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FRACTIONAL STATISTICAL DYNAMICS AND FRACTIONAL KINETICS

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Abstract. We apply the subordination principle to construct kinetic fractional statistical dynamics in the continuum in terms of solutions to Vlasov-type hierarchies. As a by-product we obtain the evolution of the density of particles in the fractional kinetics in terms of a non-linear Vlasov- type kinetic equation. As an application we study the intermittency of the fractional mesoscopic dynamics.

1. Introduction

A general scheme for the study of Markov dynamics for interacting particle systems (IPS for short) in the continuum includes the following steps. We start with a heuristic Markov generator L defined on functions over a configuration space of the system. Associated with this generator is a forward Kolmogorov equation for states of the system (a.k.a. a Fokker–Planck equation (FPE)). A solution to this equation gives the so-called statistical dynamics of the model under consideration [11]. A constructive approach to the existence and uniqueness problem for solution of the FPE and for the analysis of its properties exploits the possibility of writing this equation as a hierarchical chain of evolution equations for time dependent correlation functions [11, 12]. This step corresponds to a microscopic description of the system.

A mesoscopic level of the study is related with a Vlasov-type scaling limit for the dynamics that leads to a kinetic or Vlasov hierarchy for correlation functions. This scaling limit destroys the Markov property of the evolution of the limiting Vlasov–Fokker–Planck equation (VFPE): for an initial probability measure the solution, in general, is no longer a measure. Still, the resulting dynamics has a conditional Markov property in the following sense. If we start with a Poisson initial state then the solution of the VFPE will be given by a flow of Poisson measures on the configuration space. In theoretical physics this fact is known as the chaos propagation property.

The Poisson flow which appears in the Vlasov limit is completely characterized by the density function $\rho_t(x)$, which corresponds to the Poisson measure from the flow at time $t \geq 0$. A specific feature of the mesoscopic limit is a non-linear Vlasov-type kinetic equation for this density. In most cases this equation may be informally derived directly from the form of the generator L. However, a rigorous realization of the above scheme is a non-trivial task for each particular model [10, 11, 9]. The study of the resulting kinetic equations for concrete Markov dynamics of interacting particle systems in the continuum belongs to the general theory of non-local non-linear evolution equations which has been under active development in recent years.

The aim of the present paper is to extend the concept of statistical dynamics and related structures to the case of fractional time derivatives. From the probabilistic point of view this means that we leave the Markov dynamical framework by introducing a

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random time change in the corresponding Markov process – see for example [28, 23]. In the language of functional analysis we are no more in the arena of semigroup evolutions.

Below we discuss the concept of a fractional Fokker–Plank equation (FPE) and the related fractional statistical dynamics, which is still an evolution in the space of probability measures on the configuration space. The mesoscopic scaling of the generator of this evolutions leads to the same result as for the initial FPE. The latter leads us to the concept of a fractional VFPE. A subordination principle provides for the representation of the solution to this equation as a flow of measures that is a transformation of a Poisson flow for the initial VFPE. Note that the density function for the fractal kinetics is a subordination of the solution to the initial Vlasov equation. This density characterizes the kinetic behavior of the fractional statistical dynamics, but it is not the same as the solution to the Vlasov equation with a fractional time derivative, as is typically assumed in theoretical physics.

In this paper we leave open the problem of rigorous realization of scaling approach for particular models. Instead, our considerations are focused on questions about the properties of subordinated flows. In particular, we clarify the possibility of having time-dependent random point processes with an asymptotic intermittency property as a result of subordination of Poisson flows.

2. Preliminaries

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$ and let $\mathcal{B}_b(\mathbb{R}^d)$ denote the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_Y^{(n)} := \big\{\eta \subset Y \big|\, |\eta| = n\big\}, \quad n \in \mathbb{N}.$$

We also set $\Gamma_Y^{(0)} := \{\emptyset\}$. As a set, $\Gamma_Y^{(n)}$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \big\{(x_1, \ldots, x_n) \in Y^n \big| x_k \neq x_l \text{ if } k \neq l\big\}.$$

The configuration space over the space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$, namely,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \big\{\gamma \subset \mathbb{R}^d \big| |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\big\}. \tag{1}$$

The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. Note that the summation in $\sum_{x\in\gamma} f(x)$ is taken over only finitely many points of γ belonging to the support of f. It was shown in [18] that with the vague topology Γ may be metrizable and it becomes a Polish space (i.e., a complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings

$$\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

are measurable for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ the cardinality of a finite set.

