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ON THE CARLEMAN ULTRADIFFERENTIABLE VECTORS OF A SCALAR TYPE SPECTRAL OPERATOR

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To Academician Yu. M. Berezansky in honor of his 90th jubilee

Abstract. A description of the Carleman classes of vectors, in particular the Gevrey classes, of a scalar type spectral operator in a reflexive complex Banach space is shown to remain true without the reflexivity requirement. A similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space, is generalized to the case of a scalar type spectral operator in a complex Banach space.

Never cut what you can untie.
Joseph Joubert

1. Introduction

The description of the *Carleman classes* of ultradifferentiable vectors, in particular the *Gevrey classes*, of a *normal operator* in a complex Hilbert space in terms of its *spectral measure* established in [10] (see also [12] and [11]) is generalized in [15, Theorem 3.1] to the case of a *scalar type spectral operator* in a complex *reflexive* Banach space.

Here, the reflexivity requirement is shown to be superfluous and a similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space (see, e.g., [12]), is generalized to the case of a scalar type spectral operator in a complex Banach space.

2. Preliminaries

For the reader's convenience, we shall outline in this section certain essential preliminaries.

2.1. **Scalar type spectral operators.**

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ and $E_A(\cdot)$ to be its *spectral measure* (the *resolution of the identity*), the operator's *spectrum* $\sigma(A)$ being the *support* for the latter [1, 4].

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the *normal* ones [23].

A scalar type spectral operator in complex Banach space has an *operational calculus* analogous to that of a *normal operator* in a complex Hilbert space [1, 3, 4]. To any Borel measurable function $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$, $\mathbb{C}$ is the *complex plane*), there corresponds a scalar type spectral operator

$$F(A) := \int_{\mathbb{C}} F(\lambda)\, dE_A(\lambda) = \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda)$$

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defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \,\middle|\, \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded* scalar type spectral operators on X defined in the same manner as for a *normal operator* (see, e.g., [3, 21]).

In particular,

$$A^n = \int_{\mathbb{C}} \lambda^n \, dE_A(\lambda) = \int_{\sigma(A)} \lambda^n \, dE_A(\lambda), \quad n \in \mathbb{Z}_+, \tag{2.1}$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of *nonnegative integers*).

The properties of the *spectral measure* $E_A(\cdot)$ and the *operational calculus*, exhaustively delineated in [1, 4], underly the entire subsequent discourse. Here, we shall outline a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [2, 4], i.e., there is such an $M > 0$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \le M. \tag{2.2}$$

The notation $\|\cdot\|$ has been recycled here to designate the norm in the space $L(X)$ of all bounded linear operators on X. We shall adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the *dual space* X^* as well.

For any $f \in X$ and $g^* \in X^*$, the *total variation* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space X and its dual X^*) is a *finite* positive Borel measure with

$$v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \le 4M\|f\|\|g^*\| \tag{2.3}$$

(see, e.g., [15]). Also (Ibid.), $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$) being an arbitrary Borel measurable function, for any $f \in D(F(A))$, $g^* \in X^*$, and an arbitrary Borel set $\sigma \subseteq \mathbb{C}$,

$$\int_\sigma |F(\lambda)| \, dv(f, g^*, \lambda) \le 4M\|E_A(\sigma)F(A)f\|\|g^*\|. \tag{2.4}$$

In particular,

$$\int_{\mathbb{C}} |F(\lambda)| \, dv(f, g^*, \lambda) = \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) \le 4M\|F(A)f\|\|g^*\|. \tag{2.5}$$

The constant $M > 0$ in (2.3)–(2.5) is from (2.2).

The following statement allowing to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures is also fundamental for our discussion.

Proposition 2.1. ([14, Proposition 3.1]) *. Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ and $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$) be Borel measurable function. Then $f \in D(F(A))$ iff*

(i) *For any $g^* \in X^*$, $\int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) < \infty$.*

(ii) $\sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \,|\, |F(\lambda)|>n\}} |F(\lambda)|\, dv(f, g^*, \lambda) \to 0 \quad as \quad n \to \infty.$

Subsequently, the frequent terms *"spectral measure"* and *"operational calculus"* will be abbreviated to *s.m.* and *o.c.*, respectively.

