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SPECTRAL AND PSEUDOSPECTRAL FUNCTIONS OF HAMILTONIAN SYSTEMS: DEVELOPMENT OF THE RESULTS BY AROV-DYM AND SAKHNOVICH

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Dedicated with respect to Yuri M. Berezansky on the occasion of his anniversary

Abstract. The main object of the paper is a Hamiltonian system $Jy' - B(t)y = \lambda\Delta(t)y$ defined on an interval $[a, b)$ with the regular endpoint a. We define a pseudospectral function of a singular system as a matrix-valued distribution function such that the generalized Fourier transform is a partial isometry with the minimally possible kernel. Moreover, we parameterize all spectral and pseudospectral functions of a given system by means of a Nevanlinna boundary parameter. The obtained results develop the results by Arov-Dym and Sakhnovich in this direction.

1. Introduction

Let H be a finite dimensional Hilbert space and let $[H]$ be the set of linear operators in H. We study the Hamiltonian differential system [3, 16]

$$Jy' - B(t)y = \lambda\Delta(t)y, \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

where $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ are $[H \oplus H]$-valued functions defined on an interval $\mathcal{I} = [a, b),\ b \leq \infty$, and integrable on each compact subinterval $\mathcal{I}_\beta = [a, \beta] \subset \mathcal{I}$ and

$$J = \begin{pmatrix} 0 & -I_H \\ I_H & 0 \end{pmatrix} : H \oplus H \to H \oplus H. \tag{1.2}$$

System (1.1) is called canonical if $B(t) = 0,\ t \in \mathcal{I}$.

As is known a spectral function is a basic concept in the theory of eigenfunction expansions of differential operators (see e.g. [5] and references therein). In the case of a Hamiltonian system definition of a spectral function requires a certain modification. Namely, let $\mathfrak{H} = L^2_\Delta(\mathcal{I})$ be the Hilbert space of functions $f : \mathcal{I} \to H \oplus H$ satisfying

$$\int_{\mathcal{I}} (\Delta(t)f(t), f(t))\, dt < \infty$$

and let $\varphi(\cdot, \lambda)$ be the $[H, H \oplus H]$-valued solution of the system (1.1) such that $\varphi(0, \lambda) = (0, I_H)^\top$. Assume that system is regular, i.e., $b < \infty$ and the functions $B(\cdot)$ and $\Delta(\cdot)$ are integrable on $\mathcal{I}$. Then the generalized Fourier transform of a function $f \in \mathfrak{H}$ is

$$\widehat{f}(s) = \int_{\mathcal{I}} \varphi^*(t, s)\Delta(t)f(t)\, dt, \quad s \in \mathbb{R} \tag{1.3}$$

and according to [2, 32, 33] an $[H]$-valued distribution function $\sigma(\cdot)$ is a pseudospectral function of the system (1.1) if the equality $Vf = \widehat{f},\ f \in \mathfrak{H}$, defines a partial isometry

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$V \in [\mathfrak{H}, L^2(\sigma; H)]$ with $\ker V = L_0$, where

$$L_0 = \{f \in \mathfrak{H} : \widehat{f}(s) = 0,\ s \in \mathbb{R}\}. \tag{1.4}$$

Moreover, $\sigma(\cdot)$ is a spectral function if V is an isometry.

The following assertion concerning the subspace L_0 is obvious.

Assertion 1.1. *If $\sigma(\cdot)$ is an $[H]$-valued distribution function such that the generalized Fourier transform V is a partial isometry from $\mathfrak{H}$ to $L^2(\sigma; H)$, then $L_0 \subset \ker V$. Hence $\sigma(\cdot)$ is a pseudospectral function, if V is a partial isometry with the minimally possible kernel $\ker V = L_0$.*

Moreover, in view of [33, Lemma A.18] the following assertion holds.

Assertion 1.2. *The subspace L_0 is the set of all functions $f \in \mathfrak{H}$ such that the solution y of the inhomogeneous system*

$$Jy' - B(t)y = \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{1.5}$$

with $y(b) = 0$ satisfies $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) and $(I_H, 0)y(a) = 0$.

Let $Y(\cdot, \lambda)$ be the $[H \oplus H]$-valued solution of (1.1) with $Y(a, \lambda) = J$, let $W(\lambda) := Y(b, \lambda)$ be the monodromy matrix and let

$$W(\lambda) = \begin{pmatrix} w_1(\lambda) & w_2(\lambda) \\ w_3(\lambda) & w_4(\lambda) \end{pmatrix} : H \oplus H \to H \oplus H, \quad \lambda \in \mathbb{C} \tag{1.6}$$

be the block-matrix representation of $W(\lambda)$. A description of all pseudospectral functions of the regular system is given by the following theorem obtained by D. Arov and H. Dym [2] and A. Sakhnovich [32, 33].

Theorem 1.3. *Let system* (1.1) *be regular and canonical and let $\bigcap_{\lambda \in \mathbb{C}} \ker w_1(\lambda) = \{0\}$. Then the equalities*

$$m_\tau(\lambda) = (C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.7}$$

$$\sigma_\tau(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_\tau(u + i\varepsilon)\, du \tag{1.8}$$

establish a bijective correspondence between all Nevanlinna pairs $\tau = \{C_0(\lambda), C_1(\lambda)\}$, $C_j(\lambda) \in [H]$, $j \in \{0, 1\}$, (see Definition 2.9) satisfying a certain admissibility condition (see [33, (A.134)]*) and all pseudospectral functions $\sigma(\cdot) = \sigma_\tau(\cdot)$. Moreover, in the case $L_0 = \{0\}$ (and only in this case) the set of spectral functions is not empty and the above statement holds for spectral functions.*

It was also shown in [2] that under certain additional conditions on $W(\lambda)$ statements of Theorem 1.3 hold with arbitrary (not necessarily admissible) Nevanlinna pairs τ.

Note that $m_\tau(\cdot)$ in (1.7) is an $[H]$-valued Nevanlinna function and (1.8) is the Perron-Stieltjes formula for $m_\tau(\cdot)$. Observe also that the condition $\bigcap_{\lambda \in \mathbb{C}} \ker w_1(\lambda) = \{0\}$ in Theorem 1.3 is equivalent to $\ker w_1(\lambda) = \{0\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (see Proposition 6.21).

Assume now that system (1.1) is singular. Let $\mathfrak{H}_b$ be the set of all functions $f \in \mathfrak{H}$ that vanish in a neighborhood of b. Then the Fourier transform (1.3) is defined for $f \in \mathfrak{H}_b$ and an $[H]$-valued distribution function $\sigma(\cdot)$ is called a spectral function of the system if the operator $Vf = \widehat{f}$, $f \in \mathfrak{H}_b$, extends to an isometry $V \in [\mathfrak{H}, L^2(\sigma; H)]$ or, equivalently, if $\sigma(\cdot)$ is a spectral function for restriction of the system onto each subinterval $\mathcal{I}_\beta \subset \mathcal{I}$. Moreover, a pseudospectral function is defined in [2] by analogy with a spectral one. Namely, according to [2] a distribution function $\sigma_0(\cdot)$ is a pseudospectral function of the singular system (1.1) if $\sigma_0(\cdot)$ is a pseudospectral function of its restriction onto each compact subinterval $\mathcal{I}_\beta \subset \mathcal{I}$. Observe also that certain sufficient conditions for existence of a function $\sigma_0(\cdot)$ are given in [2].

In the present paper we offer another definition of a pseudospectral function for a singular system. Namely let $\mathcal{D}$ be the lineal of absolutely continuous functions $y \in \mathfrak{H}$ satisfying (1.5) with some $f \in \mathfrak{H}$. Moreover, let L_0 be the set of all functions $f \in \mathfrak{H}$ such that there exists $y \in \mathcal{D}$ satisfying (1.5) and the equalities

$$\Delta(t)y(t) = 0 \quad \text{(a.e. on } \mathcal{I}), \quad (I_H, 0)y(a) = 0, \quad \lim_{t \uparrow b}(Jy(t), z(t)) = 0, \quad z \in \mathcal{D}. \tag{1.9}$$

Note that in the case of a regular system the last condition in (1.9) is equivalent to $y(b) = 0$ and by Assertion 1.2 L_0 admits the representation (1.4).

We prove the following statements: (1) L_0 is a closed subspace in $\mathfrak{H}$; (2) if $\sigma(\cdot)$ is an $[H]$-valued distribution function such that the generalized Fourier transform $Vf = \widehat{f},\ f \in \mathfrak{H}_b$, extends to a partial isometry $V \in [\mathfrak{H}, L^2(\sigma; H)]$, then $L_0 \subset \ker V$. These facts together with Assertions 1.1 and 1.2 make natural the following definition.

Definition 1.4. An $[H]$-valued distribution function $\sigma(\cdot)$ is a pseudospectral function of the system (1.1) if the generalized Fourier transform extends to a partial isometry $V \in [\mathfrak{H}, L^2(\sigma; H)]$ with the minimally possible kernel $\ker V = L_0$.

It is easily seen that a rather restrictive necessary (but not sufficient) condition for existence of a pseudospectral function $\sigma_0(\cdot)$ in the sense of [2] is

$$\mathfrak{H}_{\beta_1} = (\mathfrak{H}_{0,\beta_2} \cap \mathfrak{H}_{\beta_1}) \oplus (L_{0,\beta_2} \cap \mathfrak{H}_{\beta_1}), \quad \beta_1 < \beta_2, \tag{1.10}$$

where $\mathfrak{H}_\beta = L^2_\Delta(\mathcal{I}_\beta)$, $L_{0,\beta} \subset \mathfrak{H}_\beta$ is the subspace (1.4) for the restriction of the system onto $\mathcal{I}_\beta$ and $\mathfrak{H}_{0,\beta} = \mathfrak{H}_\beta \ominus L_{0,\beta},\ \beta \in \mathcal{I}$. If (1.10) does not hold, then a pseudospectral function $\sigma_0(\cdot)$ does not exist even for a regular system. At the same time a pseudospectral function $\sigma(\cdot)$ in the sense of Definition 1.4 exists for any system (1.1). Moreover, in the case of a regular system Definition 1.4 turns into the definition of a pseudospectral function in [2, 33]. Observe also that each pseudospectral function $\sigma_0(\cdot)$ in the sense of [2] is a pseudospectral function $\sigma(\cdot)$ in the sense of Definition 1.4 (see Proposition 6.9). Therefore a pseudospectral function $\sigma(\cdot)$ seems to be more general and convenient object than $\sigma_0(\cdot)$.

Denote by $\mathcal{N}_\lambda$ the linear space of solutions of the system (1.1) belonging to $\mathfrak{H}$ and let $N_\pm = \dim \mathcal{N}_\lambda,\ \lambda \in \mathbb{C}_\pm$, be the formal deficiency indices of the system (1.1) [22]. The main result of the paper is a parametrization of all pseudospectral and spectral functions of a (regular or singular) Hamiltonian system, which is given by the following theorem.

Theorem 1.5. *Let $N_+ = N_-$ and assume that there exists only the trivial solution $y = 0$ of the system* (1.1) *satisfying $(I_H, 0)y(a) = 0$ and $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$). Then*

(1) *There exist an auxiliary finite-dimensional Hilbert space $\mathcal{H}_b$ and a Nevanlinna operator function*

$$M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H \oplus \mathcal{H}_b \to H \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{1.11}$$

such that the equality

$$m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{1.12}$$

together with (1.8) *establishes a bijective correspondence between all Nevanlinna pairs $\tau = \{C_0(\lambda), C_1(\lambda)\},\ C_j(\lambda) \in [\mathcal{H}_b],\ j \in \{0, 1\}$, satisfying the admissibility conditions*

$$\lim_{y \to \infty} \tfrac{1}{iy}(C_0(iy) - C_1(iy)M_4(iy))^{-1}C_1(iy) = 0, \tag{1.13}$$

$$\lim_{y \to \infty} \tfrac{1}{iy}M_4(iy)(C_0(iy) - C_1(iy)M_4(iy))^{-1}C_0(iy) = 0, \tag{1.14}$$

and all pseudospectral functions $\sigma(\cdot) = \sigma_\tau(\cdot)$ of the system. Moreover, the above statement holds for arbitrary (not necessarily admissible) Nevanlinna pairs τ if and only if

$$\lim_{y \to \infty} \tfrac{1}{iy}M_4(iy) = 0 \quad \text{and} \quad \lim_{y \to \infty} y \cdot \mathrm{Im}(M_4(iy)h, h) = +\infty, \quad h \in \mathcal{H}_b, \quad h \neq 0. \tag{1.15}$$

(2) *In the case* $L_0 = \{0\}$ *(and only in this case) the set of spectral functions is not empty and statement* (1) *holds for spectral functions.*

Recall that system (1.1) is called quasi-regular, if $N_+ = N_- = 2\dim H$. It turns out that for a quasi-regular (in particular regular) system there exists the operator function $W(\lambda)$ of the form (1.6) such that the equality (1.12) admits the representation (1.7); moreover, the admissibility conditions (1.13), (1.14) and the criterion (1.15) can be reformulated in terms of $W(\lambda)$ as well. These results cover Theorem 1.3 and some other results obtained for regular canonical systems in [2, 32, 33] (for more details see Theorems 6.16, 6.17 and Remark 6.24 below).

It is worth noting that matrices $M(\lambda)$ and $W(\lambda)$ are defined in terms of the boundary values of respective operator solutions of (1.1) at the endpoints a and b (for more details see Proposition 4.9, equality (6.9) and Remark 4.2).

The above results are obtained in the framework of the extension theory of symmetric linear relations. To this end one associates with system (1.1) the minimal (symmetric) linear relation $T_{\min}$ and the maximal relation $T_{\max}(= T_{\min}^*)$ in $\mathfrak{H}$ [31, 19, 25, 27]. Then $L_0 = \operatorname{mul} T$, where $\operatorname{mul} T$ is the multivalued part of a certain symmetric extension T of $T_{\min}$, and pseudospectral functions are characterized in terms of orthogonal spectral functions of self-adjoint extensions $\widetilde{T}$ of T satisfying $\operatorname{mul}\widetilde{T} = \operatorname{mul} T$. With application of this method pseudospectral and spectral functions of Hamiltonian systems were studied in [13, 14, 19, 23, 24]. Our approach is based on concepts of a boundary triplet (boundary pair) and the corresponding Weyl function (see [17, 10, 26, 7, 9] and references therein). In the framework of this approach the matrix $M(\lambda)$ in (1.11) is the Weyl function of an appropriate boundary pair for $T_{\max}$ and the conditions (1.13), (1.14) are implied by results on Π-admissibility from [7]. Observe also that general (not necessarily Hamiltonian) symmetric systems were studied by means of boundary triplets in recent papers [1, 28, 29, 30].

2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on the Hilbert space $\mathcal{H}_1$ with values in the Hilbert space $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $P_{\mathcal{L}}$ is the orthoprojection in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plane of the complex plane.

Recall that a linear relation $T : \mathcal{H}_0 \to \mathcal{H}_1$ from a Hilbert space $\mathcal{H}_0$ to a Hilbert space $\mathcal{H}_1$ is a linear manifold in the Hilbert $\mathcal{H}_0 \oplus \mathcal{H}_1$. If $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ one speaks of a linear relation T in $\mathcal{H}$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ and $\operatorname{mul} T$ the domain, range, kernel and the multivalued part of T respectively. Recall that $\operatorname{mul} T$ is a subspace in $\mathcal{H}_1$ defined by

$$\operatorname{mul} T := \{h_1 \in \mathcal{H}_1 : \{0, h_1\} \in T\}. \tag{2.1}$$

Clearly, $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is an operator if and only if $\operatorname{mul} T = \{0\}$.

For $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we will denote by $T^{-1}(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ and $T^*(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ the inverse and adjoint linear relations of T respectively. Moreover, for a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T)$ the resolvent set of T, i.e., the set of all $\lambda \in \mathbb{C}$ such that $(T - \lambda)^{-1} \in [\mathcal{H}]$.

Recall that an operator function $\Phi(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H}]$ is called a Nevanlinna function if it is holomorphic and satisfies $\operatorname{Im}\lambda \cdot \operatorname{Im}\Phi(\lambda) \geq 0$ and $\Phi^*(\lambda) = \Phi(\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. A

Nevanlinna function $\Phi(\cdot)$ is called uniformly strict if $0 \in \rho(\mathrm{Im}\Phi(\lambda))$. We denote by $R[H]$ and $R_u[H]$ the set of $[\mathcal{H}]$-valued Nevanlinna and uniformly strict Nevanlinna functions respectively.

2.2. **Symmetric relations and generalized resolvents.** As is known a linear relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $A \subset A^*$ (resp. $A = A^*$). For each symmetric relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ the following decompositions hold:

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathrm{mul}\, A, \quad A = \mathrm{gr}\, A_0 \oplus \widehat{\mathrm{mul}}\, A,$$

where $\widehat{\mathrm{mul}}\, A = \{0\} \oplus \mathrm{mul}\, A$ and A_0 is a closed symmetric not necessarily densely defined operator in $\mathfrak{H}_0$ (the operator part of A). Moreover, $A = A^*$ if and only if $A_0 = A_0^*$.

Let $A = A^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$, let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$ and let $E_0(\cdot) : \mathcal{B} \to [\mathfrak{H}_0]$ be the orthogonal spectral measure of A_0. Then the spectral measure $E_A(\cdot) : \mathcal{B} \to [\mathfrak{H}]$ of A is defined as $E_A(B) = E_0(B)P_{\mathfrak{H}_0}$, $B \in \mathcal{B}$.

Definition 2.1. Let $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ and let $\mathfrak{H}$ be a subspace in $\widetilde{\mathfrak{H}}$. The relation $\widetilde{A}$ is called $\mathfrak{H}$-minimal if there is no a nontrivial subspace $\mathfrak{H}' \subset \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ such that $E_{\widetilde{A}}(\delta)\mathfrak{H}' \subset \mathfrak{H}'$ for each bounded interval $\delta = [\alpha, \beta) \subset \mathbb{R}$.

Definition 2.2. The relations $T_j \in \widetilde{\mathcal{C}}(\mathfrak{H}_j)$, $j \in \{1, 2\}$, are said to be unitarily equivalent (by means of a unitary operator $U \in [\mathfrak{H}_1, \mathfrak{H}_2]$) if $T_2 = \widetilde{U}T_1$ with $\widetilde{U} = U \oplus U \in [\mathfrak{H}_1^2, \mathfrak{H}_2^2]$.

Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a symmetric relation. Recall the following definitions and results.

Definition 2.3. A relation $\widetilde{A} = \widetilde{A}^*$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ satisfying $A \subset \widetilde{A}$ is called an exit space self-adjoint extension of A. Moreover, such an extension $\widetilde{A}$ is called minimal if it is $\mathfrak{H}$-minimal.

In what follows we denote by $\widetilde{\mathrm{Self}}(A)$ the set of all minimal exit space self-adjoint extensions of A. Moreover, we denote by $\mathrm{Self}(A)$ the set of all extensions $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$ of A (such an extension is called canonical). As is known, for each A one has $\widetilde{\mathrm{Self}}(A) \neq \emptyset$. Moreover, $\mathrm{Self}(A) \neq \emptyset$ if and only if A has equal deficiency indices, in which case $\mathrm{Self}(A) \subset \widetilde{\mathrm{Self}}(A)$.

Definition 2.4. Exit space extensions $\widetilde{A}_j = \widetilde{A}_j^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_j)$, $j \in \{1, 2\}$, of A are called equivalent (with respect to $\mathfrak{H}$) if there exists a unitary operator $V \in [\widetilde{\mathfrak{H}}_1 \ominus \mathfrak{H}, \widetilde{\mathfrak{H}}_2 \ominus \mathfrak{H}]$ such that $\widetilde{A}_1$ and $\widetilde{A}_2$ are unitarily equivalent by means of $U = I_{\mathfrak{H}} \oplus V$.

Definition 2.5. The operator functions $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathfrak{H}]$ and $F(\cdot) : \mathbb{R} \to [\mathfrak{H}]$ are called a generalized resolvent and a spectral function of A respectively if there exists an exit space self-adjoint extension $\widetilde{A}$ of A (in a certain Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$) such that

$$R(\lambda) = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.2}$$

$$F(t) = P_{\mathfrak{H}} E((-\infty, t)) \upharpoonright \mathfrak{H}, \quad t \in \mathbb{R}. \tag{2.3}$$

Here $P_{\mathfrak{H}}$ is the orthoprojection in $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$ and $E(\cdot)$ is the spectral measure of $\widetilde{A}$.

In the case $\widetilde{A} \in \mathrm{Self}(A)$ the equality (2.2) defines a canonical resolvent

$$R(\lambda) = (\widetilde{A} - \lambda)^{-1}$$

of A.