It follows that one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}(\Gamma_Y^{(n)})$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_{0,Y} := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_Y^{(n)}.$$

This space is equipped with the topology of disjoint unions. Therefore one can introduce the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_{0,Y})$. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the notation, thus $\Gamma_0 := \Gamma_{0,\mathbb{R}^d}$ $\Gamma^{(n)} := \Gamma_{\mathbb{R}^d}^{(n)}$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\big(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\big)$ will be denoted by $m^{(n)}$, and we set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2}$$

For any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the restriction of λ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will be also denoted by λ. The space $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is the projective limit of the family of spaces $\big\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\big\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$. The Poisson measure π on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\big(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)\big)$. Here $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a *Lebesgue–Poisson exponent*

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0; \quad e_\lambda(f, \emptyset) := 1. \tag{3}$$

Then, by (2), for $f \in L^1(\mathbb{R}^d, dx)$ we obtain $e_\lambda(f) \in L^1(\Gamma_0, d\lambda)$ and

$$\int_{\Gamma_0} e_\lambda(f, \eta)\, d\lambda(\eta) = \exp\left(\int_{\mathbb{R}^d} f(x)\, dx \right). \tag{4}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. We will make use of the following classes of functions on Γ_0: (i) $L^0_{\mathrm{ls}}(\Gamma_0)$ is the set of all measurable functions on Γ_0 which have local support, i.e., $G \in L^0_{ls}(\Gamma_0)$, if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G \restriction_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$, while (ii) $B_{bs}(\Gamma_0)$ is the set of bounded measurable functions with bounded support, i.e., $G \restriction_{\Gamma_0 \setminus B} = 0$ for some bounded $B \in \mathcal{B}(\Gamma_0)$.

In fact, any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0 is a sequence of functions $\big\{G^{(n)}\big\}_{n \in \mathbb{N}_0}$, where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$.

On Γ we consider the set of cylinder functions $\mathcal{F}_{cyl}(\Gamma)$. These functions are characterized by the relation $F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda)$.

The following mapping from $L^0_{ls}(\Gamma_0)$ into $\mathcal{F}_{cyl}(\Gamma)$ which plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{5}$$

where $G \in L^0_{ls}(\Gamma_0)$. (See, for example, [17], [21, 22]). The summation in (5) is taken over all finite sub-configurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; this relationship is represented symbolically by $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{6}$$

Here and in the sequel, inclusions like $\xi \subset \eta$ hold for $\xi = \emptyset$ as well as for $\xi = \eta$. We denote the restriction of K onto functions on Γ_0 by K_0.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to (w.r.t.) a Poisson measure π if for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the projection of μ onto Γ_Λ is absolutely continuous w.r.t. projection of π onto Γ_Λ. By [17], there exists in this case a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ such that the following equality holds for any $G \in B_{bs}(\Gamma_0)$:

$$\int_\Gamma (KG)(\gamma)\, d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta)\, d\lambda(\eta). \tag{7}$$

Restrictions $k_\mu^{(n)}$ of this functional on $\Gamma_0^{(n)}$, $n \in \mathbb{N}_0$, are called *correlation functions* of the measure μ. Note that $k_\mu^{(0)} = 1$.

3. Mesoscopic statistical dynamics

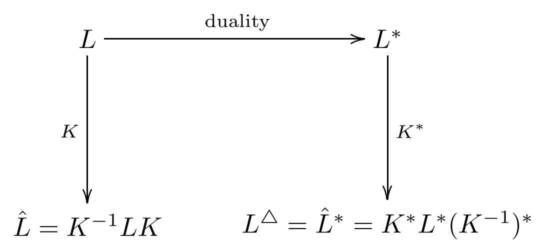
In this section we introduce the general scheme of Vlasov scaling for the Markov dynamics of IPS – interacting particle systems – on configuration space. Thus we assume that the initial distribution (the state of particles) in our system is a probability measure $\mu_0 \in \mathcal{M}^1(\Gamma)$ with corresponding correlation function $k_0 = (k_0^{(n)})_{n=0}^\infty$. The distribution of particles at time $t > 0$ is the measure $\mu_t \in \mathcal{M}^1(\Gamma)$, and $k_t = (k_t^{(n)})_{n=0}^\infty$ its correlation function. If the evolution of states $(\mu_t)_{t\geq 0}$ is determined a priori by a heuristic Markov generator L, then μ_t is the solution of the forward Kolmogorov equation (or Fokker–Plank equation (FPE)),

