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LEVEL SETS OF ASYMPTOTIC MEAN OF DIGITS FUNCTION FOR 4-ADIC REPRESENTATION OF REAL NUMBER

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Abstract. We study topological, metric and fractal properties of the level sets

$$S_\theta = \{x : r(x) = \theta\}$$

of the function r of asymptotic mean of digits of a number $x \in [0;1]$ in its 4-adic representation,

$$r(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \alpha_i(x)$$

if the asymptotic frequency $\nu_j(x)$ of at least one digit does not exist, were

$$\nu_j(x) = \lim_{n\to\infty} n^{-1}\#\{k : \alpha_k(x) = j, k \leqslant n\},\ \ j = 0, 1, 2, 3.$$

1. Introduction

Let $2 \leqslant s \in N$ and $\mathcal{A}_s = \{0, 1, \dots, s-1\}$ be an alphabet of s-adic number system. By $\Delta^s_{\alpha_1(x)\alpha_2(x)\dots\alpha_k(x)\dots}$ denote the s-adic representation of a number $x \in [0;1]$, i.e.,

$$x = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \dots + \frac{\alpha_n}{s^n} + \dots \equiv \Delta^s_{\alpha_1\alpha_2\dots\alpha_k\dots},$$

where $\mathcal{A}_s \ni \alpha_k = \alpha_k(x)$ is the *kth s-adic digit of the number x*. Note that some numbers have two s-adic representations such that

$$\Delta^s_{c_1\dots c_{k-1}c_k(0)} = \Delta^s_{c_1\dots c_{k-1}[c_k-1](s-1)},$$

where (i) is a period in the number representation. These numbers are called *s-adic–rational*. The rest of numbers have unique representations and are called *s-adic–irrational*. The kth digit of the number, as its function, is well defined after agreement to use the first s-adic representation only, i.e., the representation with period (0).

An asymptotic mean (or simply *mean*) *of digits of the number x* is a value $r(x)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \alpha_i(x) \equiv r(x),$$

(if the limit exists), where $\mathcal{A}_s \ni \alpha_i$ are digits of the s-adic representation of the number $x \in [0;1]$. The value $n^{-1}\sum\limits_{i=1}^{n} \alpha_i(x) \equiv r_n(x)$ is called *relative mean of digits* in the s-adic representation of x.

In this paper we study properties of the function r of asymptotic mean of digits, in particular, topological, metric, and fractal properties of number sets with a preassigned

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asymptotic mean of digits. Namely, we investigate sets

$$S_\theta \equiv \left\{ x : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i(x) = \theta \in [0; s-1] \right\},$$

that are level sets of the function r (indeed, $r^{-1}(\theta) = S_\theta$). If $\theta \notin [0; s-1]$ then it is easily proved that the set S_θ is empty.

Asymptotic mean of digits of a number x is closely related to the concept of digit frequency of the number.

Let $N_i(x,k)$ be the number the digits "i"$\in \mathcal{A}_s$ appears in the s-adic representation $\Delta^s_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$ of the real number $x \in [0;1]$ to kth place including, i.e.

$$N_i(x,k) = \#\{j : \alpha_j(x) = i,\ j \leqslant k\}.$$

The *frequency (asymptotic frequency) of a digit "i"* in the s-adic representation of a number $x \in [0;1]$ is the limit (if it exists) such that

$$\nu_i(x) = \lim_{k\to\infty} v_i^{(k)},$$

where $v_i^{(k)} = k^{-1} N_i(x,k)$ is called *relative frequency of the digit "i"* in the s-adic representation of a number x.

The frequency function $\nu_i(x)$ of a digit "i" in the s-adic representation of a number $x \in [0;1]$ is well defined for s-adic–irrational numbers, and, for s-adic–rational numbers, it is well defined after agreement to use representation with period (0) only.

Different mathematical objects with fractal properties were defined and studied in terms of frequencies. First of all it is Besicovitch–Eggleston's sets [3, 6]

$$E[\tau_0, \tau_1, \ldots, \tau_{s-1}] = \{x : x = \Delta^s_{\alpha_1\alpha_2\ldots\alpha_k\ldots},\ \nu_i(x) = \tau_i \geqslant 0,\ i = \overline{0, s-1}\},$$

the Hausdorff–Besicovitch dimension of the sets is equal to [4] to

$$\alpha_0(E[\tau_0, \tau_1, \ldots, \tau_{s-1}]) = -\frac{\ln \tau_0^{\tau_0} \tau_1^{\tau_1} \ldots \tau_{s-1}^{\tau_{s-1}}}{\ln s}.$$

The number x is called *normal for basis s* if the value $\nu_i(x)$ exists for all $i \in \mathcal{A}_s$ and equals to s^{-1}. The set of all normal for basis s numbers is the only Besicovitch–Eggleston's set of positive and even full Lebesgue measure.

Normal for every natural basis $s \geqslant 2$ number x is called *normal.* According to the famous Borel's theorem [5] we see that *Lebesgue measure of the set of normal numbers is equal to 1.*

In the papers [1], [11] it was proved that the Hausdorff–Besicovitch dimension of abnormal and essentially abnormal number sets (i.e. number sets having not frequency of at least one digit or having not frequencies of all digits respectively) is equal to 1.

If a number x has all digits frequencies then the relationship between asymptotic mean of digits and digits frequencies of the number x is the following:

$$r(x) = \nu_1(x) + 2\nu_2(x) + \cdots + (s-1)\nu_{s-1}(x).$$

When $s = 2$, it is obvious that the asymptotic mean of digits is equal to the frequency of digit "1". So we do not examine this case. The case $s = 3$ was studied in papers [8, 10]. It is unique since it is the only case where the set S_θ is a union of two disjoint sets Θ_1 and Θ_2 such that

$$\begin{aligned} \Theta_1 &\equiv \{x : \text{frequencies of all digits exist}\}, \\ \Theta_2 &\equiv \{x : \text{frequency of any digit does not exist}\}. \end{aligned}$$

In cases $s > 3$ the set S_θ is a union of tree disjoint sets such that

$\Theta_1 \equiv \{x\colon$ frequencies of all digits exist$\}$,
$\Theta_2 \equiv \{x\colon$ frequency of at least one digit exists and of at least one digit does not exist$\}$,
$\Theta_3 \equiv \{x\colon$ frequency of any digit does not exist$\}$.

In this paper we study the case $s = 4$ since it is the easiest and modeling in the last class. Our previous paper [9] was devoted to studying properties of the set Θ_1, and this one deals with the sets Θ_2 and Θ_3.

