Methods of Functional Analysis and Topology
Vol. 21 (2015), no. 3, pp. 256–265

INVERSE SPECTRAL PROBLEMS FOR JACOBI MATRIX WITH FINITE PERTURBED PARAMETERS

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To 90-th birthday of my Teacher Yurii M. Berezansky

Abstract. For Jacobi matrices with finitely perturbed parameters, we get an explicit representation of the Weyl function, and solve inverse spectral problems, that is, we recover Jacobi matrices from spectral data. For the spectral data, we take the following: the spectral density of the absolutely continuous spectrum, with or without all the eigenvalues; the numerical parameters of the representation of one component of the vector-eigenfunction in terms of Chebyshev polynomials. We prove that these inverse problems have a unique solution, or only a finite number of solutions.

1. Introduction

Presently, the spectral theory of Jacobi matrices is well developed and has various applications both in mathematics (orthogonal polynomials, moment problem, etc. [1, 2, 20, 33]), and in modern mathematical physics [18, 19, 17, 34]. In the development of a spectral theory for Jacobi matrices, an important contribution has been made by Ukrainian mathematicians, — N. I. Ahiezer, Yu. M. Berezansky, M. G. Krein, their pupils and followers [1, 2, 20, 21, 22]. In particular, Yu. M. Berezansky in the monograph [2] has outlined a spectral theory of Jacobi matrices in terms of the general spectral theory of self-adjoint operators, which he has greatly elaborated. He has also started to investigate block Jacobi matrices [2, 8, 9], proposed and implemented the method of inverse spectral problem of integration of various nonlinear evolutionary chains, such as Toda chains [3, 4, 5, 6, 7, 10, 11, 12]. New and most unexpected applications of methods and results of the spectral theory of Jacobi matrices are still being discovered. For example, they have been widely used recently in the spectral theory of infinite graphs [14, 23, 24, 26, 27, 31, 36].

Inverse problems play an important role in the spectral theory of Jacobi matrices. An effective procedure of recovering a Jacobi matrix from its spectral measure is well-known [1, 2]. However, as compared to the research on the inverse spectral problem and the inverse scattering problem for the Schrödinger equation [25, 28, 29, 30, 32, 36], the inverse problems that appear in various statements for Jacobi matrices, which make a difference analog, are much less investigated and start recently to attract more interest. For instance, a development of the soliton theory has inspired the research on the inverse scattering problem for various difference equations [13, 15, 16, 18, 17, 34].

The purpose of this paper is to investigate inverse spectral problems for Jacobi matrices with finitely perturbed parameters, that is, the Jacobi matrices which have only a finite number of elements that differ from the corresponding elements of a canonical Jacobi matrix J_0 (it has a zero main diagonal and units on two secondary diagonals). It is well-known [33] that the spectra of such Jacobi matrices consist of the absolutely continuous

2000 *Mathematics Subject Classification.* Primary 47A10.

Key words and phrases. Jacobi matrix, direct and inverse spectral problems, generalized eigenvector, spectral function, m-function, Weyl solution.

component, which coincides with the interval $[-2,2]$, and not more than a finite number of real eigenvalues lying outside the interval $[-2,2]$. All spectral characteristics of such a Jacobi matrix can be explicitly represented in terms of parameters of this matrix. In particular, below an explicit expression for the Weyl function (Stieltjes transform of the spectral measure) is given. This allows to show that only a finite number of Jacobi matrices from a considered class can have the same density f of the spectral measure of the absolutely continuous spectrum and solve the inverse problem, given the spectral density f and eigenvalues.

2. Spectral analysis of Jacobi matrix

Semi-infinite symmetric tridiagonal matrix of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where $a_k > 0$, b_k are real numbers, is called a Jacobi matrix, and the numbers $(a_k, b_k)_{k=1}^{\infty}$ are called its parameters.