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L^* \mu_t, \\ \mu_t|t_{=0} &= \mu_0, \end{cases} \tag{8}$$

where L^* is the adjoint operator. In terms of the time-dependent correlation functions $(k_t)_{t\geq 0}$ corresponding to $(\mu_t)_{t\geq 0}$, the FPE may be rewritten as an infinite system of evolution equations

$$\begin{cases} \frac{\partial k_t^{(n)}}{\partial t} &= (L^\triangle k_t)^{(n)}, \\ k_t^{(n)}|_{t=0} &= k_0^{(n)}, \quad n \geq 0, \end{cases} \tag{9}$$

where $L^\triangle$ is the image of L^* in a Fock-type space of vector-functions $k_t = (k_t^{(n)})_{n=0}^\infty$. In applications to concrete models, the expression for the operator $L^\triangle$ is obtained from the operator L via combinatorial calculations (cf. [17]). The following diagram



describes the relationships.

The evolution equation (9) is nothing but a hierarchical system of equations to the Markov generator L. This system is the analogue of the BBGKY-hierarchy of the Hamiltonian dynamics [4].

Our interest now turns to Vlasov-type scaling of stochastic dynamics for the IPS in a continuum. This scaling leads to so-called kinetic description of the considered model. In the language of theoretical physics we are dealing with a mean-field type scaling which is adopted to preserve the spatial structure. In addition, this scaling will lead to the limiting hierarchy, which possesses a chaos propagation property. In other words, if the initial distribution is Poisson (non-homogeneous) then the time evolution of states will maintain this property. We refer to [10] for a general approach, concrete examples, and additional references.

There exists a standard procedure for deriving Vlasov scaling $L_V^\triangle$ from the generator $L^\triangle$ in (9). Heuristically, $L_V^\triangle$ corresponds to a (non-Markov) generator L_V on observables which may be reconstructed form $L_V^\triangle$ just on the level of combinatorial calculations. All together, it gives us the following chain of transformed operators:

$$L \longrightarrow L_V \longrightarrow L_V^* \longrightarrow L_V^\triangle.$$

The specific type of scaling is dictated by the model in question. The process leading from $L^\triangle$ to $L_V^\triangle$ produces a non-Markovian generator L_V since it lacks the positivity-preserving property. Therefore instead of (8) we consider the following kinetic FPE,

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L_V^* \mu_t, \\ \mu_t | t_{=0} &= \mu_0, \end{cases} \tag{10}$$

and observe that if the initial distribution satisfies $\mu_0 = \pi_{\rho_0}$, then the solution is of the same type, i.e., $\mu_t = \pi_{\rho_t}$.

In terms of correlation functions, the kinetic FPE (10) gives rise to the following Vlasov-type hierarchical chain (Vlasov hierarchy)

$$\begin{cases} \frac{\partial k_t^{(n)}}{\partial t} &= (L_V^\triangle k_t)^{(n)}, \\ k_t^{(n)}|_{t=0} &= k_0^{(n)}, \quad n \geq 0. \end{cases} \tag{11}$$

Remark 1. 1. In applications it is important to consider the Lebesgue–Poisson exponents $k_0(\eta) = e_\lambda(\rho_0, \eta) = \prod_{x\in\eta} \rho_0(x)$ as the initial condition. The scaling $L_V^\triangle$ should be such that the dynamics $k_0 \mapsto k_t$ preserves this structure, or more precisely, k_t should be of the same type

$$k_t(\eta) = e_\lambda(\rho_t, \eta) = \prod_{x\in\eta} \rho_t(x), \quad \eta \in \Gamma_0. \tag{12}$$

2. Relation (12) is known as the *chaos preservation property* of the Vlasov hierarchy. It turns out that equation (12) implies, in general, a non-linear differential equation

$$\frac{\partial \rho_t(x)}{\partial t} = \vartheta(\rho_t)(x), \quad x \in \mathbb{R}^d, \tag{13}$$

for ρ_t, which is called the *Vlasov-type kinetic equation.*

Remark 2. In general, if one does not start with a Poisson measure, the solution will leave the space $\mathcal{M}^1(\Gamma)$. To have a bigger class of initial measures, we may consider the cone inside $\mathcal{M}^1(\Gamma)$ generated by convex combinations of Poisson measures, denoted by $\mathbb{P}(\Gamma)$.