2. The object of study

Lemma 1. *If in the* 4*-adic representation of a real number* $x \in [0;1]$ *the frequency of one digit does not exist, then the frequency of at least one more digit does not exist.*

Proof. Suppose the frequency $\nu_k(x_0)$ does not exist, i.e., $\lim\limits_{n\to\infty} \frac{N_k(x_0,n)}{n}$ does not exist. Since

$$\frac{N_k(x_0,n)}{n} = 1 - \frac{N_j(x_0,n)}{n} - \frac{N_m(x_0,n)}{n} - \frac{N_l(x_0,n)}{n},$$

we see that

$$\lim_{n\to\infty}\left(\frac{N_j(x_0,n)}{n} + \frac{N_m(x_0,n)}{n} + \frac{N_l(x_0,n)}{n}\right)$$

does not exist. It means that at least one of the limits $\lim\limits_{n\to\infty} \frac{N_j(x_0,n)}{n}$, $\lim\limits_{n\to\infty} \frac{N_m(x_0,n)}{n}$ or $\lim\limits_{n\to\infty} \frac{N_l(x_0,n)}{n}$, where $\{j,k,l,m\} = \{0,1,2,3\}$ do not exist. □

Lemma 2. *If in the* 4*-adic representation of a real number* $x \in [0;1]$ *the asymptotic mean of digits,* $r(x)$*, and at least two* 4*-adic digits frequencies* $\nu_i(x)$*,* $\nu_j(x)$*, where* $i,j \in \{0,1,2,3\}$*, exist, then the remaining two* 4*-adic digits frequencies of the number* x *exist.*

Proof. Consider the system

$$\text{(1)}\qquad \begin{cases} v_0^{(n)} + v_1^{(n)} + v_2^{(n)} + v_3^{(n)} = 1, \\ v_1^{(n)} + 2v_2^{(n)} + 3v_3^{(n)} = r_n. \end{cases}$$

Let $i,j \in \{1,2,3\}$. Since $\lim\limits_{n\to\infty} v_i^{(n)} = \nu_i(x)$, $\lim\limits_{n\to\infty} r_n = \theta$, we see that from the second equation of system (1) it follows that $\lim\limits_{n\to\infty} v_k^{(n)}$, $k \in \{1,2,3\} \setminus \{i,j\}$, exists, i.e. the frequency $\nu_k(x)$ exists. Then from the first equation of system (1) it follows that $\nu_0(x)$ exists.

Let $i = 0$, $j = 1$. Then from system (1) we have

$$\begin{cases} v_3^{(n)} = r_n + v_1^{(n)} - 2v_0^{(n)} - 2, \\ v_2^{(n)} = 1 - v_0^{(n)} - v_1^{(n)} - v_3^{(n)} = 3 + v_0^{(n)} - 2v_1^{(n)} - r_n, \end{cases}$$

which implies existence of the frequencies $\nu_2(x)$ and $\nu_3(x)$.

Let $i = 0$, $j = 2$. Then from system (1) we obtain

$$\begin{cases} v_3^{(n)} = \dfrac{1}{2}(r_n - v_2^{(n)} + v_0^{(n)} - 1), \\ v_1^{(n)} = 1 - v_0^{(n)} - v_2^{(n)} - v_3^{(n)} = \dfrac{3}{2} - \dfrac{3}{2}v_0^{(n)} - \dfrac{3}{2}v_2^{(n)} - \dfrac{r_n}{2}, \end{cases}$$

which implies existence of the frequencies $\nu_1(x)$ and $\nu_3(x)$.

Let $i=0$, $j=3$. Then from system (1) we have

$$\begin{cases} v_2^{(n)} = r_n - 2v_3^{(n)} + v_0^{(n)} - 1, \\ v_1^{(n)} = 1 - v_0^{(n)} - v_2^{(n)} - v_3^{(n)} = 2 - r_n - 2v_0^{(n)} + 2v_3^{(n)}, \end{cases}$$

which implies existence of the frequencies $\nu_1(x)$ and $\nu_2(x)$. □

Corollary 1. (from Lemmas 1 and 2). *A number $x \in S_\theta$ can not have frequencies of only two or of only three 4-adic digits.*

From Lemma 1 it follows that if a number does not have frequency of at least one 4-adic digit then it does not have frequency of one more digit, therefore, the number $x \in S_\theta$ can not have frequencies of only three digits. According to Lemma 2, if a number $x \in S_\theta$ has frequencies of at least two digits then it has frequencies of all digits, therefore, the number x can not have frequencies of only two 4-adic digits.

Hence, the set S_θ can be represented as a union of three disjoint sets Θ_1, Θ_2 and Θ_3 such that

$$\begin{aligned} \Theta_1 &\equiv \{x : \nu_i(x) \text{ exist}, \forall i \in \mathcal{A}_4\}, \\ \Theta_2 &\equiv \{x : \text{exist frequency of only one 4-adic digit } \nu_i(x), i \in \mathcal{A}_4\}, \\ \Theta_3 &\equiv \{x : \nu_i(x) \text{ do not exist}, \forall i \in \mathcal{A}_4\}. \end{aligned}$$

In the following sections we study properties of the sets Θ_2 and Θ_3.

3. Abnormal numbers that have asymptotic mean of digits

Theorem 1. *If $\theta = 0$ or $\theta = 3$, then $\Theta_2 = \Theta_3 = \varnothing$.*

Proof. Let $\theta = 0$. If $\lim\limits_{n\to\infty} r_n(x) = 0$, then for any $i \in \{1,2,3\}$ the following inequality holds: $0 \leqslant v_i^{(n)}(x) \leqslant v_1^{(n)}(x) + 2v_2^{(n)}(x) + 3v_3^{(n)}(x) = r_n(x) \to 0$, as $n \to \infty$. Therefore, $\nu_i(x) = \lim\limits_{n\to\infty} v_i^{(n)}(x) = 0$ and $\nu_0(x) = 1$. Hence, $\Theta_2 = \Theta_3 = \varnothing$.