Proposition 2.6. *Each generalized resolvent $R(\lambda)$ of A is generated by some (minimal) extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$. Moreover, the extensions $\widetilde{A}_1$, $\widetilde{A}_2 \in \widetilde{\mathrm{Self}}(A)$ inducing the same generalized resolvent $R(\cdot)$ are equivalent.*

In the sequel we suppose that a generalized resolvent $R(\cdot)$ and a spectral function $F(\cdot)$ are generated by an extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$. Moreover, we identify equivalent extensions. Then by Proposition 2.6 the equality (2.2) gives a bijective correspondence between generalized resolvents $R(\lambda)$ and extensions $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$, so that each $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ is uniquely defined by the corresponding generalized resolvent (2.2) (spectral function (2.3)).

It follows from (2.2) and (2.3) that the generalized resolvent $R(\cdot)$ and the spectral function $F(\cdot)$ generated by an extension $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ are related by the equality

$$((F(\beta)-F(\alpha))f,f) = \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{\alpha-\delta}^{\beta-\delta}\mathrm{Im}(R(u+i\varepsilon)f,f)\,du, \quad f\in\mathfrak{H}, \tag{2.4}$$

which holds for any finite interval $[\alpha,\beta)\subset\mathbb{R}$. Moreover, setting $\widetilde{\mathfrak{H}}_0 = \widetilde{\mathfrak{H}} \ominus \mathrm{mul}\,\widetilde{A}$ one gets from (2.3) that

$$F(\infty)(:= s-\lim_{t\to+\infty}F(t)) = P_{\mathfrak{H}}P_{\widetilde{\mathfrak{H}}_0}\restriction\mathfrak{H}. \tag{2.5}$$

2.3. **The spaces $\mathcal{L}^2(\sigma;\mathcal{H})$ and $L^2(\sigma;\mathcal{H})$.** Let $\mathcal{H}$ be a finite dimensional Hilbert space. A non-decreasing operator function $\sigma(\cdot):\mathbb{R}\to[\mathcal{H}]$ is called a distribution function if it is left continuous and satisfies $\sigma(0)=0$.

Theorem 2.7. ([15, ch. 3.15], [18]). *Let $\sigma(\cdot):\mathbb{R}\to[\mathcal{H}]$ be a distribution function. Then*

(1) *There exist a scalar measure ν on Borel sets of $\mathbb{R}$ and a function $\Psi:\mathbb{R}\to[\mathcal{H}]$ (uniquely defined by ν up to ν-a.e.) such that $\Psi(s)\geq 0$ ν-a.e. on $\mathbb{R}$, $\nu([\alpha,\beta))<\infty$ and $\sigma(\beta)-\sigma(\alpha)=\int_{[\alpha,\beta)}\Psi(s)\,d\nu(s)$ for any finite interval $[\alpha,\beta)\subset\mathbb{R}$.*

(2) *The set $\mathcal{L}^2(\sigma;\mathcal{H})$ of all Borel-measurable functions $f(\cdot):\mathbb{R}\to\mathcal{H}$ satisfying*

$$||f||^2_{\mathcal{L}^2(\sigma;\mathcal{H})} = \int_{\mathbb{R}}(d\sigma(s)f(s),f(s)) := \int_{\mathbb{R}}(\Psi(s)f(s),f(s))_{\mathcal{H}}\,d\nu(s)<\infty$$

is a semi-Hilbert space with the semi-scalar product

$$(f,g)_{\mathcal{L}^2(\sigma;\mathcal{H})} = \int_{\mathbb{R}}(d\sigma(s)f(s),g(s)) := \int_{\mathbb{R}}(\Psi(s)f(s),g(s))_{\mathcal{H}}\,d\nu(s), \quad f,g\in\mathcal{L}^2(\sigma;\mathcal{H}).$$

Moreover, different measures ν from statement (1) *give rise to the same space $\mathcal{L}^2(\sigma;\mathcal{H})$.*

Definition 2.8. ([15, 18]). The Hilbert space $L^2(\sigma;\mathcal{H})$ is a Hilbert space of all equivalence classes in $\mathcal{L}^2(\sigma;\mathcal{H})$ with respect to the seminorm $||\cdot||_{\mathcal{L}^2(\sigma;\mathcal{H})}$.

In the following we denote by π_σ the quotient map from $\mathcal{L}^2(\sigma;\mathcal{H})$ onto $L^2(\sigma;\mathcal{H})$.

With a distribution function $\sigma(\cdot)$ one associates the multiplication operator $\Lambda=\Lambda_\sigma$ in $L^2(\sigma;\mathcal{H})$ defined by

$$\mathrm{dom}\,\Lambda_\sigma = \{\widetilde{f}\in L^2(\sigma;\mathcal{H}) : sf(s)\in\mathcal{L}^2(\sigma;\mathcal{H}) \text{ for some (and hence for all) } f(\cdot)\in\widetilde{f}\}$$

$$\Lambda_\sigma\widetilde{f} = \pi_\sigma(sf(s)), \quad \widetilde{f}\in\mathrm{dom}\,\Lambda_\sigma, \quad f(\cdot)\in\widetilde{f}. \tag{2.6}$$

As is known, $\Lambda_\sigma^*=\Lambda_\sigma$ and the spectral measure E_σ of Λ_σ is given by

$$E_\sigma(B)\widetilde{f} = \pi_\sigma(\chi_B(\cdot)f(\cdot)), \quad B\in\mathcal{B}, \quad \widetilde{f}\in L^2(\sigma;\mathcal{H}), \quad f(\cdot)\in\widetilde{f}, \tag{2.7}$$

where $\chi_B(\cdot)$ is the indicator of the Borel set B.

Let $\mathcal{K}$, $\mathcal{K}'$ and $\mathcal{H}$ be finite dimensional Hilbert spaces and let $\sigma(s)(\in[\mathcal{H}])$ be a distribution function. For Borel measurable functions $Y(s)(\in[\mathcal{H},\mathcal{K}])$ and $g(s)(\in\mathcal{H})$, $s\in\mathbb{R}$, we let

$$\int_{\mathbb{R}}Y(s)\,d\sigma(s)g(s) := \int_{\mathbb{R}}Y(s)\Psi(s)g(s)\,d\nu(s)\,(\in\mathcal{K}), \tag{2.8}$$

where ν and $\Psi(\cdot)$ are defined in Theorem 2.7, (1).

2.4. **The class $\widetilde{R}(\mathcal{H})$.** Recall the following definition.

Definition 2.9. A pair $(C_0(\lambda), C_1(\lambda))$ of holomorphic operator functions $C_j(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H}], j\in\{0,1\}$, is said to be a Nevanlinna pair if $\mathrm{Im}\lambda\cdot\mathrm{Im}(C_1(\lambda)C_0^*(\lambda)\geq 0,\ C_1(\lambda)C_0^*(\overline{\lambda})-C_0(\lambda)C_1^*(\overline{\lambda})=0$ and $0\in\rho(C_0(\lambda)-\lambda C_1(\lambda)),\ \lambda\in\mathbb{C}\setminus\mathbb{R}$ (in the case $\dim\mathcal{H}<\infty$ the last condition can be replaced with $\mathrm{ran}\,(C_0(\lambda), C_1(\lambda))=\mathcal{H}$).

Two Nevanlinna pairs $(C_0^{(j)}(\cdot),\ C_1^{(j)}(\cdot)),\ j\in\{1,2\}$, are called equivalent if there exists a holomorphic operator function $\varphi(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H}]$ such that $0\in\rho(\varphi(\lambda))$ and $C_j^{(2)}(\lambda)=\varphi(\lambda)C_j^{(1)}(\lambda),\ \lambda\in\mathbb{C}\setminus\mathbb{R},\ j\in\{1,2\}$. Clearly, the set of all Nevanlinna pairs splits into disjoint equivalence classes; moreover, the equality

$$\tau(\lambda)=\{(C_0(\lambda),C_1(\lambda))\}:=\{\{h,h'\}\in\mathcal{H}\oplus\mathcal{H}: C_0(\lambda)h+C_1(\lambda)h'=0\},\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{2.9}$$

allows us to identify such a class with the holomorphic function $\tau(\cdot):\mathbb{C}\setminus\mathbb{R}\to\widetilde{\mathcal{C}}(\mathcal{H})$ satisfying

$$\tau^*(\lambda)=\tau(\overline{\lambda}),\quad 0\in\rho(\tau(\lambda)+\lambda),\quad \tfrac{1}{\mathrm{Im}\lambda}\cdot\mathrm{Im}(h',h)\geq 0,\quad \{h,h'\}\in\tau(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{2.10}$$

(see [7]). In the following we denote by $\widetilde{R}(\mathcal{H})$ the set of all equivalence classes of Nevanlinna pairs $(C_0(\cdot),C_1(\cdot))$ (or equivalently the set of all holomorphic functions $\tau(\cdot):\mathbb{C}\setminus\mathbb{R}\to\widetilde{\mathcal{C}}(\mathcal{H})$ satisfying (2.10)). Moreover, we denote by $\widetilde{R}^0(\mathcal{H})$ the set of all $\tau\in\widetilde{R}(\mathcal{H})$ admitting the constant-valued representation

$$\tau(\lambda)\equiv\{(C_0,C_1)\}=\theta,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{2.11}$$

with some $\theta=\theta^*\in\widetilde{\mathcal{C}}(\mathcal{H})$.

The following assertion is well known.

Assertion 2.10. *If $\tau(\lambda)=\{(C_0(\lambda),C_1(\lambda))\}\in\widetilde{R}(\mathcal{H})$ and $\Phi(\cdot)\in R_u[\mathcal{H}]$, then $0\in\rho(\tau(\lambda)+\Phi(\lambda)),\ 0\in\rho(C_0(\lambda)-C_1(\lambda)\Phi(\lambda))$ and*

$$-(\tau(\lambda)+\Phi(\lambda))^{-1}=(C_0(\lambda)-C_1(\lambda)\Phi(\lambda))^{-1}C_1(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{2.12}$$

2.5. **Boundary triplets and Weyl functions.** Here we recall some facts about boundary triplets and corresponding Weyl functions of symmetric relations following [17, 10, 26].

Let A be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$, let $\mathfrak{N}_\lambda(A)=\ker(A^*-\lambda)$ $(\lambda\in\mathbb{C})$ be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A)=\{\{f,\lambda f\}: f\in\mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A):=\dim\mathfrak{N}_\lambda(A)\leq\infty,\ \lambda\in\mathbb{C}_\pm$, be deficiency indices of A.

Definition 2.11. A collection $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma_j: A^*\to\mathcal{H},\ j\in\{0,1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma:\widehat{f}\to\{\Gamma_0\widehat{f},\Gamma_1\widehat{f}\},\widehat{f}\in A^*$, from A^* into $\mathcal{H}\oplus\mathcal{H}$ is surjective and the following Green's identity holds:

$$(f',g)-(f,g')=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g}),\quad \widehat{f}=\{f,f'\},\quad \widehat{g}=\{g,g'\}\in A^*.$$

Proposition 2.12. *Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Then the equality*

$$\Gamma_1\restriction\widehat{\mathfrak{N}}_\lambda(A)=M(\lambda)\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{2.13}$$

correctly defines the operator function $M(\cdot)\in R_u[\mathcal{H}]$ (the Weyl function of the triplet Π).

A connection between the Weyl function $M(\lambda)$ and classical Weyl functions for various differential and difference boundary problems is discussed e.g. in [11, 26].

Theorem 2.13. ([6, 10, 26]). *Let A be a closed symmetric linear relation in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. If $\tau = \{(C_0(\cdot), C_1(\cdot))\} \in \widetilde{R}(\mathcal{H})$ (see (2.9)), then for every $g \in \mathfrak{H}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the abstract boundary value problem*

$$\{f, \lambda f + g\} \in A^*, \tag{2.14}$$

$$C_0(\lambda)\Gamma_0\{f, \lambda f + g\} - C_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.15}$$

has a unique solution $f = f(g, \lambda)$ and the equality $R(\lambda)g := f(g, \lambda)$ defines a generalized resolvent $R(\lambda) = R_\tau(\lambda)$ of A. Conversely, for each generalized resolvent $R(\lambda)$ of A there exists a unique $\tau \in \widetilde{R}(\mathcal{H})$ such that $R(\lambda) = R_\tau(\lambda)$. Moreover, $R_\tau(\lambda)$ is a canonical resolvent if and only if $\tau \in \widetilde{R}^0(\mathcal{H})$.

2.6. **Boundary pairs and their Weyl functions.** Let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces and let Γ be a linear relation from $\mathfrak{H}^2$ into $\mathcal{H}^2$. Then an element $\hat{\varphi} \in \Gamma$ is a pair $\hat{\varphi} = \{\hat{f}, \hat{h}\}$, where $\hat{f} = \{f, f'\} \in \mathfrak{H}^2$ (f, $f' \in \mathfrak{H}$) and $\hat{h} = \{h, h'\} \in \mathcal{H}^2$ ($h, h' \in \mathcal{H}$). It is convenient to write

$$\hat{\varphi} = \{\hat{f}, \hat{h}\} = \left\{\hat{f}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\} = \left\{\begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\}.$$

Definition 2.14. ([8]). Let A be a closed symmetric linear relation in $\mathfrak{H}$. A pair $\{\mathcal{H}, \Gamma\}$ with a Hilbert space $\mathcal{H}$ and a linear relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ is called a boundary pair for A^* if

(1) $\operatorname{dom} \Gamma$ is dense in A^* and the abstract Green's identity

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', x)_{\mathcal{H}} - (h, x')_{\mathcal{H}} \tag{2.16}$$

holds for every $\{\binom{f}{f'}, \binom{h}{h'}\}$, $\{\binom{g}{g'}, \binom{x}{x'}\} \in \Gamma$.

(2) if $\hat{\varphi} = \{\binom{g}{g'}, \binom{x}{x'}\} \in \mathfrak{H}^2 \oplus \mathcal{H}^2$ satisfies (2.16) for every $\{\binom{f}{f'}, \binom{h}{h'}\} \in \Gamma$, then $\hat{\varphi} \in \Gamma$.

Proposition 2.15. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for A^* with $\dim \mathcal{H} < \infty$. Then $n_+(A) = n_-(A)$ and Γ is a closed relation with $\operatorname{dom} \Gamma = A^*$ and $\ker \Gamma = A$.*

Proposition 2.16. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for A^* with $\dim \mathcal{H} < \infty$. Moreover, let $\Gamma_j : A^* \to \mathcal{H}$, $j \in \{0, 1\}$, be the linear relations, given by $\Gamma_0 = P_{\mathcal{H} \oplus \{0\}} \Gamma \restriction A^*$ and $\Gamma_1 = P_{\{0\} \oplus \mathcal{H}} \Gamma \restriction A^*$ and let $\mathcal{K}_\Gamma$ be the linear manifold in $\mathcal{H}$ defined by*

$$\mathcal{K}_\Gamma = \operatorname{mul}(\operatorname{mul} \Gamma) = \left\{h' \in \mathcal{H} : \left\{0, \begin{pmatrix} 0 \\ h' \end{pmatrix}\right\} \in \Gamma\right\}. \tag{2.17}$$

If $\mathcal{K}_\Gamma = \{0\}$, then $\ker \Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(A) = \{0\}$, $\operatorname{ran} \Gamma_0 \restriction \widehat{\mathfrak{N}}_\lambda(A) = \mathcal{H}$ and the equality

$$\operatorname{gr} M(\lambda) = \left\{\{h, h'\} \in \mathcal{H}^2 : \left\{\begin{pmatrix} f \\ \lambda f \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\} \in \Gamma \text{ for some } f \in \mathfrak{N}_\lambda(A)\right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.18}$$

define the operator function $M(\cdot) \in R[\mathcal{H}]$ (the Weyl function of the pair $\{\mathcal{H}, \Gamma\}$). Moreover, if the relations Γ_0 and Γ_1 are operators, then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $M(\cdot)$ is the Weyl function of Π (in the sense of Definition 2.11 and Proposition 2.12).

Propositions 2.15 and 2.16 are immediate from [27, Section 3]. Moreover, the following proposition is implied by [27] and [9, Proposition 4.1].

Proposition 2.17. *Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for A^* with $\dim \mathcal{H} < \infty$, let $\mathcal{K}_\Gamma = \{0\}$ and let $M(\cdot)$ be the Weyl function of $\{\mathcal{H}, \Gamma\}$. Moreover, let $\mathcal{H}$ be decomposed as $\mathcal{H} = \widehat{\mathcal{H}} \oplus \dot{\mathcal{H}}$. Then*

(1) *The equalities*

$$\widetilde{A} = \left\{\widehat{f} \in A^* : \left\{\widehat{f}, \begin{pmatrix} 0 \\ h' \end{pmatrix}\right\} \in \Gamma,\ h' \in \widehat{\mathcal{H}}\right\}, \quad \widetilde{A}^* = \left\{\widehat{f} \in A^* : \left\{\widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\} \in \Gamma,\ h \in \dot{\mathcal{H}}\right\}$$

define a symmetric extension $\widetilde{A}$ *of* A *and its adjoint* $\widetilde{A}^*$.

(2) *A pair* $\{\dot{\mathcal{H}}, \dot{\Gamma}\}$ *with a linear relation* $\dot{\Gamma} : \mathfrak{H}^2 \to \dot{\mathcal{H}}^2$ *of the form*

$$\dot{\Gamma} = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ P_{\dot{\mathcal{H}}} h' \end{pmatrix} \right\} : h \in \dot{\mathcal{H}}, \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}$$

is a boundary pair for $\widetilde{A}^*$. *Moreover, for this pair* $\mathcal{K}_{\dot{\Gamma}} = \{0\}$ *and the corresponding Weyl function is* $\dot{M}(\lambda) = P_{\dot{\mathcal{H}}} M(\lambda) \upharpoonright \dot{\mathcal{H}}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

3. Pseudospectral and spectral functions of Hamiltonian systems

3.1. **Notations.** Let $\mathcal{I} = [a, b\rangle$ $(-\infty < a < b \leq \infty)$ be an interval of the real line (the symbol $\rangle$ means that the endpoint $b < \infty$ might be either included to $\mathcal{I}$ or not). For a given finite-dimensional Hilbert space H denote by $AC(\mathcal{I}; H)$ the set of functions $f(\cdot) : \mathcal{I} \to H$ which are absolutely continuous on each segment $[a, \beta] \subset \mathcal{I}$.

Next assume that $\Delta(\cdot)$ is an $[H]$-valued Borel measurable function on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$ and such that $\Delta(t) \geq 0$. Denote by $\mathcal{L}^2_\Delta(\mathcal{I})$ the semi-Hilbert space of Borel measurable functions $f(\cdot) : \mathcal{I} \to H$ satisfying $||f||^2_\Delta := \int_{\mathcal{I}} (\Delta(t) f(t), f(t))_H \, dt < \infty$ (see e.g. [15, Chapter 13.5]). The semi-definite inner product $(\cdot, \cdot)_\Delta$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ is defined by $(f, g)_\Delta = \int_{\mathcal{I}} (\Delta(t) f(t), g(t))_H \, dt$, $f, g \in \mathcal{L}^2_\Delta(\mathcal{I})$. Moreover, let $L^2_\Delta(\mathcal{I})$ be the Hilbert space of the equivalence classes in $\mathcal{L}^2_\Delta(\mathcal{I})$ with respect to the semi-norm $|| \cdot ||_\Delta$. Denote also by π_Δ the quotient map from $\mathcal{L}^2_\Delta(\mathcal{I})$ onto $L^2_\Delta(\mathcal{I})$ and let $\widetilde{\pi}_\Delta = \pi_\Delta \oplus \pi_\Delta : (\mathcal{L}^2_\Delta(\mathcal{I}))^2 \to (L^2_\Delta(\mathcal{I}))^2$, so that $\widetilde{\pi}_\Delta \{f, g\} = \{\pi_\Delta f, \pi_\Delta g\}$, $f, g \in \mathcal{L}^2_\Delta(\mathcal{I})$.

For a given finite-dimensional Hilbert space $\mathcal{K}$ we denote by $\mathcal{L}^2_\Delta[\mathcal{K}, H]$ the set of all Borel measurable operator-functions $F(\cdot) : \mathcal{I} \to [\mathcal{K}, H]$ such that $F(t)h \in \mathcal{L}^2_\Delta(\mathcal{I})$, $h \in \mathcal{K}$.

3.2. **Hamiltonian systems.** Let as above $\mathcal{I} = [a, b\rangle$ $(-\infty < a < b \leq \infty)$ be an interval in $\mathbb{R}$. Moreover, let H be a finite-dimensional Hilbert space. In the sequel we put $p = \dim H$ and $m = \dim(H \oplus H) = 2p$.

Next assume that $B(\cdot)$ and $\Delta(\cdot)$ are $[H \oplus H]$-valued Borel measurable functions on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$ and satisfying $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ a.e. on $\mathcal{I}$ and let $J \in [H \oplus H]$ be operator (1.2).

