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DYNAMICAL SYSTEMS OF CONFLICT IN TERMS OF STRUCTURAL MEASURES

VOLODYMYR KOSHMANENKO AND INGA VERYGINA

This paper is dedicated to Yu. M. Berezansky on his 90th birthday

ABSTRACT. We investigate the dynamical systems modeling conflict processes between a pair of opponents. We assume that opponents are given on a common space by distributions (probability measures) having the similar or self-similar structure. Our main result states the existence of the controlled conflict in which one of the opponents occupies almost whole conflicting space. Besides, we compare conflicting effects stipulated by the rough structural approximation under controlled redistributions of starting measures.

1. INTRODUCTION

We study the emergence of space redistributions arising due to the conflict interaction between opponents (opposite sides, players). The roles of opponents may be played by various natural entities with alternative trends (for examples see [5, 7, 9, 11]). In turn, the conflicting space may appear as a territory, a living resource, an ordering queue, in general, any value admitting division (see [2, 6, 10, 16, 21]).

We begin with an observation that opponents (competing sides) of conflict processes usually are presented in a form of a similar or self-similar structure: cells, bacterias, trees, peoples, etc. That is why the description of opponents in many-dimensional terms is a more adequate in comparison with single-meaning values of their mutual powers. By this reason we propose in constructions of the conflict theory to use the non-deterministic ideology of Quantum Mechanics. In particular, we will describe the states of opponents in terms of distributions of probability measures. Furthermore, we will assume that these distributions have a certain similar or self-similar geometrical structure consistent with a preassigned division of the conflicting space.

Let us explain our approach in more details (see also [1, 4, 5, 11]–[15],[20]).

Denote by A, B a couple of opponents (alternative sides) living on a common resource space Ω. In what follows Ω is a compact of some metric space with the Borel σ-algebra $\mathcal{B}$ of its subsets. Let λ be a fixed σ-additive measure on $\mathcal{B}$ such that $\lambda(\Omega) = 1$. In the simplest case one can think that $\Omega = [0, 1]$ and λ denotes the usual Lebesgue measure.

We denote by $\mathcal{M}(\Omega)$ a family of all σ-additive finite signed measures on $\mathcal{B}$. The subset of positive measures is denoted by $\mathcal{M}^+(\Omega)$. For a probability measure μ we write $\mu \in \mathcal{M}_1^+(\Omega)$.

We suppose that opponents A, B at the initial moment of time are presented on Ω by a couple of different probability measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$. The conflict interaction

between A, B is represented by a discrete (or continuous) binary mapping $\divideontimes$ in $\mathcal{M}_1^+(\Omega)$:

$$\mu \divideontimes \nu = \mu', \quad \nu \divideontimes \mu = \nu',$$

which usually is non-commutative and nonlinear. We call each triple $\{\Omega, \mathcal{M}_1^+(\Omega), \divideontimes\}$ the *dynamical system of conflict* (DSC).

The main problem of the conflict theory is to study and describe the behavior of trajectories of DSC in terms of couples of probability measures:

$$(1) \qquad \left\{ \begin{array}{c} \mu \\ \nu \end{array} \right\} \xrightarrow{\divideontimes, t} \left\{ \begin{array}{c} \mu(t) \\ \nu(t) \end{array} \right\}, \quad \mu, \nu \in \mathcal{M}_1^+, \quad t \geq 0.$$

We suppose that the time evolution of DSC in the general case is governed by the system of differential equations

$$(2) \qquad \frac{d\mu}{dt} = \mu\Theta - \tau, \quad \frac{d\nu}{dt} = \nu\Theta - \tau,$$

where the conflict exponent $\Theta = \Theta(\mu, \nu)$ is a positive quadratic form which describes the power of conflict interaction at whole, and the measure-valued function $\tau = \tau(\mu, \nu)$ corresponds to the local confrontation between A, B. In what follows we come to the discrete time $t = N = 0, 1, \ldots$ and use the system of difference equations

$$(3) \qquad \left\{ \begin{array}{l} \mu^{N+1}(E) = \mu^N(E) + \mu^N(E)\Theta^N - \tau^N(E), \\ \nu^{N+1}(E) = \nu^N(E) + \nu^N(E)\Theta^N - \tau^N(E), \quad E \in \mathcal{B}, \end{array} \right.$$

where we omit normalization denominators.

In [18] it was proved (see also [12, 13, 19]) that each trajectory $\{\mu^N, \nu^N\}, N \geq 1$ starting with any couple of probability measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$, $\mu \neq \nu$, converges in the weak sense to a limit fixed point $\{\mu^\infty, \nu^\infty\}$. This point creates an equilibrium state for the system and is a compromise in the sense that $\mu^\infty \perp \nu^\infty$. Moreover, for each dynamical system of conflict $\{\Omega, \mathcal{M}_1^+(\Omega), \divideontimes\}$ given by (3), there exists the limit ω-set Γ^∞ [22]. It is an attractor consisting of all couples of mutually singular measures from $\mathcal{M}_1^+(\Omega)$. Thus,

$$\Gamma^\infty = \{\{\mu^\infty, \nu^\infty\} \mid \mu^\infty, \nu^\infty \in \mathcal{M}_1^+(\Omega), \ \mu^\infty \perp \nu^\infty\}.$$

It was proved also that each limit state is uniquely determined by the starting couple $\{\mu, \nu\}$, and moreover,

$$\mu^\infty = \mu_+, \quad \nu^\infty = \nu_-,$$

where μ_+, ν_- denote the normalized components of a classic Hanh–Jordan decomposition [3, 8, 23] of the signed measure $\omega = \mu - \nu = \omega_+ - \omega_-$. Thus

$$(4) \qquad \mu^\infty = \frac{\omega_+}{\omega_+(\Omega)} =: \mu_+, \quad \nu^\infty = \frac{\omega_-}{\omega_-(\Omega)} =: \nu_-.$$

In the simplest situation the dynamical system of conflict can be written in terms of coordinates of stochastic vectors $p, r \in \mathbb{R}_+^n, n \geq 2$ corresponding to opponent sides,

$$p_i^{N+1} = 1/z^N(p_i^N\Theta^N - \tau_i^N), \quad r_i^{N+1} = 1/z^N(r_i^N\Theta^N - \tau_i^N), \quad i = 1, \ldots, n.$$