Let $\theta = 3$. If $\lim\limits_{n\to\infty} r_n(x) = 3$, then multiplying the first equation of system (1) by 3 and subtracting the second equation of the system, we obtain that $3v_0^{(n)} + 2v_1^{(n)} + v_2^{(n)} = 3 - r_n$. Hence $0 \leqslant v_i^{(n)}(x) \leqslant 3v_0^{(n)}(x) + 2v_1^{(n)}(x) + v_2^{(n)}(x) = 3 - r_n(x) \to 0$ as $n \to \infty$. Therefore, $\nu_i(x) = 0$, for all $i \in \{0,1,2\}$. Hence $\nu_3(x) = 1$ and $\Theta_2 = \Theta_3 = \varnothing$. □

Lemma 3. *Let (s_k) be a sequence of positive integers and the following conditions hold: $\lim\limits_{k\to\infty} s_k = \infty$, $\lim\limits_{k\to\infty} \frac{k}{\sum_{i=1}^{k} s_i} = 0$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \geqslant 0$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. Then there exist sequences $a_n(\alpha_1, \alpha_2)$ and $b_n(\beta_1, \beta_2)$ such that $a_n(\alpha_1, \alpha_2) \in \{\alpha_1, \alpha_2\}$ and $b_n(\beta_1, \beta_2) \in \{\beta_1, \beta_2\}$ for all $n \in N$ and the limits*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} [a_i(\alpha_1,\alpha_2) \cdot s_i]}{\sum_{i=1}^{k} s_i} \quad \textit{and} \quad \lim_{k\to\infty} \frac{\sum_{i=1}^{k} [b_i(\beta_1,\beta_2) \cdot s_i]}{\sum_{i=1}^{k} s_i}$$

do not exist.

Proof. Without loss of generality let $\alpha_2 > \alpha_1$, $\beta_2 > \beta_1$,

$$\frac{\sum_{i=1}^{k} [\lambda s_i]}{\sum_{i=1}^{k} s_i} \leqslant \frac{\sum_{i=1}^{k} \lambda s_i}{\sum_{i=1}^{k} s_i} = \lambda \quad \text{and} \quad \frac{\sum_{i=1}^{k} [\lambda s_i]}{\sum_{i=1}^{k} s_i} > \frac{\sum_{i=1}^{k} \lambda (s_i - 1)}{\sum_{i=1}^{k} s_i} = \lambda - \frac{k}{\sum_{i=1}^{k} s_i} \to \lambda \quad \text{as} \quad k \to \infty.$$

Hence, $\lim\limits_{k\to\infty}\dfrac{\sum\limits_{i=1}^{k}[\lambda s_i]}{\sum\limits_{i=1}^{k}s_i}=\lambda.$

Suppose that $\varepsilon>0$ satisfies $\alpha_2-\varepsilon>\alpha_1+\varepsilon$ and $\beta_2-\varepsilon>\beta_1+\varepsilon$. Let $r_1,\ l_1$ be smallest positive integers such that for any $n>r_1$ and $m>l_1$ the following inequalities hold:

$$\frac{\sum\limits_{i=1}^{n}[\alpha_2 s_i]}{\sum\limits_{i=1}^{n}s_i}>\alpha_2-\varepsilon \quad\text{and}\quad \frac{\sum\limits_{i=1}^{m}[\beta_2 s_i]}{\sum\limits_{i=1}^{n}s_i}>\beta_2-\varepsilon.$$

Denote $n_1=\max(r_1,l_1)$.

Let $r_2,\ l_2$ be smallest positive integers such that for all $r_2>n_1$, $l_2>n_1$ and $n>r_2$, $m>l_2$,

$$\frac{\sum\limits_{i=1}^{n_1}[\alpha_2 s_i]+\sum\limits_{i=n_1+1}^{n}[\alpha_1 s_i]}{\sum\limits_{i=1}^{n}s_i}<\alpha_1+\varepsilon \quad\text{and}\quad \frac{\sum\limits_{i=1}^{n_1}[\beta_2 s_i]+\sum\limits_{i=n_1+1}^{m}[\beta_1 s_i]}{\sum\limits_{i=1}^{n}s_i}<\beta_1+\varepsilon.$$

Denote $n_2=\max(r_2,l_2)$.

Let $r_3,\ l_3$ be smallest positive integers such that for any $r_3>n_2$, $l_3>n_2$ and $n>r_3$, $m>l_3$, the following inequalities hold:

$$\frac{\sum\limits_{i=1}^{n_1}[\alpha_2 s_i]+\sum\limits_{i=n_1+1}^{n_2}[\alpha_1 s_i]+\sum\limits_{i=n_2+1}^{n}[\alpha_2 s_i]}{\sum\limits_{i=1}^{n}s_i}>\alpha_2-\varepsilon$$

and

$$\frac{\sum\limits_{i=1}^{n_1}[\beta_2 s_i]+\sum\limits_{i=n_1+1}^{n_2}[\beta_1 s_i]+\sum\limits_{i=n_2+1}^{m}[\beta_2 s_i]}{\sum\limits_{i=1}^{n}s_i}>\beta_2-\varepsilon.$$

Denote $n_3=\max(r_3,l_3)$. And so on.

Let $a_n(\alpha_1,\alpha_2)=\alpha_1$ if $n\in\{1,\dots,n_1-1\}$, $a_n(\alpha_1,\alpha_2)=\alpha_2$ if $n\in\{n_k,\dots,n_{k+1}-1\}$ and k is not an even integer; $a_n(\alpha_1,\alpha_2)=\alpha_1$ if $n\in\{n_k,\dots,n_{k+1}-1\}$ and k is an even integer.

Let $b_n(\beta_1,\beta_2)=\beta_1$ if $n\in\{1,\dots,n_1-1\}$, $b_n(\beta_1,\beta_2)=\beta_2$ if $n\in\{n_k,\dots,n_{k+1}-1\}$ and k is not an even integer; $b_n(\beta_1,\beta_2)=\beta_1$ if $n\in\{n_k,\dots,n_{k+1}-1\}$ and k is an even integer. This is possible since for fixed p following relations hold:

$$\lim_{k\to\infty}\frac{\sum\limits_{i=p}^{k}[\lambda s_i]}{\sum\limits_{i=1}^{k}s_i}=\lim_{k\to\infty}\left(\frac{\sum\limits_{i=1}^{k}[\lambda s_i]}{\sum\limits_{i=1}^{k}s_i}-\frac{\sum\limits_{i=1}^{p-1}[\lambda s_i]}{\sum\limits_{i=1}^{k}s_i}\right)=\lambda-0=\lambda.$$

Denote $x_n=\frac{\sum_{i=1}^{n}[a_i(\alpha_1,\alpha_2)s_i]}{\sum_{i=1}^{k}s_i}$, $y_n=\frac{\sum_{i=1}^{n}[b_i(\beta_1,\beta_2)s_i]}{\sum_{i=1}^{k}s_i}$. Suppose the limits $\lim\limits_{k\to\infty}x_n$ and $\lim\limits_{k\to\infty}y_n$ exist. Let $\delta<\min(\alpha_2-\alpha_1-2\varepsilon,\beta_2-\beta_1-2\varepsilon)$. From the Cauchy criterion, it follows that there are $N_1,N_2\in N$ such that for any $a,b>N_1$ and $c,d>N_2$ the inequalities