A Hamiltonian system on an interval $\mathcal{I}$ (with the regular endpoint a) is a system of differential equations of the form

$$Jy' - B(t)y = \lambda \Delta(t) y + \Delta(t) f(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}, \tag{3.1}$$

where $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$. Together with (3.1) we consider also the homogeneous system

$$Jy' - B(t)y = \lambda \Delta(t) y, \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}, \tag{3.2}$$

and the system

$$Jy' - B(t)y = \Delta(t) f(t), \quad t \in \mathcal{I}. \tag{3.3}$$

A function $y \in AC(\mathcal{I}; H \oplus H)$ is a solution of (3.1) if it satisfies (3.1) a.e. on $\mathcal{I}$. A function $Y(\cdot, \lambda) : \mathcal{I} \to [\mathcal{K}, H \oplus H]$ is an operator solution of equation (3.2) if $y(t) = Y(t, \lambda)h$ is a (vector) solution of this equation for every $h \in \mathcal{K}$ (here $\mathcal{K}$ is a Hilbert space with $\dim \mathcal{K} < \infty$). In the sequel we denote by $\mathcal{N}_\lambda$, $\lambda \in \mathbb{C}$, the linear space of solutions of the homogeneous system (3.2) belonging to $\mathcal{L}^2_\Delta(\mathcal{I})$. It is clear that $\dim \mathcal{N}_\lambda \leq m$.

The following obvious lemma will be useful in the sequel.

Lemma 3.1. *If* $Y(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{K}, H \oplus H]$ *is an operator solution of* (3.2), *then the relation*

$$\mathcal{K} \ni h \to (Y(\lambda)h)(t) = Y(t, \lambda)h \in \mathcal{N}_\lambda \tag{3.4}$$

defines the linear mapping $Y(\lambda) : \mathcal{K} \to \mathcal{N}_\lambda$ *and, conversely, for each such a mapping* $Y(\lambda)$ *there exists a unique operator solution* $Y(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{K}, H \oplus H]$ *of* (3.2) *such that* (3.4) *holds.*

It is easily seen that the set of all solutions y of (3.2) such that $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) does depend on λ. This enables one to introduce the following definition.

Definition 3.2. The null manifold $\mathcal{N}$ of the system (3.1) is the subspace of $\mathcal{N}_\lambda$ given by

$$\mathcal{N} = \{y \in AC(\mathcal{I}; H \oplus H) : Jy'(t) - B(t)y(t) = \lambda\Delta(t)y(t) \text{ and } \Delta(t)y(t) = 0 \text{ a.e. on } \mathcal{I}\}.$$

As it is known [31, 19, 25] system (3.1) gives rise to the *maximal linear relations* $\mathcal{T}_{\max}$ and $T_{\max}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $L^2_\Delta(\mathcal{I})$ respectively. They are given by

$$\mathcal{T}_{\max} = \{\{y, f\} \in (\mathcal{L}^2_\Delta(\mathcal{I}))^2 : y \in AC(\mathcal{I}; H \oplus H) \text{ and } Jy'(t) - B(t)y(t) = \Delta(t)f(t) \text{ a.e. on } \mathcal{I}\}$$

and $T_{\max} = \widetilde{\pi}_\Delta \mathcal{T}_{\max} = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\max}\}$. Moreover the Lagrange's identity

$$(f, z)_\Delta - (y, g)_\Delta = [y, z]_b - (Jy(a), z(a)), \quad \{y, f\}, \ \{z, g\} \in \mathcal{T}_{\max} \tag{3.5}$$

holds with

$$[y, z]_b := \lim_{t \uparrow b} (Jy(t), z(t)), \quad y, z \in \operatorname{dom} \mathcal{T}_{\max}. \tag{3.6}$$

Next, define the linear relation $\mathcal{T}_a$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and the minimal linear relation $T_{\min}$ in $L^2_\Delta(\mathcal{I})$ by setting

$$\mathcal{T}_a = \{\{y, f\} \in \mathcal{T}_{\max} : y(a) = 0 \text{ and } [y, z]_b = 0 \text{ for every } z \in \operatorname{dom} \mathcal{T}_{\max}\}$$

and $T_{\min} = \widetilde{\pi}_\Delta \mathcal{T}_a = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\min}\}$. Then $T_{\min}$ is a closed symmetric linear relation in $L^2_\Delta(\mathcal{I})$ and $T^*_{\min} = T_{\max}$ [31, 19, 25, 27]. Moreover, by [27, (4.47)]

$$\ker(\widetilde{\pi}_\Delta \upharpoonright \mathcal{T}_{\max}) = \{\{y, f\} \in (\mathcal{L}^2_\Delta(\mathcal{I}))^2 : y \in \mathcal{N} \text{ and } \Delta(t)f(t) = 0 \text{ a.e. on } \mathcal{I}\}. \tag{3.7}$$

Definition 3.3. ([22]). The numbers $N_+ = \dim \mathcal{N}_i$ and $N_- = \dim \mathcal{N}_{-i}$ are called the formal deficiency indices of the system (3.1).

Proposition 3.4. ([22, 25]). *Given a Hamiltonian system* (3.1). *Then* $N_\pm = \dim \mathcal{N}_\lambda$, $\lambda \in \mathbb{C}_\pm$ *(i.e.,* $\dim \mathcal{N}_\lambda$ *does not depend on* λ *in either* $\mathbb{C}_+$ *or* $\mathbb{C}_-$*) and*

$$p \leq N_\pm \leq m. \tag{3.8}$$

Moreover, the deficiency indices of $T_{\min}$ *are* $n_\pm(T_{\min}) = N_\pm - \dim \mathcal{N}$.

Next assume that

$$U = (u_1, u_2) : H \oplus H \to H \tag{3.9}$$

is an operator satisfying the relations

$$u_1 u_2^* - u_2 u_1^* = 0 \quad \text{and} \quad \operatorname{ran} U = H \tag{3.10}$$

(this means that (u_1, u_2) is a self-adjoint operator pair). As is known such an operator U admits an extension to an operator

$$\widetilde{U} = \begin{pmatrix} u_3 & u_4 \\ u_1 & u_2 \end{pmatrix} : H \oplus H \to H \oplus H, \tag{3.11}$$

satisfying $\widetilde{U}^* J \widetilde{U} = J$ (this means that $\widetilde{U}$ is a J-unitary operator).

Clearly, each function $y \in AC(\mathcal{I}; H \oplus H)$ admits the representation

$$y(t) = \{y_0(t), \, y_1(t)\} (\in H \oplus H), \quad t \in \mathcal{I}. \tag{3.12}$$

Using (3.11) and the representation (3.12) of y we introduce the linear mappings $\Gamma_{ja} : AC(\mathcal{I}; H \oplus H) \to H, \ j \in \{0, 1\}$, by setting

$$\Gamma_{0a} y = u_3 y_0(a) + u_4 y_1(a), \tag{3.13}$$

$$\Gamma_{1a} y = u_1 y_0(a) + u_2 y_1(a), \quad y \in AC(\mathcal{I}; H \oplus H). \tag{3.14}$$

Clearly, the mapping Γ_{1a} is determined by the operator U, while Γ_{0a} is determined by the extension $\widetilde{U}$. Moreover, the equality $\Gamma_a y = \widetilde{U} y(a),\ y \in AC(\mathcal{I}; H\oplus H)$, defines the surjective linear mapping $\Gamma_a : AC(\mathcal{I}; H\oplus H) \to H\oplus H$ with the block representation

$$\Gamma_a = \begin{pmatrix} \Gamma_{0a} \\ \Gamma_{1a} \end{pmatrix} : AC(\mathcal{I}; H\oplus H) \to H\oplus H. \tag{3.15}$$

Since $\widetilde{U}$ is J-unitary, it follows that

$$(Jy(a), z(a)) = (J\Gamma_a y, \Gamma_a z) = (\Gamma_{0a} y, \Gamma_{1a} z) - (\Gamma_{1a} y, \Gamma_{0a} z), \quad y, z \in \operatorname{dom} \mathcal{T}_{\max}. \tag{3.16}$$

In the following we associate with each operator U (see (3.9)) the operator solution $\varphi_U(\cdot, \lambda)(\in [H, H\oplus H]),\ \lambda \in \mathbb{C}$, of (3.2) satisfying the initial condition

$$\varphi_U(a, \lambda) = \begin{pmatrix} u_2^* \\ -u_1^* \end{pmatrix} : H \to H\oplus H. \tag{3.17}$$

One can easily verify that for each J-unitary extension $\widetilde{U}$ of U one has

$$\widetilde{U}\varphi_U(a, \lambda) = \begin{pmatrix} I_H \\ 0 \end{pmatrix} : H \to H\oplus H. \tag{3.18}$$

3.3. **q-pseudospectral and spectral functions.** In this subsection we assume that U is the operator (3.9) and that Γ_{1a} is the linear mapping (3.14).

In what follows we put $\mathfrak{H} := L^2_\Delta(\mathcal{I})$ and denote by $\mathfrak{H}_b$ the set of all $\widetilde{f} \in \mathfrak{H}$ with the following property: there exists $\beta_{\widetilde{f}} \in \mathcal{I}$ such that for some (and hence for all) function $f \in \widetilde{f}$ the equality $\Delta(t)f(t) = 0$ holds a.e. on $(\beta_{\widetilde{f}}, b)$.

With each $\widetilde{f} \in \mathfrak{H}_b$ we associate the function $\widehat{f}(\cdot) : \mathbb{R} \to H$ given by

$$\widehat{f}(s) = \int_{\mathcal{I}} \varphi_U^*(t, s)\Delta(t) f(t)\, dt, \quad f(\cdot) \in \widetilde{f}. \tag{3.19}$$

One can easily prove that $\widehat{f}(\cdot)$ is a continuous (and even holomorphic) function on $\mathbb{R}$.

Recall that an operator $V \in [\mathfrak{H}_1, \mathfrak{H}_2]$ is called a partial isometry if $||Vf|| = ||f||$ for all $f \in \mathfrak{H}_1 \ominus \ker V$.

Definition 3.5. A distribution function $\sigma(\cdot) : \mathbb{R} \to [H]$ will be called a q-pseudospectral function of the system (3.1) (corresponding to the operator U) if $\widehat{f} \in \mathcal{L}^2(\sigma; H)$ for all $\widetilde{f} \in \mathfrak{H}_b$ and the operator $V\widetilde{f} := \pi_\sigma \widehat{f},\ \widetilde{f} \in \mathfrak{H}_b$, admits a continuation to a partial isometry $V = V_\sigma \in [\mathfrak{H}, L^2(\sigma; H)]$.

The operator $V = V_\sigma$ will be called the (generalized) Fourier transform corresponding to $\sigma(\cdot)$.

Clearly, if $\sigma(\cdot)$ is a q-pseudospectral function, then for each $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$ the integral in (3.19) converges in the norm of $\mathcal{L}^2(\sigma; H)$. This means that there exists a unique $\widetilde{g}(= V_\sigma \pi_\Delta f) \in L^2(\sigma; H)$ such that for any function $g(\cdot) \in \widetilde{g}$ one has

$$\lim_{\beta \uparrow b} \Big|\Big| g(\cdot) - \int_{[a,\beta)} \varphi_U^*(t, \cdot)\Delta(t) f(t)\, dt \Big|\Big|_{\mathcal{L}^2(\sigma; H)} = 0.$$

Moreover, similarly to [15, 34](see also [29, Proposition 4.2]) one proves that for each q-pseudospectral function $\sigma(\cdot)$

$$V_\sigma^* \widetilde{g} = \pi_\Delta \left(\int_{\mathbb{R}} \varphi_U(\cdot, s)\, d\sigma(s) g(s) \right), \quad \widetilde{g} \in L^2(\sigma; H), \quad g(\cdot) \in \widetilde{g}, \tag{3.20}$$

where the integral converges in the seminorm of $\mathcal{L}^2_\Delta(\mathcal{I})$.

Assertion 3.6. *Let T and T_* be linear relations in $\mathfrak{H}$ given by*

$$T = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\max}, \ \ \Gamma_{1a} y = 0 \ \text{ and } \ [y, z]_b = 0, \ \ z \in \operatorname{dom} \mathcal{T}_{\max}\}, \tag{3.21}$$

$$T_* = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\max} \ \text{ and } \ \Gamma_{1a} y = 0\}. \tag{3.22}$$

Then

(1) *T is a (closed) symmetric extension of $T_{\min}$ and $T_* \subset T^*$.*

(2) *The multivalued part $\operatorname{mul} T$ of T is the set of all $\widetilde f \in \mathfrak{H}$ such that for some (and hence for all) $f \in \widetilde f$ there exists a solution y of the system* (3.3) *satisfying*

$$\Delta(t) y(t) = 0 \quad (\text{a.e. on } \mathcal{I}), \quad \Gamma_{1a} y = 0 \quad \text{and} \quad [y, z]_b = 0, \quad z \in \operatorname{dom} \mathcal{T}_{\max}. \tag{3.23}$$

Proof. Statement (1) is immediate from the Lagrange's identity (3.5). Statement (2) is implied by (3.21) and (2.1). □

Let $\sigma(\cdot)$ be a q-pseudospectral function, let V_σ be the corresponding Fourier transform and let $\mathfrak{H}_0' = \mathfrak{H} \ominus \ker V_\sigma$, $L_0 = V_\sigma \mathfrak{H} (= V_\sigma \mathfrak{H}_0')$ and $L_0^\perp = L^2(\sigma; H) \ominus L_0$. Then

$$\mathfrak{H} = \ker V_\sigma \oplus \mathfrak{H}_0', \quad L^2(\sigma; H) = L_0 \oplus L_0^\perp. \tag{3.24}$$

Assume also that

$$\widetilde{\mathfrak{H}}_0' := \mathfrak{H}_0' \oplus L_0^\perp, \quad \widetilde{\mathfrak{H}}' := \overbrace{\ker V_\sigma \oplus \mathfrak{H}_0'}^{\mathfrak{H}} \oplus L_0^\perp = \mathfrak{H} \oplus L_0^\perp = \ker V_\sigma \oplus \widetilde{\mathfrak{H}}_0' \tag{3.25}$$

and let $\widetilde V' \in [\widetilde{\mathfrak{H}}_0', L^2(\sigma; H)]$ be a unitary operator of the form

$$\widetilde V' = (V_\sigma \upharpoonright \mathfrak{H}_0', \, I_{L_0^\perp}) : \mathfrak{H}_0' \oplus L_0^\perp \to L^2(\sigma; H), \tag{3.26}$$

where $I_{L_0^\perp}$ is an embedding operator from $L_0^\perp$ to $L^2(\sigma; H)$. Since $\mathfrak{H} \subset \widetilde{\mathfrak{H}}'$, one may consider T as a linear relation in $\widetilde{\mathfrak{H}}'$.

Lemma 3.7. *Let $\sigma(\cdot)$ be a q-pseudospectral function of the system* (3.1), *let $\widetilde V'$ be unitary operator* (3.26) *and let T be symmetric relation* (3.21). *Moreover, let $T^*_{\widetilde{\mathfrak{H}}'} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}')$ be the linear relation adjoint to T in $\widetilde{\mathfrak{H}}'$ and let $\Lambda = \Lambda_\sigma$ be the multiplication operator in $L^2(\sigma; H)$. Then the equalities*

$$\widetilde f = (\widetilde V')^* \widetilde g, \quad \widetilde T_0 \widetilde f = (\widetilde V')^* \Lambda \widetilde g, \quad \widetilde g \in \operatorname{dom} \Lambda \tag{3.27}$$

define a self-adjoint operator $\widetilde T_0$ in $\widetilde{\mathfrak{H}}_0'$ such that $\widetilde T_0 \subset T^*_{\widetilde{\mathfrak{H}}'}$.

Proof. Clearly, $T^*_{\widetilde{\mathfrak{H}}'} = T^* \oplus (L_0^\perp)^2$ and by (3.26) the equalities (3.27) can be written as

$$\widetilde f = V_\sigma^* \widetilde g + P_{L_0^\perp} \widetilde g, \quad \widetilde T_0 \widetilde f = V_\sigma^* \Lambda \widetilde g + P_{L_0^\perp} \Lambda \widetilde g, \quad \widetilde g \in \operatorname{dom} \Lambda.$$

Hence to prove the inclusion $\widetilde T_0 \subset T^*_{\widetilde{\mathfrak{H}}'}$ it is sufficient to show that $\{V_\sigma^* \widetilde g, V_\sigma^* \Lambda \widetilde g\} \in T^*$ for all $\widetilde g \in \operatorname{dom} \Lambda$.

Let $\widetilde g \in \operatorname{dom} \Lambda$, $g(\cdot) \in \widetilde g$ and let $E(\cdot) = E_\sigma(\cdot)$ be the spectral measure of Λ. Then by (2.6) and (2.7) for each compact interval $\delta \subset \mathbb{R}$ one has $E(\delta) \widetilde g = \pi_\sigma(\chi_\delta(\cdot) g(\cdot))$ and $\Lambda E(\delta) \widetilde g = \pi_\sigma(s \chi_\delta(s) g(s))$. Therefore in view of (3.20) $V_\sigma^* E(\delta) \widetilde g = \pi_\Delta y(\cdot)$ and $V_\sigma^* \Lambda E(\delta) \widetilde g = \pi_\Delta f(\cdot)$, where

$$y(t) = \int_{\mathbb{R}} \varphi_U(t, s) \, d\sigma(s) \chi_\delta(s) g(s), \quad f(t) = \int_{\mathbb{R}} s \varphi_U(t, s) \, d\sigma(s) \chi_\delta(s) g(s). \tag{3.28}$$

It was shown in the proof of [29, Lemma 4.4] that $\{y, f\} \in \mathcal{T}_{\max}$. Moreover, by the first equality in (3.28) and (3.17)

$$y(a) = \begin{pmatrix} u_2^* \\ -u_1^* \end{pmatrix} \int_{\mathbb{R}} d\sigma(s) \chi_\delta(s) g(s),$$

which in view of (3.14) and (3.10) yields $\Gamma_{1a}y = 0$. Therefore by (3.22)

$$\{V_\sigma^* E(\delta)\widetilde{g}, V_\sigma^* \Lambda E(\delta)\widetilde{g}\}(= \{\pi_\Delta y(\cdot), \pi_\Delta f(\cdot)\}) \in T_* \subset T^*$$

and passage to the limit when $\delta \to \mathbb{R}$ yields the required inclusion $\{V_\sigma^*\widetilde{g}, V_\sigma^*\Lambda\widetilde{g}\} \in T^*$. □

Proposition 3.8. *For each q-pseudospectral function $\sigma(\cdot)$ of the system (3.1) the corresponding Fourier transform V_σ satisfies*

$$\operatorname{mul} T \subset \ker V_\sigma \tag{3.29}$$

(for $\operatorname{mul} T$ see Assertion 3.6, (2)).

Proof. Let $\widetilde{T}_0 = \widetilde{T}_0^*$ be the operator in $\widetilde{\mathfrak{H}}_0'$ defined in Lemma 3.7 and let $(\widetilde{T}_0)^*_{\widetilde{\mathfrak{H}}'}$ be the linear relation adjoint to $\widetilde{T}_0$ in $\widetilde{\mathfrak{H}}'$. Then $(\widetilde{T}_0)^*_{\widetilde{\mathfrak{H}}'} = \widetilde{T}_0 \oplus (\ker V_\sigma)^2$ and the inclusion $\widetilde{T}_0 \subset T^*_{\widetilde{\mathfrak{H}}'}$ yields

$$T \subset \widetilde{T}_0 \oplus (\ker V_\sigma)^2. \tag{3.30}$$

Let $\widetilde{n} \in \operatorname{mul} T$. Then $\{0, \widetilde{n}\} \in T$ and by (3.30) $\{0, \widetilde{n}\} \in \widetilde{T}_0 \oplus (\ker V_\sigma)^2$. Therefore there exist $\widetilde{f} \in \operatorname{dom} \widetilde{T}_0$ and $\widetilde{g}$, $\widetilde{g}' \in \ker V_\sigma$ such that

$$\widetilde{f} + \widetilde{g} = 0, \quad \widetilde{T}_0\widetilde{f} + \widetilde{g}' = \widetilde{n}.$$

Since $\widetilde{f} \in \widetilde{\mathfrak{H}}_0'$, $\widetilde{g} \in \ker V_\sigma$ and $\widetilde{\mathfrak{H}}_0' \perp \ker V_\sigma$ (see (3.25)), it follows that $\widetilde{f} = \widetilde{g} = 0$. Therefore $\widetilde{T}_0\widetilde{f} = 0$ and hence $\widetilde{n} = \widetilde{g}' \in \ker V_\sigma$. This yields the inclusion (3.29). □

Definition 3.9. A q-pseudospectral function $\sigma(\cdot)$ of the system (3.1) with $\ker V_\sigma = \operatorname{mul} T$ will be called a pseudospectral function (corresponding to the operator U).

Definition 3.10. A distribution function $\sigma(\cdot) : \mathbb{R} \to [H]$ is called a spectral function of the system (3.1) (corresponding to the operator U) if for every $\widetilde{f} \in \mathfrak{H}_b$ the inclusion $\widehat{f} \in \mathcal{L}^2(\sigma; H)$ holds and the Parseval equality $||\widehat{f}||_{\mathcal{L}^2(\sigma;H)} = ||\widetilde{f}||_{\mathfrak{H}}$ is valid (for $\widehat{f}$ see (3.19)).

