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ON APPROXIMATION OF SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS WITH THEIR ENTIRE SOLUTIONS OF EXPONENTIAL TYPE

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Abstract. We consider an equation of the form $y'(t)+Ay(t)=0,\ t\in[0,\infty)$, where A is a nonnegative self-adjoint operator in a Hilbert space. We give direct and inverse theorems on approximation of solutions of this equation with its entire solutions of exponential type. This establishes a one-to-one correspondence between the order of convergence to 0 of the best approximation of a solution and its smoothness degree. The results are illustrated with an example, where the operator A is generated by a second order elliptic differential expression in the space $L_2(\Omega)$ (the domain $\Omega\subset\mathbb{R}^n$ is bounded with smooth boundary) and a certain boundary condition.

1. Let A be a nonnegative self-adjoint operator in a Hilbert space $\mathfrak{H}$ with a scalar product $(\cdot,\cdot)$. Denote by $C_{\{1\}}(A)$ the set of all its exponential type entire vectors (see [4]), namely,

$$C_{\{1\}}(A)=\Big\{f\in C^\infty(A)=\bigcap_{n=1}^{\infty}\mathcal{D}(A^n)\big|\exists\alpha>0,\ \exists c=c(f)>0:\\ \|A^nf\|\le c\alpha^n,\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\Big\}$$

(everywhere in the sequel c denotes various numerical constants corresponding to the situations under consideration, $\mathcal{D}(A)$ is a domain of A and $\|f\|=\sqrt{(f,f)}$). The number

$$\sigma(f,A)=\inf\big\{\alpha>0\big|\exists c>0,\ \forall n\in\mathbb{N}_0:\|A^nf\|\le c\alpha^n\big\}$$

is called the type of the vector f with respect to the operator A.

As has been shown in [3] that

$$C_{\{1\}}(A)=\big\{f\in\mathfrak{H}\big|f=E(\lambda)g,\ \forall\lambda>0,\ \forall g\in\mathfrak{H}\big\},$$

where $E(\lambda)=E([0,\lambda])$ is the spectral measure of A.

Now, consider the equation

$$y'(t)+Ay(t)=0,\quad t\in\mathbb{R}_+=[0,\infty). \tag{1}$$

By a weak solution of this equation we mean a continuous vector-valued function $y(t):\mathbb{R}_+\mapsto\mathfrak{H}$ such that for any $t\in\mathbb{R}_+$,

$$\int_0^t y(s)\,ds\in\mathcal{D}(A)\quad\text{and}\quad y(t)=-A\int_0^t y(s)\,ds+y(0).$$

Denote by S the set of all weak solutions of (1). As it was established in [1],

$$S=\big\{y(t):\mathbb{R}_+\mapsto\mathfrak{H}\big|\ y(t)=e^{-At}f,f\in\mathfrak{H}\big\}, \tag{2}$$

where

$$e^{-At}f=\int_0^\infty e^{-\lambda t}\,dE(\lambda)f.$$

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Note that the set of all strong (or simply) solutions of (1) is given by formula (2) with f ranging over whole $\mathcal{D}(A)$.

It is not difficult to verify that S is a Hilbert space with the norm

$$\|y\|_S = \sup_{t\in\mathbb{R}_+} \left\|e^{-At}f\right\| = \|f\|. \tag{3}$$

If the operator A is bounded, then each weak solution $y(t)$ of equation (1) can be extended to an entire $\mathfrak{H}$-valued vector function $y(z)$ of exponential type,

$$\sigma(y) = \inf\left\{\alpha > 0 : \|y(z)\| \le ce^{\alpha|z|}\right\}.$$

But it is not the case if A is unbounded. The set S_0 of all weak solutions of (1) admitting an extension to an entire vector function of exponential type is described by the following theorem.

