Methods of Functional Analysis and Topology
Vol. 21 (2015), no. 3, pp. 246–255

ON FOURIER ALGEBRA OF A LOCALLY COMPACT HYPERGROUP

A. A. KALYUZHNYI, G. B. PODKOLZIN, AND YU. A. CHAPOVSKY

The paper is dedicated to the 90th birthday anniversary of Yu. M. Berezansky

Abstract. We give sufficient conditions for the Fourier and the Fourier-Stieltjes spaces of a locally compact hypergroup to be Banach algebras.

1. Introduction

The Fourier and the Fourier-Stieltjes algebras for a general locally compact group have been considered in [1]. Following a similar procedure for a locally compact hypergroup, one can introduce Banach spaces that are not Banach algebras in general. This has been done in [2], where the author has also considered the Fourier spaces of some commutative hypergroups, proving that the Fourier spaces are Banach algebras with respect to a certain norm. In [3], the author has introduced a special class of hypergroups, he calls ultraspherical, that also possess the property of the Fourier space being a Banach algebra.

In this paper, we give a sufficient condition for the Fourier space to be a Banach algebra in the case of a general locally compact hypergroup. The double coset hypergroup of a locally compact group satisfies this condition.

2. Preliminary

2.1. **Main notations and definitions.** Let Q be a Hausdorff locally compact topological space. The set of all compact subsets of Q is denoted by $\mathcal{K}$.

The linear space of complex-valued continuous functions on Q is denoted by $\mathscr{C}(Q)$, the subspace of $\mathscr{C}(Q)$ of bounded functions (resp., functions approaching zero at infinity) is denoted by $\mathscr{C}_b(Q)$ (resp., $\mathscr{C}_0(Q)$). The space $\mathscr{C}_b(Q)$ is endowed with the norm $\|f\|_\infty = \sup_{t\in Q}|f(t)|$. By $\mathscr{K}(Q)$, we denote the linear subspace of $\mathscr{C}_0(Q)$ of functions with compact supports. By 1_Q, we denote the constant function, $1_Q(s) = 1$ for all $s \in Q$.

A measure is understood as a complex Radon measure [4] on Q. The linear space of complex Radon measures, over the field $\mathbf{C}$ of complex numbers, is denoted by $\mathscr{M}(Q)$. For a measure μ, its norm is $\|\mu\|_1 = \sup_{f\in\mathscr{K}(Q),\ \|f\|_\infty\le 1}|\mu(f)|$. The subspace of $\mathscr{M}(Q)$ of bounded (resp., compactly supported) measures is denoted by $\mathscr{M}_b(Q)$ (resp., $\mathscr{M}_c(Q)$). The subset of $\mathscr{M}(Q)$ of nonnegative (resp., probability) measures is denoted by $\mathscr{M}_+(Q)$ (resp., $\mathscr{M}_p(Q)$). For a measure $\mu \in \mathscr{M}_+(Q)$, its support is denoted by $S(\mu)$. The set of measures μ such that $S(\mu)$ is compact is denoted by $\mathscr{M}_c(Q)$. If $\mu \in \mathscr{M}_+(Q) \cap \mathscr{M}_b(Q)$, then $\|\mu\|_1 = \mu(1_Q)$. The Dirac measure at a point $s \in Q$ is denoted by ε_s. The integral of $f \in \mathscr{K}(F)$ with respect to a measure $\mu \in \mathscr{M}$ is denoted by $\mu(f) = \langle f, \mu\rangle = \int_F \langle f, \varepsilon_t\rangle\, d\mu(t) = \int_F f(t)\, d\mu(t)$.

2000 *Mathematics Subject Classification.* Primary: 20N20; Secondary: 22D15, 22D35.

Key words and phrases. Fourier algebra, Fourier-Stieltjes algebra, DJS-hypergroup, locally compact hypergroup, dual algebras, Pontryagin duality.

A (locally compact) hypergroup is a locally compact Hausdorff topological space Q such that $\mathscr{M}_b(Q)$ is endowed with a multiplication, called composition and denoted by $*$, satisfying the following conditions [5]:

(H_1) $(\mathscr{M}_b(Q), *)$ is an algebra over $\mathbf{C}$.
(H_2) For all $s, t \in Q$, $\varepsilon_s * \varepsilon_t \in \mathscr{M}_p(Q)$ and $S(\varepsilon_s * \varepsilon_t)$ is compact.
(H_3) The mapping $(s,t) \mapsto \varepsilon_s * \varepsilon_t$ of $Q \times Q$ into $\mathscr{M}_p(Q)$ is continuous with respect to the weak topology $\sigma(\mathscr{M}_p(Q), \mathscr{C}_0(Q))$ on $\mathscr{M}_p(Q)$.
(H_4) The mapping $(s,t) \mapsto S(\varepsilon_s * \varepsilon_t)$ of $Q \times Q$ into $\mathcal{K}$ is continuous with respect to the Michael topology on $\mathcal{K}$.
(H_5) There exists a (necessarily unique) element $e \in Q$ such that $\varepsilon_e * \varepsilon_s = \varepsilon_s * \varepsilon_e = \varepsilon_s$ for all $s \in Q$.
(H_6) There exists a (necessarily unique) homeomorphism $s \mapsto \check{s}$ of Q into Q such that $\check{\check{s}} = s$ and $(\varepsilon_s * \varepsilon_t)\check{} = \varepsilon_{\check{t}} * \varepsilon_{\check{s}}$, where $\check{\mu}$ denotes the image of the measure μ with respect to homeomorphism $s \mapsto \check{s}$, i.e. $\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle$, where $\check{f}(s) = f(\check{s})$.
(H_7) For $s, t \in Q$, $e \in S(\varepsilon_s * \varepsilon_t)$ if and only if $s = \check{t}$.