It follows from Proposition 3.8 that a pseudospectral function is a q-pseudospectral function $\sigma(\cdot)$ with the minimally possible $\ker V_\sigma$. Moreover, the same proposition yields the following assertion.

Assertion 3.11. *A distribution function $\sigma(\cdot) : \mathbb{R} \to [H]$ is a spectral function of the system (3.1) if and only if it is a pseudospectral function with $\ker V_\sigma (= \operatorname{mul} T) = \{0\}$.*

In the following we put $\mathfrak{H}_0 := \mathfrak{H} \ominus \operatorname{mul} T$, so that

$$\mathfrak{H} = \operatorname{mul} T \oplus \mathfrak{H}_0. \tag{3.31}$$

Moreover, for a pseudospectral function $\sigma(\cdot)$ we denote by $V_0 = V_{0,\sigma}$ the isometry from $\mathfrak{H}_0$ to $L^2(\sigma; H)$ given by

$$V_{0,\sigma} := V_\sigma \upharpoonright \mathfrak{H}_0. \tag{3.32}$$

4. m-functions and generalized resolvents of Hamiltonian systems

4.1. **Boundary pairs for Hamiltonian systems.** The following lemma is immediate from [27, Lemma 5.1 and Proposition 5.5].

Lemma 4.1. *If system (3.1) satisfies $N_+ = N_- =: d$, then there exist a finite dimensional Hilbert space $\mathcal{H}_b$ and a surjective linear mapping*

$$\Gamma_b = \begin{pmatrix} \Gamma_{0b} \\ \Gamma_{1b} \end{pmatrix} : \operatorname{dom} \mathcal{T}_{\max} \to \mathcal{H}_b \oplus \mathcal{H}_b \tag{4.1}$$

such that for all $y, z \in \operatorname{dom} \mathcal{T}_{\max}$ *the following identity is valid:*

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z) - (\Gamma_{1b} y, \Gamma_{0b} z). \tag{4.2}$$

Moreover, $\dim \mathcal{H}_b = d - p$.

Remark 4.2. If Γ_b is the mapping (4.1), then $\Gamma_b y$ is a singular boundary value of a function $y \in \operatorname{dom} \mathcal{T}_{\max}$ at the endpoint b in the sense of [15, Chapter 13.2] (for more details see Remark 3.5 in [1]).

Below within this section we suppose the following assumptions:

(A1) System (3.1) has equal formal deficiency indices $N_+ = N_- =: d$.
(A2) U is the operator (3.9) satisfying (3.10) and Γ_{1a} is the linear mappings (3.14).
(A3) $\mathcal{H}_b$ is a finite dimensional Hilbert space and Γ_b is a surjective linear mapping (4.1) satisfying (4.2).
(A4) For each $y \in \mathcal{N}$ the equality $\Gamma_{1a} y = 0$ yields $y = 0$.

Proposition 4.3. *Let* $\widetilde{U}$ *be a* J*-unitary extension* (3.11) *of* U *and let* Γ_{0a} *be linear mapping* (3.13). *Then a pair* $\{H \oplus \mathcal{H}_b, \Gamma\}$ *with a linear relation* $\Gamma : \mathfrak{H}^2 \to (H \oplus \mathcal{H}_b)^2$ *defined by*

$$\Gamma = \left\{ \left\{ \begin{pmatrix} \pi_\Delta y \\ \pi_\Delta f \end{pmatrix}, \begin{pmatrix} (-\Gamma_{1a} y) \oplus \Gamma_{0b} y \\ \Gamma_{0a} y \oplus (-\Gamma_{1b} y) \end{pmatrix} \right\} : \{y, f\} \in \mathcal{T}_{\max} \right\} \tag{4.3}$$

is a boundary pair for $T_{\max}$ *such that* $\mathcal{K}_\Gamma = \{0\}$ *(for* $\mathcal{K}_\Gamma$ *see* (2.17)*).*

Proof. The fact that $\{H \oplus \mathcal{H}_b, \Gamma\}$ is a boundary pair for $T_{\max}$ directly follows from [27, Corollary 5.6]. Next, by (4.3) and (3.7) one has

$$\mathcal{K}_\Gamma = \{\Gamma_{0a} y \oplus (-\Gamma_{1b} y) : y \in \mathcal{N} \text{ and } \Gamma_{1a} y = 0,\ \Gamma_{0b} y = 0\}. \tag{4.4}$$

This and the assumption (A4) yields $\mathcal{K}_\Gamma = \{0\}$. □

Definition 4.4. The boundary pair $\{H \oplus \mathcal{H}_b, \Gamma\}$ constructed in Proposition 4.3 will be called a decomposing boundary pair for $T_{\max}$.

Proposition 4.5. *Let* T *be a symmetric extension* (3.21) *of* $T_{\min}$. *Then*

(1) *The adjoint* T^* *of* T *coincides with* T_* *(see* (3.22)*), that is*

$$T^* = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\max} \text{ and } \Gamma_{1a} y = 0\}. \tag{4.5}$$

(2) *For every* $\{\widetilde{y}, \widetilde{f}\} \in T^*$ *there exists a unique* $y \in \operatorname{dom} \mathcal{T}_{\max}$ *such that* $\Gamma_{1a} y = 0$, $\pi_\Delta y = \widetilde{y}$ *and* $\{y, f\} \in \mathcal{T}_{\max}$ *for any* $f \in \widetilde{f}$.

(3) *The collection* $\dot{\Pi} = \{\mathcal{H}_b, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ *with operators* $\dot{\Gamma}_j : T^* \to \mathcal{H}_b$ *of the form*

$$\dot{\Gamma}_0 \{\widetilde{y}, \widetilde{f}\} = \Gamma_{0b} y, \quad \dot{\Gamma}_1 \{\widetilde{y}, \widetilde{f}\} = -\Gamma_{1b} y, \quad \{\widetilde{y}, \widetilde{f}\} \in T^* \tag{4.6}$$

is a boundary triplet for T^*. *In* (4.6) $y \in \operatorname{dom} \mathcal{T}_{\max}$ *is uniquely defined by* $\{\widetilde{y}, \widetilde{f}\}$ *in accordance with statement* (2).

Proof. (1) Let $\widetilde{U}$ be J-unitary extension (3.11) of U, let Γ_{0a} be operator (3.13) and let $\{H \oplus \mathcal{H}_b, \Gamma\}$ be the decomposing boundary pair (4.3) for $T_{\max}$. Applying to this pair Proposition 2.17, (1) (with $\dot{\mathcal{H}} = \mathcal{H}_b$) one obtains statement (1).

(2) If $\{\widetilde{y}, \widetilde{f}\} \in T^*$, then there exists $\{y, f\} \in \mathcal{T}_{\max}$ such that $\pi_\Delta y = \widetilde{y}$, $\pi_\Delta f = \widetilde{f}$ and $\Gamma_{1a} y = 0$. Hence y has the required properties. To prove uniqueness of such y assume that $y_1 \in \operatorname{dom} \mathcal{T}_{\max}$, $\Gamma_{1a} y_1 = 0$, $\pi_\Delta y_1 = \widetilde{y}$ and $\{y_1, f_1\} \in \mathcal{T}_{\max}$ with some $f_1 \in \widetilde{f}$. Then $\widetilde{\pi}_\Delta \{y_1, f_1\} = \{\widetilde{y}, \widetilde{f}\}$ and, consequently, $\widetilde{\pi}_\Delta \{y_1 - y, f_1 - f\} = 0$. Therefore by (3.7) $y_1 - y \in \mathcal{N}$. Moreover, $\Gamma_{1a}(y - y_1) = 0$, which in view of the assumption (A4) yields the equality $y_1 = y$.

(3) Application of Proposition 2.17,(2) to the decomposing boundary pair $\{H\oplus\mathcal{H}_b,\Gamma\}$ gives a boundary pair $\{\mathcal{H}_b,\dot\Gamma\}$ for T^* with the linear relation $\dot\Gamma:\mathfrak{H}^2\to(\mathcal{H}_b)^2$ of the form

$$\dot\Gamma=\left\{\left\{\begin{pmatrix}\pi_\Delta y\\ \pi_\Delta f\end{pmatrix},\begin{pmatrix}\Gamma_{0b}y\\ -\Gamma_{1b}y\end{pmatrix}\right\}:\{y,f\}\in\mathcal{T}_{\max}\ \text{ and }\ \Gamma_{1a}y=0\right\}. \tag{4.7}$$

Therefore by statement (2) linear relations $\dot\Gamma_0=P_{\mathcal{H}_b\oplus\{0\}}\dot\Gamma\restriction T^*$ and $\dot\Gamma_1=P_{\{0\}\oplus\mathcal{H}_b}\dot\Gamma\restriction T^*$ are the operators (4.6), which in view of Proposition 2.16 yields statement (3). □

4.2. $\mathcal{L}^2_\Delta$-solutions of boundary problems.

Definition 4.6. A boundary parameter τ (at the endpoint b) is an equivalent class of operator pairs $\tau(\lambda)\in\widetilde{R}(\mathcal{H}_b)$. According to Sect. 2.4 such a pair is of the form

$$\tau=\tau(\lambda)=\{(C_0(\lambda),C_1(\lambda))\},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{4.8}$$

where $C_j(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H}_b]$, $j\in\{0,1\}$, are holomorphic operator functions satisfying

$$\mathrm{Im}\lambda\cdot\mathrm{Im}(C_1(\lambda)C_0^*(\lambda)\geq 0,\quad C_1(\lambda)C_0^*(\overline{\lambda})-C_0(\lambda)C_1^*(\overline{\lambda})=0,\quad \mathrm{ran}\,(C_0(\lambda),C_1(\lambda))=\mathcal{H}_b.$$

In the case $\tau\in\widetilde{R}^0(\mathcal{H}_b)$ a boundary parameter τ will be called self-adjoint. According to (2.11) such a boundary parameter admits the representation in the form of a self-adjoint operator pair

$$\tau(\lambda)\equiv\{(C_0,C_1)\},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{4.9}$$

with operators $C_j\in[\mathcal{H}_b]$, $j\in\{0,1\}$, satisfying $\mathrm{Im}(C_1C_0^*)=0$ and $\mathrm{ran}\,(C_0,C_1)=\mathcal{H}_b$.

Let τ be a boundary parameter (4.8). For a given function $f\in\mathcal{L}^2_\Delta(\mathcal{I})$ consider the following boundary value problem:

$$Jy'-B(t)y=\lambda\Delta(t)y+\Delta(t)f(t),\quad t\in\mathcal{I}, \tag{4.10}$$

$$\Gamma_{1a}y=0,\quad C_0(\lambda)\Gamma_{0b}y+C_1(\lambda)\Gamma_{1b}y=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.11}$$

If τ is a self-adjoint boundary parameter (4.9), then (4.11) becomes self-adjoint separated boundary conditions

$$\Gamma_{1a}y=0,\quad C_0\Gamma_{0b}y+C_1\Gamma_{1b}y=0. \tag{4.12}$$

A function $y(\cdot,\cdot):\mathcal{I}\times(\mathbb{C}\setminus\mathbb{R})\to H\oplus H$ is called a solution of the problem (4.10), (4.11) if for each $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the function $y(\cdot,\lambda)$ belongs to $AC(\mathcal{I};H\oplus H)\cap\mathcal{L}^2_\Delta(\mathcal{I})$ and satisfies the equation (4.10) a.e. on $\mathcal{I}$ (so that $y\in\mathrm{dom}\,\mathcal{T}_{\max}$) and the boundary conditions (4.11).

Theorem 4.7. *Let T be a symmetric relation* (3.21)*. If τ is a boundary parameter* (4.8)*, then for every $f\in\mathcal{L}^2_\Delta(\mathcal{I})$ the boundary problem* (4.10), (4.11) *has a unique solution $y(t,\lambda)=y_f(t,\lambda)$ and the equality*

$$R(\lambda)\widetilde{f}=\pi_\Delta(y_f(\cdot,\lambda)),\quad \widetilde{f}\in\mathfrak{H},\quad f\in\widetilde{f},\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.13}$$

defines a generalized resolvent $R(\lambda)=:R_\tau(\lambda)$ of T. Conversely, for each generalized resolvent $R(\lambda)$ of T there exists a unique boundary parameter τ such that $R(\lambda)=R_\tau(\lambda)$. Moreover, $R_\tau(\lambda)$ is a canonical resolvent if and only if τ is a self-adjoint boundary parameter (4.9)*. In this case $R_\tau(\lambda)=(\widetilde{T}^\tau-\lambda)^{-1}$ with*

$$\widetilde{T}^\tau=\{\{\widetilde{y},\widetilde{f}\}\in T_{\max}:\Gamma_{1a}y=0,\ C_0\Gamma_{0b}y+C_1\Gamma_{1b}y=0\}. \tag{4.14}$$

Proof. Let $\dot\Pi=\{\mathcal{H}_b,\dot\Gamma_0,\dot\Gamma_1\}$ be the boundary triplet (4.6) for T^*. Then for each boundary parameter τ the problem (4.10), (4.11) is equivalent to an abstract problem (2.14), (2.15) written in terms of the triplet $\dot\Pi$. Applying now Theorem 2.13, we arrive at the required statements. □

Proposition 4.8. *For any $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique operator solution $v_0(\cdot,\lambda)\in\mathcal{L}_\Delta^2[H,H\oplus H]$ of the homogeneous system* (3.2) *satisfying*

$$\Gamma_{1a}v_0(\lambda)=-I_H,\quad \Gamma_{0b}v_0(\lambda)=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.15}$$

Moreover, for any $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique operator solution $u(\cdot,\lambda)\in\mathcal{L}_\Delta^2[\mathcal{H}_b,H\oplus H]$ of (3.2) *satisfying*

$$\Gamma_{1a}u(\lambda)=0,\quad \Gamma_{0b}u(\lambda)=I_{\mathcal{H}_b},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.16}$$

In formulas (4.15) *and* (4.16) *$v_0(\lambda)$ and $u(\lambda)$ denote linear mappings from Lemma 3.1 corresponding to the solutions $v_0(\cdot,\lambda)$ and $u(\cdot,\lambda)$, respectively.*

Proof. Assume that $\widetilde U$ is a J-unitary extension (3.11) of U, Γ_{0a} is the operator (3.13), $\{H\oplus\mathcal{H}_b,\Gamma\}$ is the decomposing boundary pair (4.3) for $\mathcal{T}_{\max}$ and $\Gamma_0:\mathfrak{H}^2\to H\oplus\mathcal{H}_b$ is the linear relation corresponding to Γ (see Proposition 2.16). Moreover, let

$$\Gamma_0'=\begin{pmatrix}-\Gamma_{1a}\\ \Gamma_{0b}\end{pmatrix}:\operatorname{dom}\mathcal{T}_{\max}\to H\oplus\mathcal{H}_b,\quad \Gamma_1'=\begin{pmatrix}\Gamma_{0a}\\ -\Gamma_{1b}\end{pmatrix}:\operatorname{dom}\mathcal{T}_{\max}\to H\oplus\mathcal{H}_b, \tag{4.17}$$

so that the relation Γ_0 admits the representation

$$\Gamma_0=\{\{\widetilde\pi_\Delta\{y,f\},\Gamma_0'y\}:\{y,f\}\in\mathcal{T}_{\max}\}. \tag{4.18}$$

First we show that

$$\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(T_{\min})=\{\{\widetilde\pi_\Delta\{y,\lambda y\},\Gamma_0'y\}:\ y\in\mathcal{N}_\lambda\},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.19}$$

Since obviously $\pi\mathcal{N}_\lambda=\mathfrak{N}_\lambda(T_{\min})$, it follows from (4.18) that $\{\widetilde\pi_\Delta\{y,\lambda y\},\Gamma_0'y\}\in\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(T_{\min})$ for each $y\in\mathcal{N}_\lambda$. Conversely, let $\{\{\widetilde y,\lambda\widetilde y\},\widetilde h\}\in\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(T_{\min})$ with some $\widetilde y\in\mathfrak{N}_\lambda(T_{\min})$ and $\widetilde h\in H\oplus\mathcal{H}_b$. Then according to (4.18) there exists $\{y,f\}\in\mathcal{T}_{\max}$ such that $\pi_\Delta y=\widetilde y$, $\pi_\Delta f=\lambda\widetilde y$ and $\Gamma_0'y=\widetilde h$. Hence $\pi_\Delta(f-\lambda y)=0$ and consequently $\Delta(t)f(t)=\Delta(t)(\lambda y(t))$ a.e.on $\mathcal{I}$. Thus $Jy'(t)-B(t)y(t)=\Delta(t)f(t)=\lambda\Delta(t)y(t)$ (a.e. on $\mathcal{I}$), so that $y\in\mathcal{N}_\lambda$. Therefore $\{\{\widetilde y,\lambda\widetilde y\},\widetilde h\}=\{\widetilde\pi_\Delta\{y,\lambda y\},\Gamma_0'y\}$ with some $y\in\mathcal{N}_\lambda$, which yields (4.19).

It follows from Proposition 2.16 that

$$\ker\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(T_{\min})=\{0\},\quad \operatorname{ran}\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(T_{\min})=H\oplus\mathcal{H}_b. \tag{4.20}$$

If $y\in\mathcal{N}_\lambda$ and $\Gamma_0'y=0$, then by (4.19) and the first equality in (4.20) one has $\pi_\Delta y=0$. Hence $y\in\mathcal{N}$ and by (4.17) $\Gamma_{1a}y=0$, which in view of the assumption (A4) yields $y=0$. Therefore $\ker\Gamma_0'\restriction\mathcal{N}_\lambda=\{0\}$. Moreover, in view of (4.19) and the second equality in (4.20) $\Gamma_0'\restriction\mathcal{N}_\lambda=H\oplus\mathcal{H}_b$. Thus the operator $\Gamma_0'\restriction\mathcal{N}_\lambda$ isomorphically maps $\mathcal{N}_\lambda$ onto $H\oplus\mathcal{H}_b$ and, consequently, the equality $Z(\lambda)=(\Gamma_0'\restriction\mathcal{N}_\lambda)^{-1}$ correctly defines the isomorphism $Z(\lambda)$ from $H\oplus\mathcal{H}_b$ onto $\mathcal{N}_\lambda$. Let

$$Z(\lambda)=(v_0(\lambda),\,u(\lambda)):H\oplus\mathcal{H}_b\to\mathcal{N}_\lambda,\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.21}$$

be the block representations of $Z(\lambda)$. Since $\Gamma_0'Z(\lambda)=I_{H\oplus\mathcal{H}_b}$, it follows from (4.17) that

$$\begin{pmatrix}-\Gamma_{1a}\\ \Gamma_{0b}\end{pmatrix}(v_0(\lambda),\,u(\lambda))=\begin{pmatrix}I_H&0\\0&I_{\mathcal{H}_b}\end{pmatrix}. \tag{4.22}$$

Let $v_0(\cdot,\lambda)\in\mathcal{L}_\Delta^2[H,H\oplus H]$ and $u(\cdot,\lambda)\in\mathcal{L}_\Delta^2[\mathcal{H}_b,H\oplus H]$ be operator solutions of (3.2) corresponding to $v_0(\lambda)$ and $u(\lambda)$ respectively (see Lemma 3.1). Then (4.22) yields (4.15) and (4.16) for $v_0(\cdot,\lambda)$ and $u(\cdot,\lambda)$. Finally, uniqueness of $v_0(\cdot,\lambda)$ and $u(\cdot,\lambda)$ is implied by uniqueness of the solution of the boundary problem (4.10), (4.11). ☐

Proposition 4.9. *Let $\widetilde U$ be a J-unitary extension* (3.11) *of U and let Γ_{0a} be the mapping* (3.13). *Moreover, let $\{H\oplus\mathcal{H}_b,\Gamma\}$ be the decomposing boundary pair* (4.3) *and let*

$$M(\lambda)=\begin{pmatrix}m_0(\lambda)&M_2(\lambda)\\M_3(\lambda)&M_4(\lambda)\end{pmatrix}:H\oplus\mathcal{H}_b\to H\oplus\mathcal{H}_b,\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.23}$$

be the block matrix representation of the corresponding Weyl function $M(\cdot)$ (see (2.18)). Then the entries of the matrix (4.23) are connected with solutions $v_0(\cdot,\lambda)$ and $u(\cdot,\lambda)$ via

$$m_0(\lambda)=\Gamma_{0a}v_0(\lambda),\quad M_2(\lambda)=\Gamma_{0a}u(\lambda) \tag{4.24}$$

$$M_3(\lambda)=-\Gamma_{1b}v_0(\lambda),\quad M_4(\lambda)=-\Gamma_{1b}u(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.25}$$