Proposition 1. *For every Jacobi matrix J of the form* (1) *there is a vector-valued function $\varphi(z) = (\varphi_1(z), \dots, \varphi_j(z), \dots)$ such that all its components $\varphi_j(z)$ are polynomials in z of degree $j-1$ and it is an eigenfunction of J, that is, $J\varphi(z) = z\varphi(z)$. The eigenfunction $\varphi(z)$ is uniquely determined by the Jacobi matrix J and the number φ_1. If two consequent polynomials $\varphi_s(z)$ and $\varphi_{s+1}(z)$ are given, then all the parameters $a_1, a_2, \dots, a_s$; $b_1, b_2, \dots, b_s$ are uniquely determined.*

Spectral Theorem ([2])**.** *Every Jacobi matrix J of the form* (1) *gives rise to a self-adjoint operator $\mathbb{J}$ on the Hilbert space $l_2(\mathbb{N})$ (not necessarily unique) such that there exists a measure $d\sigma(\lambda)$ (which is the spectral measure of J) on its spectrum $\sigma(\mathbb{J})$ with respect to which the components $\{\varphi_j(\lambda)\}_{j=1}^{\infty}$ of the vector-valued eigenfunction $\varphi(z)$ make a complete system of orthogonal polynomials in the space $L_2 = L_2(\sigma(\mathbb{J}), d\sigma(\lambda))$; they are defined on $\sigma(\mathbb{J})$ and are square integrable with respect to $d\sigma(\lambda)$. In other words, the set of vector-valued functions $\varphi(\lambda)$, $\lambda \in \sigma(\mathbb{J})$ forms a complete system of eigenfunctions (regular and generalized) of the operator $\mathbb{J}$. we also have the following representations with respect to these eigen functions.*

A direct Fourier transform F defined by

$$\tilde{x}(\lambda) = Fx = \sum_{k=1}^{\infty} x_k \varphi_k(\lambda), \quad \lambda \in \sigma(\mathbb{J}), \quad x = (x_1, \dots, x_k, \dots) \in l_2(\mathbb{N}), \quad \tilde{x} \in L_2,$$

is an isometric operator from the space l_2 onto the space L_2.

An inverse Fourier transform defined by

$$x = F^{-1}\tilde{x} = \int \varphi(\lambda)\tilde{x}(\lambda)\, d\sigma(\lambda),$$

is an isometric operator from the space L_2 onto the space l_2.

The Parseval identity holds true,

$$(x, y)_{l_2} = (\tilde{x}, \tilde{y})_{L_2}, \qquad x, y \in l_2.$$

Remark 1. If the we take the vector-valued eigenfunction $\varphi(z)$ of the Jacobi matrix J such that $\varphi_1 = 1$, then the spectral measure $d\sigma(\lambda)$ is a probability measure ($\int d\sigma(\lambda) = 1$). The Jacobi matrix J is uniquely determined by the measure $d\sigma(\lambda)$ via an effective procedure (see, for instance, [2, p. 525]).

Remark 2. If the parameters (a_k, b_k) of the Jacobi matrix rapidly stabilize for $k \to \infty$, that is, $\sum_{k=1}^{\infty} \left(|a_k - 1| + |b_k|^2\right) < \infty$, then the spectrum of the operator $\mathbb{J}$ consists of absolutely continuous part, which is the interval $[-2, 2]$, and multiplicity one eigenvalues $\{\lambda_k\}$ lying outside the interval $[-2, 2]$ with the only possible limit points $\lambda = \pm 2$ (see [33, Th. 1.10.1]).

Under the assumptions of Remark 2, the product $\prod_{k=1}^{\infty} a_k$ exists, and the eigenfunction $\varphi(z)$ of the Jacobi matrix may be conveniently "normalized" with the condition $\varphi_1 = \prod_{k=1}^{\infty} a_k$. Then the coefficients of the highest degree λ^{j-1} of the polynomials $\varphi_j(z)$ converge to 1 for $j \to \infty$. In this case, we will denote the spectral measure by $d\rho(\lambda)$ and it may be represented in the following form:

$$d\rho(\lambda) = f(\lambda)\chi(\lambda; [-2, 2])\, d\lambda + \left[\sum_k \rho_k \delta(\lambda - \lambda_k)\right] d\lambda. \tag{2}$$

Here $\chi(\lambda; [-2, 2])$ is the characteristic function of the interval $[-2, 2]$, $f(\lambda)$ is a spectral density of the absolutely continuous spectrum, $\rho_k \delta(\lambda - \lambda_k)\, d\lambda$ is an atomic measure with mass ρ_k at the point λ_k that is an eigenvalue of the matrix J, and δ is the Dirac delta-function. Note that the spectral measure $d\sigma(\lambda)$ with the normalization $\varphi_0 = 1$ is connected to $d\rho(\lambda)$ by the relation $d\rho(\lambda) = \left(\prod_{k=1}^{\infty} a_k\right)^{-2} d\sigma(\lambda)$.