Theorem 1. *A weak solution $y(t)$ of equation (1) belongs to S_0 if and only if it can be represented in form (2) with $f \in C_{\{1\}}(A)$. The set S_0 is dense in S, and $\sigma(y) = \sigma(f, A)$.*

Proof. Let $f \in C_{\{1\}}(A)$. Then $f = E(\alpha)f$, where $\alpha = \sigma(f, A)$. By (2),

$$y(t) = \int_0^\alpha e^{-\lambda t}\, dE(\lambda)f.$$

From this it follows that $y(t)$ can be extended to an entire vector-valued function $y(z)$ and

$$\|y(z)\|^2 = \int_0^\alpha e^{-2\mathrm{Re}z\lambda}\, d(E(\lambda)f, f) \le e^{2\sigma(f,A)|z|}\|f\|^2,$$

that is, $y(z)$ is an entire $\mathfrak{H}$-valued function of exponential type $\sigma(y) \le \sigma(f, A)$.

Conversely, if $y(t) = e^{-At}f$ admits an extension to an entire vector-valued function of exponential type $\sigma(y)$, then, by virtue of

$$\|y(-t)\|^2 = \int_0^\infty e^{2\lambda t}\, d(E(\lambda)f, f) \le ce^{2\sigma(y)t}, \quad t \ge 0,$$

we have

$$\int_{\sigma(y)}^\infty e^{2t(\lambda-\sigma(y))}\, d(E(\lambda)f, f) \le c.$$

Passing to the limit under the integral sign as $t \to \infty$, we conclude, on the basis of the Fatou theorem, that the measure generated by the monotone function $(E(\lambda)f, f)$ is concentrated on the interval $[0, \sigma(y)]$. Hence, $\sigma(f, A) \le \sigma(y)$.

Density of S_0 in S follows from the density in $\mathfrak{H}$ of the set $\{E([0, \alpha])f, \forall\alpha > 0, \forall f \in \mathfrak{H}\}$. □

In view of Theorem 1, it is reasonable to ask whether it is possible to approximate an arbitrary weak solution of equation (1) with its exponential type entire solutions. An answer to the question is given below. We prove direct an inverse theorems which ascertain the relationship between the degree of smoothness of a solution and the rate of convergence to 0 of its best approximation. In doing so, the operator approach developed in [7, 4, 5] plays an important role.

2. Recall some definitions and notations of the approximation theory required to formulate further results.

For $y \in S$ and a number $r > 0$, we put

$$\mathcal{E}_r(y) = \inf_{y_0\in S_0:\sigma(y_0)\le r} \|y - y_0\|_S.$$

Thus, $\mathcal{E}_r(y)$ is the best approximation of a weak solution $y(t)$ of equation (1) with its entire solutions of exponential type not exceeding r. If y is fixed, the function $\mathcal{E}_r(y)$ does not increase and, since $\overline{S_0} = S$, $\mathcal{E}_r(y) \to 0$ as $r \to \infty$.

For $f \in \mathfrak{H}$, set also

$$\mathcal{E}_r(f, A) = \inf_{f_0 \in C_{\{1\}}(A): \sigma(f_0, A) \le r} \|f - f_0\| = \|(I - E(r))f\|.$$

If $y(t) = e^{-At}f$, then, by virtue of (3),

$$\mathcal{E}_r(y) = \mathcal{E}_r(f, A). \tag{4}$$

Besides, for any $k \in \mathbb{N}_0$, we introduce the function

$$\omega_k(t, y) = \sup_{|h| \le t} \sup_{s \in \mathbb{R}_+} \Big\| \sum_{j=0}^{k} (-1)^{k-j} C_k^j y(s + jh) \Big\|, \quad k \in \mathbb{N}; \quad \omega_0(t, y) \equiv \|y\|_S, \quad t > 0.$$

Taking into account (2) and the equality $e^{-As}y(t) = y(t + s)$, we conclude that

$$\forall k \in \mathbb{N}_0 : \omega_k(t, y) = \sup_{|h| \le t} \left\| \left(e^{-Ah} - I\right)^k y \right\|_S$$

(I is the identity operator).