2.2. **The Carleman classes of vectors.** Let A be a densely defined closed linear operator in a complex Banach space $(X, \|\cdot\|)$ and $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers and

$$C^{\infty}(A) := \bigcap_{n=0}^{\infty} D(A^n).$$

The subspaces of $C^{\infty}(A)$

$$C_{\{m_n\}}(A) := \left\{ f \in C^{\infty}(A) \middle| \exists \alpha > 0 \ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n \in \mathbb{Z}_+ \right\},$$
$$C_{(m_n)}(A) := \left\{ f \in C^{\infty}(A) \middle| \forall \alpha > 0 \ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n,\ n \in \mathbb{Z}_+ \right\}$$

are called the *Carleman classes* of ultradifferentiable vectors of the operator A corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$ of *Roumieu* and *Beurling type*, respectively.

The inclusions

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \subseteq C^{\infty}(A) \subseteq X \tag{2.6}$$

are obvious.

If two sequences of positive numbers $\{m_n\}_{n=0}^{\infty}$ and $\{m'_n\}_{n=0}^{\infty}$ are related as follows:

$$\forall \gamma > 0 \ \exists c = c(\gamma) > 0 : \ m'_n \le c\gamma^n m_n, \quad n \in \mathbb{Z}_+,$$

we also have the inclusion

$$C_{\{m'_n\}}(A) \subseteq C_{(m_n)}(A), \tag{2.7}$$

the sequences being subject to the condition

$$\exists \gamma_1, \gamma_2 > 0,\ \exists c_1, c_2 > 0 : \ c_1\gamma_1^n m_n \le m'_n \le c_2\gamma_2^n m_n, \quad n \in \mathbb{Z}_+,$$

their corresponding Carleman classes coincide

$$C_{\{m_n\}}(A) = C_{\{m'_n\}}(A), \quad C_{(m_n)}(A) = C_{(m'_n)}(A). \tag{2.8}$$

Considering *Stirling's formula* and the latter,

$$\mathcal{E}^{\{\beta\}}(A) := C_{\{[n!]^{\beta}\}}(A) = C_{\{n^{\beta n}\}}(A),$$
$$\mathcal{E}^{(\beta)}(A) := C_{([n!]^{\beta})}(A) = C_{(n^{\beta n})}(A)$$

with $\beta \ge 0$ are the well-known *Gevrey classes* of strongly ultradifferentiable vectors of A of order β of Roumieu and Beurling type, respectively (see, e.g., [10, 11, 12]). In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the well-known classes of *analytic* and *entire* vectors of A, respectively [6, 19]; $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ (i.e., the classes $C_{\{1\}}(A)$ and $C_{(1)}(A)$ corresponding to the sequence $m_n \equiv 1$) are the classes of *entire* vectors of *exponential* and *minimal exponential type*, respectively (see, e.g., [22, 12]).

If the sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfies the condition

$$\textbf{(WGR)}\ \forall \alpha > 0 \ \exists c = c(\alpha) > 0 : \ c\alpha^n \le m_n, \quad n \in \mathbb{Z}_+, \tag{2.9}$$

the scalar function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad \lambda \ge 0, \quad (0^0 := 1) \tag{2.10}$$

first introduced by S. Mandelbrojt [13], is well-defined (cf. [12]). The function is *continuous*, *strictly increasing*, and $T(0) = 1$.