The main subject of research in this paper are the Jacobi matrices with finitely perturbed parameters. Such matrices have trivial parameters (a_k, b_k) for $k > r$, that is, $a_k = 1$, $b_k = 0$ for $k > r$. The least integer r satisfying this condition is called the rank of the perturbation. The unperturbed Jacobi matrix is J_0, which has $a_k = 1$ and $b_k = 0$ for all k.

It is well-known that all spectral properties of Jacobi matrices with finitely perturbed parameters are explicitly expressed in terms of the parameters (a_k, b_k) of the Jacobi matrix [33]. For instance, the spectrum of the canonical matrix J_0 is absolutely continuous and $d\sigma(\lambda) = d\rho(\lambda) = f_0(\lambda)\, d\lambda = \frac{1}{2\pi}\sqrt{4 - \lambda^2}\chi(\lambda; [-2, 2])\, d\lambda$.

Components $\varphi_j(z)$ of the vector-valued eigenfunction $\varphi(z)$ of the matrix J_0 have the form $\varphi_j(z) = P_{j-1}(z)$, where the polynomials $P_j(z)$ have degree j and can be expressed in terms of the second kind Chebyshev polynomials $U_j(z) = \frac{\sin((j+1)\arccos z)}{\sin(\arccos z)}$ as $P_j(z) = U_j(\frac{z}{2})$. The polynomials $P_j(z)$ satisfy the recurrence relations

$$P_{j+1}(z) = zP_j(z) - P_{j-1}(z) \tag{3}$$

and the initial conditions $P_{-1} = 0$, $P_0 = 1$, $P_1(z) = z$.

Proposition 2. *The components $\varphi_j(z)$ of the vector-valued eigenfunction $\varphi(\lambda)$ of a Jacobi matrix J with finitely perturbed parameters of rank r can be represented, for $j > 2r$, by*

$$\varphi_j(z) = \sum_{k=0}^{m} \gamma_k P_{j-1-k}(z), \tag{4}$$

where $P_j(z)$ are Chebyshev polynomials from relation (3)*, $m \leq 2r$, and the real numbers $\gamma_0 = 1$, $\gamma_1, \dots, \gamma_m \neq 0$ are uniquely determined by the parameters (a_k, b_k) of J.*

Proof. Since J is a Jacobi matrix with finitely perturbed parameters of rank r, for $j > r$ its parameters are trivial and therefore $\varphi_j(z)$ satisfy the equation $\varphi_{j+1}(z) = z\varphi_j(z) - a_{j-1}\varphi_{j-1}(z)$.

Due to (3), a solution of this equation can be written as

$$\varphi_j(z) = \varphi_{r+1}(z)P_{j-1-r}(z) - a_r\varphi_r(z)P_{j-2-r}(z), \tag{5}$$

and, because multiplication of the polynomials $P_n(z)$ by z is equivalent to a shift of the index n, $zP_n(z) = P_{n+1}(z) + P_{n-1}(z)$, the representation (5) can be written in the form of (4). The parameters γ_k are uniquely determined by the polynomials $\varphi_r(z)$ and $\varphi_{r+1}(z)$, which are uniquely determined by the parameters (a_k, b_k) of the Jacobi matrix according to Proposition 1. $\square$

Definition 1. The parameters $\gamma_0 = 1,\ \gamma_1, \dots, \gamma_m \neq 0$ from Proposition 2 are called spectral parameters of the Jacobi matrix J with finitely perturbed parameters of rank r.

Proposition 3. *For the set of numbers $\gamma_0 = 1,\ \gamma_1, \dots, \gamma_m \neq 0$ to be spectral parameters of a Jacobi matrix J with finitely perturbed parameters (a_k, b_k), it is necessary and sufficient that all zeros of the two polynomials $\varphi_m(z)$, $\varphi_{m+1}(z)$ constructed according to formula* (4)*, be real, simple, and alternating.*

Proof. It is well-known that, in this case, $\varphi_m(z)$ and $\varphi_{m+1}(z)$ are components of an eigenfunction of a certain Jacobi matrix. But according to Proposition 1, this matrix can be uniquely restored from the pair of polynomials $\varphi_m(z)$ and $\varphi_{m+1}(z)$. $\square$