Proof. Let Γ_0' and Γ_1' be given by (4.17) and let $Z(\lambda)$ be the same as in the proof of Proposition 4.8. Then by (4.3)

$$\left\{\begin{pmatrix}\pi_\Delta Z(\lambda)\widetilde{h}\\ \lambda\pi_\Delta Z(\lambda)\widetilde{h}\end{pmatrix},\begin{pmatrix}\Gamma_0' Z(\lambda)\widetilde{h}\\ \Gamma_1' Z(\lambda)\widetilde{h}\end{pmatrix}\right\}\in\Gamma,\quad \widetilde{h}\in H\oplus\mathcal{H}_b,\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.26}$$

Since $\Gamma_0'Z(\lambda)\widetilde{h}=\widetilde{h}$, it follows from (4.26) and (2.18) that $\Gamma_1'Z(\lambda)=M(\lambda)$, which in view of (4.21) and (4.23) can be written as

$$\begin{pmatrix}\Gamma_{0a}\\ -\Gamma_{1b}\end{pmatrix}(v_0(\lambda),\, u(\lambda))=\begin{pmatrix}m_0(\lambda) & M_2(\lambda)\\ M_3(\lambda) & M_4(\lambda)\end{pmatrix}.$$

This implies (4.24) and (4.25). □

Corollary 4.10. *Let $\dot{\Pi}=\{\mathcal{H}_b,\dot{\Gamma}_0,\dot{\Gamma}_1\}$ be the boundary triplet (4.6) for T^* and let $M_4(\lambda)$ be given by (4.25). Then*

(1)$M_4(\cdot)$ is the Weyl function of $\dot{\Pi}$ and hence $M_4(\cdot)\in R_u[\mathcal{H}_b]$;

(2) for each boundary parameter τ of the form (4.8) $0\in\rho(\tau(\lambda)+M_4(\lambda))$ and $0\in\rho(C_0(\lambda)-C_1(\lambda)M_4(\lambda))$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

Proof. Let $\widetilde{U}$ be a J-unitary extension (3.11) of U and let Γ_{0a} be the mapping (3.13). Moreover, let $\{H\oplus\mathcal{H}_b,\Gamma\}$ be the decomposing boundary pair (4.3) for $T_{\max}$ and let $M(\cdot)$ be the Weyl function of this pair. Applying Proposition 2.17 to the pair $\{H\oplus\mathcal{H}_b,\Gamma\}$ one obtains, that the Weyl function $\dot{M}(\lambda)$ of $\dot{\Pi}$ is $\dot{M}(\lambda)=P_{\mathcal{H}_b}M(\lambda)\upharpoonright\mathcal{H}_b$. This and Proposition 4.9 yield statement (1). Statement (2) is implied by Assertion 2.10. □

Theorem 4.11. *Let $\tau=\tau(\lambda)$ be a boundary parameter (4.8). Then for each $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique operator solution $v_\tau(\cdot,\lambda)\in\mathcal{L}_\Delta^2[H,H\oplus H]$ of the homogeneous system (3.2) satisfying the boundary conditions*

$$\Gamma_{1a}v_\tau(\lambda)=-I_H,\quad C_0(\lambda)\Gamma_{0b}v_\tau(\lambda)+C_1(\lambda)\Gamma_{1b}v_\tau(\lambda)=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.27}$$

Moreover, $v_\tau(\cdot,\lambda)$ is connected with the solutions $v_0(\cdot,\lambda)$ and $u(\cdot,\lambda)$ by

$$v_\tau(t,\lambda)=v_0(t,\lambda)-u(t,\lambda)(\tau(\lambda)+M_4(\lambda))^{-1}M_3(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{4.28}$$

where $M_3(\lambda)$ and $M_4(\lambda)$ are given by (4.25).

Proof. It follows from Corollary 4.10, (2) that the equality (4.28) correctly defines the solution $v_\tau(\cdot,\lambda)\in\mathcal{L}_\Delta^2[H,H\oplus H]$ of (3.2) and to prove the theorem it is sufficient to show that such $v_\tau(\cdot,\lambda)$ is a unique solution of (3.2) belonging to $\mathcal{L}_\Delta^2[H,H\oplus H]$ and satisfying (4.27).

Combining (4.28) with (4.15), (4.16) and (4.25) one gets the first equality in (4.27) and the equalities

$$\Gamma_{0b}v_\tau(\lambda)=-(\tau(\lambda)+M_4(\lambda))^{-1}M_3(\lambda),$$
$$\Gamma_{1b}v_\tau(\lambda)=-(I-M_4(\lambda)(\tau(\lambda)+M_4(\lambda))^{-1})M_3(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

The last two equalities give $\{\Gamma_{0b}v_\tau(\lambda)h,\Gamma_{1b}v_\tau(\lambda)h\}\in\tau(\lambda)$, $h\in H$, which implies the second condition in (4.27). Finally, uniqueness of $v_\tau(\cdot,\lambda)$ is implied by uniqueness of the solution of the boundary problem (4.10), (4.11) (see Theorem 4.7). □

4.3. m**-functions.** In this subsection we suppose that the hypotheses (A1)–(A4) are satisfied, that $\widetilde{U}$ is a J-unitary extension (3.11) of U and that Γ_{0a} is the mapping (3.13).

Let τ be a boundary parameter (4.8) and let $v_\tau(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H, H\oplus H]$ be the corresponding operator solution of (3.2) defined in Theorem 4.11.

Definition 4.12. The operator function $m_\tau(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [H]$ defined by

$$m_\tau(\lambda) = \Gamma_{0a} v_\tau(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R} \tag{4.29}$$

is called the m-function (Titchmarsh-Weyl function) corresponding to the boundary parameter τ or, equivalently, to the boundary problem (4.10), (4.11).

From the first equality in (4.27) it follows that

$$\widetilde{U} v_\tau(a,\lambda) \left(= \begin{pmatrix} \Gamma_{0a} \\ \Gamma_{1a} \end{pmatrix} v_\tau(\lambda) \right) = \begin{pmatrix} m_\tau(\lambda) \\ -I_H \end{pmatrix} : H \to H\oplus H, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{4.30}$$

Similarly to the case of definite system (3.1) one can easily show that, for given U and τ, the m-function $m_\tau(\cdot)$ is defined uniquely up to an additive self-adjoint constant depending on $\widetilde{U}$ (cf. [1, Proposition 5.2]).

A definition of the m-function m_τ in somewhat other terms is given in the following proposition, which directly follows from (4.30) and Theorem 4.11.

Proposition 4.13. *Let τ be a boundary parameter, let $\varphi_U(\cdot,\lambda)(\in [H, H\oplus H])$, $\lambda\in\mathbb{C}$, be the operator solution of* (3.2) *defined by* (3.17) *and let $\psi(\cdot,\lambda)(\in [H,H\oplus H])$ be the operator solution of* (3.2) *with*

$$\widetilde{U}\psi(a,\lambda) = \begin{pmatrix} 0 \\ I_H \end{pmatrix} : H \to H\oplus H, \quad \lambda\in\mathbb{C}. \tag{4.31}$$

Then there exists a unique operator function $m(\cdot) = m_\tau(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [H]$ such that for any $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the operator solution $v(\cdot,\lambda) = v_\tau(\cdot,\lambda)$ of (3.2) *given by*

$$v(t,\lambda) := \varphi_U(t,\lambda) m(\lambda) - \psi(t,\lambda) \tag{4.32}$$

belongs to $\mathcal{L}^2_\Delta[H,H\oplus H]$ and satisfies the second equality in (4.27).

In the following theorem we provide a description of all m-functions immediately in terms of the boundary parameter τ.

Theorem 4.14. *Let $M(\cdot)$ be the operator function given by* (4.23)–(4.25) *(that is $M(\cdot)$ is the Weyl function of the decomposing boundary pair $\{H\oplus\mathcal{H}_b, \Gamma\}$ for $T_{\max}$). Moreover, let τ_0 be a self-adjoint boundary parameter given by $\tau_0 = \{(I_{\mathcal{H}_b}, 0)\}$. Then $m_0(\lambda) = m_{\tau_0}(\lambda)$ and for every boundary parameter τ of the form* (4.8) *the m-function $m_\tau(\cdot)$ is*

$$m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1} C_1(\lambda) M_3(\lambda), \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.33}$$

Proof. One can easily verify that $v_0(t,\lambda) = v_{\tau_0}(t,\lambda)$, so that by (4.24) $m_0(\lambda) = m_{\tau_0}(\lambda)$. Next, application of the operator Γ_{0a} to the equality (4.28) with taking (4.24) and (4.29) into account yields

$$m_\tau(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1} M_3(\lambda), \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.34}$$

This and (2.12) imply (4.33). □

Proposition 4.15. *The m-function $m_\tau(\cdot)$ belongs to the class $R[H]$ and satisfies*

$$(\operatorname{Im}\lambda)^{-1} \cdot \operatorname{Im} m_\tau(\lambda) \geq \int_{\mathcal{I}} v_\tau^*(t,\lambda)\Delta(t) v_\tau(t,\lambda)\,dt, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.35}$$

Moreover, in the case of a self-adjoint boundary parameter τ the inequality (4.35) *turns into the equality.*

Proof. It follows from (4.33) that $m_\tau(\cdot)$ is holomorphic in $\mathbb{C}\setminus\mathbb{R}$. Moreover, since the Weyl function (4.23) satisfies $M^*(\lambda)=M(\overline{\lambda})$, it follows that $m_0^*(\lambda)=m_0(\overline{\lambda})$, $M_2^*(\lambda)=M_3(\overline{\lambda})$, $M_4^*(\lambda)=M_4(\overline{\lambda})$ and (4.34) yields $m_\tau^*(\lambda)=m_\tau(\overline{\lambda})$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Now it remains to show that $m_\tau(\cdot)$ satisfies (4.35).

Let $\lambda\in\mathbb{C}\setminus\mathbb{R}$, $h\in H$ and let $y=y(t)=v_\tau(t,\lambda)h$. Then by (3.16) $(Jy(a),y(a))=-2i\,\mathrm{Im}(\Gamma_{1a}y,\Gamma_{0a}y)$ and in view of (4.2) one has $[y,y]_b=-2i\,\mathrm{Im}(\Gamma_{1b}y,\Gamma_{0b}y)$. Applying now the Lagrange's identity (3.5) to $\{y,\lambda y\}\in\mathcal{T}_{\max}$ one gets

$$(\mathrm{Im}\,\lambda)^{-1}\cdot\mathrm{Im}(\Gamma_{1a}y,\Gamma_{0a}y)=(y,y)_\Delta+(\mathrm{Im}\,\lambda)^{-1}\cdot\mathrm{Im}(\Gamma_{1b}y,\Gamma_{0b}y). \tag{4.36}$$

It follows from (4.30) that $\Gamma_{0a}y=m_\tau(\lambda)h$, $\Gamma_{1a}y=-h$ and hence $\mathrm{Im}(\Gamma_{1a}y,\Gamma_{0a}y)=\mathrm{Im}(m_\tau(\lambda)h,h)$. Moreover, by the second equality in (4.27) one has $\{\Gamma_{0b}y,\Gamma_{1b}y\}\in\tau(\lambda)$ and (2.10) yields $(\mathrm{Im}\,\lambda)^{-1}\cdot\mathrm{Im}(\Gamma_{1b}y,\Gamma_{0b}y)\ge 0$. Observe also that

$$(y,y)_\Delta=\left(\left(\int_{\mathcal{I}}v_\tau^*(t,\lambda)\Delta(t)v_\tau(t,\lambda)\,dt\right)h,h\right).$$

Therefore by (4.36) the inequality (4.35) holds. If in addition $\tau\in\widetilde{R}^0(\mathcal{H}_b)$, then in (4.36) $\mathrm{Im}(\Gamma_{1b}y,\Gamma_{0b}y)=0$ and therefore the inequality (4.35) turns into the equality. □

Since by Proposition 4.15 $m_\tau(\cdot)\in R[\mathcal{H}]$, the Stieltjes inversion formula (1.8) defines the $[H]$-valued distribution function $\sigma_\tau(\cdot)$ such that

$$\int_{\mathbb{R}}\frac{d(\sigma_\tau(s)h,h)}{1+s^2}<\infty,\quad h\in H. \tag{4.37}$$

4.4. **Green's function and generalized resolvents.**

Proposition 4.16. *Let τ be a boundary parameter and let $R_\tau(\cdot)$ be the corresponding generalized resolvent of the relation T (see Theorem 4.7). Moreover, let $G_\tau(\cdot,\cdot,\lambda):\mathcal{I}\times\mathcal{I}\to[H\oplus H]$ be the operator-function given by*

$$G_\tau(x,t,\lambda)=\begin{cases}v_\tau(x,\lambda)\,\varphi_U^*(t,\overline{\lambda}), & x>t\\ \varphi_U(x,\lambda)\,v_\tau^*(t,\overline{\lambda}), & x<t\end{cases},\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.38}$$

(the Green's function). Then

$$R_\tau(\lambda)\widetilde{f}=\pi_\Delta\left(\int_{\mathcal{I}}G_\tau(\cdot,t,\lambda)\Delta(t)f(t)\,dt\right),\quad \widetilde{f}\in\mathfrak{H},\quad f\in\widetilde{f}. \tag{4.39}$$

Proof. As in [1, Theorem 6.2] one proves that $\int_{\mathcal{I}}||G_\tau(x,t,\lambda)\Delta(t)f(t)||\,dt<\infty$ for each $f\in\mathcal{L}_\Delta^2(\mathcal{I})$ and $x\in\mathcal{I}$. This implies that formula

$$y_f=y_f(x,\lambda):=\int_{\mathcal{I}}G_\tau(x,t,\lambda)\Delta(t)f(t)\,dt,\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{4.40}$$

correctly defines the function $y_f(\cdot,\cdot):\mathcal{I}\times\mathbb{C}\setminus\mathbb{R}\to H\oplus H$ and, therefore, (4.39) is equivalent to the following statement: for each $\widetilde{f}\in\mathfrak{H}$

$$y_f(\cdot,\lambda)\in\mathcal{L}_\Delta^2(\mathcal{I})\quad\text{and}\quad R_\tau(\lambda)\widetilde{f}=\pi_\Delta(y_f(\cdot,\lambda)),\quad f\in\widetilde{f},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.41}$$

To prove this statement we first prove the equality

$$\varphi_U(x,\lambda)v_\tau^*(x,\overline{\lambda})-v_\tau(x,\lambda)\varphi_U^*(x,\overline{\lambda})=J,\quad x\in\mathcal{I},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{4.42}$$

Let $Y(x,\lambda)=(\varphi_U(x,\lambda),\,v_\tau(x,\lambda)):H\oplus H\to H\oplus H$ and let $\widetilde{U}$ be a J-unitary extension (3.11) of U. Then by (3.18) and (4.30)

$$\widetilde{U}Y(a,\lambda)=\begin{pmatrix}I_H & m_\tau(\lambda)\\ 0 & -I_H\end{pmatrix}:H\oplus H\to H\oplus H$$

and the immediate calculations with taking the equality $m_\tau^*(\lambda) = m_\tau(\overline{\lambda})$ into account give $(\widetilde{U}Y(a,\overline{\lambda}))^* J(\widetilde{U}Y(a,\lambda)) = -J$. Therefore $Y^*(a,\overline{\lambda})JY(a,\lambda) = -J$ and by the Lagrange's identity (3.5) one has

$$Y^*(x,\overline{\lambda})JY(x,\lambda) = Y^*(a,\overline{\lambda})JY(a,\lambda) = -J.$$

Hence $Y(x,\lambda)JY^*(x,\overline{\lambda}) = -J$, $x \in \mathcal{I}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which is equivalent to (4.42).

Next assume that $\widetilde{f} \in \mathfrak{H}_b$. We show that in this case the function $y_f(\cdot,\lambda)$ given by (4.40) is a solution of the boundary problem (4.10), (4.11). Using (4.42) one proves as in [1, Theorem 6.2] that $y_f(\cdot,\lambda)$ satisfies (4.10) a.e. on $\mathcal{I}$. Moreover, by (4.38)

$$y_f(a,\lambda) = \varphi_U(a,\lambda)h_1, \quad y_f(x,\lambda) = v_\tau(x,\lambda)h_2, \quad x \in (\beta_{\widetilde{f}}, b) \tag{4.43}$$

with some $h_1,\ h_2 \in H$. Combining the first equality in (4.43) with (3.18) one gets $\Gamma_{1a}y_f = 0$. Moreover, the second equality in (4.43) shows that $y_f(\cdot,\lambda) \in \mathcal{L}^2_\Delta(\mathcal{I})$ and $\Gamma_{jb}y_f = \Gamma_{jb}v_\tau(\lambda)h_2,\ j \in \{0,1\},$. Therefore by the second equality in (4.27) y_f satisfies the second boundary condition in (4.11). Thus $y_f(\cdot,\lambda)$ is a solution of the boundary problem (4.10), (4.11) and by Theorem 4.7 relations (4.41) hold (for $\widetilde{f} \in \mathfrak{H}_b$). Finally, one proves (4.41) for arbitrary $\widetilde{f} \in \mathfrak{H}$ in the same way as in [1, Theorem 6.2]. □

5. Parametrization of pseudospectral and spectral functions

As before we suppose in this section the assumptions (A1)–(A4) specified just after Remark 4.2 (unless otherwise stated).

Let T be a symmetric relation (3.21). Then according to Theorem 4.7 the boundary problem (4.10), (4.11) induces a bijective correspondence $R(\lambda) = R_\tau(\lambda)$ between all boundary parameters τ and all generalized resolvents $R(\lambda)$ of T. In the following we denote by $\widetilde{T}^\tau (\in \widetilde{\mathrm{Self}}(T))$ the extension of T generating $R_\tau(\lambda)$ and by $F_\tau(\cdot)$ the spectral function of T generated by $\widetilde{T}^\tau$ (see Definition 2.5). Clearly, the equalities $\widetilde{T} = \widetilde{T}^\tau$ and $F(\cdot) = F_\tau(\cdot)$ give a parametrization of all extensions $\widetilde{T} \in \widetilde{\mathrm{Self}}(T)$ and all spectral functions $F(\cdot)$ of T respectively by means of the boundary parameter τ.

In what follows we assume that a certain J-unitary extension $\widetilde{U} \supset U$ of the form (3.11) is fixed and hence the m-function $m_\tau(\cdot)$ is defined by (4.29). Note that in view of the remark just after (4.30) a choice of $\widetilde{U}$ does not matter in our further considerations.

Definition 5.1. An extension $\widetilde{T} \in \widetilde{\mathrm{Self}}(T)$ ($\widetilde{T} \in \mathrm{Self}(T)$) is referred to the class $\widetilde{\mathrm{Self}}_0(T)$ (resp. $\mathrm{Self}_0(T)$) if $\mathrm{mul}\,\widetilde{T} = \mathrm{mul}\,T$.

Note that $\mathrm{Self}_0(T) \neq \emptyset$. Moreover, if $\mathrm{mul}\,T = \{0\}$, then $\widetilde{\mathrm{Self}}_0(T)$ ($\mathrm{Self}_0(T)$) is the set of all extensions $\widetilde{T} \in \widetilde{\mathrm{Self}}(T)$ (resp. $\widetilde{T} \in \mathrm{Self}(T)$) which are operators.

Definition 5.2. Let $M_4(\lambda)$ be given by the second equality in (4.25). A boundary parameter τ of the form (4.8) is called admissible if the equalities (1.13) and (1.14) hold.

Proposition 5.3. (1) *An extension $\widetilde{T}^\tau$ belongs to $\widetilde{\mathrm{Self}}_0(T)$ if and only if the boundary parameter τ is admissible.*

(2) *If $\lim_{y\to\infty} \frac{1}{iy} M_4(iy) = 0$, then a boundary parameter τ is admissible if and only if (1.13) is satisfied.*

Proof. Let $\dot{\Pi} = \{\mathcal{H}_b, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ be the boundary triplet (4.6) for T^*. Then by Corollary 4.10 the Weyl function of $\dot{\Pi}$ coincides with $M_4(\lambda)$. Moreover, it was mentioned in the proof of Theorem 4.7 that the problem (4.10), (4.11) is equivalent to the abstract problem (2.14), (2.15) for the triplet $\dot{\Pi}$. Now the required statements are implied by the results of [7]. □

Theorem 5.4. *Let τ be an admissible boundary parameter and let $F_\tau(\cdot)$ be the corresponding spectral function of T. Then there exists a unique pseudospectral function $\sigma_\tau(\cdot)$ of the system* (3.1) *satisfying*

$$((F_\tau(\beta)-F_\tau(\alpha))\widetilde{f},\widetilde{f})=\int_{[\alpha,\beta)}(d\sigma_\tau(s)\widehat{f}(s),\widehat{f}(s)),\quad \widetilde{f}\in\mathfrak{H}_b,\quad -\infty<\alpha<\beta<\infty. \tag{5.1}$$

This pseudospectral function is defined by the Stieltjes inversion formula (1.8).