$|x_a - x_b| < \varepsilon$ and $|y_c - y_d| < \varepsilon$ hold. For a sufficiently large k we have $n_k > N_j,\ j \in \{1,2\}$, whence

$$|x_{n_{k+1}} - x_{n_k}| = \alpha_2 - \alpha_1 - 2\varepsilon > \delta \quad \text{and} \quad |y_{n_{k+1}} - y_{n_k}| = \beta_2 - \beta_1 - 2\varepsilon > \delta.$$

This contradiction proves the lemma. □

4. Properties of the set Θ_2

Let (s_k) be a sequence of positive integers such that

$$\lim_{k\to\infty} s_k = \infty, \quad \lim_{k\to\infty} \frac{s_{k+1}}{\sum_{i=1}^{k} s_i} = 0, \quad \lim_{k\to\infty} \frac{k}{\sum_{i=1}^{k} s_i} = 0.$$

Let $\|\tau_{in}\|$ be a matrix of dimension $(4\times\infty)$. Consider the following form of a real number $x \in [0;1]$:

$$\hat{x} = \Delta^4_{\underbrace{\underbrace{0\ldots0}_{[\tau_{01}s_1]}\underbrace{1\ldots1}_{[\tau_{11}s_1]}\underbrace{2\ldots2}_{[\tau_{21}s_1]}\underbrace{3\ldots3}_{[\tau_{31}s_1]}}_{\text{1st block}}\ldots\underbrace{\underbrace{0\ldots0}_{[\tau_{0k}s_k]}\underbrace{1\ldots1}_{[\tau_{1k}s_k]}\underbrace{2\ldots2}_{[\tau_{2k}s_k]}\underbrace{3\ldots3}_{[\tau_{3k}s_k]}}_{\text{kth block}}\ldots}$$

In paper [9] we proved the following three theorems.

Theorem 2. *If $\|\tau_{in}\|$ is a matrix of dimension $(4\times\infty)$ such that for all $n \in N$ the following conditions hold: $\tau_{0n} + \tau_{1n} + \tau_{2n} + \tau_{3n} = 1$, $\tau_{1n} + 2\tau_{2n} + 3\tau_{3n} = \theta$, then*

$$\lim_{n\to\infty} r_n(\hat{x}) = \theta.$$

Theorem 3. *If $\|\tau_{in}\|$ is a stochastic matrix of dimension $(4\times\infty)$ and for any fixed $j \in \{0,1,2,3\}$, $\lim_{n\to\infty} \tau_{jn} = \lambda$, then*

$$\nu_j(\hat{x}) = \lambda.$$

Theorem 4. *Let $(s_k^{(1)})$, $(s_k^{(2)})$ be sequences of positive numbers such that $\lim_{k\to\infty} s_k^{(r)} = \infty$, $r \in \{1,2\}$ and $\|p^{(1)}\| = \|p_{in}^{(1)}\|$, $\|p^{(2)}\| = \|p_{in}^{(2)}\|$ be stochastic matrices of dimension $(4\times\infty)$. Let*

$$x(\|p^{(r)}\|;\|s_k^{(j)}\|) = \Delta^4_{\underbrace{\underbrace{0\ldots0}_{[p_{01}^{(r)}s_1^{(j)}]}\ \underbrace{1\ldots1}_{[p_{11}^{(r)}s_1^{(j)}]}\ \underbrace{2\ldots2}_{[p_{21}^{(r)}s_1^{(j)}]}\ \underbrace{3\ldots3}_{[p_{31}^{(r)}s_1^{(j)}]}}_{\text{1st block}}\ldots\underbrace{\underbrace{0\ldots0}_{[p_{0k}^{(r)}s_k^{(j)}]}\ \underbrace{1\ldots1}_{[p_{1k}^{(r)}s_k^{(j)}]}\ \underbrace{2\ldots2}_{[p_{2k}^{(r)}s_k^{(j)}]}\ \underbrace{3\ldots3}_{[p_{3k}^{(r)}s_k^{(j)}]}}_{\text{kth block}}\ldots}$$

If $\lim_{k\to\infty} |s_k^{(1)} - s_k^{(2)}| = \infty$, then $x(\|p^{(1)}\|;\|s_k^{(1)}\|) \neq x(\|p^{(2)}\|;\|s_k^{(2)}\|)$.

If $\overline{\lim_{n\to\infty}} \sum_{i=0}^{3} |p_{in}^{(1)} - p_{in}^{(2)}| > 0$, then $x(\|p^{(1)}\|;\|s_k^{(1)}\|) \neq x(\|p^{(2)}\|;\|s_k^{(2)}\|)$.

Theorem 5. *If $\theta \in (0;3)$, then the set Θ_2 is an everywhere dense, continuum set of zero Lebesgue measure.*

Proof. The well-known Borel's theorem states that $\nu_0 = \nu_1 = \nu_2 = \nu_3 = \frac{1}{4}$ for almost all in the sense of Lebesgue measure numbers of $[0;1]$. From this fact it follows that Lebesgue measure of the set Θ_2 is equal to zero.

Continuality. Construct a continuum subset of Θ_2 such that frequency of the digit 0 exists for all elements (similarly we can construct a continuum subset of Θ_2 such that the frequency of a fixed digit "i", $i \in \{1,2,3\}$, exists for all elements). Let $s_k = k$ and $\overline{p} = (p_0, p_1, p_2, p_3)$, $\overline{q} = (q_0, q_1, q_2, q_3)$ be stochastic vectors such that $p_0 = q_0$,

$p_1 + 2p_2 + 3p_3 = q_1 + 2q_2 + 3q_3 = \theta$, $p_1 \neq q_1$, then $\lim\limits_{k\to\infty} s_k = \infty$, $\lim\limits_{k\to\infty} \frac{k}{\sum\limits_{i=1}^{k} s_i} = 0$, $\lim\limits_{k\to\infty} \frac{s_{k+1}}{\sum\limits_{i=1}^{k} s_i} = 0$. From Lemma 3, where $\alpha_1 = \beta_1 = p_1$, $\alpha_2 = \beta_2 = q_1$, it follows that there exists a sequence $a_n(p_1, q_1)$ such that $a_n(p_1, q_1) = p_1$ or $a_n(p_1, q_1) = q_1$ for any $n \in N$ and $\lim\limits_{k\to\infty} \frac{\sum_{i=1}^{k}[a_i(p_1,q_1)s_i]}{\sum_{i=1}^{k} s_i}$ does not exist. Denote $\tau_{0k} = p_0$, $\tau_{1k} = a_k(p_1, q_1)$. Using the system