Theorem 1. *The spectral measure $d\rho(\lambda)$ of the form* (2) *for a Jacobi matrix with finitely perturbed parameters of rank r has the density*

$$f(\lambda) = \frac{1}{2\pi}\frac{\sqrt{4-\lambda^2}}{P(\lambda)}, \tag{6}$$

where the polynomial $P(z)$ has degree no more than $2r$ and is expressed in terms of the components $\varphi_j(z)$ of the eigenfunction of the Jacobi matrix as

$$\begin{aligned} P(z) = \det\begin{vmatrix} \varphi_j(z) & \varphi_{j+1}(z) \\ \varphi_{j-1}(z) & \varphi_j(z) \end{vmatrix} &= \varphi_j^2(z) - \varphi_{j-1}(z)\varphi_{j+1}(z) \\ &= \varphi_j^2(z) + \varphi_{j-1}^2(z) - z\varphi_{j-1}(z)\varphi_j(z) \end{aligned} \tag{7}$$

for $j > r+1$. The eigenvalues $\lambda_k = \mu_k + \mu_k^{-1}$ are zeros of the polynomial $P(z)$ if the absolute values of the numbers

$$\mu_k = \frac{\varphi_j(\lambda_k)}{\varphi_{j-1}(\lambda_k)} \tag{8}$$

are less than 1. The numbers ρ_k are determined by the derivative $P'(z)$ as

$$\rho_k = \frac{\left|\sqrt{\lambda_k^2 - 4}\right|}{|P'(\lambda_k)|}. \tag{9}$$

Proof. Since, for $j > r+1$, the components $\varphi_j(z)$ satisfy the difference equation

$$\varphi_{j-1}(z) - z\varphi_j(z) + \varphi_{j+1}(z) = 0, \tag{10}$$

and $P(\lambda) = \varphi_j^2(z) - \varphi_{j-1}(z)\varphi_{j+1}(z)$ is a Wronskian of solutions of this equation, the polynomial $P(z)$ does not depend on $j > r+1$. If λ_k is an eigenvalue, then $\varphi(\lambda_k) \in l_2(\mathbb{N})$. This is possible only if $|\lambda_k| > 2$ and components of the vector $\varphi(\lambda_k)$, for $j > r$, have the form $\varphi_j(\lambda_k) = c\mu_k^j$, where $\mu_k + \mu_k^{-1} = \lambda_k$. Therefore, $\frac{\varphi_j(\lambda_k)}{\varphi_{j-1}(\lambda_k)} = \frac{\varphi_{j+1}(\lambda_k)}{\varphi_j(\lambda_k)} = \mu_k$, which is equivalent to λ_k being a zero of the polynomial $P(z)$. In addition, the condition $\varphi(\lambda_k) \in l_2$ is equivalent to the condition $|\mu_k| < 1$.

Suppose we have a vector $x = (x_1, \dots, x_j, \dots) \in l_2(\mathbb{N})$. Then its Fourier transform is $\tilde{x}(\lambda) \in L_2(d\rho(\lambda))$. Because the system of Chebyshev polynomials $\{P_j(\lambda)\}_{j=0}^{\infty}$ is orthonormal in the space $L_2\left([-2,2], f_0(\lambda)d\lambda\right)$, and due to the representation (4) for $\varphi_j(\lambda)$, we have, for $j \geq 2r$, that

$$\int_{-2}^{2} \tilde{x}(\lambda)P_j(\lambda)f_0(\lambda)\,d\lambda = \sum_{\nu=0}^{m} \gamma_\nu x_{j+1+\nu}. \tag{11}$$

If the vector x is orthogonal to all eigenvectors of the Jacobi matrix J, then the inverse Fourier transform has the form

$$x_k = \int_{-\infty}^{\infty} \tilde{x}(\lambda)\varphi_k(\lambda)f(\lambda)\,d\lambda. \tag{12}$$

If we substitute (12) into (11) we have, due to (4) and (14), that

$$\int_{-2}^{2} \tilde{x}(\lambda)P_j(\lambda)f_0(\lambda)\,d\lambda = \int_{-2}^{2} \tilde{x}(\lambda)P_j(\lambda)P(\lambda)f(\lambda)\,d\lambda, \tag{13}$$

where $P(\lambda)$ is the spectral polynomial. From (13), considering that $\tilde{x}(\lambda)$ runs over the space $L_2\left([-2,2], f(\lambda)\,d\lambda\right)$ and $j \geq 2m$ is an arbitrary number, and since $f_0(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$, we conclude that (6) holds. Property (9) will be proved below in the proof of Theorem 3. □