Here we set $\Theta^N = (p^N, r^N)$ to be the inner product between the vectors p^N, r^N and $\tau_i^N = p_i^N r_i^N$. It was proved in [12, 13] that each trajectory $\{p^N, r^N\}_{N=0}^\infty$ starting with a couple of stochastic vectors $\{p^0 = p, r^0 = r\}$, $p \neq r$ converges with $N \to \infty$ to a fixed point $\{p^\infty, r^\infty\}$ which creates a compromise state, $p^\infty \perp r^\infty$. This state is uniquely determined by the starting couple $\{p, r\}$ and has an explicit coordinate representation

$$(5) \qquad p_i^\infty = \frac{d_i}{D} > 0, \quad i \in N_+, \quad r_k^\infty = -\frac{d_k}{D} > 0, \quad k \in N_-, \quad D = 1/2 \sum_{i=1}^n |d_i|,$$

$$p_i^\infty = 0, \quad i \notin N_+, \quad r_k^\infty = 0, \quad k \notin N_-,$$

where $d_i = p_i - r_i$ and $N_+ = \{i : d_i > 0\}$, $N_- = \{k : d_k < 0\}$. In [11, 15, 17, 16] we generalized the above constructions to cases of piece-wise uniformly distributed measures, self-similar, and similar structure measures.

In the present paper we study more specific questions connected with the property of similar structure measures μ, ν to have the weak approximation in terms of piece-wise uniformly distributed measures

$$\mu = \lim_{k \to \infty} \mu_k, \quad \nu = \lim_{k \to \infty} \nu_k.$$

We analyze the effects which are produced by the rough structural approximation and controlled redistribution (see below) of the starting measures. In other words we are interesting in all possible spatial and valued changes of the limit measures when μ_k, ν_k are subject to the structural redistributions, $\mu_k \to \tilde{\mu}_k, \nu_k \to \tilde{\nu}_k$, so that the limits

$$\tilde{\mu}^\infty = \lim_{k \to \infty} \tilde{\mu}_k^\infty, \quad \tilde{\nu}^\infty = \lim_{k \to \infty} \tilde{\nu}_k^\infty$$

become essentially different from μ^∞, ν^∞.

Our main result (it is hypothetical in the general case) reads as follows.

Given a couple of similar structure measures $\mu, \nu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$ with $\mathrm{supp}\mu = \Omega = \mathrm{supp}\nu$, for any $0 < \varepsilon < 1$ there exists a controlled structural redistribution of the measure μ such that on $1 \leq k < \infty$ step of the rough approximation, $\mu_k \to \tilde{\mu}_k$, The limit conflict state $\{\tilde{\mu}_k^\infty, \nu_k^\infty\}$ obeys the properties

$$\lambda(\mathrm{supp}\tilde{\mu}_k^\infty) \geq 1 - \varepsilon, \quad \tilde{\mu}_k^\infty \perp \nu_k^\infty.$$

It means that the controlled conflict under an appropriate strategy may lead to an expansion over most part of the territory.

In the paper we prove only a simplest version of this observation (see Theorem 4).

The paper consists of five sections. In Section 2 we briefly recall a general picture of DSC in terms of probability measures, in Section 3 the notions of similar and self-similar structure measures are presented, Section 4 contains the main results, and finally, in the last section we discuss the interpretation of the obtained results and their possible applications.

2. On dynamical systems of conflict in terms of abstract measures

Let us recall a general scheme of our approach to the conflict theory in terms of abstract measures (for more details, see [18, 19]). We will deal with dynamical systems $\{\Omega, \mathcal{M}_{1,\mathrm{ac}}^+(\Omega), \divideontimes\}$ of natural conflict (here $\mathcal{M}_{1,\mathrm{ac}}^+$ denotes a class of absolutely continuous measures). The term "natural" means that a conflict composition $\divideontimes$ is defined by some fixed law of the conflict interaction between opponents and their strategies do not change during the time evolution. In Sec. 4 we will discuss the dynamical systems with rough approximations and controlled redistributions when $\divideontimes$ is subjected to the strategical changes.

Let us consider an abstract variant of DSC with the discrete time

$$\mu^{N+1} = \mu^N \divideontimes \nu^N, \quad \nu^{N+1} = \nu^N \divideontimes \mu^N, \quad N = 0, 1, \ldots$$

Their state trajectories

(6)
$$\left\{ \begin{array}{c} \mu^N \\ \nu^N \end{array} \right\} \xrightarrow{\divideontimes} \left\{ \begin{array}{c} \mu^{N+1} \\ \nu^{N+1} \end{array} \right\}, \quad N = 0, 1, \ldots$$

are governed by the following *law of conflict dynamic*:

(7)
$$\left\{ \begin{array}{l} \mu^{N+1}(E) = \frac{1}{z^N}[\mu^N(E)(\Theta^N + 1) - \tau^N(E)], \\ \nu^{N+1}(E) = \frac{1}{z^N}[\nu^N(E)(\Theta^N + 1) - \tau^N(E)], \end{array} \right. \quad E \in \mathcal{B},$$

where the measures $\mu^0 = \mu$, $\nu^0 = \nu$ correspond to the initial state. The conflict exponent Θ^N in (7) is defined as

$$\Theta^N = \int_\Omega \int_\Omega \mathcal{K}(x,y)\varphi^N(x)\psi^N(y)\,dxdy,$$

where $\mathcal{K}(x,y)$ denotes the kernel of some positive bounded operator K in $L^2(\Omega,\ d\lambda)$ and

$$\varphi^N(x) = \sqrt{\rho^N(x)}, \quad \psi^N(x) = \sqrt{\sigma^N(x)},$$

where $\rho^N(x), \sigma^N(x)$ are the Radon-Nikodym derivatives of μ^N, ν^N with respect to λ. Thus,

$$\Theta_N = (K\varphi^N, \psi^N)_{L^2(\Omega,\ d\lambda)},$$

Further, τ^N in (7) stands for the occupation measure. Its values characterize the presence of opponents on opposite territories. By definition,

$$(8) \qquad \tau^N(E) = \nu^N(E_+) + \mu^N(E_-), \quad E_+ = E\bigcap\Omega_+, \quad E_- = E\bigcap\Omega_-,$$

where $\Omega = \Omega_-\bigcup\Omega_+$ corresponds to the Hahn–Jordan decomposition (see [8, 23]) of the starting signed measure $\omega = \mu - \nu$. Finally, the normalizing denominator in (7) is defined as

$$z^N = \Theta^N + 1 - W^N, \quad W^N = \mu^N(\Omega_-) + \nu^N(\Omega_+).$$

It is easy to see that all measures μ^N, ν^N, $N \geq 1$ in (7) are absolutely continuous and probability, i.e., $\mu^N, \nu^N \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$.