For a measure $\mu \in \mathscr{M}(Q)$ and $h \in \mathscr{C}(Q)$, the measure $h\mu$ is defined by $\langle f, h\mu \rangle = \langle fh, \mu \rangle$ for $f \in \mathscr{K}$; it is clear that $h\mu \in \mathscr{M}_c(Q)$ for $h \in \mathscr{K}(Q)$.

Everywhere in the sequel, we assume that the hypergroup possesses a left invariant measure, denoted by m, which means that

$$\varepsilon_s * m = m$$

for all $s \in Q$.

For $\mu \in \mathscr{M}_b(Q)$, denote by $\mu^\star$ the bounded measure defined by $\mu^\star(f) = \overline{\check{\mu}(\bar{f})}$ for $f \in \mathscr{K}(Q)$. It follows from the axiom (H_6) of a hypergroup that * is an involution on the algebra $(\mathscr{M}_b(Q), *)$. It is well known [5] that $(\mathscr{M}_b(Q), *, ^\star)$ is an involutive Banach algebra.

For C^*-algebras A and B, the tensor product $A \otimes B$ is the completion of the algebraic tensor product $A \odot B$ with respect to the min-C^*-norm on $A \odot B$,

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{\min} = \sup_{\pi_A \in \Sigma_A, \pi_B \in \Sigma_B} \Big\| \sum_{i=1}^n \pi_A(a_i) \otimes \pi_B(b_i) \Big\|,$$

where Σ_A (resp., Σ_B) is the set of all representations of A (resp., B) [6].

For a C^*-algebra A, the C^*-algebra of multipliers of A is denoted by $M(A)$, see [7] for details.

2.2. **Fourier–Stieltjes and Fourier spaces.**

Definition 1. Let Q be a locally compact hypergroup. Let $\mathscr{H}$ be a Hilbert space, $B(\mathscr{H})$ the C^*-algebra of all linear bounded operators on $\mathscr{H}$, and $\pi \colon \mathscr{M}_b(Q) \to B(\mathscr{H})$ a linear map. Then the pair $(\mathscr{H}, \pi)$ is called a *representation* of $\mathscr{M}_b(Q)$ if π is an involutive homomorphism of the involutive Banach algebra $(\mathscr{M}_b(Q), *, ^\star)$ into $B(\mathscr{H})$.

A left invariant measure m on Q gives rise to an inner product $(\,\cdot \mid \cdot\,)$ and norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathscr{K}(Q)$ as usual,

$$(\,f \mid g\,) = \int_Q f(t)\overline{g}(t)\, dm(t),$$

$$\|f\|_1 = (\,|f| \mid 1_Q\,), \qquad \|f\|_2 = (\,f \mid f\,)^{1/2}.$$

The completions of $\mathscr{K}(Q)$ with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are denoted by $L_1(Q)$ and $L_2(Q)$, respectively.

We denote $L_1(m) = \{fm : f \in L_1(Q)\}$. It is well known that $(L_1(m), *, ^\star)$ is a closed two-sided ideal of $(\mathscr{M}_b(Q), *, ^\star)$.

Identifying each $f \in L_1(Q)$ with the measure $fm \in L_1(m)$ yields an involutive Banach algebra structure on $L_1(Q)$, denoted by $(L_1(Q), *, {}^\star)$, where

$$(f * g)(s) = \int_Q f(t) \langle g, \check{\varepsilon}_t * \varepsilon_t \rangle \, dm(t), \tag{1}$$

$$f^\star(s) = \Delta^{-1}(s) \overline{f}(\check{s}), \tag{2}$$

where $\Delta : Q \to \mathbf{R}$ is the modular function, $m * \varepsilon_s = \Delta(s) m$, see [5]. Also denote by

$$f^\dagger(s) = \overline{f(\check{s})}. \tag{3}$$

A function $f \in \mathscr{C}_b$ is called *positive definite* if $\langle f, \mu^\star * \mu \rangle \geq 0$ for all $\mu \in \mathscr{M}_c(Q)$. It is well known that the function $f * f^\dagger$, $f \in \mathscr{K}$, is positive definite, see [5].

Each representation $(\mathscr{H}, \pi)$ of $(\mathscr{M}_b(Q), *, {}^\star)$, being restricted to $(L_1(m), *, {}^\star)$, defines a representation of the involutive Banach algebra $(L_1(Q), *, {}^\star)$, which we also denote by $(\mathscr{H}, \pi)$.

A representation $(\mathscr{H}, \pi)$ of $(\mathscr{M}_b(Q), *, {}^\star)$ is called a representation of Q if it gives rise to a nondegenerate representation of $(L_1(m), *, {}^\star)$, hence of $(L_1(Q), *, {}^\star)$. The set of all representation of Q will be denoted by Σ.

The left regular representation $(L_2(Q), \lambda)$ of $(L_1(Q), *, {}^\star)$ is defined by

$$\lambda(f).\, \xi = f * \xi$$

for $f \in L_1(Q)$ and $\xi \in L_2(Q)$.

Each representation $(\mathscr{H}, \pi)$ of Q induces a seminorm $\| \cdot \|_\pi$ on $(L_1(Q), *, {}^\star)$, defined by $\|f\|_\pi = \|\pi(f)\|_{B(\mathscr{H})}$, $f \in L_1(Q)$, where $\| \cdot \|_{B(\mathscr{H})}$ denotes the operator norm in $B(\mathscr{H})$.