The following theorem establishes a relation between $\mathcal{E}_r(y)$ and $\omega_k(t, y)$, and it is an analog of the well-known Jackson's theorem on approximation of a continuous periodic function by trigonometric polynomials.

Theorem 2. *Let $y \in S$. Then*

$$\forall k \in \mathbb{N},\ \exists c_k > 0 : \mathcal{E}_r(y) \le c_k \omega_k \left(\frac{1}{r}, y\right), \quad r > 0. \tag{5}$$

Proof. By (2), $y(t) = e^{-At}f,\ f \in \mathfrak{H}$. From (3), (4), it follows that

$$\begin{aligned}
\omega_k^2(t, y) &= \sup_{0 \le s \le t} \left\| \left(e^{-As} - I\right)^k y \right\|_S^2 \ge \left\| \left(e^{-At} - I\right)^k y \right\|_S^2 = \sup_{s \in \mathbb{R}_+} \left\| \left(e^{-At} - I\right)^k e^{-As} f \right\|^2 \\
&= \left\| \left(e^{-At} - I\right)^k f \right\|^2 = \int_0^\infty \left(e^{-\lambda t} - 1\right)^{2k} d(E(\lambda)f, f) \\
&\ge \int_{\frac{1}{t}}^\infty \left(e^{-\lambda t} - 1\right)^{2k} d(E(\lambda)f, f) \ge \left(1 - e^{-1}\right)^{2k} \mathcal{E}_{\frac{1}{t}}(y).
\end{aligned}$$

So,

$$\forall t > 0 : \mathcal{E}_{\frac{1}{t}}(y) \le \left(1 - e^{-1}\right)^k \omega_k(t, y).$$

Setting $r = \frac{1}{t}$ and $c_k = \left(1 - e^{-1}\right)^k$, we obtain (5). □

Denote by $C^n(\mathbb{R}_+, \mathfrak{H})$ the set of all n times continuously differentiable on $\mathbb{R}_+$ $\mathfrak{H}$-valued vector-valued functions. Since the operator A is closed, the inclusion $y \in S \cap C^n(\mathbb{R}_+, \mathfrak{H})$ implies that $y^{(k)} \in S,\ k = 1, 2, \ldots, n$.

Theorem 3. *Suppose that $y \in C^n(\mathbb{R}_+, \mathfrak{H}),\ n \in \mathbb{N}_0$. Then*

$$\forall r > 0,\ \forall k \in \mathbb{N}_0 : \mathcal{E}_r(y) \le \frac{c_{k+n}}{r^n} \omega_k \left(\frac{1}{r}, y^{(n)}\right),$$

where the constants c_k are the same as in Theorem 2.

Proof. Let $y \in C^n(\mathbb{R}_+, \mathfrak{H}), r > 0$ and $0 \le t < \frac{1}{r}$. Using properties of a contraction C_0-semigroup, we get

$$\begin{aligned}
\left\| \left(e^{-At} - I\right)^{k+n} y(s) \right\| &= \left\| \left(e^{-At} - I\right)^n \left(e^{-At} - I\right)^k y(s) \right\| \\
\le \int_0^t \cdots \int_0^t &\left\| e^{-A(s_1 + \ldots s_n)} \right\| \left\| \left(e^{-At} - I\right)^k A^n y(s) \right\| ds_1 \ldots ds_n \\
&\le t^n \left\| \left(e^{-At} - I\right)^k y^{(n)}(s) \right\|,
\end{aligned}$$

whence

$$\omega_{k+n}\left(\frac{1}{r},y\right)\le\frac{1}{r^n}\omega_k\left(\frac{1}{r},y^{(n)}\right)$$

and, because of (5),

$$\mathcal{E}_r(y)\le c_{k+n}\omega_{k+n}\left(\frac{1}{r},y\right)\le\frac{c_{k+n}}{r^n}\omega_k\left(\frac{1}{r},y^{(n)}\right),$$

which is what had to be proved. □

Setting, in Theorem 3, $k=0$ and taking into account that $\omega_0(t,y^{(n)})=\|y^{(n)}\|_S$, we arrive at the following assertion.