The DSC defined by (7) has two separate sets of fixed points. The first set contains all couples of identical measures from $\mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$. Indeed, if $\mu = \nu$, then Θ^N is a constant for all N and $\mu^N(E) = \mu(E) = \nu^N(E) = \nu(E)$ for each $E \in \mathcal{B}$. The second set is composed of measures $\mu, \nu \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$ which are orthogonal, $\mu \perp \nu$. In this case $\tau^N = 0 = W^N$ and $\Theta^N + 1 = z^N$ for all N. Due to (7) we find that $\mu^N = \mu$, $\nu^N = \nu$.

In all other cases, when the starting measures are different, $\mu \neq \nu$, and mutually non-singular the following theorem is true.

Theorem 1. *Let $\{\Omega, \mathcal{M}^+_{1,\mathrm{ac}}(\Omega), \divideontimes\}$ be a DSC generated by the system of difference equations (7). Then each its trajectory (6) starting with a couple of probability measures $\mu^0 = \mu, \nu^0 = \nu \in \mathcal{M}^+_{1,\mathrm{ac}}(\Omega)$, $\mu \neq \nu$ converges to a fixed point corresponding to a limit state $\{\mu^\infty, \nu^\infty\}$ with*

$$(9) \qquad \mu^\infty(E) = \lim_{N\to\infty}\mu^N(E), \quad \nu^\infty(E) = \lim_{N\to\infty}\nu^N(E), \quad E \in \mathcal{B},$$

where

$$(10) \qquad \mu^\infty(E) = \frac{\mu(E_+) - \nu(E_+)}{D} = \mu_+(E), \quad \nu^\infty(E) = -\frac{\mu(E_-) - \nu(E_-)}{D} = \nu_-(E)$$

with μ_+, ν_- defined by (4).

In (10) $D = 1/2\int_\Omega |\rho(x) - \sigma(x)|dx$ stands for the total difference between measures μ, ν.

If Ω is a finite set and μ, ν are stochastic vectors $p, r \in \mathbb{R}^n_{+,1}$, then coordinates of the limit states $\{p^\infty, r^\infty\}$ are described by (5). We refer to [12, 13, 18]) for the proof of this theorem.

3. The similar and self-similar structure measures

Let Ω be a compact set of some metric space and let λ be a fixed σ-additive measure on the Borel algebra of subsets. We suppose that $\lambda(\Omega) = 1$. Consider a specific class of measures as follows.

Fix $n > 1$, assume Ω is consecutively divided onto non-empty subsets (regions) on each kth level

$$
(11) \qquad \Omega = \bigcup_{i_1=1}^{n} \Omega_{i_1} = \bigcup_{i_1,i_2=1}^{n} \Omega_{i_1 i_2} = \bigcup_{i_1,i_2,\ldots,i_k=1}^{n} \Omega_{i_1 i_2 \ldots i_k} = \cdots, \quad k = 1, 2, \ldots
$$

such that all ratios

$$
(12) \qquad q_{k i_k} := \frac{\lambda(\Omega_{i_1 i_2 \ldots i_k})}{\lambda(\Omega_{i_1 i_2 \ldots i_{k-1}})}, \quad k \geq 1
$$

are independent of indices $i_1, i_2, \ldots, i_{k-1}$, where $\Omega_{i_0} = \Omega$. Besides we suppose that

$$
\inf_{k,i_k} \{ q_{k i_k} \} > 0 \quad \text{and} \quad \sum_{i_k=1}^{n} q_{k i_k} = 1.
$$

In what follows we fix some division of Ω with above properties.

We say that a probability measure μ from $\mathcal{M}_1^+(\Omega)$ belongs to the class of *similar structure measures*, write $\mu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$, if, besides (12), all ratios

$$
(13) \qquad p_{k i_k} := \frac{\mu(\Omega_{i_1 \ldots i_k})}{\mu(\Omega_{i_1 \ldots i_{k-1}})}, \quad k \geq 1
$$

with $\mu(\Omega_{i_1 \ldots i_{k-1}}) \neq 0$ are independent of the indices $i_1, i_2, \ldots, i_{k-1}$. We recall that $\mu(\Omega_{i_0}) = \mu(\Omega) = 1$. By this definition,

$$
p_{k i_k} \geq 0, \quad \sum_{i_k=1}^{n} p_{k i_k} = 1, \quad k \geq 1.
$$

In the particular case, where $q_{k i_k}$ and $p_{k i_k}$ in (12) and (13) do not depend on k, we say that the measure μ has the *self-similar structure*, write $\mu \in \mathcal{M}^{\mathrm{sss}}(\Omega)$. Clearly $\mathcal{M}^{\mathrm{sss}}(\Omega) \subset \mathcal{M}^{\mathrm{ss}}(\Omega)$.

We say that a measure μ has the *partly similar structure*, write $\mu \in \mathcal{M}^{\mathrm{pss}}(\Omega)$, if conditions (13) hold only up to some finite $k < \infty$. Inside the subsets $\Omega_{i_1 \ldots i_k}$, these measures have arbitrary distributions.

Lemma 1. *Each similar structure measure $\mu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$ is uniquely associated with a matrix*

$$
P = \{p_k\}_{k=1}^{\infty} = \{p_{k i_k}\}_{i_k=1, k=1}^{n, \infty}, \quad p_k \in \mathbb{R}_{+,1}^n,
$$

such that all vectors $p_k = (p_{k1}, p_{k2}, \ldots, p_{kn})$, $k \geq 1$ are stochastic, i.e.,

$$
p_{k i_k} \geq 0, \quad p_{k1} + p_{k2} + \cdots + p_{kn} = 1.
$$

In particular, if there exists k_0 such that all p_k, $k \geq k_0$, coincide with some fixed vector $p \in \mathbb{R}_{+,1}^n$, then $\mu \in \mathcal{M}^{\mathrm{sss}}(\Omega)$.