We would now like to generalize the above general scheme to obtain the analog of *kinetic fractional statistical dynamics* (or equivalently *mesoscopic fractional statistical dynamics*). It would be tempting simply to replace the usual time derivative in equation (13) by a time fractional derivative. Because the equation (13) in general is non-linear, it is then much harder to obtain a solution. But more essential is the question of the meaning of such an equation. A naive use of the fractional derivative in the Vlasov equation is not justified by the microscopic dynamics and its scaling. Our alternative approach to realizing this generalization is described in the following section.

4. Fractional statistical dynamics

The procedure of Section 3 is suitable for describing non-Markov evolutions. More precisely, in the FPE (8) we change the usual time derivative by the Caputo–Djrbashian fractional time derivative $\mathbb{D}_t^\alpha$ (CDfd for short) and then study the corresponding fractional dynamics.

In order to proceed, we first have to define the CDfd. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be given; then the CDfd of f is given in the Laplace transform domain by

$$\big(\mathcal{L}\mathbb{D}_t^\alpha f\big)(s) = s^\alpha (\mathcal{L}f)(s) - s^{\alpha-1} f(0), \quad s > 0, \quad \alpha \in (0, 1],$$

where $\mathcal{L}f$ denotes the Laplace transform of f

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Another possible representation of the CDfd is

$$\big(\mathbb{D}_t^\alpha f\big)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau) - f(0)}{(t-\tau)^\alpha}\, d\tau, \quad 0 < \alpha < 1.$$

In case f is absolutely continuous, we have

$$\big(\mathbb{D}_t^\alpha f\big)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau, \quad 0 < \alpha < 1.$$

The definition of the CDfd has natural extensions to vector-valued or measure valued functions on $\mathbb{R}_+$. We refer to the monographs [30] and [15] for more details and references concerning the CDfd.

We will introduce the fractional statistical dynamics for a given Markov generator L by changing the time derivative in the FPE to the CDfd. The resulting fractional Fokker–Planck dynamics (if it exists!) will act in the space of states on Γ, i.e., it will preserve probability measures on Γ. The fractional Fokker–Planck equation

$$\text{(FFPE)} \qquad \begin{cases} \mathbb{D}_t^\alpha \mu_t^\alpha & = L^* \mu_t^\alpha, \\ \mu_t^\alpha|_{t=0} & = \mu_0^\alpha. \end{cases}$$

describes a dynamical system with memory in the space of measures on Γ. The corresponding evolution no longer has the semigroup property. However, if the solution μ_t of equation (10) exists, then the subordination principle (see [31], [1, 2] and references therein) gives the solution of the equation FFPE, namely

$$\mu_t^\alpha = \int_0^\infty \Phi_\alpha(\tau) \mu_{\tau t^\alpha}\, d\tau. \tag{14}$$

Here $\Phi_\alpha(z)$ is the Wright function

$$\Phi_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)},$$

a probability density function in $\mathbb{R}_+$). It is known (see, for example, [13] and [26]) that

$$\Phi_\alpha(t) \geq 0, \quad t > 0, \quad \int_0^\infty \Phi_\alpha(t)\, dt = 1,$$

and that the moments of Φ_α are given by

$$\int_0^\infty t^\delta \Phi_\alpha(t)\, dt = \frac{\Gamma(\delta+1)}{\Gamma(\alpha\delta+1)}, \quad \delta > -1. \tag{15}$$

Its Laplace transform is given by

$$\int_0^\infty e^{-\tau t} \Phi_\alpha(\tau)\, d\tau = E_\alpha(-t), \quad t > 0,$$

where E_α is the Mittag–Leffler function (see [15]):

$$E_\alpha(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}.$$

An application of the subordination principle may be justified in many particular models where the evolution of correlation functions may be constructed by means a C_0-semigroup in a proper Banach space. In general, the subordination formula may be considered as a rule for the transformation of Markov dynamics to fractional ones.