Remark 3. The condition $|\mu_k| < 1$, where $\mu_k = \frac{\varphi_j(\lambda_k)}{\varphi_{j-1}(\lambda_k)}$, which singles out an eigenvalue $\lambda_k = \mu_k + \mu_k^{-1}$ from all real zeros of the spectral polynomial $P(z)$, can be replaced by the condition

$$\operatorname{sign}\left[\frac{1}{2}z - \frac{\varphi_j(\lambda_k)}{\varphi_{j-1}(\lambda_k)}\right]\Bigg|_{z=\lambda_k} = \operatorname{sign}\lambda_k.$$

Definition 2. A polynomial $Q(z)$ of degree less than the degree of the spectral polynomial $P(z)$ is called an eigenvalue indicator of the Jacobi matrix J, if on all real zeros $\tilde{\lambda}_k = \tilde{\mu}_k + \tilde{\mu}_k^{-1}$, $|\tilde{\lambda}_k| > 2$ of the spectral polynomial the equality $|Q(\tilde{\lambda}_k)| = \frac{1}{2}|\tilde{\mu}_k - \tilde{\mu}_k^{-1}|$ holds, and $\tilde{\lambda}_k$ is an eigenvalue of J if and only if $\operatorname{sign} Q(\tilde{\lambda}_k) = \operatorname{sign}\tilde{\lambda}_k$.

The existence of indicator polynomials will be proved in Theorem 3.

Definition 3. The polynomial $P(z)$ in Theorem 1 is called a spectral polynomial of the Jacobi matrix J with finitely perturbed parameters of rank r. It defines the spectral density of the absolutely continuous spectrum of the matrix J, and all eigenvalues $\{\lambda_k\}$ of J are zeros of $P(z)$. The spectral measure $d\rho(\lambda)$ is defined by the formulas (2), (6), (9).

We establish the following important connection between the spectral polynomial $P(z)$ and the spectral parameters.

Theorem 2. *Let $P(z)$ be a spectral polynomial, and let $\gamma_0 = 1, \gamma_1, \ldots, \gamma_m \neq 0$, be the spectral parameters of a Jacobi matrix with finitely perturbed parameters. Then for every complex number z,*

$$P(z + z^{-1}) = \gamma(z)\gamma(z^{-1}), \tag{14}$$

where $\gamma(z) = \sum_{k=0}^{m} \gamma_k z^k$.

Proof. Due to formulas (5), we have for $j > r$ that

$$\varphi_j(z) = \varphi_{r+1}(z)P_{j-1-r}(z) - a_r\varphi_r(z)P_{j-2-r}(z). \tag{15}$$

Let us consider a linear operator S on the Chebyshev polynomials $P_j(z)$, which acts by the rule $SP_j(z) = P_{j+1}(z)$. Then $S^{-1}P_j(z) = P_{j-1}(z)$, and $zP_j(z) = (S + S^{-1})P_j(z)$.

Equality (15) can be written in the form

$$\varphi_j(z) = \left[\varphi_{r+1}(S + S^{-1}) - \varphi_r(S + S^{-1})S^{-1}\right] S^{-1-r} P_j(z). \tag{16}$$

Comparing (16) and (4) we obtain that

$$\gamma(S^{-1}) = \left[\varphi_{r+1}(S + S^{-1}) - \varphi_r(S + S^{-1})S^{-1}\right] S^{-1-r}. \tag{17}$$

Then $\gamma(S^{-1})\gamma(S) = P(S + S^{-1})$ due to (7). □

Example 1. Consider all spectral characteristics of a Jacobi matrix J_b^a with finitely perturbed parameters (a, b) of rank 1. Then

a) the vector-eigenfunction $\varphi(z)$ of the matrix J_b^a is
$$\varphi(z) = (a, \lambda - b, P_2(\lambda) - bP_1(\lambda) - (a^2 - 1)P_0(\lambda), \dots);$$
b) the spectral parameters are
$$\gamma_0 = 1, \quad \gamma_1 = -b, \quad \gamma_2 = 1 - a^2;$$
c) the spectral polynomial is given by
$$P(z) = a^4 + b^2 + b(a^2 - 2)z - (a^2 - 1)z^2;$$
d) the eigenvalue indicator is
$$Q(z) = \frac{1}{2}z + \frac{b - z}{a^2}.$$

