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# ON THE GENERATION OF BEURLING TYPE CARLEMAN ULTRADIFFERENTIABLE $C_0$-SEMIGROUPS BY SCALAR TYPE SPECTRAL OPERATORS

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Abstract. A characterization of the scalar type spectral generators of Beurling type Carleman ultradifferentiable $C_0$-semigroups is established, the important case of the Gevrey ultradifferentiability is considered in detail, the implementation of the general criterion corresponding to a certain rapidly growing defining sequence is observed.

## 1. Introduction

The problem of finding conditions necessary and sufficient for a densely defined closed linear operator $A$ in a complex Banach space $X$ to be the generator of a $C_0$-semigroup $\{S(t)|t \geq 0\}$ with a certain regularity property such as strong differentiability or analyticity of its orbits on $(0, \infty)$ and thus, of the *weak/mild solutions* of the associated abstract evolution equation

$$y'(t) = Ay(t), \quad t \geq 0,$$

[1, 8] is central in qualitative theory.

The well known general generation criteria of analytic and (infinite) differentiable $C_0$-semigroups [14, 26, 27, 31, 32] (cf. also [8]) contain restrictions on the location of the generator's *spectrum* in the complex plane and on its *resolvent* behavior. As is shown in [18, 19, 21], when the potential generators are selected from the class of *scalar type spectral operators* (see Preliminaries), the restrictions of the second kind can be dropped in the foregoing and other cases, which makes the results more transparent, easier to handle, and inherently qualitative.

The characterization of the scalar type spectral generators of Roumieu type Gevrey ultradifferentiable $C_0$-semigroups found in [19] is generalized in [21] to the case of the Roumieu type Carleman ultradifferentiable $C_0$-semigroups. However, neither in [19], nor in [21], the case of Beurling type ultradifferentiability has been treated.

In the present paper, we are to establish a generation criterion of a Beurling type Carleman ultradifferentiable $C_0$-semigroup corresponding to a sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ by a scalar type spectral operator, consider in detail the important case of the Gevrey ultradifferentiability, and observe the implementation of the general criterion corresponding to a certain rapidly growing defining sequence.

## 2. Preliminaries

For the reader's convenience, we shall outline here certain essential preliminaries.

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2.1. **Scalar Type Spectral Operators.** Henceforth, unless specified otherwise, $A$ is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ and $E_A(\cdot)$ to be its *spectral measure* (the *resolution of the identity*), the operator's *spectrum* $\sigma(A)$ being the *support* for the latter [4, 7].

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the *normal* ones [30].

A scalar type spectral operator in complex Banach space has an *operational calculus* analogous to that of a *normal operator* in a complex Hilbert space [4, 6, 7]. To any Borel measurable function $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$, $\mathbb{C}$ is the *complex plane*), there corresponds a scalar type spectral operator

$$F(A) := \int_{\mathbb{C}} F(\lambda)\, dE_A(\lambda) = \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda)$$

defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$
$$D(F(A)) := \left\{ f \in X \,\middle|\, \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded* scalar type spectral operators on $X$ defined in the same manner as for a *normal operator* (see, e.g., [6, 28]).

In particular,

$$A^n = \int_{\mathbb{C}} \lambda^n\, dE_A(\lambda) = \int_{\sigma(A)} \lambda^n\, dE_A(\lambda), \quad n \in \mathbb{Z}_+, \tag{2.1}$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of *nonnegative integers*).

If a scalar type spectral operator $A$ generates $C_0$-semigroup of linear operators, it is of the form

$$e^{tA} = \int_{\mathbb{C}} e^{t\lambda}\, dE_A(\lambda) = \int_{\sigma(A)} e^{t\lambda}\, dE_A(\lambda), \quad t \ge 0$$

[2, 18, 25].

The properties of the *spectral measure* $E_A(\cdot)$ and the *operational calculus*, exhaustively delineated in [4, 7], underly the entire subsequent discourse. Here, we shall outline a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [5, 7], i.e., there is such an $M > 0$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \le M. \tag{2.2}$$

The notation $\|\cdot\|$ has been recycled here to designate the norm in the space $L(X)$ of all bounded linear operators on $X$. We shall adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the *dual space* $X^*$ as well.

For any $f \in X$ and $g^* \in X^*$, the *total variation* $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the *pairing* between the space $X$ and its dual $X^*$) is a *finite* positive Borel measure with

$$v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \le 4M\|f\|\|g^*\| \tag{2.3}$$

(see, e.g., [19, 20]). Also (Ibid.), $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$) being an arbitrary Borel measurable function, for any $f \in D(F(A))$, $g^* \in X^*$, and an arbitrary Borel set $\sigma \subseteq \mathbb{C}$,

$$\int_\sigma |F(\lambda)|\, dv(f, g^*, \lambda) \le 4M \|E_A(\sigma)F(A)f\| \|g^*\|. \tag{2.4}$$

In particular,

$$\int_{\mathbb{C}} |F(\lambda)|\, dv(f, g^*, \lambda) = \int_{\sigma(A)} |F(\lambda)|\, dv(f, g^*, \lambda) \le 4M \|F(A)f\| \|g^*\|. \tag{2.5}$$

The constant $M > 0$ in (2.3)–(2.5) is from (2.2).

Subsequently, the frequent terms *"spectral measure"* and *"operational calculus"* will be abbreviated to *s.m.* and *o.c.*, respectively.

2.2. **The Carleman Classes of Functions.** Let $I$ be an interval of the real axis $\mathbb{R}$, $C^\infty(I, X)$ be the space of all $X$-valued functions strongly infinite differentiable on $I$, and $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers.

The subspaces of $C^\infty(I, X)$

$$C_{\{m_n\}}(I, X) := \big\{ g(\cdot) \in C^\infty(I, X) \bigm| \forall [a, b] \subseteq I\ \exists \alpha > 0\ \exists c > 0 : \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n m_n,\ n \in \mathbb{Z}_+ \big\},$$

$$C_{(m_n)}(I, X) := \big\{ g(\cdot) \in C^\infty(I, X) \bigm| \forall [a, b] \subseteq I\ \forall \alpha > 0\ \exists c > 0 : \max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n m_n,\ n \in \mathbb{Z}_+ \big\}$$

are called the *Carleman classes* of strongly ultradifferentiable on $I$ vector functions corresponding to the sequence $\{m_n\}_{n=0}^\infty$ of *Roumieu* and *Beurling type*, respectively (for scalar functions, see [3, 15, 16]).