Each partly similar structural measure $\mu \in \mathcal{M}^{\mathrm{pss}}(\Omega)$ is uniquely associated with the finite matrix

$$
P_{k_0} = \{p_k\}_{k=1}^{k_0} = \{p_{k i_k}\}_{i_k=1, k=1}^{n, k_0}, \quad p_k \in \mathbb{R}_{+,1}^n, \quad k_0 < \infty.
$$

Proof. Given P define the measure μ_k for each $1 \leq k < \infty$ as follows:

$$
\mu_k(\Omega_{i_1 \ldots i_l}) := p_{1 i_1} \ldots p_{l i_l} \equiv p_{i_1 \ldots i_l}, \quad 1 \leq l \leq k,
$$

and put μ_k uniformly distributed inside subsets $\Omega_{i_1 \ldots i_k}$. By this construction,

$$\frac{\mu_k(\Omega_{i_1 \ldots i_l})}{\mu_k(\Omega_{i_1 \ldots i_{l-1}})} = \frac{p_{i_1 \ldots i_l}}{p_{i_1 \ldots i_{l-1}}} = p_{l i_l}, \quad 1 \leq l \leq k$$

and therefore $\mu_k \in \mathcal{M}^{\mathrm{pss}}(\Omega)$. Thus, μ_k is a piece-wise uniformly distributed measure and one can write for any Borel set $E \in \mathcal{B}$:

$$\mu_k(E) = \sum_{i_1, \ldots, i_k = 1}^{n} \rho_{i_1 \ldots i_k} \lambda_{i_1 \ldots i_k}(E),$$

where

$$\rho_{i_1 \ldots i_k} := \frac{p_{i_1 \ldots i_k}}{q_{i_1 \ldots i_k}}, \quad q_{i_1 \ldots i_k} := \lambda(\Omega_{i_1 \ldots i_k}) = q_{1 i_1} \ldots q_{k i_k}, \quad \lambda_{i_1 \ldots i_k} = \lambda \upharpoonright \Omega_{i_1 \ldots i_k}.$$

We put $\mu(\Omega_{i_1 \ldots i_l}) := \mu_k(\Omega_{i_1 \ldots i_l})$, $1 \leq l \leq k$ and for an arbitrary E define μ by the weak limit

$$\mu(E) = \lim_{k \to \infty} \mu_k(E), \quad E \in \mathcal{B}.$$

By this construction,

$$\mu(\Omega_{i_1 \ldots i_{k-1}}) = \mu_k(\Omega_{i_1 \ldots i_{k-1}}) = \sum_{i_k = 1}^{n} \mu_k(\Omega_{i_1 \ldots i_k}) = p_{i_1 \ldots i_{k-1}} \sum_{i_k = 1}^{n} p_{k i_k}, \quad k \geq 1$$

and therefore $\mu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$.

Vice versa, if $\mu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$, then elements p_{k, i_k} (see (13)) define the matrix P with above properties. $\qquad \square$

In what follows we will consider a couple of measures $\mu, \nu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$ associated with the matrices

$$(14) \qquad P = \{p_k\}_{k=1}^{\infty} = \{p_{k i_k}\}_{k, i_k = 1}^{\infty, n} \quad \text{and} \quad R = \{r_k\}_{k=1}^{\infty} = \{r_{k i_k}\}_{k, i_k = 1}^{\infty, n}.$$

In the case where $\Omega = [0, 1]$, the above measures μ_k have densities $\rho_k(x)$ which are the simple functions:

$$\rho_k(x) = p_{i_1 \ldots i_k} \chi_{\Omega_{i_1 \ldots i_k}}(x), \quad x \in [0, 1],$$

where $\chi_{\Omega_{i_1 \ldots i_k}}(x)$ denotes the characteristic function of subsets $\Omega_{i_1 \ldots i_k}$. The corresponding distribution functions $F_k(x)$ of μ_k are continuous piece-wise linear ones increasing from zero to 1. Obviously, the sequence $F_k(x), k = 1, 2, \ldots$ is point-wise convergent, i.e., there exists a left continuous function

$$F(x) = \lim_{k \to \infty} F_k(x)$$

which defines the distribution function of the measure μ on $[0, 1]$. Moreover, for continuous μ this convergence is uniform, since all $F_k(x)$ are uniformly bounded.

4. The rough controlled conflicts

Here we consider several variants of the rough and controlled conflict interaction in terms of the structural measures. Theorem 1 gives a general result of the mathematical theory of natural (non-controlled) conflict. However, conflict confrontations in the real situation occur usually with various deformations of the starting data. So, rather often the conflict actions are based on rather rough estimates of mutual forces, their approximate distributions, and relations. That is why there appear rough redistributions of opponents positions. We are aimed to study different effects arising from these reasons and describe a more adequate picture produced by the approximation method in terms of the rough controlled conflict. The language of the similar structure measures provides

the excellent tool for this aim. Simultaneously, some kind of notion of the controlled conflict appears naturally in this way.

We begin with a short review of the abstract mathematical scheme of the conflict theory. Let at the initial moment of time opponents A, B are distributed along the resource conflicting space Ω in according with probability measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$. Assume the law of conflict interaction is fixed by equations (7). Thus, there appears the dynamical system of conflict $\{\Omega, \mathcal{M}_1^+(\Omega), \divideontimes\}$. Note that in general it is not easy to find explicitly the limit distributions μ^∞, ν^∞ described by Theorem 1. The reason is that the Hanh–Jordan decomposition $\Omega = \Omega_+ \bigcup \Omega_-$ is non-constructive. It appears as a result of some approximate procedure (see [8]).

Nevertheless, if the conflict space Ω is divided into a finite set of regions (see (11)), one can fulfill the rough conflict "program" using firstly only one step of division: $\Omega = \bigcup_{i_1=1}^n \Omega_{i_1}$ and changing the starting measures μ, ν into piece-wise uniformly distributed ones μ_1, ν_1. Then one can come to the second more deep step of the rough "program", and so on. Importantly, the just described way of the rough approximation of the conflict interaction is often realized in applications.

