positive.

Proof. Let $s \in Q = H \setminus G/H$. Denote by m_H the Haar measure on H. Then $\varepsilon_s = m_H * \varepsilon_g * m_H$ for some $g \in G$. Thus

$$\delta(\varepsilon_s) = \varepsilon_s \otimes \varepsilon_s = (m_H * \varepsilon_g * m_H) \otimes (m_H * \varepsilon_g * m_H).$$

Denoting $L_2(G, m_G)$ simply by $L_2(G)$, where m_G is a left invariant measure on G, we will identify $L_2(Q)$ with a closed subspace of $L_2(G)$,

$$L_2(Q) = \{f \in L_2(G) : m_H * f = f * m_H = f\}.$$

With such an identification,

$$\big(f \mid g\big)_{L_2(Q)} = \big(f \mid g\big)_{L_2(G)}, \qquad f, g \in L_2(Q).$$

For any $\mu \in L_1(m)$, let $\tilde{\mu} \in L_1(m_G)$ be such that $\mu = m_H * \tilde{\mu} * m_H$. Then we have

$$\lambda_Q(\mu).\, f = \lambda_G(m_H * \tilde{\mu} * m_H).\, f = \lambda_G(m_H * \tilde{\mu}).\, f, \qquad f \in L_2(Q),$$

where λ_G is the left regular representation of $L_1(m_G)$ on $L_2(G)$.

Thus, for any $\mu \in L_1(m)$ and $\tilde\mu \in L_1(m_G)$ such that $\mu = m_H * \tilde\mu * m_H$, we have

$$\begin{aligned}(\lambda_Q \otimes \lambda_Q) \circ \delta(\mu) &= \int_G (\lambda_G \otimes \lambda_G) \circ \delta(m_H * \varepsilon_g * m_H)\, d\tilde\mu(g)\\ &= \int_G \lambda_G(m_H * \varepsilon_g * m_H) \otimes \lambda_G(m_H * \varepsilon_g * m_H)\, d\tilde\mu(g)\\ &= \int_G \lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)\, d\tilde\mu(g). \qquad (11)\end{aligned}$$

To prove the proposition, using (8), it is sufficient to show that

$$\begin{aligned}&\sum_{i,j=1}^{n} \big(A_j^* \cdot (\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) \cdot A_i . F \;\big|\; F \big)_{L_2(Q)\otimes L_2(Q)}\\ &\quad= \sum_{i,j=1}^{n} \big((\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) \cdot A_i . F \;\big|\; A_j . F \big)_{L_2(Q)\otimes L_2(Q)} \geq 0 \qquad (12)\end{aligned}$$

for any $A_i \in B(L_2(Q) \otimes L_2(Q))$, $\mu_i \in L_1(m)$, $F \in L_2(Q) \otimes L_2(Q)$, $i = 1, \ldots, n$.

Hence, letting $A_i F = F_i \in L_2(Q) \otimes L_2(Q)$, and $\tilde\mu_i \in L_1(m_G)$ such that $\mu_i = m_H * \tilde\mu_i * m_H$, and using (11), we have

$$\begin{aligned}&\sum_{i,j=1}^{n} \big((\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) . F_i \;\big|\; F_j \big)_{L_2(Q)\otimes L_2(Q)}\\ &\quad= \sum_{i,j=1}^{n} \int_G \big((\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)) . F_i \;\big|\; F_j \big)_{L_2(G)\otimes L_2(G)}\\ &\qquad\qquad d(m_H * \tilde\mu_j^\star * m_H * m_H * \tilde\mu_i * m_H)(g).\end{aligned}$$

Let $m_H * \tilde\mu_i * m_H = f_i\, m_G$, $f_i \in L_1(G)$, $i = 1, \ldots, n$. Then $m_h * \tilde\mu_j^\star * m_H = f_j^\star\, m_G$ and

$$m_H * \tilde\mu_j^\star * m_H * m_H * \tilde\mu_i * m_H = (f_j^\star * f_i)\, m_G.$$