The inclusions

$$C_{(m_n)}(I, X) \subseteq C_{\{m_n\}}(I, X) \subseteq C^\infty(I, X) \tag{2.6}$$

are obvious.

If two sequences of positive numbers $\{m_n\}_{n=0}^\infty$ and $\{m'_n\}_{n=0}^\infty$ are related as follows:

$$\forall \gamma > 0\ \exists c = c(\gamma) > 0 :\ m'_n \le c\gamma^n m_n, \quad n \in \mathbb{Z}_+,$$

we also have the inclusion

$$C_{\{m'_n\}}(I, X) \subseteq C_{(m_n)}(I, X), \tag{2.7}$$

the sequences being subject to the condition

$$\exists \gamma_1, \gamma_2 > 0,\ \exists c_1, c_2 > 0 :\ c_1\gamma_1^n m_n \le m'_n \le c_2\gamma_2^n m_n, \quad n \in \mathbb{Z}_+,$$

their corresponding Carleman classes coincide:

$$C_{\{m_n\}}(I, X) = C_{\{m'_n\}}(I, X), \quad C_{(m_n)}(I, X) = C_{(m'_n)}(I, X). \tag{2.8}$$

Considering *Stirling's formula* and the latter,

$$\mathcal{E}^{\{\beta\}}(I, X) := C_{\{[n!]^\beta\}}(I, X) = C_{\{n^{\beta n}\}}(I, X),$$
$$\mathcal{E}^{(\beta)}(I, X) := C_{([n!]^\beta)}(I, X) = C_{(n^{\beta n})}(I, X)$$

with $\beta \ge 0$ are the well-known *Gevrey classes* of strongly ultradifferentiable on $I$ vector functions of order $\beta$ of Roumieu and Beurling type, respectively (for scalar functions, see [9]). In particular, $\mathcal{E}^{\{1\}}(I, X)$ and $\mathcal{E}^{(1)}(I, X)$ are the classes of *analytic* on $I$ and *entire* vector functions, respectively; $\mathcal{E}^{\{0\}}(I, X)$ and $\mathcal{E}^{(0)}(I, X)$ (i.e., the classes $C_{\{1\}}(I, X)$ and $C_{(1)}(I, X)$ corresponding to the sequence $m_n \equiv 1$) are the classes of *entire* vector functions of *exponential* and *minimal exponential type*, respectively.

2.3. **The Carleman Classes of Vectors.** Let $A$ be a densely defined closed linear operator in a complex Banach space $(X, \|\cdot\|)$ and $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers and

$$C^{\infty}(A) := \bigcap_{n=0}^{\infty} D(A^n).$$

The subspaces of $C^{\infty}(A)$

$$C_{\{m_n\}}(A) := \left\{ f \in C^{\infty}(A) \middle| \exists \alpha > 0 \ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n, \ n \in \mathbb{Z}_+ \right\},$$
$$C_{(m_n)}(A) := \left\{ f \in C^{\infty}(A) \middle| \forall \alpha > 0 \ \exists c > 0 : \|A^n f\| \le c\alpha^n m_n, \ n \in \mathbb{Z}_+ \right\}$$

are called the *Carleman classes* of ultradifferentiable vectors of the operator $A$ corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$ of *Roumieu* and *Beurling type*, respectively.

For the Carleman classes of vectors, the inclusions analogous to (2.6) and (2.7) and the equalities analogous to (2.8) are valid.

For $\beta \ge 0$,

$$\mathcal{E}^{\{\beta\}}(A) := C_{\{[n!]^{\beta}\}}(A) = C_{\{n^{\beta n}\}}(A),$$
$$\mathcal{E}^{(\beta)}(A) := C_{([n!]^{\beta})}(A) = C_{(n^{\beta n})}(A)$$

are the well-known *Gevrey classes* of strongly ultradifferentiable vectors of $A$ of order $\beta$ of Roumieu and Beurling type, respectively (see, e.g., [11, 12, 13]). In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the well-known classes of *analytic* and *entire* vectors of $A$, respectively [10, 24]; $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ (i.e., the classes $C_{\{1\}}(A)$ and $C_{(1)}(A)$ corresponding to the sequence $m_n \equiv 1$) are the classes of *entire* vectors of *exponential* and *minimal exponential type*, respectively (see, e.g., [13, 29]).

2.4. **Conditions on the Sequence $\{m_n\}_{n=0}^{\infty}$.** If a sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfies the condition

$$\textbf{(WGR)} \ \forall \alpha > 0 \ \exists c = c(\alpha) > 0 : \ c\alpha^n \le m_n, \quad n \in \mathbb{Z}_+,$$

the scalar function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad \lambda \ge 0 \quad (0^0 := 1) \tag{2.9}$$

first introduced by S. Mandelbrojt [16], is well-defined (cf. [13]). The function is *continuous*, *strictly increasing*, and $T(0) = 1$.

Hence, the function

$$M(\lambda) := \ln T(\lambda), \quad \lambda \ge 0, \tag{2.10}$$

is *continuous*, *strictly increasing* and $M(0) = 0$. Its *inverse* $M^{-1}(\cdot)$ is defined on $[0, \infty)$ and inherits all the aforementioned properties of $M(\cdot)$.