The eigenfunctions $\varphi(\lambda)$ can be considered as a solution of the scattering problem for the semi-infinite chain given by the difference equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$ with the boundary condition $\varphi_0(\lambda) = 0$. Here for $k \to \infty$ taking $\lambda = 2\cos\theta$ we have

$$\varphi_k(\lambda) = e^{ik\theta}A - e^{-ik\theta}B, \tag{18}$$

where A is the amplitude of the incident wave and B is the amplitude of the scattered wave. The ratio of these amplitudes defines the scattering operator $S(\lambda)$: $B = SA$. The explicit representation (4) of the eigenfunctions $\varphi(\lambda)$ yields

$$S = \frac{\gamma(e^{i\theta})}{\gamma(e^{-i\theta})}, \tag{*}$$

where the polynomial $\gamma(z) = \sum_{k=0}^{m} \gamma_k z^k$ is explicitly expressed in terms of the spectral parameters $\gamma_1, \dots, \gamma_m$ of the matrix J.

Note that, for difference equations on the axis, there are several papers on direct and inverse scattering problem, which are applying the method of the inverse scattering problem of integration of the nonlinear evolution Toda chains (see [13, 15, 16, 17] and references therein). Here the factorization properties of the scattering matrix play an important role in an effective solution of the inverse problem.

3. Weyl function

Suppose $d\sigma(\lambda)$ is a spectral measure of a Jacobi matrix J, then its Stieltjes transform,

$$m(z) = \int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{\lambda - z}, \tag{19}$$

is called the Weyl function of the Jacobi matrix J. There is a well-known Stieltjes inversion formula for the recovery of the measure $d\sigma(\lambda)$ from the function $m(z)$.

In the case of the canonical Jacobi matrix J_0, the spectral measure has the form $d\sigma(\lambda) = d\rho(\lambda) = \frac{1}{2\pi}\sqrt{\lambda^2 - 4}\chi(\lambda; [-2, 2])\, d\lambda$, and the Weyl function $m_0(z)$ can be written as

$$m_0(z) = \frac{1}{2}\left(\sqrt{z^2 - 4} - z\right), \tag{20}$$

where for the square root branch of the analytic function is selected in such a way that $m_0(z) \to 0$ for $|z| \to \infty$. For instance, if z is real, $|z| > 2$, then $\operatorname{sign}\sqrt{z^2 - 4} = \operatorname{sign} z$, and for $z \to \infty$ we have $\sqrt{z^2 - 4} = z + O(\frac{1}{|z|})$. In what follows, the branch of the function $\sqrt{z^2 - 4}$ is always chosen in this way. Therefore,

$$\operatorname{Im}\left(\sqrt{z^2 - 4}\Big|_{z=\lambda+i0} - \sqrt{z^2 - 4}\Big|_{z=\lambda-i0} \right) = \begin{cases} 2|\sqrt{4 - \lambda^2}|, & |\lambda| < 2, \\ 0, & |\lambda| \geq 0. \end{cases} \tag{21}$$

Further on we assume that the measure $d\sigma(\lambda)$ in (19) has the form (2). In this case, $\{\lambda_k\}$ are poles of the function $m(z)$, and the numbers ρ_k are determined as residues of $m(z)$ in the points λ_k,

$$\rho_k = \lim_{z\to\lambda_k} \left[m(z)(\lambda_k - z)\right]. \tag{22}$$

The density $\rho(\lambda)$ on interval $(-2,2)$ is determined by the jump of $\operatorname{Im} m(z)$ as z crosses the value of $\lambda \in (-2,2)$,

$$\rho(\lambda) = \frac{1}{2\pi} \operatorname{Im}\left[m(\lambda + i0) - m(\lambda - i0)\right]. \tag{23}$$

Theorem 3. *Let J be a Jacobi matrix with finitely perturbed parameters. Then its Weyl function can be represented as*

$$m(z) = \frac{Q(z) + \frac{1}{2}\sqrt{z^2-4}}{P(z)} = \frac{\left(\prod_{k=1}^{\infty} a_k\right)^{-4} \widehat{P}(z)}{Q(z) - \frac{1}{2}\sqrt{z^2-4}}, \tag{24}$$

where $P(z)$ is a spectral polynomial of the matrix J, $\widehat{P}(z)$ is a spectral polynomial of once-stripped matrix J, that is, the Jacobi matrix $\widehat{J}$ which is obtained from J if one removes the first row and the first column, the polynomial $Q(z)$ is an indicator of the eigenvalues λ_k of the matrix J, which means that the real zero $\hat{\lambda}$ of the spectral polynomial $P(z)$ is an eigenvalue if and only if $\operatorname{sign} Q(\hat{\lambda}) = \operatorname{sign} \hat{\lambda}$.