For a subset $\Sigma' \subset \Sigma$, we define a seminorm $\| \cdot \|_{\Sigma'}$ on $(L_1(Q), *, {}^\star)$ by

$$\|f\|_{\Sigma'} = \sup_{\pi \in \Sigma'} \|f\|_\pi, \qquad f \in L_1(Q).$$

Definition 2. The enveloping C^*-algebra of $(L_1(Q), *, {}^\star)$ with respect to the norm $\| \cdot \|_\Sigma$ (resp., $\| \cdot \|_\lambda$) will be called the *full* (resp., the *reduced*) C^**-algebra* of the hypergroup Q. The full (resp., the reduced) C^*-algebra of Q will be denoted by $C^*(Q)$ (resp., $C^*_r(Q)$).

Definition 3. The Banach space dual to the full C^*-algebra $C^*(Q)$ is called the *Fourier-Stieltjes space* and will be denoted by $\mathscr{B}(Q)$.

The Banach space dual to the reduced C^*-algebra $C^*_r(Q)$ will be denoted by $\mathscr{B}_\lambda(Q)$.

It is known, see [2], that for each $\alpha \in \mathscr{B}(Q)$ (resp., $\alpha \in \mathscr{B}_\lambda(Q)$) there is a representation $(\mathscr{H}, \pi)$ of Q (resp., weakly contained in $(L_2(Q), \lambda)$) and two vectors $\xi, \eta \in \mathscr{H}$ such that the function $a \in \mathscr{C}_b(Q)$ given by

$$a(s) = \big(\pi(\varepsilon_s).\, \xi \bigm| \eta \big)_{\mathscr{H}}, \qquad s \in Q, \tag{4}$$

defines the functional α, namely,

$$\alpha(f) = \int_Q a(s) f(s) \, dm(s), \qquad f \in L_1(Q), \tag{5}$$

and

$$\|\alpha\| = \sup_{f \in L_1(Q),\, \|f\|_{\Sigma'} = 1} \Big| \int_Q a(s) f(s) \, dm(s) \Big| = \|\xi\|_{\mathscr{H}} \, \|\eta\|_{\mathscr{H}}, \tag{6}$$

where $\Sigma' = \Sigma$ (resp., $\Sigma' = \lambda$).

Henceforth, we identify $\mathscr{B}(Q)$ (resp., $\mathscr{B}_\lambda(Q)$) with a linear space of functions $a \in \mathscr{C}_b(Q)$ given by (4), where $(\mathscr{H}, \pi)$ is a representation of Q (resp., a representation weakly contained in $(L_2(Q), \lambda)$), and endow this space with the norm

$$\|a\|^\circ = \sup_{\mu \in L_1(m),\, \|\mu\|_{\Sigma'} = 1} |\mu(a)|, \tag{7}$$

where $\Sigma' = \Sigma$ (resp, $\Sigma' = \lambda$).

Definition 4. The closure of the subspace spanned by the elements $f * f^\dagger$, $f \in \mathscr{K}(Q)$, is in $\mathscr{B}_\lambda(Q)$ is called the *Fourier space* of Q and is denoted by $\mathscr{A}(Q)$.

3. Main results

Definition 5. For an involutive algebra $(B, *, ^\star)$ and a C^*-algebra $(A, \cdot, ^*)$, a linear map $\varphi\colon B \to A$ will be called *positive*, if $\varphi(b^\star * b)$ is a nonnegative element of the C^*-algebra A for any $b \in B$, and is called *completely positive*, if the linear map $\mathrm{id} \otimes \varphi\colon M_n(\mathbf{C}) \otimes B \to M_n(\mathbf{C}) \otimes A$ is positive for all $n \in \mathbf{N}$, where $M_n(\mathbf{C})$ denotes the C^*-algebra of complex $(n \times n)$-matrices.

It follows from [6] that a map $\varphi\colon B \to A$ is completely positive if and only if, for any $n \in \mathbf{N}$, $b_i \in B$ and $a_i \in A$, $i = 1, \dots, n$, we have

$$\sum_{i,j=1}^{n} a_j^* \varphi(b_j^\star * b_i) a_i \geq 0. \tag{8}$$

For a hypergroup Q, consider the product hypergroup $Q \times Q$ [5], and let $\delta\colon \mathscr{M}_b(Q) \to \mathscr{M}_b(Q \times Q)$ denote a linear extension of the map defined by

$$\delta(\varepsilon_s) = \varepsilon_s \otimes \varepsilon_s, \qquad s \in Q, \tag{9}$$

that is, for $\mu \in \mathscr{M}_b(Q)$ and $F \in \mathscr{K}(Q \times Q)$,

$$\langle\, \delta(\mu), F\,\rangle = \int_Q F(s,s)\, d\mu(s). \tag{10}$$

Let $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. For $\tilde{\mu} \in \mathscr{M}_b(Q \times Q)$, we define $(\pi_1 \otimes \pi_2)(\tilde{\mu}) \in B(\mathscr{H}_1 \otimes \mathscr{H}_2)$ by

$$(\pi_1 \otimes \pi_2)(\tilde{\mu}) = \int \pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_t)\, d\tilde{\mu}(s,t).$$

Proposition 1. *Let G be a locally compact group, H a compact subgroup of G, and $Q = H \setminus G/H$. Denote by λ_Q the left regular representation of $L_1(m)$ on $L_2(Q)$, and let δ be defined by (9). Then the linear map*

$$(\lambda_Q \otimes \lambda_Q) \circ \delta\colon L_1(m) \to B(L_2(Q) \otimes L_2(Q))$$

is completely positive.