Let us describe this approach in more detail. Let $\mu, \nu \in \mathcal{M}^{ss}(\Omega)$, $\mu \neq \nu$. Let us consider at the first step of the rough approximation the piece-wise uniformly distributed measures μ_1, ν_1 defined as follows:

$$(15) \qquad \mu_1(\Omega_{i_1}) = \mu(\Omega_{i_1}) = p_{1i_1}, \quad \nu_1(\Omega_{i_1}) = \nu(\Omega_{i_1}) = r_{1i_1}, \quad i_i = 1, 2, \ldots, n,$$

where p_{1i_1}, r_{1i_1} are coordinates of the vectors p_1, r_1 (see (14)). The Hahn–Jordan decomposition $\Omega = \Omega_{+,1} \bigcup \Omega_{-,1}$ corresponding to the signed measure $\omega_1 = \mu_1 - \nu_1$ has the form

$$\Omega_{+,1} = \bigcup_{i_1 \in N_{+,1}} \Omega_{i_1}, \quad \Omega_{-,1} = \bigcup_{i_1 \in N_{-,1}} \Omega_{i_1},$$

where

$$N_{+,1} = \{i_1 \mid \mu(\Omega_{i_1}) > \nu(\Omega_{i_1}), \quad N_{-,1} = \{i_1 \mid \mu(\Omega_{i_1}) < \nu(\Omega_{i_1})\}\}.$$

We assume for simplicity that the regions Ω_{i_1} with $\mu(\Omega_{i_1}) = \nu(\Omega_{i_1})$ are absent. Thus, due to Theorem 1 we have

$$(16) \quad \mu_1^\infty(\Omega_{i_1}) = \begin{cases} \frac{p_{1i_1} - r_{1i_1}}{D_1}, & i_1 \in N_{+,1} \\ 0, & i_1 \notin N_{+,1} \end{cases}, \quad \nu_1^\infty(\Omega_{i_1}) = \begin{cases} 0, & i_1 \notin N_{-,1} \\ -\frac{p_{1i_1} - r_{1i_1}}{D_1}, & i_1 \in N_{-,1} \end{cases},$$

where $D_1 = 1/2 \sum_{i_1} |p_{1i_1} - r_{1i_1}|$.

Let $\mu_1(\Omega_s) > 0$ for some $i_1 = s$ such that $s \in N_{-,1}$. Then $\mu_1^\infty(\Omega_s) = 0$. Thus, the region Ω_s is played over for the opponent A if the conflict game occurred at the level of the first rough approximation. This zero distribution for μ_1^∞ on Ω_s appears due to the starting inequality

$$\mu_1(\Omega_s) < \nu_1(\Omega_s), \quad s \in N_{-,1}.$$

Nevertheless, possibly there exists a subset $\tilde{\Omega}_s \subset \Omega_s$ with the opposite inequality

$$\mu_1(\tilde{\Omega}_s > \nu_1(\tilde{\Omega}_s).$$

We are interested in the folowing question. What is the maximal Lebesgue measure of such a subset $\tilde{\Omega}_s$? This subset can be saved for the opponent A under the next steps of approximation with a more thin division of the conflict space or under using the controlled redistribution (for the definition see below) of the μ_1 inside Ω_s. More precisely, we are interested in the following question. What is the biggest (in the sense of Lebesgue measure) subset $\tilde{\Omega}_s \subset \Omega_s$ such that at the second step of approximation, when Ω_s is subjected to the division $\Omega_s = \tilde{\Omega}_s \bigcup \tilde{\Omega}_s^c$, for the controlled redistribution $\mu_2 \to \tilde{\mu}_2$, the inequality

$$\tilde{\mu}_2(\tilde{\Omega}_s) > \nu(\tilde{\Omega}_s)$$

holds?

We define the procedure of the controlled redistribution as follows. We say that a measure $\mu \in \mathcal{M}_1^+(\Omega)$ is subjected to a controlled redistribution along a set $\Omega_s \subset \Omega$, if it is replaced by the measure $\tilde{\mu} \in \mathcal{M}_1^+(\Omega)$ which differs from μ only inside Ω_s.

One of the effects under the rough approximation with the consequent controlled redistributions we formulate as follows.

Theorem 2. *Let $\mu, \nu \in \mathcal{M}^{ss}(\Omega)$ and μ_1, ν_1 be defined by (15) at the first step of the rough structural approximation. Assume that*

$$0 < \mu_1(\Omega_s) < \nu_1(\Omega_s), \quad s = i_1 \in N_{-,1}$$

and therefore $\mu_1^\infty(\Omega_s) = 0$, $\nu_1^\infty(\Omega_s) > 0$. Let for the division

$$(17) \qquad\qquad \Omega_s = \tilde{\Omega}_s \bigcup \tilde{\Omega}_s^c,$$

$\tilde{\mu}_2, \nu_2 = \nu_1$ denote the measures at the second step of the rough structural approximation with the controlled redistribution of μ_2 obeying the inequalities

$$(18) \qquad\qquad \mu_1(\Omega_s) \geq \tilde{\mu}_2(\tilde{\Omega}_s) > \nu_2(\tilde{\Omega}_s).$$

Then

$$\tilde{\mu}_2^\infty(\tilde{\Omega}_s) > 0, \quad \nu_2^\infty(\tilde{\Omega}_s) = 0$$

and moreover

$$(19) \qquad \lambda(\tilde{\Omega}_s) \leq \sigma_s(\mu, \nu)\lambda(\Omega_s) \quad with \quad \sigma_s(\mu, \nu) = \frac{\mu_1(\Omega_s)}{\nu_1(\Omega_s)}.$$

In the extremal case, where the value $\lambda(\tilde{\Omega}_s)$ is maximal, both limiting distributions on $\tilde{\Omega}_s$ for opponents A, B are zero,

$$\tilde{\mu}_2^\infty(\tilde{\Omega}_s) = \nu_2^\infty(\tilde{\Omega}_s) = 0.$$

Proof. To prove inequality (19) we will apply the geometrical reasoning using the uniform distribution of ν_2 on Ω_s. It is easy to see that there exists a non-unique division (17) such that $\nu_2(\tilde{\Omega}_s) \equiv \nu_1(\tilde{\Omega}_s) < \tilde{\mu}_2(\tilde{\Omega}_s) = \mu_1(\Omega_s)$, where we produce a controlled redistribution $\mu_1 \to \tilde{\mu}_2$ such that $\tilde{\mu}_2(\tilde{\Omega}_s^c) = 0$ and $\tilde{\mu}_2(\tilde{\Omega}_s) = \mu_1(\Omega_s)$. Now the estimate (19) appears by the linear geometrical interpolation. Indeed, take any subset $\tilde{\Omega}_s \subset \Omega_s$ such that $\nu_1(\tilde{\Omega}_s) \leq \mu_1(\tilde{\Omega}_s)$ and put $\tilde{\mu}_2(\tilde{\Omega}_s) = \mu_1(\Omega_s)$, $\tilde{\mu}_2(\tilde{\Omega}_s^c) = 0$. Denote $\sigma_s = \sigma_s(\nu, \lambda) := \nu_1(\Omega_s)/\lambda(\Omega_s)$. Then obviously

$$\nu_2(\tilde{\Omega}_s) = \nu_1(\tilde{\Omega}_s) = \sigma_s \lambda(\tilde{\Omega}_s) \leq \tilde{\mu}(\tilde{\Omega}_s) = \mu_1(\Omega_s).$$