As is shown in [11] (see also [13] and [12]), the sequence $\{m_n\}_{n=0}^{\infty}$ satisfying the condition **(WGR)**, for a *normal operator* $A$ in a complex Hilbert space $X$, the equalities

$$\begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)) \end{aligned} \tag{2.11}$$

are true, the operators $T(t|A|)$, $t > 0$, defined in the sense of the operational calculus for a normal operator (see, e.g., [6, 28]) and the function $T(\cdot)$ being replaceable with any *nonnegative*, *continuous*, and *increasing* on $[0, \infty)$ function $F(\cdot)$ satisfying

$$c_1 F(\gamma_1 \lambda) \le T(\lambda) \le c_2 F(\gamma_2 \lambda), \quad \lambda \ge R, \tag{2.12}$$

with some $\gamma_1, \gamma_2, c_1, c_2 > 0$ and $R \geq 0$, in particular, with

$$S(\lambda) := m_0 \sup_{n \geq 0} \frac{\lambda^n}{m_n}, \quad \lambda \geq 0, \quad \text{or} \quad P(\lambda) := m_0 \left[ \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{m_n^2} \right]^{1/2}, \quad \lambda \geq 0,$$

(cf. [13]).

In [20, Theorem 3.1], the above is generalized to the case of a *scalar type spectral operator* $A$ in a *reflexive* complex Banach space $X$, the reflexivity requirement shown to be superfluous in [23, Theorem 3.1].

In [21], the sequence $\{m_n\}_{n=0}^{\infty}$ is subject to the following conditions:

$$\textbf{(GR)}\ \exists \alpha > 0\ \exists c > 0 : \ c\alpha^n n! \leq m_n, \quad n \in \mathbb{Z}_+,$$

and

$$\textbf{(SBC)}\ \exists h, H > 1\ \exists l, L > 0 : \ lh^n \leq \sum_{k=0}^{n} \frac{m_n}{m_k m_{n-k}} \leq LH^n, \quad n \in \mathbb{Z}_+.$$

The former is a stronger version of **(WGR)**, both **(WGR)** and **(GR)** being restrictions on the *growth* of $\{m_n\}_{n=0}^{\infty}$, which explains the names. The latter resembles the fundamental property of the *binomial coefficients*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad n \in \mathbb{Z}_+,$$

which also explains the name, and is precisely arrived at for $m_n = n!$.

Both **(GR)** and **(SBC)** are satisfied for $m_n = [n!]^{\beta}$ with $\beta \geq 1$ (see [21] for details).

Here, the sequence $\{m_n\}_{n=0}^{\infty}$ will be subject to a stronger version of **(GR)**

$$\textbf{(SGR)}\ \forall \alpha > 0\ \exists c = c(\alpha) > 0 : \ c\alpha^n n! \leq m_n, \quad n \in \mathbb{Z}_+,$$

and a weaker version of **(SBC)**

$$\textbf{(BC)}\ \exists h > 1\ \exists l > 0 : \ lh^n \leq \sum_{k=0}^{n} \frac{m_n}{m_k m_{n-k}}, \quad n \in \mathbb{Z}_+.$$

Both **(SGR)** and **(BC)** are satisfied for $m_n = [n!]^{\beta}$ with $\beta > 1$, also for $m_n = e^{n^2}$ (see [22] for details).

Observe that there are examples demonstrating the independence of the conditions **(GR)** and **(BC)** [22] (cf. [21]).

As is shown in [21], the conditions **(GR)** and **(SBC)** have the following implications for the function $M(\cdot)$ defined in (2.10) and its inverse $M^{-1}(\cdot)$:

$$\exists \alpha > 0\ \exists R > 0 : \ 2\alpha^{-1} M^{-1}(\lambda) \geq \lambda, \quad \lambda \geq M(R),$$

and

$$\exists h, H > 1\ \exists l, L > 0 \text{ (the constants from the condition } \textbf{(SBC)}\text{)} :$$
$$2^{-n} M(h^n \lambda) + [1 - 2^{-n}] \ln(m_0 l) \leq M(\lambda) \leq 2^{-n} M(H^n \lambda) + [1 - 2^{-n}] \ln(m_0 L),$$
$$n \in \mathbb{N}, \quad \lambda \geq 0.$$

Observe that, from **(SBC)** with $n = 0$, the estimates

$$\ln(m_0 l) \leq 0 \leq \ln(m_0 L)$$

are inferred immediately.

The conditions **(SGR)** and **(BC)** imply

$$\forall \alpha > 0\ \exists R = R(\alpha) > 0 : \ 2\alpha^{-1} M^{-1}(\lambda) \geq \lambda, \quad \lambda \geq M(R), \tag{2.13}$$

and

$$\exists h > 1\ \exists l > 0 \text{ (the constants of the condition } \mathbf{(BC)}) : \tag{2.14}$$
$$2^{-n}M(h^n\lambda) + [1 - 2^{-n}]\ln(m_0 l) \le M(\lambda), \quad n \in \mathbb{N}, \quad \lambda \ge 0$$

(see [22] for details).

Substituting $h^{-n}\lambda$ for $\lambda$, we obtain the following equivalent version:

$$\exists h > 1\ \exists l > 0 \text{ (the constants from the condition } \mathbf{(BC)}) : \tag{2.15}$$
$$M(\lambda) \le 2^n M(h^{-n}\lambda) - [2^n - 1]\ln(m_0 l), \quad \lambda \ge 0, \quad n \in \mathbb{N}.$$

## 3. Beurling type Carleman ultradifferentiable $C_0$-semigroups

**Definition 3.1.** Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. We shall call a $C_0$-semigroup $\{S(t)|t \ge 0\}$ in a complex Banach space $(X, \|\cdot\|)$ a Roumieu (Beurling) type Carleman ultradifferentiable $C_0$-semigroup corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$, or a $C_{\{m_n\}}$-semigroup (a $C_{(m_n)}$-semigroup), if each orbit $S(\cdot)f$, $f \in X$, belongs to the Roumieu (Beurling) type Carleman class of vector functions

$$C_{\{m_n\}}((0,\infty), X) \quad (C_{(m_n)}((0,\infty), X), \text{ respectively})$$

(cf. [21]).

Recall that in [21], we have proved the following statements.