Proof. Let $s \in Q = H \setminus G/H$. Denote by m_H the Haar measure on H. Then $\varepsilon_s = m_H * \varepsilon_g * m_H$ for some $g \in G$. Thus

$$\delta(\varepsilon_s) = \varepsilon_s \otimes \varepsilon_s = (m_H * \varepsilon_g * m_H) \otimes (m_H * \varepsilon_g * m_H).$$

Denoting $L_2(G, m_G)$ simply by $L_2(G)$, where m_G is a left invariant measure on G, we will identify $L_2(Q)$ with a closed subspace of $L_2(G)$,

$$L_2(Q) = \{f \in L_2(G) : m_H * f = f * m_H = f\}.$$

With such an identification,

$$\big(\, f \;\big|\; g\,\big)_{L_2(Q)} = \big(\, f \;\big|\; g\,\big)_{L_2(G)}, \qquad f, g \in L_2(Q).$$

For any $\mu \in L_1(m)$, let $\tilde{\mu} \in L_1(m_G)$ be such that $\mu = m_H * \tilde{\mu} * m_H$. Then we have

$$\lambda_Q(\mu).\, f = \lambda_G(m_H * \tilde{\mu} * m_H).\, f = \lambda_G(m_H * \tilde{\mu}).\, f, \qquad f \in L_2(Q),$$

where λ_G is the left regular representation of $L_1(m_G)$ on $L_2(G)$.

Thus, for any $\mu \in L_1(m)$ and $\tilde{\mu} \in L_1(m_G)$ such that $\mu = m_H * \tilde{\mu} * m_H$, we have

$$\begin{aligned}(\lambda_Q \otimes \lambda_Q) \circ \delta(\mu) &= \int_G (\lambda_G \otimes \lambda_G) \circ \delta(m_H * \varepsilon_g * m_H)\, d\tilde{\mu}(g) \\ &= \int_G \lambda_G(m_H * \varepsilon_g * m_H) \otimes \lambda_G(m_H * \varepsilon_g * m_H)\, d\tilde{\mu}(g) \\ &= \int_G \lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)\, d\tilde{\mu}(g). \end{aligned} \tag{11}$$

To prove the proposition, using (8), it is sufficient to show that

$$\begin{aligned}&\sum_{i,j=1}^{n} \big(\, A_j^* \cdot (\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) \cdot A_i.\, F \;\big|\; F \,\big)_{L_2(Q) \otimes L_2(Q)} \\ &\quad = \sum_{i,j=1}^{n} \big(\, (\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i) \cdot A_i.\, F \;\big|\; A_j.\, F \,\big)_{L_2(Q) \otimes L_2(Q)} \geq 0 \end{aligned} \tag{12}$$

for any $A_i \in B(L_2(Q) \otimes L_2(Q))$, $\mu_i \in L_1(m)$, $F \in L_2(Q) \otimes L_2(Q)$, $i = 1, \ldots, n$.

Hence, letting $A_i F = F_i \in L_2(Q) \otimes L_2(Q)$, and $\tilde{\mu}_i \in L_1(m_G)$ such that $\mu_i = m_H * \tilde{\mu}_i * m_H$, and using (11), we have

$$\begin{aligned}&\sum_{i,j=1}^{n} \big(\, (\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i).\, F_i \;\big|\; F_j \,\big)_{L_2(Q) \otimes L_2(Q)} \\ &\quad = \sum_{i,j=1}^{n} \int_G \big(\, (\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)).\, F_i \;\big|\; F_j \,\big)_{L_2(G) \otimes L_2(G)} \\ &\qquad\qquad d(m_H * \tilde{\mu}_j^\star * m_H * m_H * \tilde{\mu}_i * m_H)(g). \end{aligned}$$

Let $m_H * \tilde{\mu}_i * m_H = f_i\, m_G$, $f_i \in L_1(G)$, $i = 1, \ldots, n$. Then $m_h * \tilde{\mu}_j^\star * m_H = f_j^\star\, m_G$ and

$$m_H * \tilde{\mu}_j^\star * m_H * m_H * \tilde{\mu}_i * m_H = (f_j^\star * f_i)\, m_G.$$

Using left invariance of m_G and (2) we thus have

$$\begin{aligned}&\sum_{i,j=1}^{n} \big(\, (\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i).\, F_i \;\big|\; F_j \,\big)_{L_2(Q) \otimes L_2(Q)} \\ &\quad = \sum_{i,j=1}^{n} \int_G \big(\, (\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)).\, F_i \;\big|\; F_j \,\big)_{L_2(G) \otimes L_2(G)} \\ &\qquad\qquad \cdot (f_j^\star * f_i)(g)\, dm_G(g) \\ &\quad = \sum_{i,j=1}^{n} \int_{G^2} \big(\, (\lambda_G(m_H * \varepsilon_g) \otimes \lambda_G(m_H * \varepsilon_g)).\, F_i \;\big|\; F_j \,\big)_{L_2(G) \otimes L_2(G)} \\ &\qquad\qquad \cdot f_j^*(p) f_i(p^{-1}g)\, dm_G(p) dm_G(g) \\ &\quad = \sum_{i,j=1}^{n} \int_{G^2} \big(\, (\lambda_G(m_H * \varepsilon_{pg}) \otimes \lambda_G(m_H * \varepsilon_{pg})).\, F_i \;\big|\; F_j \,\big)_{L_2(G) \otimes L_2(G)} \\ &\qquad\qquad \cdot f_j^*(p) f_i(g)\, dm_G(p) dm_G(g) \\ &\quad = \sum_{i,j=1}^{n} \int_{G^2} \big(\, (\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)).\, F_i \;\big|\; (\lambda_G(\varepsilon_{p^{-1}}) \otimes \lambda_G(\varepsilon_{p^{-1}}).\, F_j \,\big)_{L_2(G) \otimes L_2(G)} \\ &\qquad\qquad \cdot f_j^*(p) f_i(g)\, dm_G(p) dm_G(g) 0 \end{aligned}$$