Corollary 1. *Let $y\in C^n(\mathbb{R}_+,\mathfrak{H})$, $n\in\mathbb{N}$. Then*

$$\forall r>0:\mathcal{E}_r(y)\le\frac{c_n}{r^n}\|y^{(n)}\|_S.$$

For numbers $h>0$ and $k\in\mathbb{N}_0$, we put

$$\Delta_h^k=\left(e^{-Ah}-I\right)^k=\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-Ajh}.$$

Lemma 1. *If $y\in S_0$ and $\sigma(y)=\alpha$, then*

(6) $$\forall h>0,\ \forall k,n\in\mathbb{N}_0:\left\|\Delta_h^ky^{(n)}\right\|_S\le(\alpha h)^k\alpha^n\|y\|_S.$$

Proof. It follows from the inequality

$$1-\lambda h-e^{-\lambda h}\le 0\quad(\lambda\ge 0,\ h>0)$$

and the representation $y(t)=e^{-At}f$ that

$$\begin{aligned}\left\|\Delta_h^ky^{(n)}\right\|^2&=\int_0^\alpha\left(1-e^{-\lambda h}\right)^{2k}e^{-2\lambda t}\lambda^{2n}\,d(E(\lambda)f,f)\\&\le\int_0^\alpha(\lambda h)^{2k}\lambda^{2n}\,d(E(\lambda)f,f)\le(\alpha h)^{2k}\alpha^{2n}\|f\|.\end{aligned}$$

This and (3) imply

$$\left\|\Delta_h^ky^{(n)}\right\|_S\le(\alpha h)^k\alpha^n\|f\|=(\alpha h)^k\alpha^n\|y\|_S.$$ □

Taking in (6) $k=0$, we arrive at an analog of Bernstein's inequality, namely

(7) $$\forall n\in\mathbb{N}:\left\|y^{(n)}\right\|\le\alpha^n\|y\|_S.$$

Putting there $n=0$, we obtain

$$\forall n\in\mathbb{N}:\left\|\Delta_h^ky\right\|_S\le(\alpha h)^k\|y\|_S=(\alpha h)^k\alpha^n\|y\|_S.$$

It should be noted that the inequality

(8) $$\mathcal{E}_r(y)\le\frac{c}{r^n},\quad r>0,\quad n\in\mathbb{N},$$

does not yet imply the inclusion $y\in C^n(\mathbb{R}_+,\mathfrak{H})$. Nevertheless, the following statement, inverse to Theorem 3, is valid.

Theorem 4. *Suppose that $y\in S$, and let $\omega(t)$ be a continuity module type function, i.e.,*

1) *$\omega(t)$ is continuous and nondecreasing on $\mathbb{R}_+$;*
2) *$\omega(0)=0$;*
3) *$\exists c>0,\ \forall t>0:\omega(2t)\le c\omega(t)$.*

In order that $y\in C^n(\mathbb{R}_+,\mathfrak{H})$, it is sufficient that there exist a number $m>0$ such that

(9) $$\forall r>0,\ \forall n\in\mathbb{N}:\mathcal{E}_r(y)\le\frac{m}{r^n}\omega\left(\frac{1}{r}\right).$$

Proof. Assume that, for $y \in S$, condition (9) is fulfilled. Then there exists a sequence $y_i \in S : \sigma(y_i) < 2^i$ $(i \in \mathbb{N})$ such that

$$\|y - y_i\|_S \to 0 \quad \text{as} \quad i \to \infty.$$

In view of (7) and the inequality $\sigma(y_i - y_{i-1}) < 2^i$ $(i \in \mathbb{N})$, we have

$$\begin{aligned}\left\|y_i^{(n)} - y_{i-1}^{(n)}\right\|_S \leq & 2^{in}\|y_i - y_{i-1}\|_S \leq 2^{in}(\|y - y_i\|_S + \|y - y_{i-1}\|_S) \\ \leq & 2^{in}\left(\frac{m}{2^{in}}\omega\left(\frac{1}{2^i}\right) + \frac{m}{2^{(i-1)n}}\omega\left(\frac{1}{2^{i-1}}\right)\right).\end{aligned}$$