**Proposition 3.1.** ([21, Proposition 4.1]). *Let $A$ be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ generating a $C_0$-semigroup $\{e^{tA}|t \ge 0\}$ and $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. Then the restriction of an orbit $e^{tA}f$, $t \ge 0$, $f \in X$, to a subinterval $I \subseteq [0,\infty)$ belongs to the Carleman class $C_{\{m_n\}}(I, X)$ ($C_{(m_n)}(I, X)$) iff*

$$e^{tA}f \in C_{\{m_n\}}(A) \ (C_{(m_n)}(A), \text{ respectively}), \quad t \in I.$$

**Theorem 3.1.** ([21, Theorem 5.1]). *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the conditions* **(GR)** *and* **(SBC)**. *Then a scalar type spectral operator $A$ in a complex Banach space $(X, \|\cdot\|)$ generates a $C_{\{m_n\}}$-semigroup iff there are $b > 0$ and $a \in \mathbb{R}$ such that*

$$\operatorname{Re}\lambda \le a - bM(|\operatorname{Im}\lambda|), \quad \lambda \in \sigma(A),$$

*where $M(\lambda) = \ln T(\lambda)$, $0 \le \lambda < \infty$, and the function $T(\cdot)$ defined by* (2.9) *is replaceable with any nonnegative, continuous, and increasing on $[0,\infty)$ function $F(\cdot)$ satisfying* (2.12).

Now, we are going to prove the following

**Theorem 3.2.** *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the conditions* **(SGR)** *and* **(BC)**. *Then a scalar type spectral operator $A$ in a complex Banach space $(X, \|\cdot\|)$ generates a $C_{(m_n)}$-semigroup iff, for any $b > 0$, there is an $a \in \mathbb{R}$ such that*

$$\operatorname{Re}\lambda \le a - bM(|\operatorname{Im}\lambda|), \ \lambda \in \sigma(A),$$

*where $M(\lambda) = \ln T(\lambda)$, $0 \le \lambda < \infty$, and the function $T(\cdot)$ defined by* (2.9) *is replaceable with any nonnegative, continuous, and increasing on $[0,\infty)$ function $F(\cdot)$ satisfying* (2.12).

*Proof.* **"If"** Part. By the hypothesis,

$$\operatorname{Re}\lambda \le a, \quad \lambda \in \sigma(A),$$

with some $a \in \mathbb{R}$, which, by [18, Proposition 3.1] implies that $A$ does generate a $C_0$-semigroup of its exponentials $\{e^{tA}|t \ge 0\}$ (see [18], cf. also [2, 25]).

Consider an arbitrary *orbit* $e^{tA}f$, $t \ge 0$, $f \in X$.

By Proposition 3.1, we are to show that

$$e^{tA}f \in C_{(m_n)}(A), \quad t > 0.$$

For arbitrary $t > 0$ and $s > 0$, let us fix a sufficiently large $N \in \mathbb{N}$ so that

$$h^{-N}2s \le 1,$$

where $h > 1$ is the constant from the condition **(BC)**, and set

$$b := 2^{N+1}t^{-1} > 0.$$

Since, due to the condition **(SGR)**, $\alpha > 0$ in (2.13) is arbitrary, we can assume that $\alpha := b > 0$.

For any $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad= \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad+ \int_{\{\lambda\in\sigma(A)\,|\,\min(-2^{-1}bM(R),a)<\operatorname{Re}\lambda\le a\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) < \infty,
\end{aligned}$$

where $R = R(\alpha) > 0$ is the constant from (2.13).

Indeed, the latter of the two integrals in the right side of the equality is finite due to the *boundedness* of the set $\{\lambda \in \sigma(A) \mid \min(-2^{-1}bM(R),a) < \operatorname{Re}\lambda \le a\}$ (for $a \le -2^{-1}bM(R)$, the set is, obviously, empty), the *continuity* of the integrand on $\mathbb{C}$, and the *finiteness* of the measure $v(f,g^*,\cdot)$ (see (2.3)).

For the former one, there are the two possibilities

$$a \le 0 \quad \text{or} \quad a > 0.$$

If $a \le 0$,

$$\begin{aligned}
(3.16)\quad &\int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&= \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} e^{M(s|\lambda|)}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\le \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} e^{M(s[|\operatorname{Re}\lambda|+|\operatorname{Im}\lambda|])}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\qquad\qquad \text{for } \lambda\in\sigma(A) \text{ with } \operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a),\\
&\qquad\qquad \operatorname{Re}\lambda\le -2^{-1}bM(R)\le 0 \text{ and } |\operatorname{Im}\lambda|\le M^{-1}(b^{-1}[a-\operatorname{Re}\lambda]);\\
&\le \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} e^{M(s[-\operatorname{Re}\lambda+M^{-1}(b^{-1}[a-\operatorname{Re}\lambda])])}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\qquad \text{since } a\le 0,\ a-\operatorname{Re}\lambda\le -2\operatorname{Re}\lambda \text{ whenever } \operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\le 0;\\
&\le \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} e^{M(s[-\operatorname{Re}\lambda+M^{-1}(2b^{-1}[-\operatorname{Re}\lambda])])}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\text{by (2.13), } 2b^{-1}[-\operatorname{Re}\lambda]\le 2\alpha^{-1}M(2b^{-1}[-\operatorname{Re}\lambda]) \text{ whenever } \operatorname{Re}\lambda\le -2^{-1}bM(R);\\
&\qquad \text{since } \alpha := b,\ -\operatorname{Re}\lambda\le M(2b^{-1}[-\operatorname{Re}\lambda]) \text{ whenever } \operatorname{Re}\lambda\le -2^{-1}bM(R);\\
&\le \int_{\{\lambda\in\sigma(A)\,|\,\operatorname{Re}\lambda\le\min(-2^{-1}bM(R),a)\}} e^{M(2sM^{-1}(2b^{-1}[-\operatorname{Re}\lambda]))}e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\qquad\qquad\qquad\qquad \text{by (2.15);}
\end{aligned}$$

$$= \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\}} e^{2^N M(h^{-N}2sM^{-1}(2b^{-1}[-\mathrm{Re}\,\lambda]))-[2^N-1]\ln(m_0 l)} \times e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

$$= (m_0L)^{-[2^N-1]} \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\}} e^{2^N M(h^{-N}2sM^{-1}(2b^{-1}[-\mathrm{Re}\,\lambda]))} \times e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

by choice, $h^{-N}2s \le 1$;