$$= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)).\, F_i \bigm| (\lambda_G(\varepsilon_{p^{-1}}) \otimes \lambda_G(\varepsilon_{p^{-1}}).\, F_j \big)_{L_2(G)\otimes L_2(G)} \cdot \overline{f}_j(p^{-1})\Delta_G^{-1}(p) f_i(g)\, dm_G(p) dm_G(g)$$

$$= \sum_{i,j=1}^{n} \int_{G^2} \big((\lambda_G(\varepsilon_g) \otimes \lambda_G(\varepsilon_g)).\, F_i \bigm| (\lambda_G(\varepsilon_p) \otimes \lambda_G(\varepsilon_p).\, F_j \big)_{L_2(G)\otimes L_2(G)} \cdot \overline{f}_j(p) f_i(g)\, dm_G(p) dm_G(g)$$

$$= \Big\| \sum_{i=1}^{n} \int_G f_i(g) (\lambda_G(\varepsilon_g) \otimes \lambda_G(g))\, dm_G(g).\, F_i \Big\|^2_{L_2(G)\otimes L_2(G)} \geq 0. \qquad \square$$

Theorem 1. *Let the map δ defined by (9) be completely positive. Let $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. Then we have the following.*

(i) *There is a representation $(\mathscr{H}, \pi)$ of Q such that*

$$\big\| (\pi_1 \otimes \pi_2)(\delta(\mu)) \big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \leq \big\| \pi(\mu) \big\|_{B(\mathscr{H})}, \qquad \mu \in \mathscr{M}_b(Q). \tag{13}$$

If $\pi_1 = \pi_2 = \lambda$ and $\mathscr{H}_1 = \mathscr{H}_2 = L_2(Q)$, then

$$\big\| (\lambda \otimes \lambda)(\delta(\mu)) \big\|_{B(L_2(Q)\otimes L_2(Q))} \leq \big\| \lambda(\mu) \big\|_{B(L_2(Q))}, \qquad \mu \in \mathscr{M}_b(Q). \tag{14}$$

(ii) *Let $\xi_i, \eta_i \in \mathscr{H}_i$ be arbitrary vectors in the corresponding spaces, and*

$$a_i(s) = \big(\pi_i(\varepsilon_s).\, \xi_i \bigm| \eta_i \big)_{\mathscr{H}_i}, \qquad i = 1, 2.$$

Then there are vectors $\xi, \eta \in \mathscr{H}$ such that

$$a_1(s) a_2(s) = \big(\pi(\varepsilon_s).\, \xi \bigm| \eta \big)_{\mathscr{H}}. \tag{15}$$

Proof. The construction of the representation $(\mathscr{H}, \pi)$ uses the Stinespring construction [8].

Consider the linear space $\mathscr{H}_0 = \mathscr{M}_b(Q) \odot (\mathscr{H}_1 \otimes \mathscr{H}_2)$, where $\odot$ denotes the algebraic tensor product, and for each $\mu \in \mathscr{M}_b(Q)$, define a linear operator $\pi_0(\mu)$ on $\mathscr{H}_0$ by setting

$$\pi_0(\mu).\, \nu \otimes \tilde{v} = (\mu * \nu) \otimes \tilde{v},$$

where $\nu \otimes \tilde{v} \in \mathscr{H}_0$, with $\nu \in \mathscr{M}_b(Q)$ and $\tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$.

Using δ, introduce a sesquilinear form $\big(\cdot \bigm| \cdot \big)_{\mathscr{H}_0}$ on $\mathscr{H}_0$ defined by

$$\big(\mu \otimes \tilde{u} \bigm| \nu \otimes \tilde{v} \big)_{\mathscr{H}_0} = \big((\pi_1 \otimes \pi_2)\delta(\nu^\star * \mu).\, \tilde{u} \bigm| \tilde{v} \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \tag{16}$$

for $\mu, \nu \in \mathscr{M}_b(Q)$ and $\tilde{u}, \tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$, and bilinearly extended to $\mathscr{H}_0$. Since δ is completely positive,

$$\big(w \bigm| w \big)_{\mathscr{H}_0} \geq 0, \qquad w \in \mathscr{H}_0,$$

hence, this sesquilinear form gives rise to a seminorm $\| \cdot \|_{\mathscr{H}_0}$ on $\mathscr{H}_0$.

If $\mathscr{N} = \{ w \in \mathscr{H}_0 | \, \|w\|_{\mathscr{H}_0} = 0 \}$, then $\pi_0(\mu).\, \mathscr{N} \subset \mathscr{N}$ for any $\mu \in \mathscr{M}_b(Q)$. If now $\mathscr{H}$ is the completion of $\mathscr{H}_0 / \mathscr{N}$ with respect to the norm defined by the sesquilinear form (16) and $\pi(\mu)$ is the operator on $\mathscr{H}$ corresponding to the operator $\pi_0(\mu)$, then it follows from [8] that $(\mathscr{H}, \pi)$ is an involutive representation of $\mathscr{M}_b(Q)$.

To prove (13), let $\tilde{u}, \tilde{v} \in \mathscr{H}_1 \otimes \mathscr{H}_2$ and $\mu \in \mathscr{M}_b$. Then

$$\begin{aligned} \big| \big((\pi_1 \otimes \pi_2)(\delta(\mu)).\, \tilde{u} \bigm| \tilde{v} \big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \big| &= \big| \big(\mu \otimes \tilde{u} \bigm| \varepsilon_e \otimes \tilde{v} \big)_{\mathscr{H}} \big| \\ &= \big| \big(\pi(\mu).\, (\varepsilon_e \otimes \tilde{u}) \bigm| \varepsilon_e \otimes \tilde{v} \big)_{\mathscr{H}} \big| \\ &\leq \big\| \pi(\mu) \big\|_{B(\mathscr{H})} \big\| \varepsilon_e \otimes \tilde{u} \big\|_{\mathscr{H}} \big\| \varepsilon_e \otimes \tilde{v} \big\|_{\mathscr{H}} \\ &= \big\| \pi(\mu) \big\|_{B(\mathscr{H})} \big\| \tilde{u} \big\|_{\mathscr{H}_1 \otimes \mathscr{H}_2} \big\| \tilde{v} \big\|_{\mathscr{H}_1 \otimes \mathscr{H}_2}, \end{aligned}$$

which proves (13).