It is easy to see that μ_t^α is a measure. Actually, positivity follows from that fact that for any measurable set A we have

$$\mu_t^\alpha(A) = \int_0^\infty \Phi_\alpha(t,s)\mu_s(A)\,ds \geq 0,$$

since μ_s is a measure and Φ_α is a pdf. The σ-additivity property may be verified using the standard procedure. The FFPE equation may be written in terms of time-dependent correlation functions as an infinite system of evolution equations, the so-called *hierarchical chain*:

$$\begin{cases} \mathbb{D}_t^\alpha k_{\alpha,t}^{(n)} & = (L^\triangle k_{\alpha,t})^{(n)}, \\ k_{\alpha,t}^{(n)}|_{t=0} & = k_{\alpha,0}^{(n)}, \quad n \geq 0. \end{cases}$$

The evolution of the correlation functions should also be given by the subordination principle. More precisely, if the solution k_t of equation (11) exists, then we have

$$k_{\alpha,t} = \int_0^\infty \Phi_\alpha(\tau) k_{\tau t^\alpha}\,ds.$$

5. Fractional kinetics and Poisson flows

As in the case of Markov statistical dynamics addressed above, we may consider Vlasov-type scaling in the framework of the FFPE. We know that the kinetic statistical dynamics for a Poisson initial state π_{ρ_0} is given by a flow of Poisson measures

$$\mathbb{R}_+ \ni t \mapsto \mu_t = \pi_{\rho_t} \in \mathcal{M}^1(\Gamma),$$

where ρ_t is the solution to the corresponding Vlasov kinetic equation. Then the fractional kinetic dynamics of states may be defined as the subordination of this flow (see comments above). Specifically, for $0 < \alpha < 1$ we consider the subordinated flow

$$\mu_t^\alpha := \int_0^\infty \Phi_\alpha(\tau)\mu_{\tau t^\alpha}\,d\tau = \int_0^\infty \Phi_\alpha(\tau)\pi_{\rho_{\tau t^\alpha}}\,d\tau.$$

The family of measures μ_t^α is no longer a Poisson flow. We would like to analyze the properties of these subordinated flows to distinguish the effects of fractional evolution. Note first that the density of the fractional kinetic state is given by the formula

$$\rho_t^\alpha(x) = \int_0^\infty \Phi_\alpha(\tau)\rho_{\tau t^\alpha}(x)\,d\tau.$$

The latter is the subordination of the solution to the Vlasov equation and is not related to a fractional Vlasov equation as it is expected in several heuristic considerations in physics.

It is reasonable to study the properties of subordinated flows from a more general point of view when the evolution of densities $\rho_t(x)$ is not necessarily related to a particular Vlasov-type kinetic equation. Similar transformations of Poisson flows do appear due to completely different motivations in several applications. See, for example, [8, 29] and, for the related fractional Poisson process, [32, 20, 24, 25, 33, 27], and references therein.

Below we will study certain properties of the resulting flows affected by fractional dynamics.

5.1. **Front propagation for the density.** Let us consider a density evolution of the form

$$\rho_t(x) = 1\!\!1_{[-1-vt,1+vt]}(x), \quad t \geq 0, \quad x \in \mathbb{R},$$

where $v > 0$ is the constant speed of the density front. The subordinated density has the following representation

$$\rho_t^\alpha(x) = \int_0^\infty \Phi_\alpha(\tau) 1\!\!1_{[-1-vt^\alpha\tau,1+vt^\alpha\tau]}(x)\,d\tau,$$

and for $|x| > 1$

$$\rho_t^\alpha(x) = \int_{A(x,t)}^\infty \Phi_\alpha(\tau)\, d\tau,$$

where

$$A(x,\tau) = \frac{|x|-1}{vt^\alpha}.$$

We have $\rho_t^\alpha(x) \to 1, t \to \infty, |x| > 1$ and $\rho_t^\alpha(x) \to 0, x \to \infty, t \geq 0$. Consider

$$\Psi_\alpha(s) = \int_s^\infty \Phi_\alpha(\tau)\, d\tau.$$

Due to monotonicity we may find a unique s_α s.t. $\Psi_\alpha(s_\alpha) = 1/2$. Define the front of ρ_t^α for given $t > 0$ as $x \in \mathbb{R}$, for which $\rho_t^\alpha(x) = 1/2$. The motion of the front is then given by the formula

$$|x| = 1 + s_\alpha vt^\alpha.$$

The latter result means that in the subordinated dynamics the density will be expanded sub-linearly and more slower for smaller $\alpha \in (0,1)$.