From this it follows that

$$\left\|y_i^{(n)} - y_{i-1}^{(n)}\right\|_S \leq m2^n\left(\frac{1}{2^n} + c\right)\omega\left(\frac{1}{2^i}\right),$$

and therefore

$$\left\|y_i^{(n)} - y_{i-1}^{(n)}\right\|_S \to 0 \quad \text{as} \quad i \to \infty.$$

Since the space S is complete, there exists $\widetilde{y} \in S$ such that

$$\left\|y_i^{(n)} - \widetilde{y}\right\|_S \to 0 \quad \text{when} \quad i \to \infty.$$

Thus, $y_i \to y$, $y_i^{(n)} \to \widetilde{y}$ $(i \to \infty)$ in the space S. Taking into account that the operator $\frac{d^n}{dt^n}$ is closed in S, we conclude that $y \in C^n(\mathbb{R}_+, \mathfrak{H})$ and $y^{(n)}(t) \equiv \widetilde{y}(t)$. ☐

Replacing in inequality (8) n by $n + \varepsilon$ and thus strengthening it, we shall arrive at the following consequence.

Corollary 2. *Let, for $y \in S$,*

$$\exists c > 0,\ \exists \varepsilon > 0 : \mathcal{E}_r(y) \leq \frac{c}{r^{n+\varepsilon}}.$$

Then $y \in C^n(\mathbb{R}_+, \mathfrak{H})$.

3. Now let $\{m_n\}_{n \in \mathbb{N}_0}$ be a nondecreasing sequence of numbers (there is no loss of generality in assuming that $m_0 = 1$). We put

$$C_{\{m_n\}} = C_{\{m_n\}}(\mathbb{R}_+, \mathfrak{H}) = \bigcup_{\alpha > 0} C_{m_n}^{\alpha}, \quad C_{(m_n)} = C_{(m_n)}(\mathbb{R}_+, \mathfrak{H}) = \bigcap_{\alpha > 0} C_{m_n}^{\alpha},$$

where

$$\begin{aligned}C_{m_n}^{\alpha} &= C_{m_n}^{\alpha}(\mathbb{R}_+, \mathfrak{H}) \\ &= \left\{y \in C^{\infty}(\mathbb{R}_+, \mathfrak{H}) \middle| \exists c = c(y) > 0,\ \forall k \in \mathbb{N}_0 : \sup_{t \in \mathbb{R}_+}\left\|y^{(k)}(t)\right\| \leq c m_k \alpha^k\right\}\end{aligned}$$

is a Banach space with respect to the norm

$$\|y\|_{C_{m_n}^{\alpha}} = \sup_{k \in \mathbb{N}} \frac{\sup_{t \in \mathbb{R}_+}\left\|y^{(k)}(t)\right\|}{\alpha^k m_k}.$$

The spaces $C_{\{m_n\}}$ and $C_{(m_n)}$ are equipped with the topologies of inductive and projective limits of the spaces $C_{m_n}^{\alpha}$, respectively. Note that the spaces $C_{\{n!\}}$, $C_{(n!)}$ $(m_n = n!)$ and $C_{\{1\}}$ $(m_n \equiv 1)$ are nothing that, respectively, the spaces of bounded on $\mathbb{R}_+$ with all their derivatives analytic, entire, and entire of exponential type $\mathfrak{H}$-valued vector functions.