To prove (14), we identify $L_2(Q)\otimes L_2(Q)$ with $L_2(Q\times Q)$ with respect to the product measure $m\otimes m$. To shorten the notations, we write $F(\mu_1,\mu_2)$ instead of $\langle F,\mu_1\otimes\mu_2\rangle$ for $F\in L_2(Q\times Q)$ and $\mu_1,\mu_2\in\mathscr{M}_b$.

Let $F,G\in L_2(Q\times Q)$, $\mu\in\mathscr{M}_b$. Using left invariance of m and the Cauchy inequality, we have

$$\begin{aligned}\Big|\big((\lambda\otimes\lambda)\delta(\mu).F \;\big|\; G\big)_{L_2(Q\times Q)}\Big| &= \Big|\big(\int_Q(\lambda(\varepsilon_u)\otimes\lambda(\varepsilon_u)).F \;\big|\; G\big)_{L_2(Q\times Q)}\,d\mu(u)\Big|\\ &= \Big|\int_{Q^3}F(\check\varepsilon_u*\varepsilon_s,\check\varepsilon_u*\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_t)}\,dm(s)dm(t)d\mu(u)\Big|\\ &= \Big|\int_{Q^3}F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_u*\varepsilon_t)}\,dm(s)dm(t)d\mu(u)\Big|\\ &\le \int_{Q^2}\Big|\int_Q F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\overline{G(\varepsilon_s,\varepsilon_u*\varepsilon_t)}\,dm(t)\Big|dm(s)d\mu(u)\\ &\le \int_{Q^2}\Big(\int_Q\big|F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}\\ &\qquad\cdot\Big(\int_Q\big|G(\varepsilon_s,\varepsilon_u*\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}\,dm(s)d\mu(u).\end{aligned}$$

Denoting

$$f(s) = \Big(\int_Q\big|F(\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12}, \tag{17}$$

$$g(s) = \Big(\int_Q\big|G(\varepsilon_s,\varepsilon_u*\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12} = \Big(\int|G(\varepsilon_s,\varepsilon_t)|^2\,dm(t)\Big)^{\frac12}, \tag{18}$$

where the last equality is due to left invariance of m, we get

$$(\lambda(\varepsilon_u).f)(s) = \Big(\int_Q\big|F(\check\varepsilon_u*\varepsilon_s,\varepsilon_t)\big|^2\,dm(t)\Big)^{\frac12},$$

hence

$$\begin{aligned}\Big|\big((\lambda\otimes\lambda)\delta(\mu).F \;\big|\; G\big)_{L_2(Q\times Q)}\Big| &\le \int_{Q^2}(\lambda(\varepsilon_u).f)(s)g(s)\,dm(s)d\mu(u)\\ &= \big(\lambda(\mu).f \;\big|\; g\big)_{L_2(Q)},\end{aligned}$$

where, as it follows from (17) and (18),

$$\|f\|_{L_2(Q)} = \|F\|_{L_2(Q\times Q)}, \qquad \|g\|_{L_2(Q)} = \|G\|_{L_2(Q\times Q)}.$$

This finishes the proof of (14).

Let us now prove (ii).

Define $\xi_0,\eta_0\in\mathscr{H}_0$ by

$$\xi_0 = \varepsilon_e\otimes(\xi_1\otimes\xi_2), \qquad \eta_0 = \varepsilon_e\otimes\eta_1\otimes\eta_2.$$

Then

$$\big(\pi_0(\varepsilon_s).\xi_0 \;\big|\; \eta_0\big)_{\mathscr{H}_0} = \big(\pi_0(\varepsilon_s).\varepsilon_e\otimes(\xi_1\otimes\xi_2) \;\big|\; \varepsilon_e\otimes(\eta_1\otimes\eta_2)\big)_{\mathscr{H}_0}0$$

$$\begin{aligned}
&= \big(\, \varepsilon_s \otimes (\xi_1 \otimes \xi_2) \;\big|\; \varepsilon_e \otimes (\eta_1 \otimes \eta_2) \,\big)_{\mathscr{H}_0} \\
&= \big(\, (\pi_1 \otimes \pi_2)\delta(\varepsilon_s).\, (\xi_1 \otimes \xi_2) \;\big|\; \eta_1 \otimes \eta_2 \,\big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \\
&= \big(\, (\pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_s)).\, (\xi_1 \otimes \xi_2) \;\big|\; \eta_1 \otimes \eta_2 \,\big)_{\mathscr{H}_1 \otimes \mathscr{H}_2} \\
&= \big(\, \pi_1(\varepsilon_s).\, \xi_1 \;\big|\; \eta_1 \,\big)_{\mathscr{H}_1} \big(\, \pi_2(\varepsilon_s).\, \xi_2 \;\big|\; \eta_2 \,\big)_{\mathscr{H}_2} \\
&= a_1(s)a_2(s). \qquad \square
\end{aligned}$$

Proposition 2. *Let $\mu \in L_1(m)$, and $(\mathscr{H}_i, \pi_i)$, $i = 1, 2$, be representations of Q. Then $(\pi_1 \otimes \pi_2)(\delta(\mu)) \in M\big(\pi_1(L_1(m)) \otimes \pi_2(L_1(m))\big)$.*