Therefore

$$\lambda(\tilde{\Omega}_s) \leq \mu(\Omega_s)/\sigma_s = \frac{\mu_1(\Omega_s)}{\nu_1(\Omega_s)}\lambda(\Omega_s).$$

This proves (19).

Clearly, in (19) we have the equality iff $\nu_2(\tilde{\Omega}_s) = \mu(\Omega_s) = \tilde{\mu}_2(\tilde{\Omega}_s)$. In this extremal case

$$(20) \qquad\qquad \sup \lambda(\tilde{\Omega}_s) = \frac{\mu_1(\Omega_s)}{\nu_1(\Omega_s)}\lambda(\Omega_s),$$

where the supremum is taken over all divisions (17) satisfying conditions (18). Then both sequences $\tilde{\mu}_2^N(\tilde{\Omega}_s)$, $\nu_1^N(\tilde{\Omega}_s)$, $N = 1, 2, ...$ converge to zero due to (16) (see the vector version of Theorem 1). $\qquad\square$

Therefore, if the opponent A associated with a measure μ at the first step of the structural approximation looses some region Ω_s, i.e., $\mu_1^\infty(\Omega_s) = 0$, then at the second step of structural approximation (under an additional division of Ω_s) using the controlled redistribution inside Ω_s it can return a part $\tilde{\Omega}_s$ of this region. The size of the returned

subset (in the sense Lebesgue measure is estimated by (19)). In turn, the opponent B wins the whole region Ω_s at the first step, however at the second step it gets zero inside $\tilde{\Omega}_s$ since it does not produce any preserving actions: $\nu_2^\infty(\tilde{\Omega}_s) = 0$, $\tilde{\mu}_2^\infty(\tilde{\Omega}_s) > 0$.

It is not easy to generalize Theorem 2 to the case of arbitrary similar structure measures since the local densities $\sigma_{i_1\ldots i_k} := \nu(\Omega_{i_1\ldots i_k})/\lambda(\Omega_{i_1\ldots i_k})$ take in general any values. That is why it is so hard to predict in applications the evolution of space redistributions when both opponents use different individual strategies under the controlled conflict interactions.

Let us consider a couple of examples.

Example 1. *From losses to gains, or an expansion on a new territory.*

Let $\Omega = \bigcup_{i=1}^3 \Omega_{1_i}$, $\lambda(\Omega_{1_i}) = 1/3$. Let us put in correspondence to opponents A, B the measures μ_1, ν_1 such that vectors $p_1 = \{\mu_1(\Omega_{i_1})\}, r_1 = \{\nu_1(\Omega_{i_1})\}$ from $\mathbb{R}_{+,1}^3$ have the following coordinates:

$$p_{i_1} = 1/3, \quad i_1 = 1, 2, 3, \quad r_1 = (2-\varepsilon)/9, \quad r_2 = (1+\varepsilon)/3, \quad r_3 = 4/9, \quad \varepsilon > 0.$$

Then, by using Theorem 1, we find by direct calculation that

$$\Omega_- = \Omega_3, \quad \nu_1^\infty(\Omega_-) = 1, \quad \lambda(\Omega_-) = 1/3.$$

In particular, the opponent A has a priority for two regions $\Omega_+ = \Omega_1 \bigcup \Omega_2$ with $\lambda(\Omega_+) = 2/3$. However, at the second step of partition, $\Omega = \bigcup_{i_1, i_2=1}^3 \Omega_{i_1 i_2}$, the measures μ_2, ν_2 have new, different signs of their priorities and therefore the Hanh–Jordan decomposition is changed

$$\Omega_- = \Omega_{22} \bigcup \Omega_{23} \bigcup \Omega_{32} \bigcup \Omega_{33}, \quad \lambda(\Omega_-) = 4/9.$$

Thus, $\nu_2^\infty(\Omega_-) = 1$ and the area of priority for the opponent B becomes larger. We can go to the next step of approximation and put

$$r_1 = (9-3\varepsilon)/27, \quad r_2 = (9+\varepsilon)/27, \quad r_3 = (9+2\varepsilon)/27.$$

It leads to a greater extension of the area of priority for B.

This example shows that the strategy of the directed priority: $r_1 < r_2 < r_3$ in comparison with the strategy of uniform distribution $p_i = 1/3$ leads to an extension of the occupation area under the conflict interaction on the way of increasing steps of the structural approximation.

Example 2. *Spectral gaps as a result of conflict interactions.*

Let $\mu, \nu \in \mathcal{M}^{\mathrm{sss}}(\Omega)$. Assume that at the first step of the rough approximation the measures μ_1, ν_1 are presented by vectors $p_1, r_1 \in \mathbb{R}_{+,1}^n$, $n \geq 3$ with the coordinates

$$p_{i_1} = 1/n, \quad i_1 = 1, \ldots, n, \quad r_s = (n-1)/n, \quad r_{i_1 \neq s} = 1/(n(n-1)), \quad 1 \leq s \leq n.$$

By this assumption, all $p_{i_1} > r_{i_1}, i_1 \neq s$, and $p_s < r_s$. Thus, $D_1 = r_s - p_s = (n-2)/n$ and

$$p_{i_1}^\infty = 1/(n-1), \quad r_{i_1}^\infty = 0, \quad \text{if} \quad i_1 \neq s, \quad \text{and} \quad p_s = 0, \quad r_s = 1.$$

Therefore, the opponent A loses the region Ω_s and B wins this region with probability 1. The spectral support of μ^∞ coincides with $\Omega_s^c = \Omega \backslash \Omega_s$.