In what follows, we assume in addition that $\{m_n\}_{n \in \mathbb{N}_0}$ satisfies the condition

$$\forall \alpha > 0,\ \exists c = c(\alpha) : m_n \geq c\alpha^n \tag{10}$$

and put

$$\tau(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}. \tag{11}$$

It is clear that $\tau(\lambda)$ is entire, $\tau(\lambda) \geq 1$ for $\lambda \geq 0$, and $\tau(\lambda) \uparrow \infty$ as $\lambda \to \infty$.

Theorem 5. *Suppose the condition*

$$\exists c > 0,\ \exists h > 1,\ \forall n \in \mathbb{N}_0 : m_{n+1} \le c h^n m_n \tag{12}$$

to be fulfilled for the sequence $\{m_n\}_{n\in\mathbb{N}_0}$. *Then the following equivalence relations hold:*

$$\begin{aligned} y \in C^\infty(\mathbb{R}_+, \mathfrak{H}) &\iff \forall \alpha > 0 : \mathcal{E}_r(y) = O\left(\tfrac{1}{r^\alpha}\right) \quad (r \to \infty),\\ y \in C_{\{m_n\}} &\iff \exists \alpha > 0 : \mathcal{E}_r(y) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty),\\ y \in C_{(m_n)} &\iff \forall \alpha > 0 : \mathcal{E}_r(y) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty) \end{aligned}$$

($\tau(\lambda)$ *is defined by (11)).*

Proof. Let $C^\infty(A)$ denote the set of all infinitely differentiable vectors of the operator A,

$$C^\infty(A) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n).$$

For a number $\alpha > 0$, we put

$$C^\alpha_{m_n}(A) = \left\{ f \in C^\infty(A) \middle| \exists c = c(f) > 0,\ \forall n \in \mathbb{N}_0 : \|A^n f\| \le c\alpha^n m_n \right\}.$$

The set $C^\alpha_{m_n}(A)$ is a Banach space with respect to the norm

$$\|f\|_{C^\alpha_{m_n}(A)} = \sup_{n\in\mathbb{N}_0} \frac{\|A^n f\|}{\alpha^n m_n}.$$

Then

$$C_{\{m_n\}}(A) = \bigcup_{\alpha>0} C^\alpha_{m_n}(A) \text{ and } C_{(m_n)}(A) = \bigcap_{\alpha>0} C^\alpha_{m_n}(A)$$

are linear locally convex spaces with topologies of the inductive and the projective limits, respectively.

Let

$$\mathcal{E}_r(f, A) = \inf_{f_0 \in C_{\{1\}}(A)} \|f - f_0\|.$$

As it has been shown in [4], the following equivalence relations take place:

$$\begin{aligned} f \in C^\infty(A) &\iff \forall \alpha > 0 : \mathcal{E}_r(f, A) = O\left(\tfrac{1}{r^\alpha}\right) \quad (r \to \infty),\\ f \in C_{\{m_n\}}(A) &\iff \exists \alpha > 0 : \mathcal{E}_r(f, A) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty),\\ f \in C_{(m_n)}(A) &\iff \forall \alpha > 0 : \mathcal{E}_r(f, A) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty). \end{aligned} \tag{13}$$

Consider the map $F : \mathfrak{H} \mapsto S$,

$$Ff = e^{-At} f.$$

Since, for $y \in S$, there exists a unique vector $f \in \mathfrak{H}$ such that $y = e^{-At} f$, this transformation is one-to-one. From (3) it follows that F maps $\mathfrak{H}$ onto S isometrically and

$$F\left(C^\infty(A)\right) = C^\infty(\mathbb{R}_+, \mathfrak{H}), \quad F\left(C_{\{m_n\}}(A)\right) = C_{\{m_n\}}, \quad F\left(C_{(m_n)}(A)\right) = C_{(m_n)}. \tag{14}$$

The proof of the theorem follows from (13), (14) because of $\mathcal{E}_r(f, A) = \mathcal{E}_r\left(e^{-At} f\right)$. □

If $m_n = n^{n\beta}$ ($\beta > 0$), then $\tau(r) = e^{-r^{1/\beta}}$ and Theorem 5 yields the following assertion.