Proof. (1) Let us show that (1.8) defines a pseudospectral function $\sigma_\tau(\cdot)$. To this end we first prove that $\sigma_\tau(\cdot)$ satisfies (5.1) (we give only the sketch of the proof, because it is similar to that of alike results in [14, 34]). It follows from Proposition 4.13 that the Green's function (4.38) admits the representation

$$G_\tau(x,t,\lambda)=\varphi_U(x,\lambda)m_\tau(\lambda)\varphi_U^*(t,\overline{\lambda})+G_0(x,t,\lambda), \tag{5.2}$$

where $G_0(x,t,\lambda)$ is an entire function of λ such that $G_0^*(x,t,\lambda)=G_0(t,x,\overline{\lambda})$. Let $\widehat{f}(\cdot)$ be an entire function given by $\widehat{f}(\lambda)=\int_{\mathcal{I}}\varphi_U^*(t,\overline{\lambda})\Delta(t)f(t)\,dt$. Then by (4.39) and (5.2) the generalized resolvent $R_\tau(\cdot)$ satisfies

$$(R_\tau(\lambda)\widetilde{f},\widetilde{f})=(m_\tau(\lambda)\widehat{f}(\lambda),\widehat{f}(\overline{\lambda}))+S(\lambda),\quad \widetilde{f}\in\mathfrak{H}_b,\quad \lambda\in\mathbb{C}\setminus\mathbb{R},$$

where $S(\cdot)$ is a certain continuous function on $\mathbb{C}$ with real values on $\mathbb{R}$. Therefore in view of (2.4) for each finite interval $[\alpha,\beta)$ one has

$$((F_\tau(\beta)-F_\tau(\alpha))\widetilde{f},\widetilde{f})=\lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{\alpha-\delta}^{\beta-\delta}\operatorname{Im}(m_\tau(u+i\varepsilon)\widehat{f}(u+i\varepsilon),\widehat{f}(u-i\varepsilon))\,du.$$

Now by using the Stieltjes-Livšic inversion formula [20, 34] one derives (5.1).

Next assume that $\widetilde{T}^\tau$ is a (self-adjoint) relation in a certain Gilbert space $\widetilde{\mathfrak{H}}$ and let $\widetilde{\mathfrak{H}}_0=\widetilde{\mathfrak{H}}\ominus\operatorname{mul}T$, so that

$$\widetilde{\mathfrak{H}}=\operatorname{mul}T\oplus\widetilde{\mathfrak{H}}_0. \tag{5.3}$$

Since by Proposition 5.3 $\operatorname{mul}\widetilde{T}^\tau=\operatorname{mul}T$, it follows from (5.1) and (2.5) that $\widehat{f}\in\mathcal{L}^2(\sigma_\tau;H)$ and $||\widehat{f}||_{\mathcal{L}^2(\sigma_\tau;H)}=||P_{\widetilde{\mathfrak{H}}_0}\widetilde{f}||_{\widetilde{\mathfrak{H}}}\le||\widetilde{f}||_{\mathfrak{H}}$, $\widetilde{f}\in\mathfrak{H}_b$. Hence the operator $V_b\widetilde{f}:=\pi_{\sigma_\tau}\widehat{f}$, $\widetilde{f}\in\mathfrak{H}_b$, admits a continuation to an operator $V\in[\mathfrak{H},L^2(\sigma_\tau;H)]$ satisfying

$$||V\widetilde{f}||_{L^2(\sigma_\tau;H)}=||P_{\widetilde{\mathfrak{H}}_0}\widetilde{f}||_{\widetilde{\mathfrak{H}}},\quad \widetilde{f}\in\mathfrak{H}. \tag{5.4}$$

Let $\mathfrak{H}$ be decomposed as in (3.31). Then in view of (5.4), (5.3) and the inclusion $\mathfrak{H}_0\subset\widetilde{\mathfrak{H}}_0$ one has $V\widetilde{f}=0$, $\widetilde{f}\in\operatorname{mul}T$, and $||V\widetilde{f}||_{L^2(\sigma;H)}=||\widetilde{f}||_{\widetilde{\mathfrak{H}}}=||\widetilde{f}||_{\mathfrak{H}}$, $\widetilde{f}\in\mathfrak{H}_0$. Thus V is a partial isometry with $\ker V=\operatorname{mul}T$ and, consequently, σ_τ is a pseudospectral function of the system (3.1) such that (5.1) holds.

(2) Let us prove that $\sigma_\tau(\cdot)$ is a unique pseudospectral function satisfying (5.1). Assume that V_{σ_τ} is the Fourier transform corresponding to σ_τ and let E_{σ_τ} be the spectral measure (2.7). Then by (5.1) for each finite interval $\delta=[\alpha,\beta)$ one has

$$F_\tau(\beta)-F_\tau(\alpha)=V_{\sigma_\tau}^*E_{\sigma_\tau}(\delta)V_{\sigma_\tau} \tag{5.5}$$

and (3.20) yields

$$(F_\tau(\beta)-F_\tau(\alpha))\widetilde{f}=\pi_\Delta\left(\int_\delta\varphi_U(\cdot,s)\,d\sigma_\tau(s)\widehat{f}(s)\right),\quad \delta=[\alpha,\beta)\subset\mathbb{R},\quad \widetilde{f}\in\mathfrak{H}_b. \tag{5.6}$$

Let $\sigma(\cdot)$ be a pseudospectral function such that (5.1) holds with $\sigma(\cdot)$ instead of $\sigma_\tau(\cdot)$. Then (5.6) also holds with $\sigma(\cdot)$ in place of $\sigma_\tau(\cdot)$ and, consequently,

$$\pi_\Delta\left(\int_\delta\varphi_U(\cdot,s)\,d\sigma_\tau(s)\widehat{f}(s)\right)=\pi_\Delta\left(\int_\delta\varphi_U(\cdot,s)\,d\sigma(s)\widehat{f}(s)\right),\quad \delta=[\alpha,\beta),\quad \widetilde{f}\in\mathfrak{H}_b. \tag{5.7}$$

It follows from Theorem 2.7 that there exist a scalar measure ν on Borel sets of $\mathbb{R}$ and functions $\Psi_j\cdot : \mathbb{R} \to [H]$, $j \in \{1,2\}$, such that

$$(5.8)\quad \sigma_\tau(\beta)-\sigma_\tau(\alpha) = \int_\delta \Psi_1(s)\,d\nu(s) \quad \text{and} \quad \sigma(\beta)-\sigma(\alpha) = \int_\delta \Psi_2(s)\,d\nu(s), \quad \delta = [\alpha,\beta).$$

Let $\Psi(s) := \Psi_1(s) - \Psi_2(s)$ and let μ be the Lebesgue measure on Borel sets of $\mathcal{I}$. Denote also by $\mathcal{G}$ the set of all functions $\widehat{f}(\cdot) : \mathbb{R} \to H$ admitting the representation (3.19) with some $\widetilde{f} \in \mathfrak{H}_b$. Then in view of (5.7), (2.8) and (5.8) one has

$$(5.9)\qquad \Delta(t)\int_\delta \varphi_U(t,s)\Psi(s)\widehat{f}(s)\,d\nu(s) = 0 \quad (\mu\text{-}a.e.\ on\ \mathcal{I}), \quad \widehat{f} \in \mathcal{G}, \quad \delta = [\alpha,\beta).$$

Denote by F the (countable) set of all finite intervals $\delta = [\alpha,\beta)$ with rational endpoints. It follows from (5.9) that for each $\delta \in F$ and for each $\widehat{f} \in \mathcal{G}$ there exists a Borel set $B_{\delta,\widehat{f}} \subset \mathcal{I}$ such that $\mu(\mathcal{I} \setminus B_{\delta,\widehat{f}}) = 0$ and $\int_\delta \Delta(t)\varphi_U(t,s)\Psi(s)\widehat{f}(s)d\nu(s) = 0$, $t \in B_{\delta,\widehat{f}}$. Hence $B_{\widehat{f}} := \bigcap_{\delta\in F} B_{\delta,\widehat{f}}$ is the Borel set in $\mathcal{I}$ such that

$$(5.10)\qquad \mu(\mathcal{I} \setminus B_{\widehat{f}}) = 0 \quad \text{and} \quad \nu(\{s \in \mathbb{R} : \Delta(t)\varphi_U(t,s)\Psi(s)\widehat{f}(s) \neq 0\}) = 0, \quad t \in B_{\widehat{f}}.$$

For each $\widehat{f} \in \mathcal{G}$ we put

$$\widetilde{B}_{\widehat{f}} = \{(t,s) \in \mathcal{I} \times \mathbb{R} : \Delta(t)\varphi_U(t,s)\Psi(s)\widehat{f}(s) \neq 0\}.$$

Then by (5.10) $(\mu \times \nu)(\widetilde{B}_{\widehat{f}}) = 0$ and, consequently, there is a Borel set $C_{\widehat{f}} \subset \mathbb{R}$ such that

$$(5.11)\qquad \nu(\mathbb{R} \setminus C_{\widehat{f}}) = 0 \quad \text{and} \quad \mu(\{t \in \mathcal{I} : \Delta(t)\varphi_U(t,s)\Psi(s)\widehat{f}(s) \neq 0\}) = 0, \quad s \in C_{\widehat{f}}.$$

Let $s \in C_{\widehat{f}}$ and let $y = y(t) = \varphi_U(t,s)\Psi(s)\widehat{f}(s)$. Then by (3.17) $y(a) = \begin{pmatrix} u_2^* \\ -u_1^* \end{pmatrix} \Psi(s)\widehat{f}(s)$ and (3.10) yields $\Gamma_{1a}y = 0$. Moreover, by (5.11) $\Delta(t)y(t) = 0$ (μ a.e. on $\mathcal{I}$) and hence $y \in \mathcal{N}$. Therefore by the assumption (A4) $y = 0$ and, consequently, $\Psi(s)\widehat{f}(s) = 0$. Thus for any $\widehat{f} \in \mathcal{G}$ there exists a Borel set $C_{\widehat{f}} \subset \mathbb{R}$ such that

$$(5.12)\qquad \nu(\mathbb{R} \setminus C_{\widehat{f}}) = 0 \quad \text{and} \quad \Psi(s)\widehat{f}(s) = 0, \quad s \in C_{\widehat{f}}.$$

Next we prove the following statement:

(S) for any $s \in \mathbb{R}$ and $h \in H$ there is $\widehat{f}(\cdot) \in \mathcal{G}$ such that $\widehat{f}(s) = h$.

Indeed, let $s \in \mathbb{R}$, $h' \in H$ and $(\widehat{f}(s), h') = 0$ for any $\widehat{f}(\cdot) \in \mathcal{G}$. Put $y = y(t) = \varphi_U(t,s)h'$. Then for any $\beta \in \mathcal{I}$ one has $\widehat{f}_\beta(\cdot) := \int_{[a,\beta]} \varphi_U^*(t,\cdot)\Delta(t)y(t)\,dt \in \mathcal{G}$ and,consequently,

$$0 = (\widehat{f}_\beta(s), h') = \int_{[a,\beta]} (\varphi_U^*(t,s)\Delta(t)y(t), h')\,dt = \int_{[a,\beta]} (\Delta(t)y(t), y(t))\,dt, \quad \beta \in \mathcal{I}.$$

Hence $\Delta(t)y(t) = 0$ (μ - a.e. on $\mathcal{I}$) and, therefore, $y \in \mathcal{N}$. Moreover, by (3.17) $\Gamma_{1a}y = 0$ and the assumption (A4) yields $y = 0$. Hence $h' = 0$, which proves statement (S).

Let $\{e_j\}_1^p$ be an orthonormal basis in H. Then in view of statement (S) there exists a system $\{\widehat{f}_j\}_1^p$ of functions $\widehat{f}_j \in \mathcal{G}$ such that $\widehat{f}_j(0) = e_j$. Let us put

$$D_1 = \{s \in \mathbb{R} : d(s) \neq 0\}, \quad D_2 = \bigcap_{j=1}^p C_{\widehat{f}_j}, \quad D_3 = \{s \in \mathbb{R} : \nu(\{s\}) > 0\},$$
$$D = (D_1 \cap D_2) \cup D_3,$$

where $d(s) = \det(\widehat{f}_j(s), e_k)$. If $s \in D_1 \cap D_2$, then $d(s) \neq 0$ (so that $\{\widehat{f}_j(s)\}_1^p$ is a basis in H) and $\Psi(s)\widehat{f}_j(s) = 0$, $j = \overline{1,p}$. Hence $\Psi(s) = 0$, $s \in D_1 \cap D_2$. Next, D_3 is an (at most countable) Borel set and $D_3 \subset C_{\widehat{f}}$ for any $\widehat{f} \in \mathcal{G}$. Therefore by statement (S)

$\Psi(s)=0,\ s\in D_3$. This implies that $\Psi(s)=0,\ s\in D$. Moreover, $\mathbb{R}\setminus D=D'\cup D''$, where

$$D'=(\mathbb{R}\setminus D_1)\cap(\mathbb{R}\setminus D_3),\quad D''=(\mathbb{R}\setminus D_2)\cap(\mathbb{R}\setminus D_3).$$

Since $\widehat{f}_j(\cdot)$ is an entire function, so is the function $d(\cdot)$. Moreover, $d(0)=1$ and hence the set $\mathbb{R}\setminus D_1$ is at most countable. Therefore D' is at most countable set and $\nu(\{s\})=0,\ s\in D'$, which yields the equality $\nu(D')=0$. Moreover, $\nu(\mathbb{R}\setminus D_2)=0$ and hence $\nu(D'')=0$. This implies $\nu(\mathbb{R}\setminus D)=0$ and, consequently, $\Psi(s)=0$ (ν-a.e. on $\mathbb{R}$). Thus $\Psi_1(s)=\Psi_2(s)$ (ν-a.e. on $\mathbb{R}$) and by (5.8) $\sigma_\tau(s)=\sigma(s)$. □

Proposition 5.5. *Let $\sigma(\cdot)$ be a q-pseudospectral function of the system* (3.1) *and let $L_0=V_\sigma\mathfrak{H}$. Then the multiplication operator Λ_σ is L_0-minimal (see Definition 2.1).*

Proof. Let $L_0^\perp:=\ker V_\sigma^*(=L^2(\sigma;H)\ominus L_0)$, let $\widetilde{g}\in L_0^\perp$ be an element such that $E_\sigma(\delta)\widetilde{g}\in L_0^\perp$ for each bounded interval $\delta=[\alpha,\beta)\subset\mathbb{R}$ and let $g(\cdot)\in\widetilde{g}$. Then $\Lambda_\sigma E_\sigma(\delta)\widetilde{g}\in L_0^\perp$ and, consequently, $V_\sigma^*E_\sigma(\delta)\widetilde{g}=0$ and $V_\sigma^*\Lambda_\sigma E_\sigma(\delta)\widetilde{g}=0$. Combining of these equalities with (2.6), (2.7) and (3.20) shows that the functions

$$y(t)=\int_\delta\varphi_U(t,s)\,d\sigma(s)g(s),\quad f(t)=\int_\delta s\varphi_U(t,s)\,d\sigma(s)g(s)\tag{5.13}$$

satisfy the equalities $\Delta(t)y(t)=0$ and $\Delta(t)f(t)=0$ (a.e. on $\mathcal{I}$). On the other hand, according to [29, Lemma 3.1] $y\in AC(\mathcal{I};H\oplus H)$ and

$$y'(t)=-J\int_\delta(B(t)+s\Delta(t))\varphi_U(t,s)\,d\sigma(s)g(s)\quad\text{(a.e. on }\mathcal{I}).$$

Therefore

$$Jy'(t)-B(t)y(t)=\Delta(t)\int_\delta s\varphi_U(t,s)d\sigma(s)g(s)=\Delta(t)f(t)=0\quad\text{(a.e. on }\mathcal{I})$$

and consequently $y\in\mathcal{N}$. Moreover, (3.17) and the first equality in (5.13) yields

$$y(a)=\int_\delta\varphi_U(a,s)\,d\sigma(s)g(s)=\begin{pmatrix}u_2^*\\-u_1^*\end{pmatrix}\int_\delta d\sigma(s)g(s).\tag{5.14}$$

Therefore by (3.10) $\Gamma_{1a}y=0$ and in view of the assumption (A4) one has $y=0$. Hence by (5.14) $\int_\delta d\sigma(s)g(s)=0$, which implies that $\widetilde{g}=0$. Thus the subspace L_0 satisfies the condition of Definition 2.1. □

Let $\widetilde{T}\in\widetilde{\mathrm{Self}}_0(T)$ be a linear relation in a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ and let $\widetilde{\mathfrak{H}}$ be decomposed as in (5.3). In the sequel we denote by $\widetilde{T}_0$ the operator part of $\widetilde{T}$. Since $\operatorname{mul}\widetilde{T}=\operatorname{mul}T$, it follows that $\widetilde{T}_0$ is a self-adjoint operator in $\widetilde{\mathfrak{H}}_0$.

Proposition 5.6. *For each pseudospectral function $\sigma(\cdot)$ of the system* (3.1) *there exists an extension $\widetilde{T}\in\widetilde{\mathrm{Self}}_0(T)$ such that the relative spectral function $F(\cdot)$ of T satisfies*

$$((F(\beta)-F(\alpha))\widetilde{f},\widetilde{f})=\int_{[\alpha,\beta)}(d\sigma(s)\widehat{f}(s),\widehat{f}(s)),\quad\widetilde{f}\in\mathfrak{H}_b,\quad-\infty<\alpha<\beta<\infty.\tag{5.15}$$

Moreover, there exists a unitary operator $\widetilde{V}\in[\widetilde{\mathfrak{H}}_0,L^2(\sigma;H)]$ such that $\widetilde{V}\upharpoonright\mathfrak{H}_0=V_{0,\sigma}$ and the operators $\widetilde{T}_0$ and Λ_σ are unitarily equivalent by means of $\widetilde{V}$ (for $\mathfrak{H}_0$, $\widetilde{\mathfrak{H}}_0$ and $V_{0,\sigma}$ see (3.31), (5.3) *and* (3.32) *respectively).*

Proof. For a given pseudospectral function $\sigma(\cdot)$ we put $L_0=V_\sigma\mathfrak{H}$ and $L_0^\perp=L^2(\sigma;H)\ominus L_0$, so that $L^2(\sigma;H)=L_0\oplus L_0^\perp$. Assume also that

$$\widetilde{\mathfrak{H}}_0:=\mathfrak{H}_0\oplus L_0^\perp,\quad\widetilde{\mathfrak{H}}:=\operatorname{mul}T\oplus\mathfrak{H}_0\oplus L_0^\perp=\operatorname{mul}T\oplus\widetilde{\mathfrak{H}}_0\tag{5.16}$$

and let $\widetilde{V}\in[\widetilde{\mathfrak{H}}_0,L^2(\sigma;H)]$ be a unitary operator given by

$$\widetilde{V}=(V_{0,\sigma},\,I_{L_0^\perp}):\mathfrak{H}_0\oplus L_0^\perp\to L^2(\sigma;H).\tag{5.17}$$

Since $\ker V_\sigma = \operatorname{mul} T$, it follows that $\mathfrak{H}_0' = \mathfrak{H}_0$, $\widetilde{\mathfrak{H}}_0' = \widetilde{\mathfrak{H}}_0$ and $\widetilde{V}' = \widetilde{V}$ (see (3.24), (3.25) and (3.26)). Therefore by Lemma 3.7 the equalities (3.27) with $\widetilde{V}' = \widetilde{V}$ define a self-adjoint operator $\widetilde{T}_0$ in $\widetilde{\mathfrak{H}}_0$. Moreover, the operators $\widetilde{T}_0$ and Λ_σ are unitarily equivalent by means of $\widetilde{V}$ and hence the spectral measure $E_0(\cdot)$ of $\widetilde{T}_0$ satisfies

$$E_0([\alpha,\beta)) = \widetilde{V}^* E_\sigma([\alpha,\beta))\widetilde{V}, \quad -\infty < \alpha < \beta < \infty. \tag{5.18}$$

Observe also that $\widetilde{V}\mathfrak{H}_0 = V_\sigma \mathfrak{H} = L_0$ and by Proposition 5.5 the operator Λ_σ is L_0-minimal. Therefore the operator $\widetilde{T}_0$ is $\mathfrak{H}_0$-minimal.

It follows from the second equality in (5.16) that $\widetilde{T} := (\{0\} \oplus \operatorname{mul} T) \oplus \widetilde{T}_0$ is a self-adjoint linear relation in $\widetilde{\mathfrak{H}}$ with the operator part $\widetilde{T}_0$ and $\operatorname{mul} \widetilde{T} = \operatorname{mul} T$. Moreover, $\{0\} \oplus \operatorname{mul} T \subset T \subset T^*_{\widetilde{\mathfrak{H}}}$ and by Lemma 3.7 $\widetilde{T}_0 \subset T^*_{\widetilde{\mathfrak{H}}}$. Hence $\widetilde{T} \subset T^*_{\widetilde{\mathfrak{H}}}$ and, consequently, $T \subset \widetilde{T}$. Observe also that the relation $\widetilde{T}$ is $\mathfrak{H}$-minimal, since the operator $\widetilde{T}_0$ is $\mathfrak{H}_0$-minimal. Hence $\widetilde{T} \in \widetilde{\operatorname{Self}}_0(T)$. Finally by using (5.18) and (5.17) one can easily prove the equality (5.15). $\square$

In the following theorem we describe all pseudospectral functions of the system (3.1) in terms of an admissible boundary parameter τ.