$$\begin{cases} \tau_{2k} + \tau_{3k} = 1 - p_0 - a_k(p_1, q_1), \\ 2\tau_{2k} + 3\tau_{3k} v_3^{(n)} = \theta - a_k(p_1, q_1) \end{cases}$$

we calculate τ_{2k}, τ_{3k}. Namely, $\tau_{3k} = \theta + a_k(p_1, q_1) - 2 + 2p_0$ $\tau_{2k} = 3 - 3p_0 - \theta - 2a_k(p_1, q_1)$. It is evident that τ_{ik}, where $i \in \{2, 3\}$ is equal to p_i or q_i if $a_k(p_1, q_1)$ is equal to p_1 or q_1, respectively. From Theorem 3 it follows that $\nu_0(x) = p_0$. Since $\lim\limits_{k\to\infty} \frac{\sum_{i=1}^{k}[a_i(p_1,q_1)s_i]}{\sum_{i=1}^{k} s_i}$ does not exist we obtain that the frequency $\nu_1(x)$ does not exist either. From Theorem 4 it follows that different numbers x constructed as specified above correspond to different pairs of vectors $\overline{p}$ and $\overline{q}$ with relevant properties. Since the set of such pairs is a continuum, we see that the set Θ_2 is a continuum.

Everywhere density. Since the condition $\lim\limits_{k\to\infty} r_k(x) = \theta$ does not depend on an arbitrary finite group of first symbols and for any interval $[a; b]$ there exists a cylinder $[\Delta^4_{\gamma_1\gamma_2\ldots\gamma_r(0)}; \Delta^4_{\gamma_1\gamma_2\ldots\gamma_r(3)}]$ completely contained in it, we see that Θ_2 is an everywhere dense set. □

Theorem 6. *If $\theta \in (0; 3)$, then the Hausdorff–Besicovitch dimension $\alpha_0(\Theta_2)$ of the set Θ_2 is positive, i.e., $\alpha_0(\Theta_2) > 0$.*

Proof. Let (ε_i) be an arbitrary sequence of zeros and ones, vectors (p_0, p_1, p_2, p_3) and (p_0, q_1, q_2, q_3) be stochastic vectors such that $p_0 > 0$, $p_1 \neq q_1$, $p_1 + 2p_2 + 3p_3 = \theta = q_1 + 2q_2 + 3q_3$, $x_i = [p_0k(i+1)] - [p_0ki] - r_i$, $r_i = \begin{cases} 0, & \text{if } \varepsilon_i = 1 \\ 1, & \text{if } \varepsilon_i = 0 \end{cases}$, $t_i = [p_3k(i+1)] - [p_3ki]$.

Consider the system

$$\begin{cases} x_i + y_i + z_i + t_i = k - 1, \\ y_i + 2z_i + 3t_i = [\theta k(i + 1)] - [\theta ki], \end{cases} \tag{2}$$

whence $z_i = [\theta k(i+1)] - [\theta ki] - k + 1 + x_i - 2y_i$, $y_i = 2(k-1) - ([\theta k(i+1)] - [\theta ki]) - 2x_i + t_i$. We obtain that

$$\begin{aligned} \frac{z_i}{k} &= \frac{[\{\theta ki\} + \theta k]}{k} - 1 + \frac{1}{k} + \frac{[\{p_0ki\} + p_0k]}{k} - \frac{2[\{p_3ki\} + p_3k]}{k} \\ &= \theta - 1 + p_0 - 2p_3 + \frac{[\{\theta ki\} + \{\theta k\}]}{k} + \frac{1}{k} + \frac{[\{p_0ki\} + \{p_0k\}]}{k} - \frac{2[\{p_3ki\} + \{p_3k\}]}{k}. \end{aligned}$$

Since $\theta - 1 + p_0 - 2p_3 = p_2$ and for a sufficiently large k we have $z_i \in N$ for all $i \in N$. In the same way,

$$\frac{y_i}{k} = 2 - \frac{2}{k} - \frac{[\{\theta ki\} + \theta k]}{k} - \frac{2[\{p_0 ki\} + p_0 k]}{k} + \frac{r_i}{k} + \frac{[\{p_3 ki\} + p_3 k]}{k}$$
$$= 2 - \theta - 2p_0 + 3p_3 - \frac{2}{k} - \frac{[\{\theta ki\} + \{\theta k\}]}{k} - \frac{2[\{p_0 ki\} + \{p_0 k\}]}{k}$$
$$+ \frac{r_i}{k} + \frac{[\{p_3 ki\} + \{p_3 k\}]}{k}.$$

Since $2 - \theta - 2p_0 + 3p_3 = p_1$ and for a sufficiently large k we obtain $y_i \in N$ for all $i \in N$. Similarly we prove that for a sufficiently large $k \in N$ all solutions of the system

$$(3) \qquad \begin{cases} x_i + y_i + z_i + t_i = k - 1, \\ y_i + 2z_i + 3t_i = [\theta k(i+1)] - \theta ki], \\ x_i = [p_0 k(i+1)] - [p_0 ki] - r_i, \\ t_i = [q_3 k(i+1)] - [q_3 ki], \end{cases}$$

are positive integers for all $i \in N$.

Let k be a sufficiently large positive integer. Let all solutions of systems (2) and (3) be positive integers for arbitrary sequence of zeros and ones (ε_i), $i \in N$. Construct the number $x(\varepsilon_i)$ as follows:

$$x(\varepsilon_i) = \Delta^4_{\underbrace{\varepsilon_1 \underbrace{0\ldots0}_{x_1}\underbrace{1\ldots1}_{y_1}\underbrace{2\ldots2}_{z_1}\underbrace{3\ldots3}_{t_1}}_{\text{k symbols}}\ldots\underbrace{\varepsilon_j \underbrace{0\ldots0}_{x_j}\underbrace{1\ldots1}_{y_j}\underbrace{2\ldots2}_{z_j}\underbrace{3\ldots3}_{t_j}}_{\text{k symbols}}\ldots}$$

Without loss of generality put $p_3 > q_3$, let $\delta > 0$ be such that $p_3 - \delta > q_3 - \delta$. Let r_1 be a positive integer such that (x_i, y_i, z_i, t_i) is a solution of system (2) for any $j \in \{1, 2, \ldots, r_1\}$ and

$$\frac{N_3(x, kr_1)}{kr_1} = \frac{\sum\limits_{i=1}^{r_1} t_i}{kr_1} = \frac{[p_3 k(r_1+1)]}{kr_1} > p_3 - \delta,$$

this is possible since the last value tends to p_3 as $r_1 \to \infty$.