Corollary 3. *The following equivalence relations are valid:*

$$\begin{aligned} y \in C_{\{n^{n\beta}\}}(A) &\iff \exists \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r^{1/\beta}}\right) \quad (r \to \infty),\\ y \in C_{(n^{n\beta})}(A) &\iff \forall \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r^{1/\beta}}\right) \quad (r \to \infty). \end{aligned}$$

Recall that an entire $\mathfrak{H}$-valued vector function $x(\lambda)$ has a finite order of growth if

$$\exists \gamma > 0,\ \forall \lambda \in \mathbb{C} : \|x(\lambda)\| \le \exp(|\lambda|^\gamma).$$

The greatest lower bound $\rho(x)$ of such γ is the order of $x(\lambda)$. The type of an entire vector-valued function $x(\lambda)$ of an order ρ is determined as

$$\sigma(x) = \inf\left\{ a > 0 : \|x(\lambda)\| \le \exp(a|\lambda|^\rho) \right\}.$$

Since the semigroup $\left\{e^{-At}\right\}_{t\geq 0}$ is analytic, every weak solution $y(t)$ of equation (1) is analytic on $(0,\infty)$. It is not difficult to show that it is analytic on $[0,\infty)$ if and only if $y \in C_{\{n^n\}}$.

By Corollary 3,

$$\exists \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r}\right) \quad (r \to \infty).$$

As for the extendability of $y(t)$ to an entire vector function of order ρ and finite type, the answer to the question gives the next theorem.

Theorem 6. *In order that a weak solution of equation (1) admit an extension to an entire $\mathfrak{H}$-valued vector function $y(z)$, it is necessary and sufficient that*

$$\forall \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r}\right) \quad (r \to \infty). \tag{15}$$

The extension $y(z)$ is of finite order ρ and finite type if and only if

$$\exists \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r^{1/\beta}}\right) \quad (r \to \infty),$$

where β and ρ are connected with each other by the formula

$$\beta = \frac{\rho - 1}{\rho} < 1.$$

Note that we may always suppose $\rho > 1$. Indeed, if $\rho \leq 1$ and the type is finite, then $y \in S_0$ and it has no sense to approximate a solution from S_0 by solutions from the same space.

Proof of Theorem 6. Let $y(t)$ be a weak solution of equation (1). By Corollary 3, $y \in C_{(n^n)}$, so $y(t)$ admits an extension to an entire vector function if and only if relation (15) is fulfilled.

Assume that $y(t)$ admits an extension to an entire $\mathfrak{H}$-valued vector function $y(z)$ which has an order ρ and a finite type σ. Then

$$\forall \sigma_1 > \sigma,\ \exists c = c(\sigma_1) : \|y(z)\| \leq c e^{\sigma_1 |z|^\rho}.$$

Hence,

$$\forall r > 0,\ \forall n \in \mathbb{N}_0 : \left\|y^{(n)}(z)\right\| \leq \frac{n!}{2\pi} \int_{|z-\zeta|=r} \frac{\|y(\zeta)\|}{|z-\zeta|^{n+1}}\, d\zeta \leq \frac{c\sigma_1 n!}{r^n} \exp\left(2\sigma_1 r^\rho\right).$$

Taking into account that the function $\frac{\exp(ar^\rho)}{r^n}$ reaches its minimum at the point $\left(\frac{n}{a\rho}\right)^{1/\rho}$ and using Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right) \quad (n \to \infty),$$

we get

$$\left\|y^{(n)}(z)\right\| \leq c\left(2e^{1-\rho}\sigma_1\rho\right)^{\frac{1}{\rho}} n^{\frac{\rho-1}{\rho}n},$$

which shows that $y \in C_{\{n^{n\beta}\}}$, where

$$\beta \leq \frac{\rho - 1}{\rho} \Longleftrightarrow \rho \geq \frac{1}{1-\beta}, \quad \beta < 1. \tag{16}$$