Let us come to the second step of the structural approximation. Then we get

$$p_{i_1 i_2} = 1/n^2 \quad \text{for all} \quad i_1, i_2 = 1, \ldots, n$$

and

$$r_{i_1 i_2} = 1/(n-1)n, \quad \text{if both} \quad i_1, i_2 \neq s,$$
$$r_{i_1 i_2} = 1/n^2, \quad \text{if only one} \quad i_1 \text{ or } i_2 \neq s,$$
$$r_{ss} = ((n-1)/n)^2.$$

Now $D_2 = D_1 = 1 - 2/n$, $p_{i_1 i_2}^\infty = 1/(n-1)^2$, $r_{ss}^\infty = 1$, and $r_{i_1 i_2}^\infty = 0$, for the rest i_1, i_2. In particular,

$$p_{i_1 s}^\infty = p_{s i_2}^\infty = r_{s i_2}^\infty = r_{i_1 s}^\infty = 0, \quad i_1 \neq s \neq i_2.$$

Thus, at the second step of the rough approximation we observe gaps in the regions $\Omega_{i_1 s}, \Omega_{s i_2} \subset \Omega_{i_1}$, $i_1 \neq s$ for the opponent A and gaps for the opponent B in the regions $\Omega_{s i_2} \subset \Omega_s$, $i_2 \neq s$. If we will continue the rough approximation to the third step, similar gaps appear for the opponent B in regions $\Omega_{ss i_3} \subset \Omega_{ss}$, $i_3 \neq s$. And so on.

The general result in this direction reads as follows.

Theorem 3. *Let $\mu, \nu \in \mathcal{M}^{\mathrm{sss}}(\Omega)$. Assume $\mu \neq \nu$ and let Ω_s be such that*

$$0 < \mu_1(\Omega_s) < \nu_1(\Omega_s), \quad s \in N_{-,1},$$

and therefore $\mu_1^\infty(\Omega_s) = 0$, $\nu_1^\infty(\Omega_s) > 0$. Then with necessity there exist subsets $\Omega_{s i_2 \ldots i_k} \subset \Omega_s$ such that the opposite inequality holds

$$\mu_k(\Omega_{s i_2 \ldots i_k}) > \nu_k(\Omega_{s i_2 \ldots i_k}) \geq 0, \quad s i_2 \ldots i_k \in N_{+,k}$$

and therefore $\mu_k^\infty(\Omega_{s i_2 \ldots i_k}) > 0$, $\nu_k^\infty(\Omega_{s i_2 \ldots i_k}) = 0$, where

$$N_{+,k} := \{i_1 i_2 \ldots i_k | p_{i_1 i_2 \ldots i_k} > r_{i_1 i_2 \ldots i_k}\}.$$

Proof. Since $\mu \neq \nu$ and $0 < p_s < r_s$ there exists $m \neq s$ such that $p_m > r_m \geq 0$. If $r_m = 0$, then $p_{sm} = p_s \cdot p_m > r_{sm} = r_s \cdot r_m = 0$. Therefore $\mu_2(\Omega_{sm}) > \nu_2(\Omega_{sm})$ and $\mu_2^\infty(\Omega_{sm}) > 0$, $\nu_2^\infty(\Omega_{sm}) = 0$. In this case the theorem is proved.

If $p_m > r_m \geq 0$ and $r_m \neq 0$, then $p_m/r_m > 1$ and $(p_m/r_m)^k \to \infty$, with $k \to \infty$. Thus $(p_m/r_m)^k > p_s/r_s$ and $p_s \cdot (p_m)^k > r_s \cdot (r_m)^k$ for some finite k. Therefore, if for all $i_2 = m, \ldots, i_k = m$ the conditions $p_{s i_2 \ldots i_k} > r_{s i_1 \ldots i_k} > 0$ hold, then $s i_2 \ldots i_k \in N_{+,k}$, and by Theorem 1 we get

$$p_{sm \ldots m}^\infty = \mu_k^\infty(\Omega_{s i_2 \ldots i_k}) > 0, \quad r_{sm \ldots m}^\infty = \nu_k^\infty(\Omega_{s i_2 \ldots i_k}) = 0, \quad i_2 = m, \quad \ldots, \quad i_k = m.$$

$\square$

We are able to estimate the maximal value of $\mu_k^\infty(\Omega_{s i_2 \ldots i_k}) > 0$ in Theorem 3.

Proposition 1.

$$\max_k \mu_k^\infty(\Omega_{s i_2 \ldots i_k}) \leq p_s.$$

Proof. By Theorem 1,

$$\mu_k^\infty(\Omega_{s i_2 \ldots i_k}) = 1/D_k \sum_{s i_2 \ldots i_k \in N_{+,k}} (p_{s i_2 \ldots i_k} - r_{s i_2 \ldots i_k})$$

$$= 1/D_k \sum_{s i_2 \ldots i_k \in N_{+,k}} (p_s(p_{i_2} \cdots p_{i_k} - r_{i_2} \cdots r_{i_k}) - (r_s - p_s) \cdot r_{i_2} \cdots r_{i_k})$$

$$\leq p_s/D_k \sum_{i_2 \ldots i_k \in N_{+,k-1}} (p_{i_2 \ldots i_k} - r_{i_2 \ldots i_k}) \leq p_s/D_k \cdot D_{k-1} \leq p_s,$$

where we used the following lemma. $\square$

Lemma 2. *For any couple of measures $\mu, \nu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$, $\mu \neq \nu$ the variation distance $D_k = D(\mu_k, \nu_k)$ between their kth structural approximated variants μ_k, ν_k creates a non-decreasing sequence*

$$D_k \leq D_{k+1}, \quad k \geq 1.$$

Proof. By definition, $D_k = 1/2 \sum\limits_{i_1 i_2 \ldots i_k = 1}^{n} |p_{i_1 i_2 \ldots i_k} - r_{i_1 i_2 \ldots i_k}|$. Since

$$1 = p_{(k+1)i_1} + \cdots + p_{(k+1)i_n} = r_{(k+1)i_1} + \cdots + r_{(k+1)i_n}$$

we have

$$D_k = 1/2 \sum_{i_1 \ldots i_k = 1}^{n} |p_{i_1 \ldots i_k}(p_{(k+1)i_1} + \cdots + p_{(k+1)i_n}) - r_{i_1 \ldots i_k}(r_{(k+1)i_1} + \cdots + r_{(k+1)i_n})|$$

$$\leq 1/2 \sum_{i_1 \ldots i_{k+1} = 1}^{n} |p_{i_1 \ldots i_{k+1}} - r_{i_1 \ldots i_{k+1}}| = D_{k+1}$$