Before we prove Theorem 3, let us consider the following important corollaries of this Theorem.

Corollary 1. *Relations* (21), (23), (24) *imply that the spectral density satisfies* (6).

Corollary 2. *If $\lambda = \mu + \mu^{-1} > 2$ is a zero of the spectral polynomial $P(\lambda)$, then*

$$|Q(\lambda)| = \frac{1}{2}|\sqrt{\lambda^2-4}| = \frac{1}{2}|\mu - \mu^{-1}|. \tag{25}$$

Corollary 3. *Due to* (22), (25) *we have* (9) *for ρ_k from* (2),

$$\rho_k = \frac{\operatorname{sign}\lambda_k |\sqrt{\lambda_k^2-4}|}{-P'(\lambda_k)}. \tag{26}$$

Therefore the spectral density (2) is uniquely determined by the spectral polynomial $P(z)$ and the set of all eigenvalues.

Corollary 4. *Since all $\rho_k > 0$,* (26) *implies that two consecutive zeros of the spectral polynomial can be eigenvalues if and only if they have opposite signs. Therefore, we conclude that there are no more than $r+1$ eigenvalues, where r is the finite perturbation rank of the parameters of the Jacobi matrix.*

Proof of Theorem 3. A Jacobi matrix with finitely perturbed parameters can be obtained from the canonical Jacobi matrix J_0 by consecutively, step by step, adding the parameters (a_k, b_k), starting with (a_r, b_r), then (a_{r-1}, b_{r-1}), and so on ending with (a_1, b_1). If one adds (a, b) to the parameters of the Jacobi matrix $\widehat{J}$ from the beginning, then its Weyl function $\widehat{m}(z)$ is connected with the Weyl function $m(z)$ of the resulting matrix J by the following fractional linear transformation (see [33, Th. 3.2.4]):

$$m(z) = \left[(b-z)A^{-2} - \widehat{m}(z)\right]^{-1} A^{-4}, \tag{27}$$

where $A = \prod a_k$ is the product of all the parameters a_k of the matrix J. Note that in (19) we use the normalization $\varphi_1 = \prod a_k$ for the Jacobi matrices we consider, that is, we choose $d\rho(\lambda)$ of the form (2) as a spectral measure. Since for the matrix J_0 the

Weyl function $m_0(z) = -\frac{1}{2}z + \frac{1}{2}\sqrt{z^2-4}$ is well-known, by induction one can show that equality (24) holds for certain polynomials Q, P, $\widehat{P}$ satisfying the equalities

$$\begin{aligned} &Q(z) = -\widehat{Q}(z) + (b-z)A^{-2}\widehat{P}(z), \\ &P(z) = (b-z)^2\widehat{P}(z) - 2(b-z)A^2\widehat{Q}(z) + a^4\widehat{\widehat{P}}(z), \\ &Q^2(z) + 1 - \frac{1}{4}z^2 = A^{-4}P(z)\widehat{P}(z), \end{aligned} \tag{28}$$

Note that $P(z)$ is a spectral polynomial of J due to Theorem 1 and Corollary 1. In (28) $\widehat{P}$ and $\widehat{\widehat{P}}$ are spectral polynomials for the one–stripped and two–stripped matrix J.

Representation (24) implies that for real zeros λ of the polynomial $P(z)$ the equality $|Q(z)| = \frac{1}{2}|\sqrt{\lambda^2-4}|$ holds, and that λ is an eigenvalue of the Jacobi matrix J if and only if $\operatorname{sign} Q(\lambda) = \operatorname{sign} \lambda$. Thus, the polynomial $Q(z)$ is an eigenvalue indicator according to Definition 2 and the representations (26) and (9) hold. □

Problem. Find conditions for the Jacobi matrix J from Remark 2 such that its Weyl function satisfies the equality (24), where $P(z)$, $\widehat{P}(z)$, $Q(z)$ are analytic entire functions.