As is shown in [10] (see also [12] and [11]), the sequence $\{m_n\}_{n=0}^\infty$ satisfying the condition **(WGR)**, for a *normal operator* A in a complex Hilbert space X, the equalities

$$\begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)) \end{aligned} \tag{2.11}$$

are true, the normal operators $T(t|A|)$, $t > 0$, defined in the sense of the operational calculus for a normal operator (see, e.g., [3, 21]) and the function $T(\cdot)$ being replaceable with any *nonnegative*, *continuous*, and *increasing* on $[0,\infty)$ function $F(\cdot)$ satisfying

$$c_1F(\gamma_1\lambda) \le T(\lambda) \le c_2F(\gamma_2\lambda), \quad \lambda \ge R, \tag{2.12}$$

with some $\gamma_1, \gamma_2, c_1, c_2 > 0$ and $R \ge 0$, in particular, with

$$S(\lambda) := m_0 \sup_{n\ge 0} \frac{\lambda^n}{m_n}, \quad \lambda \ge 0, \quad \text{or} \quad P(\lambda) := m_0\left[\sum_{n=0}^\infty \frac{\lambda^{2n}}{m_n^2}\right]^{1/2}, \quad \lambda \ge 0,$$

(cf. [12]).

In [15, Theorem 3.1], the above is generalized to the case of a *scalar type spectral operator* A in a *reflexive* complex Banach space X. The reflexivity requirement dropped, proved were the inclusions

$$\begin{aligned} C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)) \end{aligned} \tag{2.13}$$

only, which is a deficiency for statements like [16, Theorem 5.1] and [18, Theorem 3.2].

3. The Carleman classes of a scalar type spectral operator

Theorem 3.1. *Let $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers satisfying the condition* **(WGR)** *(see (2.9)). Then, for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, equalities (2.11) are true, the scalar type spectral operators $T(t|A|)$, $t > 0$, defined in the sense of the operational calculus for a scalar type spectral operator and the function $T(\cdot)$ being replaceable with any nonnegative, continuous, and increasing on $[0,\infty)$ function $F(\cdot)$ satisfying (2.12).*

Proof. We are only to prove the inclusions inverse to (2.13), the rest, including the latter, having been proved in [15, Theorem 3.1].

Consider an arbitrary vector $f \in C_{\{m_n\}}(A)$ ($f \in C_{(m_n)}(A)$). Then necessarily, $f \in C^\infty(A)$ and for a certain $\alpha > 0$ (an arbitrary $\alpha > 0$), there is a $c > 0$ such that

$$\|A^nf\| \le c\alpha^n m_n, \quad n \in \mathbb{Z}_+, \tag{3.14}$$

(see Preliminaries).

For any $g^* \in X^*$,

$$\int_{\sigma(A)} T\left(\tfrac{1}{2\alpha}|\lambda|\right) dv(f, g^*, \lambda) = \int_{\sigma(A)} \sum_{n=0}^\infty \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f, g^*, \lambda) \tag{3.15}$$

by the *Monotone Convergence Theorem*;

$$= \sum_{n=0}^\infty \int_{\sigma(A)} \frac{|\lambda|^n}{2^n\alpha^n m_n} dv(f, g^*, \lambda) = \sum_{n=0}^\infty \frac{1}{2^n\alpha^n m_n} \int_{\sigma(A)} |\lambda|^n \, dv(f, g^*, \lambda)$$

by (2.5) and (2.1);

$$\leq \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \|A^n f\| \|g^*\|$$

by (3.14);

$$\leq 4Mc \sum_{n=0}^{\infty} \frac{1}{2^n} \|g^*\| = 8Mc\|g^*\| < \infty.$$

For an arbitrary $\varepsilon > 0$, one can fix an $N \in \mathbb{N}$ such that,

$$\frac{M^2 c}{2^{N-2}} < \varepsilon/2. \tag{3.16}$$

Due to the strong continuity of the *s.m.*, for any $n \in \mathbb{N}$,

$$\left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, there is a $K \in \mathbb{N}$ such that

$$\sum_{n=0}^{N} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| < \varepsilon/2 \tag{3.17}$$

whenever $k \geq K$.