$$\le (m_0L)^{-[2^N-1]} \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\}} e^{2^N M(M^{-1}(2b^{-1}[-\mathrm{Re}\,\lambda]))} \times e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

$$= (m_0L)^{-[2^N-1]} \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\}} e^{2^{N+1}b^{-1}[-\mathrm{Re}\,\lambda]+t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

by choice, $b := 2^{N+1}t^{-1}$;

$$= (m_0L)^{-[2^N-1]} v\left(f,g^*,\left\{\lambda\in\sigma(A)\,\middle|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\right\}\right)$$
$$\le (m_0L)^{-[2^N-1]} v(f,g^*,\sigma(A))$$

by (2.3);

$$\le (m_0L)^{-[2^N-1]} 4M\|f\|\|g^*\| < \infty.$$

If $a > 0$,

$$\int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),a)\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$
$$= \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$
$$+ \int_{\{\lambda\in\sigma(A)\,|\,\min(-2^{-1}bM(R),-a)<\mathrm{Re}\,\lambda\le-2^{-1}bM(R)\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda) < \infty.$$

Indeed, the latter of the two integrals in the right side of the equality is finite due to the *boundedness* of the set $\left\{\lambda\in\sigma(A)\,\middle|\,\min(-a,-2^{-1}bM(R)) < \mathrm{Re}\,\lambda \le -2^{-1}bM(R)\right\}$ (for $a \le 2^{-1}bM(R)$, the set is, obviously, empty), the *continuity* of the integrand on $\mathbb{C}$, and the *finiteness* of the measure $v(f,g^*,\cdot)$ (see (2.3)).

For the former one, we have

$$\int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$
$$= \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} e^{M(s|\lambda|)}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$
$$\le \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} e^{M(s[|\,\mathrm{Re}\,\lambda|+|\,\mathrm{Im}\,\lambda|])}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

for $\lambda\in\sigma(A)$ with $\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)$,

$\mathrm{Re}\,\lambda \le -2^{-1}bM(R) \le 0$ and $|\,\mathrm{Im}\,\lambda| \le M^{-1}(b^{-1}[a-\mathrm{Re}\,\lambda])$;

$$\int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} e^{M(s[-\mathrm{Re}\,\lambda+M^{-1}(b^{-1}[a-\mathrm{Re}\,\lambda])])}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

$a - \mathrm{Re}\,\lambda \le -2\,\mathrm{Re}\,\lambda$ whenever $\mathrm{Re}\,\lambda \le -a$;

$$\le \int_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\le\min(-2^{-1}bM(R),-a)\}} e^{M(s[-\mathrm{Re}\,\lambda+M^{-1}(2b^{-1}[-\mathrm{Re}\,\lambda])])}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

in the same manner as in (3.16);

$$< \infty.$$

Thus, we have proved that, for arbitrary $s>0$, $t>0$, $f\in X$, and $g^*\in X^*$,

$$\int_{\sigma(A)} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)<\infty. \tag{3.17}$$

Furthermore, for any $s>0$, $t>0$, $f\in X$,

$$\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\int_{\{\lambda\in\sigma(A)|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\to 0 \quad\text{as}\quad n\to\infty.$$

Indeed, as follows from the preceding argument, for any $s,t>0$, the spectrum $\sigma(A)$ can be partitioned into two Borel subsets $\sigma_1$ and $\sigma_2$ ($\sigma(A)=\sigma_1\cup\sigma_2$, $\sigma_1\cap\sigma_2=\emptyset$) in such a way that $\sigma_1$ is *bounded* and

$$T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\le 1,\quad \lambda\in\sigma_2.$$

Therefore,

$$\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\int_{\{\lambda\in\sigma(A)|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

$$=\sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\Bigg[\int_{\{\lambda\in\sigma_1|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)$$

$$+\int_{\{\lambda\in\sigma_2|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}} T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\Bigg]$$

since $\sigma_1$ is bounded and $T(s|\cdot|)e^{t\,\mathrm{Re}\,\cdot}$ is continuous on $\mathbb{C}$,

there is such a $C\ge 1$ that $T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}\le C$, $\lambda\in\sigma_1$;

$$\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}}\Big[Cv\left(f,g^*,\{\lambda\in\sigma_1\big|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}\right)$$

$$+\,v\left(f,g^*,\{\lambda\in\sigma_2\big|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}\right)\Big]$$

$$\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} Cv\left(f,g^*,\{\lambda\in\sigma(A)\big|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\}\right)\quad \text{by (2.4) with } F(\lambda)\equiv 1;$$

$$\le \sup_{\{g^*\in X^*\,|\,\|g^*\|=1\}} C4M\|E_A(\{\lambda\in\sigma(A)|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\})f\|\|g^*\|$$

$$=4CM\|E_A(\{\lambda\in\sigma(A)|T(s|\lambda|)e^{t\,\mathrm{Re}\,\lambda}>n\})f\| \quad\text{by the strong continuity of the } \textit{s.m.};$$

$$\to 0\quad\text{as}\quad n\to\infty.$$

According to [17, Proposition 3.1], (3.17) and (3) imply that, for any $t>0$, $f\in X$, and $s>0$,

$$e^{tA}f\in D(T(s|A|)).$$

Hence, for any $f\in X$, due to (2.11),

$$e^{tA}f\in\bigcap_{s>0}D(T(s|A|))=C_{(m_n)}(A),\quad t>0,$$

which, by Proposition 3.1, implies that, for $f\in X$,

$$e^{\cdot A}f\in C_{(m_n)}((0,\infty),X),$$

i.e., the $C_0$-semigroup $\left\{e^{tA}\big|t\ge 0\right\}$ generated by $A$ is a $C_{(m_n)}$-semigroup.

**"Only if" Part.** We shall prove this part by *contrapositive*, i.e., assuming that there is such a $b > 0$ that for any $a \in \mathbb{R}$,

$$\sigma(A) \setminus \left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda \le a - bM(|\operatorname{Im}\lambda|)\right\} \ne \emptyset,$$

we are to show that $A$ does not generate a $C_{(m_n)}$-semigroup.