□

Theorem 4. *Given a couple of the similar structure measures $\mu, \nu \in \mathcal{M}^{\mathrm{ss}}(\Omega)$ assume $\Omega_{i_1 \ldots i_k} = (1/n)^k, \forall k$ and*

(21) $$\mathrm{supp}\mu = \Omega = \mathrm{supp}\nu.$$

Then for any $0 < \varepsilon < 1$ there exist controlled structural redistributions of the starting measure μ such that on kth step, $1 \leq k < \infty$, of the rough controlled approximations $\mu_k \to \tilde{\mu}_k$, $\nu_k \to \nu_k$ the limit conflict state $\{\tilde{\mu}_k^\infty, \nu_k^\infty\}$ obeys the properties

(22) $$\mathrm{supp}\tilde{\mu}_k^\infty = \Omega_{k,+}, \quad \lambda(\Omega_{k,+}) \geq 1 - \varepsilon, \quad \mathrm{supp}\nu_k^\infty = \Omega_{k,-}, \quad \lambda(\Omega_{k,-}) \leq \varepsilon,$$

where

$$\Omega = \Omega_{k,+} \bigcup \Omega_{k,-}$$

denotes the Hahn–Jordan decomposition corresponding to the signed measure $\tilde{\omega}_k = \tilde{\mu}_k - \nu_k$.

Proof. Without loss of generality we put $q_{ki} = 1/n$ for all k, i and assume that all $p_{ki_k} \neq 0 \neq r_{ki_k}$. Moreover, we can assume that for some single $1 \leq s \leq n$

(23) $$p_{ks} < r_{ks} \quad \text{and} \quad p_{ki_k} > r_{ki_k}, \quad i_k \neq s, \quad \forall k \geq 1.$$

Then at the first step of the rough approximation we have

$$\mathrm{supp}\mu_1^\infty = \Omega_{1,+} = \bigcup_{i_1 \neq s} \Omega_{i_1}, \quad \lambda(\Omega_{i_1}) = 1 - 1/n.$$

So, the theorem is proved if $1/n < \varepsilon$. Assume $(1/n)^2 < \varepsilon < 1/n$. Show that (22) may be reached at the second step of the rough controlled approximation. Indeed, replace all p_{1i_1}, $i_i \neq s$ with $\tilde{p}_{1i_1} = r_{1i_1} + \delta/(n-1)$ where δ is chosen so that $\tilde{p}_{1s} = r_{1s} - \delta$ satisfies the conditions

$$(n-1)n^{-1}r_{1s} < \tilde{p}_{1s} < r_{1s}$$

and

$$\tilde{p}_{1s}\tilde{p}_{2i_2} > r_{1s}r_{2i_2}, \quad i_2 \neq s, \quad \tilde{p}_{1s}\tilde{p}_{2s} < r_{1s}r_{2s}.$$

Then for the controlled conflict with the division

$$\Omega = \bigcup_{i_1 \neq s} \Omega_{i_1} \bigcup_{i_2 = 1}^{n} \Omega_{si_2}$$

for $\tilde{\mu}_2, \nu_2$ we obtain

$$\mathrm{supp}(\tilde{\mu}_2^\infty) = \Omega_{2,+} = \bigcup_{i_1 \neq s} \Omega_{i_1} \bigcup_{i_2 \neq s} \Omega_{si_2}, \quad \lambda(\Omega_{2,+}) = 1 - (1/n)^2 > 1 - \varepsilon,$$

$$\mathrm{supp}(\nu_2^\infty) = \Omega_{2,-} = \Omega_{ss}, \quad \lambda(\Omega_{ss}) = (1/n)^2 < \varepsilon.$$

If $(1/n)^3 < \varepsilon < (1/n)^2$, then one can reach (22) at $k = 3$ step of the rough controlled approximation with an appropriate δ. And so on.   □

5. **Discussion**

Let us discuss some interpretation of the above results from the point of view of possible applications.

We recall that for models of DSC which describe natural conflicts its trajectories are governed by some law of conflict interaction which is independent of the time. In this case each limit state $\{\mu^\infty, \nu^\infty\}$ of the system is a fixed point (an equilibrium state) defined by the starting couple of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$ (see Theorems 1).

In other situation when the law of conflict interaction may be changed at any moment of time, we deal with the *controlled conflict*. In other words, such changes mean the choice of the local strategies. In the present paper we discussed only the simplest versions of models with the controlled conflict. They were reduced to redistributions of the starting measures $\mu, \nu \in \mathcal{M}^{ss}(\Omega)$ at kth steps of their structural approximations and mean changes of the vectors p_l, $l \leq k$ in the matrix P (see Lemma 1) to other vectors $\tilde{p}_l \in \mathbb{R}_{+,1}^n$. These changes where aimed to get the new limit states $\{\tilde{\mu}^\infty, \tilde{\nu}^\infty\}$ different from the ones in the case of natural conflict.

In Section 4 it was shown that the limit result of the natural conflict may be essentially changed. So, in the situation of complete defeat for the opponent A in a region Ω_s, when from the condition $0 < \mu(\Omega_s) < \nu(\Omega_s)$ it follows that $\mu^\infty(\Omega_s) = 0$, i.e., there appears a limit gap, one can produce some redistribution of μ inside Ω_s in such a way that after the controlled conflict interaction, A reaches a victory in a subregion $\tilde{\Omega} \subset \Omega_s$, i.e., $\tilde{\mu}^\infty(\tilde{\Omega}) > 0$. Moreover, in Theorem 2 we got estimates both for the value of $\tilde{\mu}^\infty(\tilde{\Omega})$ and for the size of area with a limit priority, i.e., for $\lambda(\tilde{\Omega})$. Clearly, these estimates depend on the structure of divisions (11) and the values of redistributions. Theorem 4 shows that under an appropriate redistribution for one of the starting measure, the limit result of the controlled conflict might be very successful for one opponent and extremely bad for the other (it looses almost whole territory, see (22)).

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