Similarly to (3.15), for $k \geq K$, we have

$$\sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \middle| T(\frac{1}{2\alpha}|\lambda|) > k\right\}} T\left(\tfrac{1}{2\alpha}|\lambda|\right) \, dv(f, g^*, \lambda)$$

$$= \sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} \int_{\left\{\lambda \in \sigma(A) \middle| T(\frac{1}{2\alpha}|\lambda|) > k\right\}} |\lambda|^n \, dv(f, g^*, \lambda)$$

by (2.4) and (2.1);

$$\leq \sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| \|g^*\|$$

$$\leq \sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \Bigg[\sum_{n=0}^{N} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| \|g^*\|$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) \right\| \|A^n f\| \|g^*\| \Bigg]$$

by (2.2) and (3.14);

$$\leq \sup_{\{g^* \in X^* \,|\, \|g^*\|=1\}} \Bigg[\sum_{n=0}^{N} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| \|g^*\|$$

$$+ 4M^2 c \sum_{n=N+1}^{\infty} \frac{1}{2^n} \|g^*\| \Bigg]$$

$$\leq \sum_{n=0}^{N} \frac{1}{2^n \alpha^n m_n} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \middle| T(\tfrac{1}{2\alpha}|\lambda|) > k\right\}\right) A^n f \right\| + \frac{M^2 c}{2^{N-2}}$$

by (3.16) and (3.17);

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and we conclude that

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda\in\sigma(A) \middle| T(\frac{1}{2\alpha}|\lambda|)>k\right\}} T\left(\tfrac{1}{2\alpha}|\lambda|\right)\, dv(f,g^*,\lambda) \to 0 \quad \text{as} \quad k\to\infty. \tag{3.18}$$

By Proposition 2.1, (3.15) and (3.18) imply

$$f \in D(T(\tfrac{1}{2\alpha}|A|)).$$

Considering that for $f \in C_{\{m_n\}}(A)$, $\alpha > 0$ is fixed and for $f \in C_{(m_n)}(A)$, $\alpha > 0$ is arbitrary, we infer that

$$f \in \bigcup_{t>0} D(T(t|A|))$$

in the former case and

$$f \in \bigcap_{t>0} D(T(t|A|))$$

in the latter.

Since $f \in C_{\{m_n\}}(A)$ ($f \in C_{(m_n)}(A)$) is arbitrary, we have proved the inclusions

$$C_{\{m_n\}}(A) \subseteq \bigcup_{t>0} D(T(t|A|)),$$
$$C_{(m_n)}(A) \subseteq \bigcap_{t>0} D(T(t|A|)),$$

which along with their inverses (2.13) imply equalities (2.11) to be true. □

4. The Gevrey classes

The sequence $m_n := [n!]^\beta$ ($m_n := n^{\beta n}$) with $\beta > 0$ satisfying the condition **(WGR)** and the corresponding function $T(\cdot)$ being replaceable with $F(\lambda) = e^{\lambda^{1/\beta}}$, $\lambda \ge 0$, (see [15] for details, cf. also [12]), in [15, Corollary 4.1] describing the Gevrey classes of vectors of a scalar type spectral operator in a *reflexive* complex Banach space, the reflexivity requirement can be dropped as well and we have the following

Corollary 4.1. *Let $\beta > 0$. Then, for a A scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$,*

$$\mathcal{E}^{\{\beta\}}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}),$$
$$\mathcal{E}^{(\beta)}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta}}).$$

Corollary 4.1 generalizes the corresponding result of [10] (see also [12, 11]) for a *normal operator* A in a complex Hilbert space and, for $\beta = 1$, gives a description of the *analytic* and *entire* vectors of a scalar type spectral operator A in a complex Banach space.

5. The entire vectors of exponential type

Observe that the sequence $m_n \equiv 1$ generating the entire vectors of exponential type does not meet the condition **(WGR)** (see (2.9)) and thus, this case falls outside the realm of Theorem 3.1.

As is known (cf., e.g., [12, 7]), for a *normal operator* A in a complex Hilbert space X,

$$\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$$

and

$$\mathcal{E}^{(0)}(A) = \bigcap_{\alpha>0} E_A(\Delta_\alpha)X = E_A(\{0\})X = \ker A := \left\{f \in X \middle| Af = 0\right\}$$

($E_A(\cdot)$ is the *spectral measure* of A) with

$$\Delta_\alpha := \left\{\lambda \in \mathbb{C} \middle| |\lambda| \le \alpha\right\}, \quad \alpha > 0.$$

We are to generalize the above to the case of a *scalar type spectral operator* A in a complex Banach space X.