Observe that the latter readily implies that the set

$$\sigma(A) \setminus \left\{\lambda \in \mathbb{C}\middle| \operatorname{Re}\lambda \le -bM(|\operatorname{Im}\lambda|)\right\}$$

is *unbounded*,

For $\sigma(A)$, there are two possibilities

$$\sup_{\lambda\in\sigma(A)} \operatorname{Re}\lambda = \infty \quad \text{or} \quad \sup_{\lambda\in\sigma(A)} \operatorname{Re}\lambda < \infty.$$

The first one implies that $A$ does not generate a $C_0$-semigroup [14], let alone a $C_{(m_n)}$-semigroup.

With

$$\sup_{\lambda\in\sigma(A)} \operatorname{Re}\lambda < \infty \tag{3.18}$$

being the case, $A$ generates a $C_0$-semigroup of its exponentials $\left\{e^{tA}\middle|t \ge 0\right\}$ [18] and one can choose a sequence of points $\{\lambda_n\}_{n=1}^\infty$ in the complex plane as follows:

$$\begin{gathered}\lambda_n \in \sigma(A), \quad n \in \mathbb{N},\\ \operatorname{Re}\lambda_n > -bM(|\operatorname{Im}\lambda_n|), \quad n \in \mathbb{N}, \quad \text{and}\\ \lambda_0 := 0, \quad |\lambda_n| > \max\bigl[n, |\lambda_{n-1}|\bigr], \quad n \in \mathbb{N}.\end{gathered}$$

The latter, in particular, indicates that the points $\lambda_n$ are *distinct*

$$\lambda_i \ne \lambda_j, \quad i \ne j.$$

Since each set

$$\left\{\lambda \in \mathbb{C}\middle| \operatorname{Re}\lambda > -bM(|\operatorname{Im}\lambda|),\ |\lambda| > \max\left[n, |\lambda_{n-1}|\right]\right\}, \quad n \in \mathbb{N},$$

is *open* in $\mathbb{C}$, there exists such an $\varepsilon_n > 0$ that, along with the point $\lambda_n$, the set contains the *open disk*

$$\Delta_n = \left\{\lambda \in \mathbb{C}\middle||\lambda - \lambda_n| < \varepsilon_n\right\},$$

i.e., for any $\lambda \in \Delta_n$,

$$\operatorname{Re}\lambda > -bM(|\operatorname{Im}\lambda|) \quad \text{and} \quad |\lambda| > \max\bigl[n, |\lambda_{n-1}|\bigr]. \tag{3.19}$$

The radii of the disks $\varepsilon_n$ can be chosen small enough so that

$$0 < \varepsilon_n < 1/n, \quad n \in \mathbb{N}, \quad \text{and} \quad \Delta_i \cap \Delta_j = \emptyset, \quad i \ne j, \tag{3.20}$$

i.e., the disks are *pairwise disjoint.*

Considering that each $\Delta_n \cap \sigma(A) \ne \emptyset$, $\Delta_n$ being an *open set*, by the properties of the *s.m.* and the latter, we infer

$$E_A(\Delta_n) \ne 0, \quad n \in \mathbb{N}, \quad \text{and} \quad E_A(\Delta_i)E_A(\Delta_j) = \delta_{ij}E_A(\Delta_i), \tag{3.21}$$

($\delta_{ij}$ is *Kronecker's delta symbol* and 0, here and whenever appropriate, designates the *zero operator*). Hence, the subspaces $E_A(\Delta_n)X$ are *nontrivial* and

$$E_A(\Delta_i)X \cap E_A(\Delta_j)X = \{0\}, \quad i \ne j.$$

Thus, choosing vectors

$$e_n \in E_A(\Delta_n)X, \quad n \in \mathbb{N}, \quad \text{with} \quad \|e_n\| = 1, \tag{3.22}$$

we obtain a vector sequence $\{e_n\}_{n=1}^\infty$ such that, by (3.21),

$$E_A(\Delta_i)e_j = \delta_{ij}e_i. \tag{3.23}$$

The latter, showing the *linear independence* of $\{e_1, e_2, \dots\}$, goes a step beyond implying the existence of an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}\left(e_n, \operatorname{span}\left(\left\{e_i \middle| i \in \mathbb{N},\ i \neq n\right\}\right)\right) \geq \varepsilon, \quad n \in \mathbb{N}. \tag{3.24}$$

Otherwise, there is a vanishing subsequence $\left\{d_{n(k)}\right\}_{k=1}^{\infty}$

$$d_{n(k)} \to 0 \quad \text{as} \quad k \to \infty,$$

and hence, for any $k \in \mathbb{N}$, there exists an

$$f_{n(k)} \in \operatorname{span}\left(\left\{e_i \middle| i \in \mathbb{N},\ i \neq n(k)\right\}\right) \quad \text{with} \quad \|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/n(k),$$

which, considering (2.2), implies

$$e_{n(k)} = E_A(\Delta_{n(k)})[e_{n(k)} - f_{n(k)}] \to 0 \quad \text{as} \quad k \to \infty$$

contradicting (3.22).

As follows from the *Hahn-Banach Theorem* (see, e.g., [5]), (3.24) implies that, for each $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1 \quad \text{and} \quad \langle e_i, e_j^* \rangle = \delta_{ij} d_i. \tag{3.25}$$

For the sequence of the real parts $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$, there are the two possibilities

$$\sup_{n \in \mathbb{N}} |\operatorname{Re} \lambda_n| < \infty \quad \text{or} \quad \sup_{n \in \mathbb{N}} |\operatorname{Re} \lambda_n| = \infty.$$

Suppose that

$$\sup_{n \in \mathbb{N}} |\operatorname{Re} \lambda_n| =: \omega < \infty. \tag{3.26}$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n \in X \quad \text{and} \quad g^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^* \in X^*,$$

the series strongly converging in $X$ and $X^*$, respectively, due to (3.22) and (3.25). By (3.25) and (3.24),

$$\langle e_n, g^* \rangle = \frac{1}{n^2} \langle e_n, e_n^* \rangle = \frac{d_n}{n^2} \geq \frac{\varepsilon}{n^2}, \quad n \in \mathbb{N}. \tag{3.27}$$