Using left invariance of m_G and (2) we thus have

$$\begin{aligned}&\sum_{i,j=1}^{n} \big((\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) . F_i \;\big|\; F_j \big)_{L_2(Q)\otimes L_2(Q)}\\ &\quad= \sum_{i,j=1}^{n} \int_G \big((\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)) . F_i \;\big|\; F_j \big)_{L_2(G)\otimes L_2(G)}\\ &\qquad\qquad \cdot (f_j^\star * f_i)(g)\, dm_G(g)\\ &\quad= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)) . F_i \;\big|\; F_j \big)_{L_2(G)\otimes L_2(G)}\\ &\qquad\qquad \cdot f_j^*(p) f_i(p^{-1}g)\, dm_G(p) dm_G(g)\\ &\quad= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(m_H * \varepsilon_{pg}) \otimes \lambda_G(m_H * \varepsilon_{pg})) . F_i \;\big|\; F_j \big)_{L_2(G)\otimes L_2(G)}\\ &\qquad\qquad \cdot f_j^*(p) f_i(g)\, dm_G(p) dm_G(g)\\ &\quad= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)) . F_i \;\big|\; (\lambda_G(\varepsilon_{p^{-1}}) \otimes \lambda_G(\varepsilon_{p^{-1}}) . F_j \big)_{L_2(G)\otimes L_2(G)}\\ &\qquad\qquad \cdot f_j^*(p) f_i(g)\, dm_G(p) dm_G(g) 0\end{aligned}$$

$$= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)).\, F_i \;\big|\; (\lambda_G(\varepsilon_{p^{-1}}) \otimes \lambda_G(\varepsilon_{p^{-1}}).\, F_j \big)_{L_2(G)\otimes L_2(G)}$$
$$\cdot\, \overline{f}_j(p^{-1}) \Delta_G^{-1}(p) f_i(g)\, dm_G(p) dm_G(g)$$
$$= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)).\, F_i \;\big|\; (\lambda_G(\varepsilon_p) \otimes \lambda_G(\varepsilon_p).\, F_j \big)_{L_2(G)\otimes L_2(G)}$$
$$\cdot\, \overline{f}_j(p) f_i(g)\, dm_G(p) dm_G(g)$$
$$= \Big\| \sum_{i=1}^{n} \int_G f_i(g) (\lambda_G(\varepsilon_g) \otimes \lambda_G(g))\, dm_G(g).\, F_i \Big\|^2_{L_2(G)\otimes L_2(G)} \geq 0. \qquad \square$$

Theorem 1. *Let the map δ defined by (9) be completely positive. Let $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. Then we have the following.*

(i) *There is a representation $(\mathscr{H}, \pi)$ of Q such that*

$$\big\| (\pi_1 \otimes \pi_2)(\delta(\mu)) \big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \leq \big\| \pi(\mu) \big\|_{B(\mathscr{H})}, \qquad \mu \in \mathscr{M}_b(Q). \tag{13}$$

If $\pi_1 = \pi_2 = \lambda$ and $\mathscr{H}_1 = \mathscr{H}_2 = L_2(Q)$, then

$$\big\| (\lambda \otimes \lambda)(\delta(\mu)) \big\|_{B(L_2(Q)\otimes L_2(Q))} \leq \big\| \lambda(\mu) \big\|_{B(L_2(Q))}, \qquad \mu \in \mathscr{M}_b(Q). \tag{14}$$

(ii) *Let $\xi_i, \eta_i \in \mathscr{H}_i$ be arbitrary vectors in the corresponding spaces, and*

$$a_i(s) = \big(\pi_i(\varepsilon_s).\, \xi_i \;\big|\; \eta_i \big)_{\mathscr{H}_i}, \qquad i = 1, 2.$$

Then there are vectors $\xi, \eta \in \mathscr{H}$ such that

$$a_1(s) a_2(s) = \big(\pi(\varepsilon_s).\, \xi \;\big|\; \eta \big)_{\mathscr{H}}. \tag{15}$$

Proof. The construction of the representation $(\mathscr{H}, \pi)$ uses the Stinespring construction [8].

Consider the linear space $\mathscr{H}_0 = \mathscr{M}_b(Q) \odot (\mathscr{H}_1 \otimes \mathscr{H}_2)$, where $\odot$ denotes the algebraic tensor product, and for each $\mu \in \mathscr{M}_b(Q)$, define a linear operator $\pi_0(\mu)$ on $\mathscr{H}_0$ by setting

$$\pi_0(\mu).\, \nu \otimes \tilde{v} = (\mu * \nu) \otimes \tilde{v},$$

where $\nu \otimes \tilde{v} \in \mathscr{H}_0$, with $\nu \in \mathscr{M}_b(Q)$ and $\tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$.