Theorem 5.7. *Let the assumptions* (A1)-(A4) *from Sect. 4.1 be satisfied and let* $M(\lambda)$ *be given by* (4.23)–(4.25). *Then the equality*

$$m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

together with formula (1.8) *establishes a bijective correspondence between all admissible boundary parameters* τ *of the form* (4.8) *and all pseudospectral functions* $\sigma(\cdot) = \sigma_\tau(\cdot)$ *of the system* (3.1) *(corresponding to the operator* U*).*

Proof. Let τ be an admissible boundary parameter (4.8). Then by Theorem 5.4 formula (1.8) defines a pseudospectral function $\sigma(\cdot) = \sigma_\tau(\cdot)$. Conversely, let $\sigma(\cdot)$ be a pseudospectral function of the system (3.1). Then by Propositions 5.6 and 5.3 there exists a unique admissible boundary parameter τ such that (5.15) holds with $F(\cdot) = F_\tau(\cdot)$. Moreover, by Theorem 5.4 $\sigma(\cdot) = \sigma_\tau(\cdot)$, where $\sigma_\tau(\cdot)$ is given by (1.8). Thus formula (1.8) gives a bijective correspondence between all admissible boundary parameters τ and all pseudospectral functions $\sigma(\cdot) = \sigma_\tau(\cdot)$. Now the statement of the theorem is implied by Theorem 4.14. $\square$

Corollary 5.8. *Let* $N_+ = N_-$ *and let the assumption (A4) be fulfilled. Then each pseudospectral function of the system* (3.1) *satisfies* (4.37).

Theorem 5.9. *There is a one to one correspondence between all extensions* $\widetilde{T}(= \widetilde{T}^\tau) \in \widetilde{\operatorname{Self}}_0(T)$ *and all pseudospectral functions* $\sigma(\cdot)(= \sigma_\tau(\cdot))$ *of the system* (3.1). *This correspondence is given by the equality* (5.15), *where* $F(\cdot)(= F_\tau(\cdot))$ *is a spectral function of* T *generated by* $\widetilde{T}$. *Moreover, the operators* $\widetilde{T}_0$ *(the operator part of* $\widetilde{T}$*) and* Λ_σ *are unitarily equivalent and hence they have the same spectral properties. In particular this implies that the spectral multiplicity of* $\widetilde{T}_0$ *does not exceed* p.

Proof. Combining of Theorems 5.4, 5.5 and Proposition 5.6 gives the required statements. $\square$

The following corollary is immediate from Theorem 5.7 and Proposition 5.6.

Corollary 5.10. *Let* $\sigma(\cdot) = \sigma_\tau(\cdot)$ *be a pseudospectral function of the system* (3.1) *and let* $V_{0,\sigma} \in [\mathfrak{H}_0, L^2(\sigma; H)]$ *be the corresponding isometry* (3.32). *Then* $V_{0,\sigma}$ *is a unitary operator if and only if* τ *is a self-adjoint (admissible) boundary parameter* (4.9). *If this condition is satisfied, then the equality* (4.14) *defines an extension* $\widetilde{T}^\tau \in \operatorname{Self}_0(T)$ *and the operators* $\widetilde{T}_0^\tau$ *(the operator part of* $\widetilde{T}^\tau$*) and* Λ_σ *are unitarily equivalent by means of* $V_{0,\sigma}$.

In the following theorem we give a criterion, that enables one to describe all pseudospectral functions in terms of an arbitrary (not necessarily admissible) boundary parameter τ.

Theorem 5.11. *Let the assumptions of Theorem 5.7 be satisfied and let T be the linear relation* (3.21)*. Then the following statements* (1)–(5) *are equivalent:*

(1) *each boundary parameter τ is admissible;*
(2) $\lim_{y\to\infty}\frac{1}{iy}M_4(iy)=0$ *and* $\lim_{y\to\infty} y\cdot \mathrm{Im}(M_4(iy)h,h)=+\infty,\ h\in\mathcal{H}_b,\ h\neq 0$*;*
(3) $\mathrm{mul}\, T=\mathrm{mul}\, T^*$*;*
(4) $\widetilde{\mathrm{Self}}(T)=\widetilde{\mathrm{Self}}_0(T)$*;*
(5) *statement of Theorem 5.7 holds for arbitrary boundary parameters τ.*

Proof. Proposition 5.3, (1) yields the equivalences (1)$\Leftrightarrow$ (4)$\Leftrightarrow$ (3). Since by Corollary 4.10 $M_4(\cdot)$ is the Weyl function of the boundary triplet $\dot{\Pi}$ for T^*, the equivalence (2)$\Leftrightarrow$ (3) is implied by [12, Remark 5.1]. Moreover, the equivalence (1) $\Leftrightarrow$(5) follows from Theorem 5.7. $\square$

Combining the results of this section with Assertion 3.11 we arrive at the following theorem.

Theorem 5.12. *The set of spectral functions of the system* (3.1) *is not empty if and only if* $\mathrm{mul}\, T=\{0\}$ *(for* $\mathrm{mul}\, T$ *see Assertion 3.6, (2)). If this condition is satisfied, then the sets of spectral and pseudospectral functions of the system* (3.1) *coincide and hence Theorems 5.7, 5.9, 5.11 and Corollary 5.10 are valid for spectral functions (instead of pseudospectral ones). Moreover, in this case equality* (4.14) *defines the operator $\widetilde{T}^\tau$ and statements of Theorem 5.9 and Corollary 5.10 hold with $\widetilde{T}=\widetilde{T}^\tau$ and V_σ in place of $\widetilde{T}_0=\widetilde{T}_0^\tau$ and $V_{0,\sigma}$ respectively.*

Corollary 5.13. *Let the operator $\Delta(t)$ be invertible a.e. on $\mathcal{I}$ and let the assumptions* (A1)-(A3) *from Sect. 4.1 be satisfied. Then the equality* (4.33) *together with formula* (1.8) *establishes a bijective correspondence between all boundary parameters τ of the form* (4.8) *and all spectral functions $\sigma(\cdot)=\sigma_\tau(\cdot)$ of the system* (3.1)*.*

Proof. If $\Delta(t)$ is invertible a.e. on $\mathcal{I}$, then the assumption (A4) is satisfied and $\mathrm{mul}\, T=\mathrm{mul}\, T^*=\{0\}$. Therefore the required statement is implied by Theorems 5.11 and 5.8. $\square$

6. Quasi-regular and regular systems

6.1. **Quasi-regular systems and their pseudospectral functions.** The following proposition directly follows from [22, 25].

Proposition 6.1. *For system* (3.1) *the following assertions are equivalent:*

(1) *System has maximal formal deficiency indices $N_+=N_-=m$.*
(2) *For any $\lambda\in\mathbb{C}$ each solution $y(\cdot,\lambda)$ of the homogeneous system* (3.2) *belongs to $\mathcal{L}^2_\Delta(\mathcal{I})$ (that is $\dim\mathcal{N}_\lambda=m,\ \lambda\in\mathbb{C}$).*
(3) *There exists $\lambda_0\in\mathbb{C}$ such that $\dim\mathcal{N}_{\lambda_0}=\dim\mathcal{N}_{\overline{\lambda}_0}=m$.*

Definition 6.2. Hamiltonian system (3.1) is said to be quasi-regular if at least one (and hence all) of the conditions (1)–(3) are satisfied.

Definition 6.3. System (3.1) is called regular if the coefficients $B(\cdot)$ and $\Delta(\cdot)$ are integrable on $\mathcal{I}$.

It follows from Definition 6.3 that system on a compact interval $\mathcal{I}=[a,b]$ is regular.

Proposition 6.4. *Assume that system* (3.1) *is regular. Then*

(1) *This system is quasi-regular and for any* $y \in \operatorname{dom} \mathcal{T}_{\max}$ *there exists the limit*

$$y(b) := \lim_{t \uparrow b} y(t). \tag{6.1}$$

Moreover, for any $\widetilde{h} \in H \oplus H$ *there exists* $y \in \operatorname{dom} \mathcal{T}_{\max}$ *such that* $y(b) = \widetilde{h}$.

(2) *Definition* (3.21) *of the linear relation* T *can be rewritten as*

$$T = \{\{\pi_\Delta y, \pi_\Delta f\} : \{y, f\} \in \mathcal{T}_{\max},\ \Gamma_{1a} y = 0,\ y(b) = 0\}. \tag{6.2}$$

Proof. Statement (1) is immediate from [4, Proposition 2.6]. Statement (2) follows from statement (1). □

Suppose that under the assumptions of Sect. 3.3 system (3.1) is quasi-regular. Then the equality (3.19) defines a function $\widehat{f}(\cdot) : \mathbb{R} \to H$ for any $\widetilde{f} \in \mathfrak{H}$ and hence Definitions 3.5, 3.9 and 3.10 can be reformulated as follows.

Definition 6.5. A distribution function $\sigma(\cdot) : \mathbb{R} \to [H]$ is called a q-pseudospectral function of the quasi-regular system (3.1) if $\widehat{f} \in \mathcal{L}^2(\sigma; H)$ for all $\widetilde{f} \in \mathfrak{H}$ and the equality $V\widetilde{f} := \pi_\sigma \widehat{f},\ \widetilde{f} \in \mathfrak{H}$, defines a partial isometry (the Fourier transform) $V = V_\sigma \in [\mathfrak{H}, L^2(\sigma; H)]$.

Definition 6.6. A q-pseudospectral function with $\ker V_\sigma = \operatorname{mul} T$ ($\ker V_\sigma = \{0\}$) is called a pseudospectral (resp. spectral) function (for $\operatorname{mul} T$ see Assertion 3.6).

Proposition 6.7. *Assume that system* (3.1) *is quasi-regular. Let* T *be a symmetric relation* (3.21) *and let*

$$L_0 = \{\widetilde{f} \in \mathfrak{H} : \widehat{f}(s) = 0,\ s \in \mathbb{R}\},$$

where $\widehat{f}(\cdot)$ *is defined by* (3.19). *Then*

$$\operatorname{mul} T = L_0. \tag{6.3}$$

Proof. Let $\widetilde{f} \in \operatorname{mul} T$ and $f(\cdot) \in \widetilde{f}$. Then according to Assertion 3.6, (2) there exists a function $y \in AC(\mathcal{I}; H \oplus H)$ such that $\{y, f\} \in \mathcal{T}_{\max}$ and (3.23) holds. Next, for fixed $s \in \mathbb{R}$ and $h \in H$ put $z = z(t) = \varphi_U(t, s)h$. Then $\{z, sz\} \in \mathcal{T}_{\max}$ and application of the Lagrange's identity (3.5) to $\{y, f\}$ and $\{z, sz\}$ gives

$$(f, z)_\Delta - s(y, z)_\Delta = [y, z]_b - (Jy(a), z(a)). \tag{6.4}$$

Here

$$(f, z)_\Delta = \int_{\mathcal{I}} (\Delta(t) f(t), \varphi_U(t, s)h)\, dt = \left(\int_{\mathcal{I}} \varphi_U^*(t, s)\Delta(t) f(t)\, dt, h \right) = (\widehat{f}(s), h)$$

and in view of the first equality in (3.23) one has $(y, z)_\Delta = 0$. Moreover, by (3.23) $\Gamma_{1a} y = 0$ and by (3.18) $\Gamma_{1a} z = 0$, which in view of (3.16) yields $(Jy(a), z(a)) = 0$. Observe also that according to (3.23) $[y, z]_b = 0$. Therefore (6.4) yields $(\widehat{f}(s), h) = 0,\ s \in \mathbb{R},\ h \in H$, and, consequently, $\widehat{f}(s) = 0,\ s \in \mathbb{R}$. Hence $\widetilde{f} \in L_0$, which proves the inclusion $\operatorname{mul} T \subset L_0$. On the other hand, for each pseudospectral function $\sigma(\cdot)$ one has $L_0 \subset \ker V_\sigma = \operatorname{mul} T$. Therefore the equality (6.3) is valid. □

Remark 6.8. It follows from (6.3) that in Definition 6.6 of a pseudo-spectral function the condition $\ker V_\sigma = \operatorname{mul} T$ can be replaced with $\ker V_\sigma = L_0$.

The following proposition holds for arbitrary (not necessarily quasi-regular) Hamiltonian system (3.1).

Proposition 6.9. *Let* $\sigma_0(\cdot) : \mathbb{R} \to [H]$ *be a distribution function such that for any compact interval* $\mathcal{I}_\beta = [a, \beta] \subset \mathcal{I}$ *it is a pseudospectral function of the restriction of the system* (3.1) *onto* $\mathcal{I}_\beta$. *Then* $\sigma_0(\cdot)$ *is a pseudospectral function of the system* (3.1) *in the sense of Definition 3.9.*

Proof. For a compact interval $\mathcal{I}_\beta = [a,\beta] \subset \mathcal{I}$ put $\mathfrak{H}_\beta := \{\widetilde{f} \in \mathfrak{H} : \Delta(t)f(t) = 0 \;\text{ a.e. on }\; (\beta, b) \;\text{ for each }\; f(\cdot) \in \widetilde{f}\}$ (in the sequel we identify $\mathfrak{H}_\beta$ with $L^2_\Delta(\mathcal{I}_\beta)$). Since each $\widetilde{f} \in \mathfrak{H}_b$ belongs to $\mathfrak{H}_\beta$ with some $\beta \in \mathcal{I}$, the equality (3.19) with $\widetilde{f} \in \mathfrak{H}_b$ defines a function $\widehat{f}(\cdot) \in \mathcal{L}^2(\sigma_0; H)$ such that $||\widehat{f}|| \le ||\widetilde{f}||$. Therefore the operator $V\widetilde{f} = \pi_{\sigma_0}\widehat{f}(\cdot)$, $\widetilde{f} \in \mathfrak{H}_b$, admits a continuation to a contraction $V \in [\mathfrak{H}, L^2(\sigma_0; H)]$. Let $L_{0,\beta} \subset \mathfrak{H}_\beta$ be the subspace (1.4) for the restriction of the system onto $\mathcal{I}_\beta$, let $L_0 = \bigcup\limits_{\beta \in \mathcal{I}} L_{0,\beta}$ and let $\mathfrak{H}_0 = \mathfrak{H} \ominus L_0$. Assume that $\widetilde{f} \in \mathfrak{H}_0$, $f(\cdot) \in \widetilde{f}$ and let $\widetilde{f}_\beta = \pi_\Delta(f(\cdot)\chi_{\mathcal{I}_\beta}(\cdot))$, $\beta \in \mathcal{I}$, where $\chi_{\mathcal{I}_\beta}(\cdot)$ is the indicator of $\mathcal{I}_\beta$. Then $\widetilde{f}_\beta \in \mathfrak{H}_\beta \ominus L_{0,\beta}$ and by Remark 6.8 $||V\widetilde{f}_\beta|| = ||\widetilde{f}_\beta||$. Now passage to the limit when $\beta \to b$ yields $||V\widetilde{f}|| = ||\widetilde{f}||$. Moreover, for any $\beta \in \mathcal{I}$ and $\widetilde{f} \in L_{0,\beta}$ one has $V\widetilde{f} = 0$ and, therefore, $V\widetilde{f} = 0$, $\widetilde{f} \in \overline{L}_0$. Hence V is a partial isometry with $\ker V = \overline{L}_0$ and, consequently, $\sigma_0(\cdot)$ is a q-pseudospectral function of the system (3.1). Therefore by Proposition 3.8 $\operatorname{mul} T \subset \overline{L}_0$. On the other hand, for each pseudospectral function $\sigma(\cdot)$ of the system (3.1) and for any $\beta \in \mathcal{I}$ one has $L_{0,\beta} \subset \ker V_\sigma = \operatorname{mul} T$ and hence $\overline{L}_0 \subset \operatorname{mul} T$. Thus $\ker V = \overline{L}_0 = \operatorname{mul} T$ and, consequently, $\sigma_0(\cdot)$ is a pseudospectral function. □

6.2. **The matrix $W(\lambda)$ and description of pseudo-spectral functions.** If system (3.1) is quasi regular, then in view of Lemma 4.1 there exists a surjective linear mapping

$$\Gamma_b = \begin{pmatrix} \Gamma_{0b} \\ \Gamma_{1b} \end{pmatrix} : \operatorname{dom} \mathcal{T}_{\max} \to H \oplus H \tag{6.5}$$

satisfying

$$[y,z]_b = (J\Gamma_b y, \Gamma_b z) = (\Gamma_{0b} y, \Gamma_{1b} z) - (\Gamma_{1b} y, \Gamma_{0b} z), \quad y, z \in \operatorname{dom} \mathcal{T}_{\max}. \tag{6.6}$$

Suppose that system (3.1) is quasi-regular and the following assumptions are fulfilled:

(Q1) U is the operator (3.9) satisfying (3.10), $\widetilde{U}$ is a J-unitary extension (3.11) of U and Γ_{ja}, $j \in \{0,1\}$, are the linear mappings (3.13) and (3.14).

(Q2) Γ_b is a surjective linear mapping (6.5) such that (6.6) holds.

Let $\varphi_U(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H, H \oplus H]$ and $\psi(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H, H\oplus H]$, $\lambda \in \mathbb{C}$, be solutions of (3.2) satisfying (3.18) and (4.31) respectively and let $Y(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H \oplus H]$ be a solution of (3.2) given by

$$Y(\cdot,\lambda) = (\varphi_U(\cdot,\lambda),\, \psi(\cdot,\lambda)) : H \oplus H \to H \oplus H. \tag{6.7}$$

Moreover, let $\varphi_U(\lambda)$ and $\psi(\lambda)$ be linear mappings from Lemma 3.1 corresponding to solutions $\varphi_U(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$. Then

$$\widetilde{U} Y(a,\lambda) = \begin{pmatrix} \Gamma_{0a}\varphi_U(\lambda) & \Gamma_{0a}\psi(\lambda) \\ \Gamma_{1a}\varphi_U(\lambda) & \Gamma_{1a}\psi(\lambda) \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ 0 & I_H \end{pmatrix} \tag{6.8}$$

and the equality

$$W(\lambda) = \begin{pmatrix} w_1(\lambda) & w_2(\lambda) \\ w_3(\lambda) & w_4(\lambda) \end{pmatrix} := \begin{pmatrix} \Gamma_{0b}\varphi_U(\lambda) & \Gamma_{0b}\psi(\lambda) \\ \Gamma_{1b}\varphi_U(\lambda) & \Gamma_{1b}\psi(\lambda) \end{pmatrix} : H \oplus H \to H \oplus H, \quad \lambda \in \mathbb{C} \tag{6.9}$$

defines an entire $[H \oplus H]$-valued function $W(\cdot)$. It is clear that

$$W(\lambda) = \Gamma_b Y(\lambda), \quad \lambda \in \mathbb{C}. \tag{6.10}$$

Proposition 6.10. *The operator function $W(\cdot)$ satisfies the identity*

$$W^*(\lambda) J W(\mu) - J = (\mu - \overline{\lambda}) \int_{\mathcal{I}} Y^*(t,\lambda)\Delta(t) Y(t,\mu)\, dt, \quad \lambda, \mu \in \mathbb{C}, \tag{6.11}$$

with $Y(\cdot,\lambda)$ given by (6.7). *Hence*

$$iJ - W^*(\lambda)(iJ)W(\lambda) \ge 0, \quad \lambda \in \mathbb{C}_+; \quad W^*(\lambda)(iJ)W(\lambda) = iJ, \quad \lambda \in \mathbb{R},$$

that is the operator $W(\lambda)$ is iJ-contractive for $\lambda \in \mathbb{C}_+$ and iJ-unitary for $\lambda \in \mathbb{R}$.

Proof. Let $\widetilde{h}\in H\oplus H,\ \lambda,\mu\in\mathbb{C}$ and let $y=Y(t,\mu)\widetilde{h},\ z=Y(t,\lambda)\widetilde{h}$. Then by (6.8) one has

$$(Jy(a),z(a))=(J\widetilde{U}Y(a,\mu)\widetilde{h},\widetilde{U}Y(a,\lambda)\widetilde{h})=(J\widetilde{h},\widetilde{h}).$$

Moreover, combining of (6.6) with (6.10) yields

$$[y,z]_b=(J\Gamma_bY(\mu)\widetilde{h},\Gamma_bY(\lambda)\widetilde{h})=(JW(\mu)\widetilde{h},W(\lambda)\widetilde{h})=(W^*(\lambda)JW(\mu)\widetilde{h},\widetilde{h}).$$

Applying now the Lagrange's identity (3.5) to $\{y,\mu y\}$ and $\{z,\lambda z\}$ one obtains

$$(\mu-\overline{\lambda})(y,z)_\Delta=(W^*(\lambda)JW(\mu)\widetilde{h},\widetilde{h})-(J\widetilde{h},\widetilde{h}). \tag{6.12}$$

Since

$$(y,z)_\Delta=\int_{\mathcal{I}}(\Delta(t)Y(t,\mu)\widetilde{h},Y(t,\lambda)\widetilde{h})\,dt=\left(\left(\int_{\mathcal{I}}Y^*(t,\lambda)\Delta(t)Y(t,\mu)\,dt\right)\widetilde{h},\widetilde{h}\right),$$

the equality (6.12) yields (6.11). □

In the following proposition we provide an explicit construction of the mapping Γ_b and the corresponding operator function $W(\lambda)$.