As can be easily deduced from (3.23),

$$E_A(\Delta_n) f = \frac{1}{n^2} e_n, \quad n \in \mathbb{N}, \quad \text{and} \quad E_A(\cup_{n=1}^{\infty} \Delta_n) f = f. \tag{3.28}$$

Considering the latter and (3.27),

$$v(f, g^*, \Delta_n) \geq |\langle E_A(\Delta_n) f, g^* \rangle| = \left\langle \frac{1}{n^2} e_n, g^* \right\rangle \geq \frac{\varepsilon}{n^4}, \quad n \in \mathbb{N}. \tag{3.29}$$

For $s = t = 1$, we have

$$\begin{aligned}
&\int_{\sigma(A)} T(|\lambda|) e^{\operatorname{Re} \lambda} \, dv(f, g^*, \lambda) && \text{by (3.28);} \\
&= \int_{\sigma(A)} T(|\lambda|) e^{\operatorname{Re} \lambda} \, dv(E_A(\cup_{n=1}^{\infty} \Delta_n) f, g^*, \lambda) && \text{by the properties of the } o.c.; \\
&= \int_{\cup_{n=1}^{\infty} \Delta_n} T(|\lambda|) e^{\operatorname{Re} \lambda} \, dv(f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\Delta_n} T(|\lambda|) e^{\operatorname{Re} \lambda} \, dv(f, g^*, \lambda) \\
&\qquad \text{for } \lambda \in \Delta_n, \text{ by (3.19), (3.26), and (3.20):} \quad |\lambda| \geq n \text{ and} \\
&\qquad \operatorname{Re} \lambda = \operatorname{Re} \lambda_n - (\operatorname{Re} \lambda_n - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_n - |\lambda_n - \lambda| \geq -\omega - \varepsilon_n \geq -\omega - 1;
\end{aligned} \tag{3.30}$$

$$\geq \sum_{n=1}^{\infty} T(n) e^{-(\omega+1)} v(f, g^*, \Delta_n) \qquad \text{by (3.29);}$$

$$\geq e^{-(\omega+1)} \sum_{n=1}^{\infty} T(n) \frac{\varepsilon}{n^4} = \infty.$$

Indeed, by definition (2.9),

$$T(n) \geq m_0 \frac{n^4}{m_4}, \quad n \in \mathbb{N}.$$

Hence, by [17, Proposition 3.1],

$$e^{tA} f\big|_{t=1} \notin D(T(|A|)).$$

Considering (2.11), the more so,

$$e^{tA} f\big|_{t=1} \notin \bigcap_{s>0} D(T(s|A|)) = C_{(m_n)}(A).$$

Hence, according to Proposition 3.1,

$$e^{\cdot A} f \notin C_{(m_n)}\left((0,\infty), X\right),$$

which implies that the $C_0$-semigroup $\left\{e^{tA}\middle| t \geq 0\right\}$ generated by $A$ is not a $C_{(m_n)}$-semigroup. Suppose that

$$\sup_{n \in \mathbb{N}} |\operatorname{Re} \lambda_n| = \infty$$

and recall that we are also acting under hypothesis (3.18). Hence, there is a sub(3.22) and (3.25)sequence $\left\{\operatorname{Re} \lambda_{n(k)}\right\}_{k=1}^{\infty}$ such that

$$\operatorname{Re} \lambda_{n(k)} \leq -k, \quad k \in \mathbb{N}. \tag{3.31}$$

Let

$$f := \sum_{k=1}^{\infty} \frac{1}{k^2} e_{n(k)} \in X \quad \text{and} \quad g^* := \sum_{k=1}^{\infty} \frac{1}{k^2} e^*_{n(k)} \in X^*,$$

the series strongly converging in $X$ and $X^*$, respectively, due to (3.22) and (3.25). By (3.25) and (3.24),

$$\langle e_{n(k)}, g^* \rangle = \frac{1}{k^2} \langle e_{n(k)}, e^*_{n(k)} \rangle = \frac{d_{n(k)}}{k^2} \geq \frac{\varepsilon}{k^2}, \quad k \in \mathbb{N}. \tag{3.32}$$

By (3.23),

$$E_A(\Delta_{n(k)}) f = \frac{1}{k^2} e_{n(k)}, \quad k \in \mathbb{N}, \quad \text{and} \quad E_A(\cup_{k=1}^{\infty} \Delta_{n(k)}) f = f. \tag{3.33}$$

Considering the latter and (3.32),

$$v(f, g^*, \Delta_{n(k)}) \geq |\langle E_A(\Delta_{n(k)}) f, g^* \rangle| = \left\langle \frac{1}{k^2} e_{n(k)}, g^* \right\rangle \geq \frac{\varepsilon}{k^4}, \quad k \in \mathbb{N}. \tag{3.34}$$

Similarly to (3.30), for $s = 1$ and $t = (2b)^{-1}$,

$$\int_{\sigma(A)} T(|\lambda|) e^{(2b)^{-1} \operatorname{Re} \lambda} \, dv(f, g^*, \lambda) = \sum_{k=1}^{\infty} \int_{\Delta_{n(k)}} T(|\lambda|) e^{(2b)^{-1} \operatorname{Re} \lambda} \, dv(f, g^*, \lambda) = \infty.$$

Indeed, for $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$, by (3.19), (3.20), and (3.31),

$$\begin{aligned} -bM(|\operatorname{Im} \lambda|) < \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} - (\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda) \leq \operatorname{Re} \lambda_{n(k)} + |\lambda_{n(k)} - \lambda| \\ &\leq \operatorname{Re} \lambda_{n(k)} + \varepsilon_{n(k)} \leq -k + 1 \leq 0 \end{aligned}$$

and hence,

$$\operatorname{Re} \lambda \leq -k + 1 \leq 0 \quad \text{and} \quad |\lambda| \geq |\operatorname{Im} \lambda| \geq M^{-1}\left(b^{-1}[-\operatorname{Re} \lambda]\right).$$