Using δ, introduce a sesquilinear form $\big(\cdot \;\big|\; \cdot \big)_{\mathscr{H}_0}$ on $\mathscr{H}_0$ defined by

$$\big(\mu \otimes \tilde{u} \;\big|\; \nu \otimes \tilde{v} \big)_{\mathscr{H}_0} = \big((\pi_1 \otimes \pi_2) \delta(\nu^\star * \mu).\, \tilde{u} \;\big|\; \tilde{v} \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \tag{16}$$

for $\mu, \nu \in \mathscr{M}_b(Q)$ and $\tilde{u}, \tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$, and bilinearly extended to $\mathscr{H}_0$. Since δ is completely positive,

$$\big(w \;\big|\; w \big)_{\mathscr{H}_0} \geq 0, \qquad w \in \mathscr{H}_0,$$

hence, this sesquilinear form gives rise to a seminorm $\|\cdot\|_{\mathscr{H}_0}$ on $\mathscr{H}_0$.

If $\mathscr{N} = \{ w \in \mathscr{H}_0 \,|\, \|w\|_{\mathscr{H}_0} = 0 \}$, then $\pi_0(\mu).\, \mathscr{N} \subset \mathscr{N}$ for any $\mu \in \mathscr{M}_b(Q)$. If now $\mathscr{H}$ is the completion of $\mathscr{H}_0/\mathscr{N}$ with respect to the norm defined by the sesquilinear form (16) and $\pi(\mu)$ is the operator on $\mathscr{H}$ corresponding to the operator $\pi_0(\mu)$, then it follows from [8] that $(\mathscr{H}, \pi)$ is an involutive representation of $\mathscr{M}_b(Q)$.

To prove (13), let $\tilde{u}, \tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$ and $\mu \in \mathscr{M}_b$. Then

$$\begin{aligned}
\big| \big((\pi_1 \otimes \pi_2)(\delta(\mu)).\, \tilde{u} \;\big|\; \tilde{v} \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \big| &= \big| \big(\mu \otimes \tilde{u} \;\big|\; \varepsilon_e \otimes \tilde{v} \big)_{\mathscr{H}} \big| \\
&= \big| \big(\pi(\mu).\, (\varepsilon_e \otimes \tilde{u}) \;\big|\; \varepsilon_e \otimes \tilde{v} \big)_{\mathscr{H}} \big| \\
&\leq \big\| \pi(\mu) \big\|_{B(\mathscr{H})} \big\| \varepsilon_e \otimes \tilde{u} \big\|_{\mathscr{H}} \big\| \varepsilon_e \otimes \tilde{v} \big\|_{\mathscr{H}} \\
&= \big\| \pi(\mu) \big\|_{B(\mathscr{H})} \big\| \tilde{u} \big\|_{\mathscr{H}_1 \otimes \mathscr{H}_2} \big\| \tilde{v} \big\|_{\mathscr{H}_1 \otimes \mathscr{H}_2},
\end{aligned}$$

which proves (13).

To prove (14), we identify $L_2(Q)\otimes L_2(Q)$ with $L_2(Q\times Q)$ with respect to the product measure $m\otimes m$. To shorten the notations, we write $F(\mu_1,\mu_2)$ instead of $\langle F,\mu_1\otimes\mu_2\rangle$ for $F\in L_2(Q\times Q)$ and $\mu_1,\mu_2\in\mathscr{M}_b$.

Let $F,G\in L_2(Q\times Q)$, $\mu\in\mathscr{M}_b$. Using left invariance of m and the Cauchy inequality, we have

$$\begin{aligned}
\Big|\big((\lambda\otimes\lambda)\delta(\mu).\,F \;\big|\; G\big)_{L_2(Q\times Q)}\Big| &= \Big|\big(\int_Q(\lambda(\varepsilon_u)\otimes\lambda(\varepsilon_u)).\,F \;\big|\; G\big)_{L_2(Q\times Q)}\,d\mu(u)\Big|\\
&= \Big|\int_{Q^3} F(\check\varepsilon_u*\varepsilon_s,\check\varepsilon_u*\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_t)}\,dm(s)dm(t)d\mu(u)\Big|\\
&= \Big|\int_{Q^3} F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_u*\varepsilon_t)}\,dm(s)dm(t)d\mu(u)\Big|\\
&\le \int_{Q^2}\Big|\int_Q F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_u*\varepsilon_t)}\,dm(t)\Big|dm(s)d\mu(u)\\
&\le \int_{Q^2}\Big(\int_Q\big|F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}\\
&\qquad\cdot\Big(\int_Q\big|G(\varepsilon_s,\varepsilon_u*\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}\,dm(s)d\mu(u).
\end{aligned}$$