Proof. Let $f, g \in L_1(Q)$. Using left invariance of m we have

$$\begin{aligned}
&(\pi_1 \otimes \pi_2)(\delta(fm) * (gm \otimes \varepsilon_e)) \\
&\quad = (\pi_1 \otimes \pi_2)(\delta(fm)) \cdot (\pi_1(gm) \otimes I) \\
&\quad = \int_Q \big(\pi_1(\varepsilon_s) \otimes \pi_2(\varepsilon_s)\big) f(s)\, dm(s) \cdot \int_Q (\pi_1(\varepsilon_u) \otimes I) g(u)\, dm(u) \\
&\quad = \int_{Q^2} \big(\pi_1(\varepsilon_s * \varepsilon_u) \otimes \pi_2(\varepsilon_s)\big) f(s) g(u)\, dm(s) dm(u) \\
&\quad = \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) f(s) \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle\, dm(t) dm(s).
\end{aligned}$$

Since the function H defined by $H(t, s) = \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle f(s)$ belongs to $L_1(Q \times Q, m \otimes m)$, for any $\epsilon > 0$ there is $H_0 \in \mathscr{K} \odot \mathscr{K} \subset L_1(Q) \odot L_1(Q)$ such that $\|H - H_0\|_1 < \varepsilon$. Then

$$\begin{aligned}
&\Big\| \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) f(s) \langle\, g, \check{\varepsilon}_s * \varepsilon_t \,\rangle\, dm(t) dm(s) \\
&\qquad\qquad - \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) H_0(t, s)\, dm(t) dm(s) \Big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \\
&\quad = \Big\| \int_{Q^2} \big(\pi_1(\varepsilon_t) \otimes \pi_2(\varepsilon_s)\big) \big(H(t, s) - H_0(t, s)\big)\, dm(t) dm(s) \Big\|_{B(\mathscr{H}_1 \otimes \mathscr{H}_2)} \\
&\quad \le \|H - H_0\|_{L_1(Q \times Q)} < \epsilon.
\end{aligned}$$

Since $(\pi_1 \otimes \pi_2)(H_0 m \otimes m) \in \pi_1(L_1(m)) \odot \pi_2(L_1(m))$, we see that

$$(\pi_1 \otimes \pi_2)(\delta(fm) * (gm \otimes \varepsilon_e)) \in \pi_1(L_1(Q)) \otimes \pi_2(L_2(Q)).$$

Substituting the functions $f^\star, g^\star \in L_1(Q)$ into the above inclusion and taking the involution, we also get

$$(\pi_1 \otimes \pi_2)\big((gm \otimes \varepsilon_e) * \delta(fm)\big) \in \pi_1(L_1(Q)) \otimes \pi_2(L_2(Q)). \qquad \square$$

Corollary 1. *Let δ defined by (9) be completely positive. Then the restriction $\delta \restriction_{L_1(m)}$ can be extended by continuity to the following maps, still denoted by δ:*

$$\delta \colon C^*(Q) \to M(C^*(Q) \otimes C^*(Q)), \qquad \delta \colon C^*_r(Q) \to M(C^*_r(Q) \otimes C^*_r(Q)).$$

Proof. The proof follows immediately from Proposition 2 and Theorem 1. $\square$

Proposition 3. *Let δ be completely positive, and $a_1, a_2 \in \mathscr{B}(Q)$. Then*

$$\|a_1 a_2\|^\circ \le \|a_1\|^\circ \|a_2\|^\circ. \tag{19}$$

Proof. We keep the notations introduced in the proof of Theorem 1. Let $(\mathscr{H}_1, \pi_1)$, $(\mathscr{H}_2, \pi_2)$ be representations of Q, and $\xi_i, \eta_i \in \mathscr{H}_i$, $i = 1, 2$, be such that

$$a_i(s) = \big(\, \pi_i(\varepsilon_s).\, \xi_i \;\big|\; \eta_i \,\big), \qquad i = 1, 2.$$

Then

$$
\begin{aligned}
\|a_1a_2\|^\circ &= \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} |\mu(a_1a_2)| \\
&= \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} \Big|\int_Q a_1(s)a_2(s)\,d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} \Big|\int_Q \big(\,\pi_1(\varepsilon_s).\,\xi_1 \;\big|\; \eta_1\,\big)_{\mathscr{H}_1}\big(\,\pi_2(\varepsilon_s).\,\xi_2 \;\big|\; \eta_2\,\big)_{\mathscr{H}_2}\,d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} \Big|\big(\,\mu\otimes\xi_1\otimes\xi_2 \;\big|\; \varepsilon_e\otimes\eta_1\otimes\eta_2\,\big)_{\mathscr{H}_0}\Big| \\
&= \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} \Big|\big(\,\pi_0(\mu).\,\varepsilon_e\otimes\xi_1\otimes\xi_2 \;\big|\; \varepsilon_e\otimes\eta_1\otimes\eta_2\,\big)_{\mathscr{H}_0}\Big| \\
&\le \sup_{\mu\in L_1(m),\,\|\mu\|_\Sigma=1} \|\pi_0(\mu)\|\,\|\varepsilon_e\otimes\xi_1\otimes\xi_2\|_{\mathscr{H}_0}\,\|\varepsilon_e\otimes\eta_1\otimes\eta_2\|_{\mathscr{H}_0} \\
&\le \|\varepsilon_e\otimes\xi_1\otimes\xi_2\|_{\mathscr{H}_0}\,\|\varepsilon_e\otimes\eta_1\otimes\eta_2\|_{\mathscr{H}_0} \\
&= \|\xi_1\|_{\mathscr{H}_1}\,\|\xi_2\|_{\mathscr{H}_2}\,\|\eta_1\|_{\mathscr{H}_1}\,\|\eta_2\|_{\mathscr{H}_2} \\
&= \|a_1\|^\circ\,\|a_2\|^\circ
\end{aligned}
$$

by (6) and (7). □

Theorem 2. *Let Q be a locally compact hypergroup such that $\delta\colon \mathscr{M}_b(Q)\to\mathscr{M}_b(Q\times Q)$ defined by (10) is completely positive. Then the Fourier-Stieltjes space $\mathscr{B}(Q)$ is a Banach algebra.*