Using these estimates, for $k \in \mathbb{N}$, we have

$$\int_{\Delta_{n(k)}} T(|\lambda|)e^{(2b)^{-1}\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \ge \int_{\Delta_{n(k)}} e^{M(|\lambda|)}e^{(2b)^{-1}\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)$$
$$\ge \int_{\Delta_{n(k)}} e^{M(M^{-1}(b^{-1}[-\operatorname{Re}\lambda]))}e^{(2b)^{-1}\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)$$
$$= \int_{\Delta_{n(k)}} e^{b^{-1}[-\operatorname{Re}\lambda]}e^{(2b)^{-1}\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) = \int_{\Delta_{n(k)}} e^{(2b)^{-1}[-\operatorname{Re}\lambda]}\,dv(f,g^*,\lambda)$$
$$\ge e^{(2b)^{-1}(k-1)}v(f,g^*,\Delta_{n(k)}) \qquad \text{by (3.34);}$$
$$\ge e^{(2b)^{-1}(k-1)}\frac{\varepsilon}{k^4} \to \infty \text{ as } k \to \infty.$$

Thus, by [17, Proposition 3.1],

$$e^{tA}f\big|_{t=(2b)^{-1}} \notin D(T(|A|)).$$

Considering (2.11), the more so,

$$e^{tA}f\big|_{t=(2b)^{-1}} \notin \bigcap_{s>0} D(T(s|A|)) = C_{(m_n)}(A).$$

Hence, according to Proposition 3.1,

$$e^{\cdot A}f \notin C_{(m_n)}\left((0,\infty),X\right),$$

which implies that the $C_0$-semigroup $\left\{e^{tA}\middle|t \ge 0\right\}$ generated by $A$ is not a $C_{(m_n)}$-semigroup.

This concludes the analysis of all the possibilities and thus, the proof of the "only if" part by *contrapositive*.

By [23, Theorem 3.1], the function $T(\cdot)$ defined by (2.9) can be replaced with any *nonnegative*, *continuous*, and *increasing* on $[0,\infty)$ function $F(\cdot)$ satisfying (2.12). □

## 4. Gevrey ultradifferentiable $C_0$-semigroups

**Definition 4.1.** Let $\beta \ge 0$. We shall call a $C_0$-semigroup $\left\{S(t)\middle|t \ge 0\right\}$ in a complex Banach space $(X,\|\cdot\|)$ a Roumieu (Beurling) type Gevrey ultradifferentiable $C_0$-semigroup of order $\beta$, or a $\mathcal{E}^{\{\beta\}}$-semigroup ($\mathcal{E}^{(\beta)}$-semigroup), if it is a $C_{\{[n!]^\beta\}}$-semigroup ($C_{([n!]^\beta)}$-semigroup, respectively) in accordance with Definition 3.1.

The sequence $m_n = [n!]^\beta$ with $\beta \ge 1$ satisfying the conditions **(GR)** and **(SBC)** [21] and the function $T(\cdot)$ being replaceable with $F(\lambda) = e^{\lambda^{1/\beta}}$, $\lambda \ge 0$, (see [20] for details), [19, Theorem 5.1] giving a characterization of the scalar type spectral generators of Roumieu type Gevrey ultradifferentiable $C_0$-semigroups of order $\beta \ge 1$ (in particular, for $\beta = 1$, of *analytic semigroups* [8, 14, 18]) immediately follows from Theorem 3.1.

The sequence $m_n = [n!]^\beta$ with $\beta > 1$ satisfying the conditions **(SGR)** and **(BC)** (see [22] for details), in the same manner, a ready consequence of Theorem 3.2 is the following

**Corollary 4.1.** *Let $\beta > 1$. Then a scalar type spectral operator $A$ in a complex Banach space $(X,\|\cdot\|)$ generates a $\mathcal{E}^{(\beta)}$-semigroup iff, for any $b > 0$, there is an $a \in \mathbb{R}$ such that*

$$\operatorname{Re}\lambda \le a - b|\operatorname{Im}\lambda|^{1/\beta}, \quad \lambda \in \sigma(A).$$

Observe that, for $0 \le \beta \le 1$, the sequence $m_n = [n!]^\beta$ fails to satisfy the condition **(SGR)**, and, for $0 \le \beta < 1$, even **(GR)**. If a scalar type spectral operator $A$ in a complex Banach space $(X,\|\cdot\|)$ generates a $\mathcal{E}^{\{\beta\}}$-semigroup with $0 \le \beta < 1$ or a $\mathcal{E}^{(\beta)}$-semigroup with $0 \le \beta \le 1$, due to inclusions (2.6) and (2.7),

$$\mathcal{E}^{(\beta)}((0,\infty),X) \subseteq \mathcal{E}^{\{\beta\}}((0,\infty),X) \subseteq \mathcal{E}^{(1)}((0,\infty),X),$$

which implies that all the orbits $e^{\cdot A}f$, $f \in X$, are entire vector functions. Hence, being defined of the whole space $X$, $A \in L(X)$ by the *Closed Graph Theorem* and generates a uniformly continuous semigroup (an *entire semigroup of exponential type*).

## 5. One more example

The rapidly growing sequence $m_n := e^{n^2}$ also satisfies the conditions **(SGR)** and **(BC)** and the function $M(\cdot)$ in this case can be replaced with

$$L(\lambda) := \begin{cases} 0 & \text{for } 0 \le \lambda < 1, \\ [\ln \lambda]^2 & \text{for } \lambda \ge 1 \end{cases}$$

(see [22] for details). Thus we have the following

**Corollary 5.1.** *A scalar type spectral operator $A$ in a complex Banach space $(X, \|\cdot\|)$ generates a $C_{(e^{n^2})}$-semigroup iff, for any $b > 0$, there is an $a \in \mathbb{R}$ such that*

$$\operatorname{Re} \lambda \le a - bL(|\operatorname{Im} \lambda|), \quad \lambda \in \sigma(A).$$

(cf. [19, Theorem 4.1]).

## 6. Final remark

Due to the scalar type spectrality of the operator $A$, Theorem 3.2 is void of restrictions on its resolvent behavior, which appear to be inevitable for the results of this nature in the general case (cf. [8, 14, 27]).

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