4. Inverse spectral problems

Inverse Spectral Problem 1. *Recover a Jacobi matrix with finitely perturbed parameters from its spectral parameters $\gamma_0 = 1, \gamma_1, \dots, \gamma_m \neq 0$.*

Proposition 4. *Proposition 1 and formula* (4) *imply that Inverse Spectral Problem 1 has a unique solution and also provide an algorithm of solving this problem.*

Inverse Spectral Problem 2. *Recover a Jacobi matrix J with finitely perturbed parameters from a given polynomial $a(z)$ if it is a component of the eigenfunction $\varphi(z)$.*

Proposition 5. *Inverse Spectral Problem 2 has a unique solution if the degree n of the given polynomial $a(z)$ is more that twice the rank r of the finite perturbation.*

Proof. If we write the polynomial $a(z)$ as $\sum_{k=0}^{m} \gamma_k P_{n-k}(z)$, we obtain the spectral parameters $\{\gamma_k\}_{k=0}^{m}$. Therefore Inverse Spectral Problem 2 is reduced to Inverse Spectral Problem 1. □

Inverse Spectral Problem 3. *Recover a Jacobi matrix J with finitely perturbed parameters from its spectral polynomial $P(z)$.*

Theorem 4. *Inverse Spectral Problem 3 can have only a finite number of solutions.*

Proof. The factorization (14) of a given polynomial $P(z)$ can have only a finite number of polynomial solutions $\gamma(z)$. This can be easily shown by expanding $P(z)$ and $\gamma(z)$ into factors which are linear with respect to z. Then every left-hand factor which is linear with respect to $z+z^{-1}$ coincides with a certain right-hand factor of (14) that is also linear with respect to $z+z^{-1}$. However, there can only be a finite number of such matches. Now the assertion of the theorem follows from Proposition 4. □

Inverse Spectral Problem 4. *Recover a Jacobi matrix J with finitely perturbed parameters by the density $f(\lambda)$ of its absolutely continuous spectrum and the set of all eigenvalues $\{\lambda_k\}$.*

Theorem 5. *The Inverse Spectral Problem 4 has a unique solution.*

Proof. Due to (6), $f(\lambda)$ uniquely determines the spectral polynomial $P(\lambda)$. The spectral measure $d\rho(\lambda)$ of the form (2) is uniquely defined by $f(\lambda)$, the set of all eigenvalues $\{\lambda_k\}$, and the set $\{\rho_k\}$ of densities of atomic measures from (26). The probability spectral measure $d\sigma(\lambda) = A^2 d\rho(\lambda)$ is uniquely defined by $d\rho(\lambda)$. But due to Remark 1, the measure $d\sigma(\lambda)$ uniquely defines the Jacobi matrix J. □

Remark 4. Theorem 5 implies Theorem 4 because the spectral polynomial $P(\lambda)$ has a finite number of zeros and, therefore, there is a finite number of different sets of these zeros that can be eigenvalues of J.

Remark 5. In Example 1 we give the spectral characteristics of the Jacobi matrix J_b^a with finitely perturbed parameters (a, b) of rank 1. From the explicit form of the spectral polynomial $P(\lambda)$ we conclude that, if $a \neq \sqrt{2}$, then the parameters a and b are uniquely determined by $P(\lambda)$ and, therefore, by the density $f(\lambda)$ of the absolutely continuous spectrum. If $a = \sqrt{2}$, then $P(\lambda) = 1 + b^2 - \lambda^2$ and there are only two Jacobi matrices $J_b^{\sqrt{2}}$ and $J_{-b}^{\sqrt{2}}$ with the same density of the absolutely continuous spectrum.

Hypothesis. *The general Jacobi matrix J with finitely perturbed parameters is uniquely determined by the density of its absolutely continuous spectrum. In other words, for a matrix J with parameters $(a_k, b_k)_{k=1}^r$ for every $\varepsilon > 0$ there is a Jacobi matrix $J(\varepsilon)$ with finitely perturbed parameters $(a_k^\varepsilon, b_k^\varepsilon)_{k=1}^r$ such that $\sum_{k=1}^r (|a_k - a_k^\varepsilon| + |b_k - b_k^\varepsilon|) < \varepsilon$ and $J(\varepsilon)$ is uniquely determined by the density of its absolutely continuous spectrum.*

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Received 28/04/2015