Denoting

$$f(s) = \Big(\int_Q\big|F(\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}, \tag{17}$$

$$g(s) = \Big(\int_Q\big|G(\varepsilon_s,\varepsilon_u*\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12} = \Big(\int|G(\varepsilon_s,\varepsilon_t)|^2\,dm(t)\Big)^{\frac12}, \tag{18}$$

where the last equality is due to left invariance of m, we get

$$(\lambda(\varepsilon_u).\,f)(s) = \Big(\int_Q\big|F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12},$$

hence

$$\begin{aligned}
\Big|\big((\lambda\otimes\lambda)\delta(\mu).\,F \;\big|\; G\big)_{L_2(Q\times Q)}\Big| &\le \int_{Q^2}(\lambda(\varepsilon_u).\,f)(s)g(s)\,dm(s)d\mu(u)\\
&= \big(\lambda(\mu).\,f \;\big|\; g\big)_{L_2(Q)},
\end{aligned}$$

where, as it follows from (17) and (18),

$$\|f\|_{L_2(Q)} = \|F\|_{L_2(Q\times Q)}, \qquad \|g\|_{L_2(Q)} = \|G\|_{L_2(Q\times Q)}.$$

This finishes the proof of (14).

Let us now prove (ii).

Define $\xi_0,\eta_0\in\mathscr{H}_0$ by

$$\xi_0 = \varepsilon_e\otimes(\xi_1\otimes\xi_2), \qquad \eta_0 = \varepsilon_e\otimes\eta_1\otimes\eta_2.$$

Then

$$\big(\pi_0(\varepsilon_s).\,\xi_0 \;\big|\; \eta_0\big)_{\mathscr{H}_0} = \big(\pi_0(\varepsilon_s).\,\varepsilon_e\otimes(\xi_1\otimes\xi_2) \;\big|\; \varepsilon_e\otimes(\eta_1\otimes\eta_2)\big)_{\mathscr{H}_0}0$$

$$\begin{aligned}
&= \big(\varepsilon_s \otimes (\xi_1 \otimes \xi_2) \;\big|\; \varepsilon_e \otimes (\eta_1 \otimes \eta_2) \big)_{\mathscr{H}_0} \\
&= \big((\pi_1 \otimes \pi_2)\delta(\varepsilon_s).\, (\xi_1 \otimes \xi_2) \;\big|\; \eta_1 \otimes \eta_2 \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \\
&= \big((\pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_s)).\, (\xi_1 \otimes \xi_2) \;\big|\; \eta_1 \otimes \eta_2 \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \\
&= \big(\pi_1(\varepsilon_s).\, \xi_1 \;\big|\; \eta_1 \big)_{\mathscr{H}_1} \big(\pi_2(\varepsilon_s).\, \xi_2 \;\big|\; \eta_2 \big)_{\mathscr{H}_2} \\
&= a_1(s)a_2(s). \qquad \square
\end{aligned}$$

Proposition 2. *Let $\mu \in L_1(m)$, and $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. Then $(\pi_1 \otimes \pi_2)(\delta(\mu)) \in M\big(\pi_1(L_1(m)) \otimes \pi_2(L_1(m))\big)$.*

Proof. Let $f, g \in L_1(Q)$. Using left invariance of m we have

$$\begin{aligned}
&(\pi_1 \otimes \pi_2)(\delta(fm) * (gm \otimes \varepsilon_e)) \\
&\quad = (\pi_1 \otimes \pi_2)(\delta(fm)) \cdot (\pi_1(gm) \otimes I) \\
&\quad = \int_Q \big(\pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_s)\big) f(s)\, dm(s) \cdot \int_Q (\pi_1(\varepsilon_u) \otimes I) g(u)\, dm(u) \\
&\quad = \int_{Q^2} \big(\pi_1(\varepsilon_s * \varepsilon_u) \otimes \pi_2(\varepsilon_s)\big) f(s) g(u)\, dm(s) dm(u) \\
&\quad = \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) f(s) \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle\, dm(t) dm(s).
\end{aligned}$$