Proof. The proof follows directly from Theorem 1 and estimate (19). □

Theorem 3. *Let Q be a locally compact hypergroup such that $\delta\colon \mathscr{M}_b(Q)\to\mathscr{M}_b(Q\times Q)$ defined by (10) is completely positive. Then the Fourier space $\mathscr{A}(Q)$ is a Banach algebra.*

Proof. Let $f_i\in\mathscr{K}(Q)$, and $a_i = f_i * f_i^\dagger$, $i=1,2$. Then

$$
\begin{aligned}
a_i(s) &= \int_Q f_i(t)\overline{\check{f}_i}(\check{\varepsilon}_t*\varepsilon_s)\,dm(t) = \int_Q f_i(t)\overline{f}_i(\check{\varepsilon}_s*\varepsilon_t)\,dm(t) \\
&= \big(\,\lambda(\varepsilon_s).\,f_i \;\big|\; f_i\,\big)_{L_2(Q)}, \qquad i=1,2,
\end{aligned}
$$

and

$$
a_1(s)a_2(s) = \big(\,\lambda(\varepsilon_s).\,f_1 \;\big|\; f_1\,\big)_{L_2(Q)}\big(\,\lambda(\varepsilon_s).\,f_2 \;\big|\; f_2\,\big)_{L_2(Q)} = \big(\,\pi(\varepsilon_s).\,\xi \;\big|\; \xi\,\big)_{\mathscr{H}},
$$

where π and $\mathscr{H}$ are as in Theorem 1 and $\xi\in\mathscr{H}$ is an equivalence class of $\varepsilon_e\otimes f_1\otimes f_2\in\mathscr{H}_0$. This implies that a_1a_2 is positive definite. It is clear that $a_1a_2\in L_2(Q)$. The same reasoning as in [9, 13.8.6] applied to the case of a locally compact hypergroup shows that there is $g\in L_2(Q)$ such that $a_1a_2 = g*g^\dagger$. And, since

$$
\|f*g^\dagger\|^\circ \le \|f\|_2\,\|g\|_2, \qquad f,g\in L_2(Q),
$$

and $\mathscr{K}(Q)$ is dense in $L_2(Q)$, we see that $a_1a_2\in\mathscr{A}(Q)$.

It follows from (14) that

$$\begin{aligned}
\|a_1a_2\|^\circ &= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} |\mu(a_1a_2)| \\
&= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \Big|\int_Q \big(\,\lambda(\varepsilon_s).\,f_1 \;\big|\; f_1\,\big)_{L_2(Q)} \big(\,\lambda(\varepsilon_s).\,f_2 \;\big|\; f_2\,\big)_{L_2(Q)}\, d\mu(s)\Big| \\
&= \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \big|\big(\,(\lambda\otimes\lambda)(\delta(\mu)).\,(f_1\otimes f_2) \;\big|\; f_1\otimes f_2\,\big)_{L_2(Q)\otimes L_2(Q)}\big| \\
&\le \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \|(\lambda\otimes\lambda)(\delta(\mu))\|_{B(L_2(Q)\otimes L_2(Q))} \big(\,f_1\otimes f_2 \;\big|\; f_1\otimes f_2\,\big)_{L_2(Q)\otimes L_2(Q)} \\
&\le \sup_{\mu\in L_1(m),\|\mu\|_\lambda=1} \|\mu\|_{B(L_2(Q))}\, \|f_1\|_2^2\, \|f_2\|_2^2 \\
&= \|f_1\|_2^2\, \|f_2\|_2^2 = \\
&= \|a_1\|^\circ \|a_2\|^\circ,
\end{aligned}$$

prooving that $\mathscr{A}(Q)$ is a Banach algebra. □

References

1. P. Eymard, *L'algébre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
2. Varadharajan Muruganandam, *Fourier algebra of a hypergroup.* I, J. Aust. Math. Soc. **82** (2007), no. 1, 59–83.
3. Varadharajan Muruganandam, *Fourier algebra of a hypergroup.* II. *Spherical hypergroups*, Math. Nachr. **281** (2008), no. 11, 1590–1603.
4. N. Bourbaki, *Integration*, vol. 1, Springer-Verlag, Berlin–Heidelberg, 2004.
5. W. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyer, Berlin—New York, 1995.
6. M. Takesaki, *Theory of Operator Algebras.* I, Springer-Verlag, New York, 1979.
7. G. K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London—New York—San Francisco, 1979.
8. W. F. Stinespring, *Positive functions on C^*-algebras*, Proceedings of the American Mathematical Society, vol. 6, no. 2, 1955, pp. 211–216.
9. J. Dixmier, *Les C^*-Algébres et leur représentations*, Gauthier-Villars, Paris, 1969.

Institute of Mathematics, National Academy of Sciences of Ukraine, vul. Tereshchinkivs'ka, 3, Kyiv, 01601, Ukraine
E-mail address: akalyuz@gmail.com

Ukrainian National Technical University ("KPI"), pr. Pobedy, 57, Kyiv, Ukraine
E-mail address: glebpodkolzin@gmail.com

Institute of Mathematics, National Academy of Sciences of Ukraine, vul. Tereshchinkivs'ka, 3, Kyiv, 01601, Ukraine
E-mail address: yc@imath.kiev.ua

Received 26/05/2015