5.2. **Intermittency for subordinated flows.** Each measure from the flow μ_t^α defines a generalized random process on $\mathbb{R}^d$ given for $f \in C_0(\mathbb{R}^d)$ by

$$X_f(\gamma) = \sum_{x\in\gamma} f(x), \quad \gamma \in \Gamma.$$

Let us consider the corresponding moments

$$m_t^p(f) = \int X_f^p\, d\mu_t^\alpha, \quad p \geq 1.$$

The notion of asymptotic intermittency is well understood for regular random fields; see for example [7, 6]. In the case of generalized random fields this notion may be formulated as follows.

Definition 1. (Intermittency via moments). The flow $\mu_t^\alpha, t \geq 0$ has the asymptotic intermittency property if for any $0 \leq f \in C_0(\mathbb{R}^d)$ and for all $p_1, \ldots, p_n \in \mathbb{N}$ with $p_1 + \cdots + p_n = p$ one had

$$\lim_{t\to\infty} \frac{m_t^p(f)}{m_t^{p_1}(f)\ldots m_t^{p_n}(f)} = \infty.$$

This property means that moments of the random field grow in time progressively with the order. In the case of random point processes the leading growth of moments is defined in terms of correlation functions of the corresponding orders.

This gives us the option of reformulating the definition of asymptotic intermittency in terms more convenient for our purposes.

Definition 2. (Intermittency via correlation functions). The flow $\mu_t^\alpha, t \geq 0$ has the asymptotic intermittency property if for any $\eta \in \Gamma_0$ and its decomposition $\eta = \eta_1 \cup \cdots \cup \eta_n$ in disjunct subsets for the correlation function $k_{\mu_t^\alpha}$, one has

$$\lim_{t\to\infty} \frac{k_{\mu_t^\alpha}(\eta)}{k_{\mu_t^\alpha}(\eta_1)\ldots k_{\mu_t^\alpha}(\eta_n)} = \infty.$$

For a detailed discussion of relations between different versions of the intermittency property for random point processes, see [16].

Let us consider the dynamics of the density given by $\rho_t(x) = e^{\beta t^\sigma}$, $\beta, \sigma > 0$. The flow of Poisson measures π_{ρ_t} has, for each $t \geq 0$, correlation functions $k_{\pi_{\rho_t}}^{(n)}(x_1, \ldots, x_n) = e^{\beta n t^\sigma}$. Therefore the intermittency is absent.

Theorem 1. *Let $0 < \alpha < 1$ be given. Consider the subordinated flow for the Poisson flow introduced above,*

$$\mu_t^\alpha := \int_0^\infty \Phi_\alpha(\tau)\pi_{\rho_{\tau t^\alpha}}\, d\tau.$$

Assume $\sigma(1-\alpha) < 1$. Then the flow μ_t^α has the asymptotic intermittency property.

Proof. The n-th correlation function of μ_t^α is given by

$$\begin{aligned}(\rho_t^\alpha)^{(n)}(x_1,\dots,x_n) &= \int_0^\infty \Phi_\alpha(\tau)(\rho_{\tau t^\alpha}(x))^n\, d\tau \\ &= \int_0^\infty \Phi_\alpha(\tau) e^{n\beta t^{\sigma\alpha}\tau^\sigma}\, d\tau = \sum_{k=0}^\infty \frac{(n\beta t^{\sigma\alpha})^k}{k!}\int_0^\infty \Phi_\alpha(\tau)\tau^{\sigma k}\, d\tau \\ &= \sum_{k=0}^\infty \frac{(n\beta t^{\sigma\alpha})^k}{k!}\frac{\Gamma(\sigma k+1)}{\Gamma(\sigma k\alpha+1)} = \sum_{k=0}^\infty \frac{n^k z^k}{k!}\frac{\Gamma(\sigma k+1)}{\Gamma(\sigma k\alpha+1)},\end{aligned}$$

where $z := \beta t^{\sigma\alpha}$. It is known [5] that the series converges for all values of z if and only if $\sigma < 1/(1-\alpha)$, and that

$$(\rho_t^\alpha)^{(n)}(x_1,\dots,x_n) \sim C(nz)^{1/(2\mu)}\exp\big(c(nz)^{1/\mu}\big),$$