Since the function H defined by $H(t, s) = \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle f(s)$ belongs to $L_1(Q \times Q, m \otimes m)$, for any $\epsilon > 0$ there is $H_0 \in \mathscr{K} \odot \mathscr{K} \subset L_1(Q) \odot L_1(Q)$ such that $\|H - H_0\|_1 < \varepsilon$. Then

$$\begin{aligned}
&\Big\| \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) f(s) \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle\, dm(t) dm(s) \\
&\qquad - \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) H_0(t, s)\, dm(t) dm(s) \Big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \\
&\quad = \Big\| \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) \big(H(t, s) - H_0(t, s)\big)\, dm(t) dm(s) \Big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \\
&\quad \le \|H - H_0\|_{L_1(Q \times Q)} < \epsilon.
\end{aligned}$$

Since $(\pi_1 \otimes \pi_2)(H_0 m \otimes m) \in \pi_1(L_1(m)) \odot \pi_2(L_1(m))$, we see that

$$(\pi_1 \otimes \pi_2)(\delta(fm) * (gm \otimes \varepsilon_e)) \in \pi_1(L_1(Q)) \otimes \pi_2(L_2(Q)).$$

Substituting the functions $f^\star, g^\star \in L_1(Q)$ into the above inclusion and taking the involution, we also get

$$(\pi_1 \otimes \pi_2)\big((gm \otimes \varepsilon_e) * \delta(fm)\big) \in \pi_1(L_1(Q)) \otimes \pi_2(L_2(Q)). \qquad \square$$

Corollary 1. *Let δ defined by (9) be completely positive. Then the restriction $\delta \restriction_{L_1(m)}$ can be extended by continuity to the following maps, still denoted by δ:*

$$\delta \colon C^*(Q) \to M(C^*(Q) \otimes C^*(Q)), \qquad \delta \colon C^*_r(Q) \to M(C^*_r(Q) \otimes C^*_r(Q)).$$

Proof. The proof follows immediately from Proposition 2 and Theorem 1. $\square$

Proposition 3. *Let δ be completely positive, and $a_1, a_2 \in \mathscr{B}(Q)$. Then*

$$\|a_1 a_2\|^\circ \le \|a_1\|^\circ \|a_2\|^\circ. \tag{19}$$

Proof. We keep the notations introduced in the proof of Theorem 1. Let $(\mathscr{H}_1, \pi_1)$, $(\mathscr{H}_2, \pi_2)$ be representations of Q, and $\xi_i, \eta_i \in \mathscr{H}_i$, $i = 1, 2$, be such that

$$a_i(s) = \big(\pi_i(\varepsilon_s).\, \xi_i \;\big|\; \eta_i \big), \qquad i = 1, 2.$$

Then

$$\begin{aligned}
\|a_1a_2\|^\circ &= \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} |\mu(a_1a_2)| \\
&= \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} \Big|\int_Q a_1(s)a_2(s)\, d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} \Big|\int_Q \big(\pi_1(\varepsilon_s).\,\xi_1 \;\big|\; \eta_1\big)_{\mathscr{H}_1} \big(\pi_2(\varepsilon_s).\,\xi_2 \;\big|\; \eta_2\big)_{\mathscr{H}_2}\, d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} \big|\big(\mu\otimes\xi_1\otimes\xi_2 \;\big|\; \varepsilon_e\otimes\eta_1\otimes\eta_2\big)_{\mathscr{H}_0}\big| \\
&= \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} \big|\big(\pi_0(\mu).\,\varepsilon_e\otimes\xi_1\otimes\xi_2 \;\big|\; \varepsilon_e\otimes\eta_1\otimes\eta_2\big)_{\mathscr{H}_0}\big| \\
&\le \sup_{\mu\in L_1(m),\, \|\mu\|_\Sigma=1} \|\pi_0(\mu)\|\, \|\varepsilon_e\otimes\xi_1\otimes\xi_2\|_{\mathscr{H}_0}\, \|\varepsilon_e\otimes\eta_1\otimes\eta_2\|_{\mathscr{H}_0} \\
&\le \|\varepsilon_e\otimes\xi_1\otimes\xi_2\|_{\mathscr{H}_0}\, \|\varepsilon_e\otimes\eta_1\otimes\eta_2\|_{\mathscr{H}_0} \\
&= \|\xi_1\|_{\mathscr{H}_1}\, \|\xi_2\|_{\mathscr{H}_2}\, \|\eta_1\|_{\mathscr{H}_1}\, \|\eta_2\|_{\mathscr{H}_2} \\
&= \|a_1\|^\circ\, \|a_2\|^\circ
\end{aligned}$$