Let $r_1 < r_2$ be a positive integer such that (x_j, y_j, z_j, t_j) is a solution of system (3) for any $j \in \{r_1 + 1, \ldots, r_2\}$ and

$$\frac{N_3(x, kr_2)}{kr_2} = \frac{\sum\limits_{i=1}^{r_2} t_i}{kr_2} = \frac{[p_3 k(r_1+1)] - [q_3 k(r_1+1)] + [q_3 k(r_2+1)]}{kr_2} < q_3 + \delta,$$

this is possible since the last value tends to q_3 as $r_2 \to \infty$.

Let $r_2 < r_3$ be a positive integer such that (x_j, y_j, z_j, t_j) is a solution of system (2) for any $j \in \{r_2 + 1, \ldots, r_3\}$ and

$$\frac{N_3(x, kr_3)}{kr_3}$$
$$= 3\frac{[p_3 k(r_1+1)] - [q_3 k(r_1+1)] + [q_3 k(r_2+1)] - [p_3 k(r_2+1)] + [p_3 k(r_3+1)]}{kr_3} >$$
$$> p_3 - \delta,$$

this is possible since the last value tends to p_3 as $r_k \to \infty$. And so on.

We obtain that $\left|\frac{N_3(x, kr_i)}{kr_i} - \frac{N_3(x, kr_{i+1})}{kr_{i+1}}\right| > p_3 - q_3 - 2\delta$ for all $i \in N$. Assume that $\lim\limits_{i\to\infty} \frac{N_3(x, kr_i)}{kr_i}$ exists. Hence, we have a contradiction with Cauchy's criterion. Thus,

$\lim\limits_{i\to\infty} \dfrac{N_3(x, kr_i)}{kr_i}$ does not exist, i.e., the frequency $\nu_3(x(\varepsilon_i))$ does not exist. On the other hand, if $kj \leqslant n \leqslant k(j+1)$ then

$$\frac{N_0(x(\varepsilon_i), n)}{n} \geqslant \frac{\sum\limits_{i=1}^{j}[p_0k(i+1)] - [p_0ki]}{k(j+1)} = \frac{[p_0k(j+1)] - [p_0k]}{k(j+1)}$$
$$= p_0 - \frac{\{p_0k(j+1)\} - [p_0k]}{k(j+1)} \to p_0, \quad \text{as} \quad j \to \infty,$$
$$\frac{N_0(x(\varepsilon_i), n)}{n} \leqslant \frac{\sum\limits_{i=1}^{j+1}[p_0k(i+1)] - [p_0ki]}{kj} = \frac{\{p_0k(j+2)\} - [p_0k]}{kj}$$
$$= p_0\frac{j+2}{j} - \frac{\{p_0k(j+2)\} - [p_0k]}{kj} \to p_0, \quad \text{as} \quad j \to \infty,$$

hence, $\nu_0(x(\varepsilon_i)) = p_0$. Also

$$r_n(x(\varepsilon_i)) \geqslant \frac{\sum\limits_{i=1}^{j}[\theta k(i+1)] - [\theta ki]}{k(j+1)} = \frac{[\theta k(j+1)] - [\theta k]}{k(j+1)}$$
$$= \theta - \frac{\{\theta k(j+1)\} - [\theta k]}{k(j+1)} \to \theta, \quad \text{as} \quad j \to \infty,$$
$$r_n(x(\varepsilon_i)) \leqslant \frac{\sum\limits_{i=1}^{j+1}[\theta k(i+1)] - [\theta ki]}{kj} = \frac{\{\theta k(j+2)\} - [\theta k]}{kj}$$
$$= \theta\frac{j+2}{j} - \frac{\{\theta k(j+2)\} - [\theta k]}{kj} \to \theta, \quad \text{as} \quad j \to \infty,$$

hence, $\lim\limits_{n\to\infty} r_n(x(\varepsilon_i)) = \theta$, and from Theorem 4, the frequencies $\nu_1(x(\varepsilon_i))$ and $\nu_2(x(\varepsilon_i))$ do not exist.

Thus, $x(\varepsilon_i) \in \Theta_2$.

Selecting an arbitrary quantity of (not necessarily consecutive) blocks of number $x(\varepsilon_i)$ and changing the order of digits (except for ε_i) inside each block we get either the "old" number $x(\varepsilon_i)$, or a new number $\widetilde{x}(\varepsilon_i)$. These numbers belong to Θ_2 since $N_l(x(\varepsilon_i), kr) = N_l(\widetilde{x}(\varepsilon_i), kr)$ for any $r \in N$ and $l \in \{0, 1, 2, 3\}$. Denote by $C(x(\varepsilon_i))$ the set of numbers $\widetilde{x}(\varepsilon_i)$ obtained from $x(\varepsilon_i)$ by choosing an arbitrary number of blocks and changing the digit order inside them. It is obvious that the set is a continuum. Denote by C_1 a union of the sets $C(x(\varepsilon_i))$ with respect to all possible sequences (ε_i) and show that $\alpha_0(C_1) = \frac{1}{2k}$.

To calculate the Hausdorff–Besicovitch dimension it is sufficient to use covering of 4-adic cylinders. Consider a covering of the set C_1 by cylinders of the same rank m. the α-volume of the covering is equal to

$$R_m^{\alpha} = \begin{cases} 2^{t-1}(4^{-(kt-j)})^{\alpha}, & \text{if} \quad m = kt - j,\ j \in \{1, \dots, k-1\}, \\ 2^t(4^{-kt})^{\alpha}, & \text{if} \quad m = kt. \end{cases}$$

It is clear that $R_{kt-1}^{\alpha} < R_{kt-j}^{\alpha}$, $j \in \{2, \dots, k-1\}$ hence, consider an α-covering of the set C_1 with cylinders of rank $n = kt - 1$.