where $C, c > 0$, $\mu = 1+\sigma(\alpha-1)$. Since $0<\mu<1$, this asymptotic behavior implies intermittency. In fact, due to Definition 2, we need to consider the limiting behavior of the ratio

$$\frac{\exp(c(nz)^{\frac{1}{\mu}}}{\exp\sum_{k=1}^m \big(c(n_k z)^{\frac{1}{\mu}}\big)}, \quad z = \beta t^{\sigma\alpha}$$

for $t\to\infty$ under the assumption $\sum_{k=1}^m n_k = n$. This limit is equal $+\infty$ due to the inequality

$$\left(\sum_{k=1}^m n_k\right)^{\frac{1}{\mu}} > \sum_{k=1}^m (n_k)^{\frac{1}{\mu}}$$

for $1/\mu > 1$ (see [3], Chapter 1, § 16). □

5.3. **Polynomially growing density.** Let us now consider the case of a polynomial density $\rho_t(x) = (1+t)^p$, $p\in\mathbb{N}$. For any $n\in\mathbb{N}$, the nth correlation function is given by

$$\begin{aligned}(\rho_t^\alpha)^{(n)}(x_1,\dots,x_n) &= \int_0^\infty \Phi_\alpha(\tau)\rho_{\tau t^\alpha}(x_1)\dots\rho_{\tau t^\alpha}(x_n)\, d\tau \\ &= \int_0^\infty \Phi_\alpha(\tau)(1+\tau t^\alpha)^{pn}\, d\tau = \sum_{j=0}^{np}\binom{np}{j} t^{\alpha j}\int_0^\infty \tau^j\Phi_\alpha(\tau)\, d\tau \\ &= \sum_{j=0}^{np}\binom{np}{j} t^{\alpha j}\frac{\Gamma(j+1)}{\Gamma(j\alpha+1)} = \sum_{j=0}^{np}\frac{(np)!}{(np-j)!}t^{\alpha j}\frac{1}{\Gamma(\alpha j+1)} \\ &= \frac{(np)!}{\Gamma(\alpha np+1)}t^{\alpha np} + o(t^{\alpha np}).\end{aligned}$$

In particular, for $n=1$, the 1st correlation function is equal to

$$(\rho_t^\alpha)^{(1)} = \rho_t^\alpha = \frac{p!}{\Gamma(\alpha p+1)}t^{\alpha p} + o(t^{\alpha p}).$$

Therefore we obtain

$$\begin{aligned}\frac{(\rho_t^\alpha)^{(n)}}{(\rho_t^\alpha)^n} &= \frac{(np)!}{\Gamma(\alpha np+1)}t^{\alpha np}\times\left(\frac{\Gamma(\alpha p+1)}{p!}\right)^n\frac{1}{t^{\alpha pn}}+o(1)\\ &= \frac{(np)!}{\Gamma(\alpha np+1)}\times\left(\frac{\Gamma(\alpha p+1)}{p!}\right)^n+o(1),\end{aligned}$$

which is constant as t goes to infinity. In conclusion, the power growth of the 1st correlation function is not sufficient to organize intermittency in the subordinated flow. Summarizing the above considerations, we conclude that subordinating a flow (which corresponds to the dynamics of a system without intermittency) is a way to *organize intermittency*.

6. Examples

In this section we apply the general scheme of the fractional statistical dynamics developed here to concrete models, namely the *contact model* and the *pure birth model*, also known as the *Surgailis pure birth model*.

Example 3. (Surgailis pure birth model). This is an example in which the kinetic fractional statistical dynamics is a mixture of Poisson measures. The Surgailis pure birth model (zero mortality) has generator given by

$$(LF)(\gamma)=z\int_{\mathbb{R}^d}[F(\gamma\cup x)-F(\gamma)]\,dx.$$

(cf. [19]). Starting from from the Poisson initial distribution $\mu_0=\pi_{\rho_0}$, the solution of the FPE

$$\begin{cases}\frac{\partial\mu_t}{\partial t} &= L^*\mu_t,\\ \mu_t|_{t=0} &= \pi_{\rho_0}\end{cases}$$

is of the same type

$$\mu_t=\pi_{zt+\rho_0}.$$

The solution of the fractional FPE

$$\begin{cases}\mathbb{D}_t^\alpha\nu_t^\alpha &= L^*\nu_t^\alpha,\\ \nu_t^\alpha|_{t=0} &= \pi_{\rho_0}\end{cases}$$