by (6) and (7). □

Theorem 2. *Let Q be a locally compact hypergroup such that $\delta\colon \mathscr{M}_b(Q)\to\mathscr{M}_b(Q\times Q)$ defined by (10) is completely positive. Then the Fourier-Stieltjes space $\mathscr{B}(Q)$ is a Banach algebra.*

Proof. The proof follows directly from Theorem 1 and estimate (19). □

Theorem 3. *Let Q be a locally compact hypergroup such that $\delta\colon \mathscr{M}_b(Q)\to\mathscr{M}_b(Q\times Q)$ defined by (10) is completely positive. Then the Fourier space $\mathscr{A}(Q)$ is a Banach algebra.*

Proof. Let $f_i\in\mathscr{K}(Q)$, and $a_i = f_i * f_i^\dagger$, $i=1,2$. Then

$$\begin{aligned}
a_i(s) &= \int_Q f_i(t)\overline{\check{f}_i}(\check{\varepsilon}_t * \varepsilon_s)\, dm(t) = \int_Q f_i(t)\overline{f}_i(\check{\varepsilon}_s * \varepsilon_t)\, dm(t) \\
&= \big(\lambda(\varepsilon_s).\,f_i \;\big|\; f_i\big)_{L_2(Q)}, \qquad i=1,2,
\end{aligned}$$

and

$$a_1(s)a_2(s) = \big(\lambda(\varepsilon_s).\,f_1 \;\big|\; f_1\big)_{L_2(Q)}\big(\lambda(\varepsilon_s).\,f_2 \;\big|\; f_2\big)_{L_2(Q)} = \big(\pi(\varepsilon_s).\,\xi \;\big|\; \xi\big)_{\mathscr{H}},$$

where π and $\mathscr{H}$ are as in Theorem 1 and $\xi\in\mathscr{H}$ is an equivalence class of $\varepsilon_e\otimes f_1\otimes f_2\in\mathscr{H}_0$. This implies that a_1a_2 is positive definite. It is clear that $a_1a_2\in L_2(Q)$. The same reasoning as in [9, 13.8.6] applied to the case of a locally compact hypergroup shows that there is $g\in L_2(Q)$ such that $a_1a_2 = g*g^\dagger$. And, since

$$\|f*g^\dagger\|^\circ \le \|f\|_2\,\|g\|_2, \qquad f,g\in L_2(Q),$$

and $\mathscr{K}(Q)$ is dense in $L_2(Q)$, we see that $a_1a_2\in\mathscr{A}(Q)$.

It follows from (14) that

$$\begin{aligned}
\|a_1 a_2\|^\circ &= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} |\mu(a_1a_2)| \\
&= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \Big|\int_Q \big(\,\lambda(\varepsilon_s).\,f_1 \;\big|\; f_1\,\big)_{L_2(Q)} \big(\,\lambda(\varepsilon_s).\,f_2 \;\big|\; f_2\,\big)_{L_2(Q)}\, d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \big|\big(\,(\lambda\otimes\lambda)(\delta(\mu)).\,(f_1\otimes f_2) \;\big|\; f_1\otimes f_2\,\big)_{L_2(Q)\otimes L_2(Q)}\big| \\
&\le \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \|(\lambda\otimes\lambda)(\delta(\mu))\|_{B(L_2(Q)\otimes L_2(Q))} \big(\,f_1\otimes f_2 \;\big|\; f_1\otimes f_2\,\big)_{L_2(Q)\otimes L_2(Q)} \\
&\le \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \|\mu\|_{B(L_2(Q))}\, \|f_1\|_2^2\, \|f_2\|_2^2 \\
&= \|f_1\|_2^2\, \|f_2\|_2^2 = \\
&= \|a_1\|^\circ \|a_2\|^\circ,
\end{aligned}$$

prooving that $\mathscr{A}(Q)$ is a Banach algebra. □

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Received 26/05/2015