The Hausdorff's box–counting α-measure of the set C_1 is equal to

$$\widehat{H}_{\alpha}(C_1) = \varliminf_{t\to\infty} \frac{2^{t-1}}{4^{(kt-1)\alpha}} = 2^{2\alpha-1} \varliminf_{t\to\infty} 2^{t(1-2k\alpha)}.$$

Whence,

$$\widehat{H}_\alpha(C_1) = \begin{cases} 0, & \text{if } \alpha > \dfrac{1}{2k}, \\ \infty, & \text{if } \alpha < \dfrac{1}{2k}. \end{cases}$$

Therefore, box–counting dimension of the set C_1 is equal to $\alpha = \frac{1}{2k}$. Let us show that $\alpha_0(C_1) = \frac{1}{2k}$. Consider an arbitrary finite covering of the set C_1 by 4-adic cylinders $\{v_j\}$, $j \in \{1, \dots, l\}$, and prove that if $\alpha = \frac{1}{2k}$ then the preceding rank covering is not improvable. Let u_i be one of cylinders of the covering. Then $|u_j| = 3^{-n}$ for some $n \in N$. Let $n = kp - r,\ r \in \{0, \dots, k-1\}$, then α-volume of covering of the set $C_1 \cap \Delta_j$ by cylinders of rank $kp - r + m$ is equal to

$$R^\alpha_{kp-r+m}(C_1 \cap \Delta_j) = \begin{cases} 2^{l-1}(4^{-(kp-r+kl-j)})^\alpha, & \text{if } m = kl + r - j,\ j \in \{1, \dots, k-1\}, \\ 2^k(4^{-(kp-r+kl)})^\alpha, & \text{if } m = kl + r,\ l \in N. \end{cases}$$

Let us show that $R^\alpha_{kp-r+m}(C_1 \cap v_j) \leqslant v_n = (4^{-(kp-r)})^\alpha$. It is obvious that $R^\alpha_{k(p+l)-1} < R^\alpha_{k(p+l)-j}$, if $j \in \{2, \dots, k\}$. Consider an α-covering $K \cap \Delta_j$ by cylinders of rank $k(p+l)-1$. Its volume is equal to

$$2^{l-1}(4^{-(kp+3l-1)})^\alpha.$$

Since $\left(\dfrac{2}{4^{kl}}\right)^l = 1$ and $\dfrac{4^{(1-r)\alpha}}{2} < 1$ we obtain if $\alpha = \dfrac{1}{2k}$ then $2^{l-1}(4^{-(kp+3l-1)})^\alpha = \left(4^{-(kp-r)}\right)^\alpha \dfrac{4^{(1-r)\alpha}}{2}\left(\dfrac{2}{4^{kl}}\right)^l \leqslant \left(4^{-(kp-r)}\right)^\alpha$. Hence if $\alpha = \dfrac{1}{2k}$ we have $\widehat{H}_\alpha(C_1) = H_\alpha(C_1) = \dfrac{1}{2k}$ and we see that the Hausdorff–Besicovitch dimension of the set C_1 is equal to the box–counting dimension of the set $C_1 \subset \Theta_2$, thus $\alpha_0(\Theta_2) \geqslant \alpha_0(C_1) = \dfrac{1}{2k} > 0$.

□

5. The set Θ_3

Theorem 7. *If $\theta \in (0;3)$, then the set Θ_3 is an everywhere dense, continuum set of zero Lebesgue measure.*

Proof. Lebesgue measure. Since almost all (in the sense of Lebesgue measure) numbers of the interval $[0;1]$ are normal, i.e., $\nu_0 = \nu_1 = \nu_2 = \nu_3 = \dfrac{1}{4}$ [5] we see that Lebesgue measure of the set Θ_3 is equal to zero.

Continuality. Let $s_k = k$, $p_0 > q_0 > 0$, $p_1 > q_1 > 0$. Suppose that all solutions of the system

$$\begin{cases} x + y + z + t = 1, \\ y + 2z + 3t = \theta, \\ x = p_0 \vee q_0, \\ y = p_1 \vee q_1 \end{cases}$$

are positive.

It is obvious that $\lim\limits_{k\to\infty} s_k = \infty$, $\lim\limits_{k\to\infty} \dfrac{k}{\sum\limits_{i=1}^{k} s_i} = 0$ $\lim\limits_{k\to\infty} \dfrac{s_{k+1}}{\sum\limits_{i=1}^{k} s_i} = 0$. From Lemma 3, where $\alpha_1 = p_0$, $\alpha_2 = q_0$, $\beta_1 = p_1$, $\beta_2 = q_1$ it follows that there exist sequences $a_n(p_0, q_0) = p_0$

or $a_n(p_0, q_0) = q_0$ and $b_n(p_1, q_1) = p_1$ or $b_n(p_1, q_1) = q_1$ such that for all $n \in N$ the limits

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k}[a_i(p_0, q_0)s_i]}{\sum_{i=1}^{k} s_i} \quad \text{and} \quad \lim_{k\to\infty} \frac{\sum_{i=1}^{k}[b_i(p_1, q_1)s_i]}{\sum_{i=1}^{k} s_i}$$

do not exist.

Denote $\tau_{0k} = a_k(p_0, q_0)$, $\tau_{1k} = b_k(p_1, q_1)$. From following system

$$\begin{cases} \tau_{0k} + \tau_{1k} + \tau_{2k} + \tau_{3k} = 1, \\ \tau_{1k} + 2\tau_{2k} + 3\tau_{3k} = \theta \end{cases}$$

we obtain τ_{2k}, τ_{3k}, i.e., $\tau_{3k} = \theta - 2 + 2\tau_{0k} + \tau_{1k}$, $\tau_{2k} = 3 - \theta - 3\tau_{0k} - \tau_{1k}$.

Since the limits

$$\lim_{k\to\infty} \frac{N_0(x, \sum_{i=1}^{k} s_i)}{\sum_{i=1}^{k} s_i} = \lim_{k\to\infty} \frac{\sum_{i=1}^{k}[a_i(p_0, q_0)s_i]}{\sum_{i=1}^{k} s_i}$$

and

$$\lim_{k\to\infty} \frac{N_1(x, \sum_{i=1}^{k} s_i)}{\sum_{i=1}^{k} s_i} = \lim_{k\to\infty} \frac{\sum_{i=1}^{k}[b_i(p_1, q_1)s_i]}{\sum_{i=1}^{k} s_i}$$

do not exist, the frequencies $\nu_0(x)$ and $\nu_1(x)$ do not exist either. Then from Theorem 2 and from Theorem 4 it follows that $\lim_{n\to\infty} r_n(x) = \theta$ and $\nu_2(x)$, $\nu_3(x)$ do not exist.

From Theorem 4 it follows that different numbers constructed as indicated above correspond to different pairs (p_0, q_0) and (p_1, q_1). Since the set of such pairs is a continuum, we obtain that set Θ_3 is a continuum.

Everywhere density. Since the condition $\lim_{k\to\infty} r_k(x) = \theta$ does not depend on an arbitrary finite group of first symbols, and for any interval $[a; b] \subset [0; 1]$ there exists a cylinder $[\Delta^4_{\gamma_1\gamma_2\ldots\gamma_r(0)}; \Delta^4_{\gamma_1\gamma_2\ldots\gamma_r(3)}]$ completely contained in it, we see that Θ_3 is an everywhere dense set. □

Theorem 8. *If $\theta \in (0; 3)$, then the Hausdorff–Besicovitch dimension $\alpha_0(\Theta_3)$ of the set Θ_3 is positive, i.e., $\alpha_0(\Theta_3) > 0$.*

Proof. Let (ε_i) be an arbitrary sequence of zeros and ones, let $p_0^{(1)} > p_0^{(2)} > 0$ and $p_1^{(1)} > p_1^{(2)} > 0$, let solutions of the system

$$\begin{cases} x + y + z + t = 1, \\ y + 2z + 3t = \theta, \\ x = p_0^{(1)} \vee p_0^{(2)}, \\ y = p_1^{(1)} \vee p_1^{(2)} \end{cases}$$

be positive.