Theorem 5.1. *For a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$,*

(i) $\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$,

(ii) $\mathcal{E}^{(0)}(A) = \bigcap_{\alpha>0} E_A(\Delta_\alpha)X = E_A(\{0\})X = \ker A := \left\{f \in X \middle| Af = 0\right\}$,

where $\Delta_\alpha := \left\{\lambda \in \mathbb{C} \middle| |\lambda| \le \alpha\right\}$, $\alpha > 0$.

Proof. Let $f \in \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$ ($f \in \bigcap_{\alpha>0} E_A(\Delta_\alpha)X$), i.e.,

$$f = E_A\left(\left\{\lambda \in \mathbb{C} \mid |\lambda| \le \alpha\right\}\right) f$$

for some (any) $\alpha > 0$. Then, by the properties of the *o.c.*,

$$f \in C^\infty(A).$$

Furthermore, as follows form the *Hahn-Banach Theorem* and the properties of the *o.c.* (in particular, (2.1)),

$$\begin{aligned}
\|A^n f\| &= \left\|\int_{\mathbb{C}} \lambda^n \, dE_A(\lambda) f\right\| = \left\|\int_{\{\lambda\in\mathbb{C} | |\lambda|\le\alpha\}} \lambda^n \, dE_A(\lambda) f\right\| \\
&= \sup_{g^*\in X^*,\, \|g^*\|=1} \left|\left\langle \int_{\{\lambda\in\mathbb{C} | |\lambda|\le\alpha\}} \lambda^n \, dE_A(\lambda) f, g^* \right\rangle\right| \\
&= \sup_{g^*\in X^*,\, \|g^*\|=1} \left|\int_{\{\lambda\in\mathbb{C} | |\lambda|\le\alpha\}} \lambda^n \, d\langle E_A(\lambda) f, g^*\rangle\right| \\
&\le \sup_{g^*\in X^*,\, \|g^*\|=1} \int_{\{\lambda\in\mathbb{C} | |\lambda|\le\alpha\}} |\lambda|^n \, dv(f, g^*, \lambda) \\
&\le \sup_{g^*\in X^*,\, \|g^*\|=1} \alpha^n v(f, g^*, \{\lambda \in \mathbb{C} | |\lambda| \le \alpha\}) \\
&\qquad \text{by (2.3);} \\
&\le \sup_{g^*\in X^*,\, \|g^*\|=1} 4M\|f\|\|g^*\|\alpha^n \le 4M\left[\|f\| + 1\right]\alpha^n, \quad \alpha > 0,
\end{aligned}$$

which implies that $f \in \mathcal{E}^{\{0\}}(A)$ ($f \in \mathcal{E}^{(0)}(A)$).

Conversely, for an arbitrary $f \in \mathcal{E}^{\{0\}}(A)$ ($f \in \mathcal{E}^{(0)}(A)$),

$$\|A^n f\| \le c\alpha^n, \quad n \in \mathbb{Z}_+,$$

with some (any) $\alpha > 0$ and some $c > 0$.

Then, for any $\gamma > \alpha$ and $g^* \in X^*$, we have

$$\begin{aligned}
&\gamma^n v(f, g^*, \{\lambda \in \mathbb{C} | |\lambda| \ge \gamma\}) \le \int_{\{\lambda\in\mathbb{C} | |\lambda|\ge\gamma\}} |\lambda|^n \, dv(f, g^*, \lambda) \\
&\le \int_{\mathbb{C}} |\lambda|^n \, dv(f, g^*, \lambda) \qquad \text{by (2.5);} \\
&\qquad \le 4M\|A^n f\|\|g^*\| \le 4Mc\|g^*\|\alpha^n, \quad n \in \mathbb{Z}_+.
\end{aligned}$$

Therefore,

$$v(f, g^*, \{\lambda \in \mathbb{C} | |\lambda| \ge \gamma\}) \le 4Mc\|g^*\|\left(\frac{\alpha}{\gamma}\right)^n, \quad n \in \mathbb{Z}_+.$$