Proposition 6.11. *Let system* (3.1) *be quasi-regular, let the assumption (Q1) be fulfilled and let $Y(\cdot,\lambda)$ be given by* (6.7). *Then for each $y\in\operatorname{dom}\mathcal{T}_{\max}$ there exists the limit*

$$\Gamma_by:=\lim_{t\uparrow b}(-JY^*(t,0)Jy(t)),\quad y\in\operatorname{dom}\mathcal{T}_{\max}, \tag{6.13}$$

and the equality (6.13) *defines a surjective linear mapping $\Gamma_b:\operatorname{dom}\mathcal{T}_{\max}\to H\oplus H$ satisfying* (6.6). *Moreover, the corresponding operator function $W(\cdot)$ is*

$$W(\lambda)=\lim_{t\uparrow b}(-JY^*(t,0)JY(t,\lambda)),\quad \lambda\in\mathbb{C} \tag{6.14}$$

and the entries of the matrix (6.9) *admit the representation*

$$w_1(\lambda)=I+\lambda\int_{\mathcal{I}}\psi^*(t,0)\Delta(t)\varphi(t,\lambda)\,dt,\quad w_2(\lambda)=\lambda\int_{\mathcal{I}}\psi^*(t,0)\Delta(t)\psi(t,\lambda)\,dt, \tag{6.15}$$

$$w_3(\lambda)=-\lambda\int_{\mathcal{I}}\varphi^*(t,0)\Delta(t)\varphi(t,\lambda)\,dt,\quad w_4(\lambda)=I-\lambda\int_{\mathcal{I}}\varphi^*(t,0)\Delta(t)\psi(t,\lambda)\,dt. \tag{6.16}$$

Proof. The first statement and the equality (6.14) directly follow from [30, Proposition 4.8]. Next, in view of (6.14) $W(0)=I_{H\oplus H}$ and (6.11) yields

$$W(\lambda)=I-\lambda J\int_{\mathcal{I}}Y^*(t,0)\Delta(t)Y(t,\lambda)\,dt.$$

Now the immediate calculations give (6.15) and (6.16). □

The following assertion directly follows from Proposition 6.4.

Assertion 6.12. *Assume that in addition to the assumptions of Proposition 6.11 system* (3.1) *is regular. Then the equality*

$$\Gamma_by:=y(b)(=\lim_{t\uparrow b}y(t)),\quad y\in\operatorname{dom}\mathcal{T}_{\max} \tag{6.17}$$

defines a surjective linear mapping $\Gamma_b:\operatorname{dom}\mathcal{T}_{\max}\to H\oplus H$ satisfying (6.6) *and the corresponding operator function $W(\cdot)$ is*

$$W(\lambda)=Y(b,\lambda):=\lim_{t\uparrow b}Y(t,\lambda),\quad \lambda\in\mathbb{C}. \tag{6.18}$$

If $\varphi_U(t,\lambda)=\begin{pmatrix}\varphi_{0,U}(t,\lambda)\\ \varphi_{1,U}(t,\lambda)\end{pmatrix}$ $(\in[H,H\oplus H])$ and $\psi(t,\lambda)=\begin{pmatrix}\psi_0(t,\lambda)\\ \psi_1(t,\lambda)\end{pmatrix}$ $(\in[H,H\oplus H])$ are the block-matrix representations of $\varphi_U(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$, then

$$W(\lambda)=\begin{pmatrix}w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix}=\begin{pmatrix}\varphi_{0,U}(b,\lambda) & \psi_0(b,\lambda)\\ \varphi_{1,U}(b,\lambda) & \psi_1(b,\lambda)\end{pmatrix}:=\lim_{t\uparrow b}\begin{pmatrix}\varphi_{0,U}(t,\lambda) & \psi_0(t,\lambda)\\ \varphi_{1,U}(t,\lambda) & \psi_1(t,\lambda)\end{pmatrix}. \tag{6.19}$$

Remark 6.13. In the case of a compact interval $\mathcal{I}=[a,b]$ the function $W(\lambda)=Y(b,\lambda)$ in Assertion 6.12 is the monodromy matrix for the system (3.1) (see e.g. [2]).

Proposition 6.14. *Assume that system* (3.1) *is quasi-regular and let the assumptions (Q1) and (Q2) be fulfilled. Moreover, let the assumption (A4) from Sect. 4.1 be satisfied and let* $M(\cdot)$ *be the operator function defined by* (4.23)–(4.25)*. Then* $0\in\rho(M_2(\lambda))$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, *and for each* $\lambda\in\mathbb{C}\setminus\mathbb{R}$ *the operator function* $W(\cdot)$ *(see* (6.9)*) admits the representation*

(6.20)
$$W(\lambda)=\begin{pmatrix} w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix}=\begin{pmatrix} M_2^{-1}(\lambda) & M_2^{-1}(\lambda)m_0(\lambda)\\ -M_4(\lambda)M_2^{-1}(\lambda) & M_3(\lambda)-M_4(\lambda)M_2^{-1}(\lambda)m_0(\lambda)\end{pmatrix}.$$

Proof. It follows from (4.24) and the first equalities in (4.15) and (4.16) that

$$v_0(t,\lambda)=\varphi_U(t,\lambda)m_0(\lambda)-\psi(t,\lambda),\quad u(t,\lambda)=\varphi_U(t,\lambda)M_2(\lambda) \tag{6.21}$$

and application of operators Γ_{0b} and Γ_{1b} to equalities (6.21) with taking (4.15), (4.16) and (4.25) into account gives

$$0=w_1(\lambda)m_0(\lambda)-w_2(\lambda),\quad -M_3(\lambda)=w_3(\lambda)m_0(\lambda)-w_4(\lambda), \tag{6.22}$$

$$I=w_1(\lambda)M_2(\lambda),\quad -M_4(\lambda)=w_3(\lambda)M_2(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{6.23}$$

It follows from the first equality in (6.23) that $0\in\rho(M_2(\lambda))$ and $w_1(\lambda)=M_2^{-1}(\lambda)$. Moreover, the second equality in (6.23) yields $w_3(\lambda)=-M_4(\lambda)M_2^{-1}(\lambda)$. Substituting these values of $w_1(\lambda)$ and $w_3(\lambda)$ into (6.22) one gets the required equalities for $w_2(\lambda)$ and $w_4(\lambda)$. □

Suppose that system (3.1) is quasi-regular, that the assumptions (Q1) and (Q2) are satisfied and that for each $y\in\mathcal{N}$ the equality $\Gamma_{1a}y=0$ yields $y=0$ (that is the assumption (A4) from Sect. 4.1 is fulfilled).

Let $W(\cdot)$ be the operator function (6.9) and let τ be a boundary parameter (4.8). Then by (6.20) $M_4(\lambda)=-w_3(\lambda)w_1^{-1}(\lambda)$ and hence

$$C_0(\lambda)-C_1(\lambda)M_4(\lambda)=(C_0(\lambda)w_1(\lambda)+C_1(\lambda)w_3(\lambda))w_1^{-1}(\lambda),\quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

Therefore by Corollary 4.10 the operator $C_0(\lambda)w_1(\lambda)+C_1(\lambda)w_3(\lambda)$ is invertible and Definition 5.2 can be reformulated as follows.

Definition 6.15. A boundary parameter τ of the form (4.8) is admissible if the following two equalities hold:

$$\lim_{y\to\infty}\tfrac{1}{iy}w_1(iy)(C_0(iy)w_1(iy)+C_1(iy)w_3(iy))^{-1}C_1(iy)=0, \tag{6.24}$$

$$\lim_{y\to\infty}\tfrac{1}{iy}w_3(iy)(C_0(iy)w_1(iy)+C_1(iy)w_3(iy))^{-1}C_0(iy)=0. \tag{6.25}$$

Moreover, it follows from Proposition 5.3, (2) that in the case

$$\lim_{y\to\infty}\tfrac{1}{iy}w_3(iy)w_1^{-1}(iy)=0$$

a boundary parameter τ is admissible if and only if (6.24) is satisfied.

In the following theorem we show that in the case of a quasi-regular system the parametrization of pseudospectral functions given in Theorem 5.7 can be represented in a somewhat other form.

Theorem 6.16. *Let the assumptions just after Proposition 6.14 be satisfied and let* $W(\cdot)$ *be the operator function* (6.9)*. Then the equality*

$$m_\tau(\lambda)=(C_0(\lambda)w_1(\lambda)+C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda)+C_1(\lambda)w_4(\lambda)),\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{6.26}$$

together with the Stieltjes inversion formula (1.8) *gives a bijective correspondence between all admissible boundary parameters* τ *of the form* (4.8) *and all pseudospectral functions*

$\sigma(\cdot) = \sigma_\tau(\cdot)$ *of the system* (3.1) *(corresponding to the operator* U*). If in addition* $\operatorname{mul} T = \{0\}$*, then the above statement holds for spectral functions instead of pseudospectral ones.*

Proof. It follows from (4.33) and (6.20) that

$$\begin{aligned} m_\tau(\lambda) &= m_0(\lambda) + (C_0(\lambda)M_2^{-1}(\lambda) - C_1(\lambda)M_4(\lambda)M_2^{-1}(\lambda))^{-1}C_1(\lambda)M_3(\lambda) \\ &= (C_0(\lambda)M_2^{-1}(\lambda) - C_1(\lambda)M_4(\lambda)M_2^{-1}(\lambda))^{-1} \\ &\quad \times [(C_0(\lambda)M_2^{-1}(\lambda) - C_1(\lambda)M_4(\lambda)M_2^{-1}(\lambda))m_0(\lambda) + C_1(\lambda)M_3(\lambda)] \\ &= (C_0(\lambda)M_2^{-1}(\lambda) - C_1(\lambda)M_4(\lambda)M_2^{-1}(\lambda))^{-1} \\ &\quad \times [C_0(\lambda)M_2^{-1}(\lambda)m_0(\lambda) + C_1(\lambda)(M_3(\lambda) - M_4(\lambda)M_2^{-1}(\lambda)m_0(\lambda))] \\ &= (C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda)), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \end{aligned}$$

Now the required statements are implied by Theorems 5.7 and 5.12. □

In the following theorem we give a criterion that guaranties the validity of Theorem 6.16 for arbitrary (not necessarily admissible) boundary parameter τ.

Theorem 6.17. *Let the assumptions of Theorem 6.16 be satisfied and let*

$$\chi(\lambda) = (iw_3^*(\overline{\lambda}) + w_1^*(\overline{\lambda}))^{-1}(iw_3^*(\overline{\lambda}) - w_1^*(\overline{\lambda})), \quad \lambda \in \mathbb{C}_+. \tag{6.27}$$

Then the following statements are equivalent:

(1) *each boundary parameter* τ *is admissible;*

(2) $\lim_{y\to\infty} \frac{1}{iy} w_3(iy)w_1^{-1}(iy) = 0$ *and* $\lim_{y\to\infty} y \cdot \operatorname{Im}(w_3(iy)w_1^{-1}(iy)h, h) = -\infty$ *for each*
$h \in H, \quad h \neq 0$*;*

(3) $\lim_{y\to+\infty} y(||h|| - ||\chi(iy)h||) = +\infty, \;\; h \in H, \;\; h \neq 0$*;*

(4) *statements of Theorem 6.16 hold for arbitrary boundary parameters* τ*.*

Proof. The equivalence (1)⇔(2) is a consequence of Theorem 5.11. The equivalence (1)⇔(4) directly follows from Theorem 6.16. Next, in view of (6.20) $w_1^*(\overline{\lambda}) = M_2^{-1*}(\overline{\lambda})$, $w_3^*(\overline{\lambda}) = -M_2^{-1*}(\overline{\lambda})M_4(\lambda)$ and (6.27) admits the representation

$$\chi(\lambda) = (M_4(\lambda) + iI)^{-1}(M_4(\lambda) - iI) = (M_4(\lambda) - iI)(M_4(\lambda) + iI)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{6.28}$$

Let as before T be symmetric relation (3.21). Since by Corollary 4.10 $M_4(\lambda)$ is the Weyl function of the boundary triplet $\dot{\Pi}$ for T^*, it follows from (6.28) and [26] that $\chi(\lambda)$ is a characteristic function of T in the sense of [35]. Therefore according to [35, Theorem 3.3] $\operatorname{mul} T = \operatorname{mul} T^*$ if and only if the condition (3) is satisfied. This and Theorem 5.11 yield the equivalence (1)⇔(3). □

Similarly to Corollary 5.13 one proves the following corollary.

Corollary 6.18. *Let system* (3.1) *be quasi-regular, let the operator* $\Delta(t)$ *be invertible a.e. on* $\mathcal{I}$*, let the assumptions* (Q1) *and* (Q2) *be fulfilled and let* $W(\lambda)$ *be given by* (6.9)*. Then the equalities* (6.26) *and* (1.8) *give a bijective correspondence between all boundary parameters* τ *of the form* (4.8) *and all spectral functions* $\sigma(\cdot) = \sigma_\tau(\cdot)$ *of the system* (3.1)*.*

Remark 6.19. Description of m-functions of quasi-regular differential operators with invertible weight in the form close to (6.26) was obtained in [21].

6.3. **The case of the canonical system.** Recall that system (3.1) is called canonical if $B(t) = 0$, i.e., if it is of the form

$$Jy' = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}. \tag{6.29}$$

In this case the corresponding homogeneous system is

$$Jy' = \lambda\Delta(t)y, \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}. \tag{6.30}$$

Assertion 6.20. *Each quasi-regular canonical system* (6.29) *is regular.*

Proof. If system (6.29) is quasi-regular, then each solution $y(t) \equiv \widetilde{h}, \ \widetilde{h} \in H \oplus H$, of the equation $Jy' = 0$ belongs to $\mathcal{L}^2_\Delta(\mathcal{I})$. Therefore $\int_{\mathcal{I}} (\Delta(t)\widetilde{h}, \widetilde{h})\, dt < \infty, \ \widetilde{h} \in H \oplus H$, and hence $\Delta(\cdot)$ is integrable on $\mathcal{I}$. $\square$

Proposition 6.21. *Assume that canonical system* (6.29) *is regular. Moreover, let* $\varphi_U(\cdot, \lambda), \ \psi(\cdot, \lambda)(\in [H, H \oplus H])$ *be the operator solutions of* (6.30) *with*

$$\varphi_U(a, \lambda) = \begin{pmatrix} 0 \\ iI_H \end{pmatrix} : H \to H \oplus H, \quad \psi(a, \lambda) = \begin{pmatrix} -iI_H \\ 0 \end{pmatrix} : H \to H \oplus H \tag{6.31}$$

and let $W(\cdot)$ *be the operator function* (6.19). *Then the following conditions are equivalent:*

(1) *If* $h \in H$ *and* $\Delta(t)\{0, h\} = 0$ *a.e. on* $\mathcal{I}$, *then* $h = 0$.
(2) *The equality* $\ker w_1(\lambda) = \{0\}$ *holds for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
(3) *The equality* $\ker w_1(\lambda) = \{0\}$ *holds for some* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
(4) $\bigcap_{\lambda \in \mathbb{C}} \ker w_1(\lambda) = \{0\}$.

Proof. Assume that

$$U = (iI_H, 0) : H \oplus H \to H, \quad \widetilde{U} = \begin{pmatrix} 0 & -iI_H \\ iI_H & 0 \end{pmatrix} : H \oplus H \to H \oplus H. \tag{6.32}$$

Then $\widetilde{U}$ is a J-unitary extension (3.11) of U and the mappings (3.13) and (3.14) are $\Gamma_{0a} y = -iy_1(a), \ \Gamma_{1a} y = iy_0(a)$. Moreover, the corresponding solutions $\varphi_U(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ of (6.30) satisfying (6.8) are defined by (6.31). If in addition Γ_b is the mapping (6.17), then the assumptions (Q1) and (Q2) are fulfilled and by Assertion 6.12 the corresponding operator function $W(\cdot)$ is of the form (6.19).

Assume that condition (1) is satisfied. Let $y \in \mathcal{N}$ and let $\Gamma_{1a} y (= iy_0(a)) = 0$. Since y is a solution of the equation $Jy' = 0$, it follows that $y(t) = \{0, h\}, \ t \in \mathcal{I}$, with some $h \in H$. Moreover, $\Delta(t)y(t) = \Delta(t)\{0, h\} = 0$ a.e. on $\mathcal{I}$ and hence $h = 0$. Therefore $y = 0$ and, consequently, assumption (A4) from Sect 4.1 is satisfied. Hence by Proposition 6.14 $\ker w_1(\lambda) = 0, \ \lambda \in \mathbb{C} \setminus \mathbb{R}$, which proves the implication (1)⇒(2). The implications (2)⇒(3) and (3)⇒(4) are obvious.

Next assume that $\bigcap_{\lambda \in \mathbb{C}} \ker w_1(\lambda) = \{0\}$. If $h \in H$ and $\Delta(t)\{0, h\} = 0$ a.e. on $\mathcal{I}$, then for any $\lambda \in \mathbb{C}$ the function $y = y(t) = \{0, ih\}, \ t \in \mathcal{I}$, satisfies (6.30) and hence $y = \varphi_U(t, \lambda)h, \ \lambda \in \mathbb{C}$. Therefore $w_1(\lambda)h = \varphi_{0,U}(b, \lambda)h = y_0(b) = 0, \ \lambda \in \mathbb{C}$, and consequently $h = 0$. This proves the implication (4)⇒(1). $\square$

Corollary 6.22. *Let the assumptions be the same as in Proposition 6.21, let U be the operator* (6.32) *and let at least one (and hence all) the conditions (1)–(4) from this proposition be satisfied. Then statements of Theorem 6.16 are valid (with* $W(\lambda)$ *of the form* (6.19)*).*

Proof. As was shown in the proof of Proposition 6.21 the condition (1) yields the assumption (A4). Therefore statements of Theorem 6.16 are valid. $\square$

Remark 6.23. Recall [4, 14, 25] that Hamiltonian system (3.1) is called definite if $\mathcal{N} = \{0\}$ (for $\mathcal{N}$ see Definition 3.2). According to [16, 22] canonical system (6.29) is definite if and only if the function $\Delta(t)$ is of positive type, that is for some compact interval $\mathcal{I}_\beta \subset \mathcal{I}$ the operator $\int_{\mathcal{I}_\beta} \Delta(t)\, dt$ is invertible. Clearly, for definite systems the assumption (A4) is satisfied and hence Theorems 5.7 and 6.16 are valid.

Remark 6.24. (1) Let the assumptions of Proposition 6.21 be satisfied. For any $\widetilde{f} \in \mathfrak{H}$ put $y_{\widetilde{f}}(x) = J \int_{[x,b)} \Delta(t) f(t)\, dt, \ f(\cdot) \in \widetilde{f}$. Then in view of (6.2) one may represent statement (2) of Assertion 3.6 in the form of the equality

$$\operatorname{mul} T = \{\widetilde{f} \in \mathfrak{H} : y_{0,\widetilde{f}}(a) = 0 \ \text{ and } \ \Delta(x) y_{\widetilde{f}}(x) = 0 \ \text{ a.e. on } \mathcal{I}\}. \tag{6.33}$$

In this connection note that for a canonical system on a compact interval $\mathcal{I} = [a, b]$ the equality (6.3) with $\operatorname{mul} T$ of the form (6.33) is proved in [33, Lemma A.18].

(2) For canonical systems (6.29) on a compact interval $\mathcal{I} = [a, b]$ Corollary 6.22 follows from [33, Theorem A.13] and [2, Theorem 7.28]. Observe also that in [33] the admissibility condition for the parameter $\tau = \{C_0(\lambda), C_1(\lambda)\}$ is more complicated than our relations (6.24) and (6.25) (cf. formula (A.134) in [33]).

(3) Let $W(\lambda)$ be the monodromy matrix (6.19) for canonical system on a compact interval $\mathcal{I} = [a, b]$ and let $\chi(\lambda)$ be given by (6.27). If

$$||\chi(iy)|| \leq \gamma < 1, \quad y > 0, \tag{6.34}$$

then according to [2, Theorem 7.30] the assumption (A4) from Sect. 4.1 is satisfied and statement of Theorem 6.16 holds for arbitrary (not necessarily admissible) boundary parameter τ. Note that condition (3) in Theorem 6.17 provides a necessary and sufficient condition for validity of Theorem 6.16 with arbitrary boundary parameter τ and this condition is weaker than (6.34). Hence our Theorem 6.17 is a stronger result than Theorem 7.30 in [2].

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