Denote

$$r_i = \begin{cases} 0, \text{ if } \varepsilon_i = 1, \\ 1, \text{ if } \varepsilon_i = 0, \end{cases} \qquad \widetilde{r_i} = \begin{cases} 0, \text{ if } \varepsilon_i = 0, \\ 1, \text{ if } \varepsilon_i = 1. \end{cases}$$

Similarly to the proof of Theorem 6, we show existence of a sufficiently large positive integer k such that all solutions of the systems

$$(4)\qquad \begin{cases} x_i + y_i + z_i + t_i = k+1, \\ y_i + 2z_i + 3t_i = [\theta k(i+1)] - [\theta k i], \\ x_i = [p_0^{(1)} k(i+1)] - [p_0^{(1)} k i] - r_i, \\ t_i = [p_1^{(1)} k(i+1)] - [p_1^{(1)} k i] - \widetilde{r}_i, \end{cases}$$

$$(5)\qquad \begin{cases} x_i + y_i + z_i + t_i = k+1, \\ y_i + 2z_i + 3t_i = [\theta k(i+1)] - [\theta k i], \\ x_i = [p_0^{(2)} k(i+1)] - [p_0^{(2)} k i] - r_i, \\ t_i = [p_1^{(2)} k(i+1)] - [p_1^{(2)} k i] - \widetilde{r}_i \end{cases}$$

are positive for all $i \in N$.

Let (ε_i) be a fixed sequence of zeros and ones. Construct a number $x(\varepsilon_i)$ as follows:

$$x(\varepsilon_i) = \Delta^4_{\underbrace{\varepsilon_1 \underbrace{0\ldots0}_{x_1}\underbrace{1\ldots1}_{y_1}\underbrace{2\ldots2}_{z_1}\underbrace{3\ldots3}_{t_1}}_{\text{k symbols}} \ldots \underbrace{\varepsilon_j \underbrace{0\ldots0}_{x_j}\underbrace{1\ldots1}_{y_j}\underbrace{2\ldots2}_{z_j}\underbrace{3\ldots3}_{t_j}}_{\text{k symbols}} \ldots}$$

Let $\delta > 0$ be such that $p_0^{(1)} - \delta > p_0^{(2)} + \delta$, $p_1^{(1)} - \delta > p_1^{(2)} + \delta$.

Let g_1 be a positive integer such that (x_j, y_j, z_j, t_j) is a solution of system (4) for any $j \in \{1, 2, \ldots, g_1\}$ and $\dfrac{N_0(x, kg_1)}{kg_1} > p_0^{(1)} - \delta$, $\dfrac{N_1(x, kg_1)}{kg_1} > p_1^{(1)} - \delta$.

Let g_2 be a positive integer such that (x_j, y_j, z_j, t_j) is a solution of system (5) for all $j \in \{g_1+1, g_1+2, \ldots, g_2\}$ and $\dfrac{N_0(x, kg_2)}{kg_2} < p_0^{(2)} + \delta$, $\dfrac{N_1(x, kg_2)}{kg_2} < p_1^{(2)} + \delta$.

Let g_3 be a positive integer such that (x_j, y_j, z_j, t_j) is a solution of system (4) for any $j \in \{g_2+1, g_2+2, \ldots, g_3\}$ and $\dfrac{N_0(x, kg_3)}{kg_3} > p_0^{(1)} - \delta$, $\dfrac{N_1(x, kg_3)}{kg_3} > p_1^{(1)} - \delta$. And so on.

Since

$$\left| \frac{N_a(x, kg_{j+1})}{kg_{j+1}} - \frac{N_a(x, kg_j)}{kg_j} \right| > p_a^{(1)} - p_a^{(2)} - 2\delta$$

for all $j \in N$, the limits $\lim\limits_{j\to\infty} \dfrac{N_a(x, kg_j)}{kg_j}$, $a \in \{0, 1\}$ do not exist (assuming the converse, we obtain a contradiction to the Cauchy criterion). Thus, $\nu_0(x(\varepsilon_i))$ and $\nu_1(x(\varepsilon_i))$ do not exist.

Let

$$kj \leqslant n < k(j+1),$$
$$r_n \geqslant \frac{[\theta k(j+1)] - [\theta k]}{k(j+1)} = \theta - \frac{\{\theta k(j+1)\} + [\theta k]}{k(j+1)} \to \theta,$$
$$r_n \leqslant \frac{[\theta k(j+2)] - [\theta k]}{kj} = \theta \cdot \frac{j+2}{j} - \frac{\{\theta k(j+2)\} - [\theta k]}{kj} \to \theta \quad (j \to \infty).$$

Hence, $r_n \to \theta$ as $n \to \infty$ and from Theorem 4, it follows that the frequencies $\nu_2(x(\varepsilon_i))$ and $\nu_3(x(\varepsilon_i))$ do not exist, i.e., $x(\varepsilon_i) \in \Theta_3$.

Selecting an arbitrary quantity of (not necessarily consecutive) blocks of number $x(\varepsilon_i)$ and changing the order of digits inside each block (except for ε_i) we obtain either the "old" number $x(\varepsilon_i)$, or a new number $\widetilde{x}(\varepsilon_i)$. These numbers are contained in Θ_3 since $N_l(x(\varepsilon_i), kr) = N_l(\widetilde{x}(\varepsilon_i), kr)$, for any $r \in N$ and $l \in \{0, 1, 2, 3\}$. Denote by $C(x(\varepsilon_i))$ the set of numbers $\widetilde{x}(\varepsilon_i)$ obtained from $x(\varepsilon_i)$ by choosing an arbitrary number of blocks and changing digit order inside them. It is evident that the set is a continuum. Denote by C_1 a union of the sets $C(x(\varepsilon_i))$ of all possible sequences (ε_i) and show that $\alpha_0(C_1) = \frac{1}{2k}$. Similarly to the proof of Theorem 6, we show that $\alpha_0(C_1) = \dfrac{1}{2k}$.

Thus, $\alpha_0(\Theta_3) \geqslant \alpha_0(C_1) > 0$. □

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