Considering that $\alpha/\gamma < 1$ and passing to the limit as $n \to \infty$, we conclude that

$$v(f, g^*, \{\lambda \in \mathbb{C} | |\lambda| \ge \gamma\}) = 0, \quad g^* \in X^*,$$

and the more so

$$\langle E_A(\{\lambda \in \mathbb{C} | |\lambda| \geq \gamma\})f, g^*\rangle = 0, \quad g^* \in X^*.$$

Whence, as follows from the *Hahn-Banach Theorem*,

$$E_A(\{\lambda \in \mathbb{C} | |\lambda| \geq \gamma\})f = 0,$$

which, considering that $\gamma > \alpha$ is arbitrary, by the *strong continuity* of the *s.m.*, implies that

$$E_A(\{\lambda \in \mathbb{C} | |\lambda| > \alpha\})f = 0.$$

Hence, by the *additivity* of the *s.m.*,

$$f = E_A(\{\lambda \in \mathbb{C} | |\lambda| \leq \alpha\})f + E_A(\{\lambda \in \mathbb{C} | |\lambda| > \alpha\})f = E_A(\{\lambda \in \mathbb{C} | |\lambda| \leq \alpha\})f,$$

which implies that $f \in \bigcup_{\alpha>0} E_A(\Delta_\alpha)X$ ($f \in \bigcap_{\alpha>0} E_A(\Delta_\alpha)X$). □

An immediate implication of Theorem 5.1 is the following generalization of the well-known result on the denseness of exponential type vectors of a normal operator in a complex Hilbert space (see, e.g., [12]), which readily follows by the *strong continuity* of the *s.m.* and joins a number of similar results of interest for approximation and qualitative theories (see [22, 12, 7, 8, 9]).

Corollary 5.1. *For a scalar-type spectral operator A in a complex Banach space $(X, \|\cdot\|)$,*

$$\overline{\mathcal{E}^{\{0\}}(A)} = X$$

($\bar{\cdot}$ is the closure of a set in the strong topology of X).

Hence, for any positive sequence $\{m_n\}_{n=0}^\infty$ satisfying the condition **(WGR)**, in particular, for $m_n = [n!]^\beta$ with $\beta > 0$, due to inclusion (2.7) with $m'_n \equiv 1$,

$$\mathcal{E}^{\{0\}}(A) \subseteq C_{(m_n)}(A),$$

which implies

$$\overline{C_{(m_n)}(A)} = X.$$

6. Final remarks

Observe that, for a normal operator in a complex Hilbert, equalities (2.11) have not only the set-theoretic but also a topological meaning [10] (see also [11, 12]). By analogy, this also appears to be true for a scalar type spectral operator in a complex Banach space, although the idea was entertained by the author neither in [15] nor here.

For a normal operator in a complex Hilbert space, Theorems 5.1 and 3.1 can be considered as generalizations of *Paley-Wiener Theorems* relating the smoothness of a square-integrable on the real axis $\mathbb{R}$ function $f(\cdot)$ to the decay of its *Fourier transform* $\hat{f}(\cdot)$ as $x \to \pm\infty$ [20], which precisely corresponds to the case of the *self-adjoint* differential operator $A = i\dfrac{d}{dx}$ (i is the *imaginary unit*) in the complex Hilbert space $L_2(\mathbb{R})$ [12]. Observe that, in $L_p(\mathbb{R})$ with $1 \leq p < \infty$, $p \neq 2$, the same operator fails to be spectral [5] (the domain of $A = i\dfrac{d}{dx}$ in $X = L_p(\mathbb{R})$, $1 \leq p < \infty$, is understood to be the subspace $W_p^1(\mathbb{R}) := \{f \in L_p(\mathbb{R}) | f(\cdot)$ is *absolutely continuous* on $\mathbb{R}$ and $f' \in L_p(\mathbb{R})\}$).

Theorem 3.1 entirely substantiates the proof of the *"only if" part* of [16, Theorem 5.1], where inclusions (2.13) turn out to be insufficient, and of [18, Theorem 3.2]. It appears to be fundamental for qualitative results of this nature (cf. [17]).

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