is then given by the subordination principle as

$$\nu_t^\alpha=\int_0^\infty\Phi_\alpha(s)\pi_{zt^\alpha s+\rho_0}\,ds.$$

Hence the solution ν_t^α, $t>0$ is a mixture of Poisson measures. The correlation function of the Poisson measure $\pi_{zt^\alpha s+\rho_0}$ is $((zt^\alpha s+\rho_0)^n)_{n=0}^\infty$, and therefore the correlation function of the mixture ν_t^α is, for $n\geq 0$,

$$\begin{aligned}r_{t,\alpha}^{(n)} &= \int_0^\infty\Phi_\alpha(s)(zt^\alpha s+\rho_0)^n\,ds\\ &= \sum_{j=0}^n\binom{n}{j}(zt^\alpha)^j\rho_0^{n-j}\int_0^\infty\Phi_\alpha(s)s^j\,ds.\end{aligned}$$

The absolute moments of Φ_α (cf. eq. (15)) satisfy

$$\int_0^\infty s^j\Phi_\alpha(s)\,ds=\frac{\Gamma(j+1)}{\Gamma(\alpha j+1)},\quad j>-1.$$

Accordingly, the nth order correlation function of the measure ν_t^α reduces to

$$r_{t,\alpha}^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \rho_0^{n-j} (zt^\alpha)^j \frac{j!}{\Gamma(\alpha j+1)} = (zt^\alpha)^n \frac{n!}{\Gamma(\alpha n+1)} + o\big((zt^\alpha)^n\big).$$

In particular

$$\big(r_{t,\alpha}^{(1)}\big)^n = \left(\frac{zt^\alpha}{\Gamma(\alpha+1)}\right)^n + o\big((zt^\alpha)^n\big),$$

and thus

$$\begin{aligned} \frac{r_{t,\alpha}^{(n)}}{\big(r_{t,\alpha}^{(1)}\big)^n} &= (zt^\alpha)^n \frac{n!}{\Gamma(n\alpha+1)} \times \left(\frac{\Gamma(\alpha+1)}{zt^\alpha}\right)^n + o((zt^\alpha)^n) \\ &= \frac{(n-1)!}{\alpha\Gamma(n\alpha)} (\Gamma(\alpha+1))^n + o((zt^\alpha)^n). \end{aligned}$$

From this we see that as $t \to \infty$ the above coefficient does not explode, which tell us that this model has no asymptotic intermittency. In other words, the power growth of the correlation function corresponding to the FPE is not sufficient to realize asymptotic intermittency of the kinetic fractional statistical dynamics. In the next example we show that under strong growth on the nth order correlation function of the FPE (exponential growth), the kinetic fractional statistical dynamics does exhibit asymptotic intermittency.

Example 4. (Contact model). The contact model is one of the simplest models in the theory of IPS. Nevertheless, it has interesting properties, e.g., its asymptotic behavior and the structure of its equilibrium measures. We refer to [10] for more details.

The generator L of the stochastic dynamics is given informally by

$$(LF)(\gamma) = \sum_{x\in\gamma} m\big(F(\gamma\backslash x) - F(\gamma)\big) + \int_{\mathbb{R}^d} b(x,\gamma)\big(F(\gamma\cup x) - F(\gamma)\big)\, dx.$$

Here $m > 0$ is a constant mortality rate and the birth rate is

$$b(x,\gamma) = \sum_{y\in\gamma} a(x-y),$$

where $0 \le a \in L^1(\mathbb{R}^d)$ is even.

In the kinetic limit, the correlation functions of the contact model in the super critical regime are given by

$$r_t^{(n)}(x_1,\dots,x_n) = C^n e^{\beta n t}$$

for certain $C, \beta > 0$ [19], [10]. The correlation functions of the solution for the fractional kinetic dynamics are then given by

$$r_{t,\alpha}^{(n)} = C^n \int_0^\infty \Phi_\alpha(s) e^{\beta n t^\alpha s}\, ds = C^n E_\alpha(\beta n t^\alpha), \quad n \in \mathbb{N}.$$

Using the asymptotic behavior of the Mittag-Leffler function E_α as $t \to \infty$ (see eq. (6.4) in [14]), we can conclude that the kinetic fractional statistical dynamics in the contact model does exhibit asymptotic intermittency. Of course, this statement is a particular case of Theorem 1 for $\sigma = 1$.

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