By Corollary 3,

$$\exists \alpha > 0 : \mathcal{E}_r(y) = O\left(e^{-\alpha r^{1/\beta}}\right) \quad (r \to \infty). \tag{17}$$

Conversely, let (17) hold true. Then Corollary 3 implies that $y \in C_{\{n^{n\beta}\}}$ $(0 < \beta < 1)$ is an entire vector-valued function and it can be represented by the series $\sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} z^k$. For its order of growth ρ we have

$$\rho = \varlimsup_{n\to\infty} \frac{n \ln n}{\ln \frac{n!}{\left\|y^{(n)}(0)\right\|}} \leq \varlimsup_{n\to\infty} \frac{n \ln n}{\ln\left(n! n^{-n\beta}\right)} = \frac{1}{1-\beta},$$

that is,

$$\rho \le \frac{1}{1-\beta}. \tag{18}$$

It follows from (16) and (18) that

$$\rho = \frac{1}{1-\beta}, \quad 0 < \beta < 1. \qquad \square$$

4. The direct and inverse theorems of the approximation theory are usually formulated for Banach space. Proving them is slightly more complex than in the case of a Hilbert space. Below we show how, for example, Theorem 5 can be reformulated in a Banach space. To this end we introduce the following notations:

$$\mathfrak{H}^n = \mathcal{D}(A^n), \|f\|_{\mathfrak{H}^n} = \left(\|f\|^2 + \|A^n f\|^2\right)^{1/2}.$$

The space $\mathfrak{H}^n$ is continuously and densely embedded into $\mathfrak{H}$. Denote by $\mathfrak{H}^{-n}$ the completion of $\mathfrak{H}$ in the norm $\|f\|_{\mathfrak{H}^{-n}} = \left\|(A+I)^{-n} f\right\|$, the so called negative space associated with the positive space $\mathfrak{H}^n$ in the chain

$$\mathfrak{H}^n \subseteq \mathfrak{H} \subseteq \mathfrak{H}^{-n}$$

(see [2]). By a suitable choice of the norms in $\mathfrak{H}^n$ and $\mathfrak{H}^{-n}$, one can attain the relation

$$\|f\|_{\mathfrak{H}^{-n}} \le \|f\| \le \|f\|_{\mathfrak{H}^n}.$$

Theorem 7. *Let $\mathfrak{B}$ be a Banach space inside the chain*

$$\mathfrak{H}^{k_1} \subseteq \mathfrak{B} \subseteq \mathfrak{H}^{-k_2} \tag{19}$$

of continuously and densely embedded into each other spaces with some $k_1, k_2 \in \mathbb{N}$, and let a sequence $\{m_n\}_{n\in\mathbb{N}_0}$ possess the properties (10) and (12). Then for a weak solution $y(t)$ of equation (1), the following equivalence relations hold true:

$$\begin{aligned} y \in C^\infty(\mathbb{R}_+, \mathfrak{H}) &\iff \forall\alpha > 0 : \mathcal{E}_r(y, \mathfrak{B}) = O\left(r^{-\alpha}\right) \quad (r \to \infty), \\ y \in C_{\{m_n\}} &\iff \exists\alpha > 0, : \mathcal{E}_r(y, \mathfrak{B}) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty), \\ y \in C_{(m_n)} &\iff \forall\alpha > 0 : \mathcal{E}_r(y, \mathfrak{B}) = O\left(\tau^{-1}(\alpha r)\right) \quad (r \to \infty), \end{aligned}$$

where

$$\mathcal{E}_r(y, \mathfrak{B}) = \inf_{y_0 \in S_0 : \sigma(y_0) \le r} \sup_{s\in\mathbb{R}_+} \|y(s) - y_0(s)\|_{\mathfrak{B}},$$

$\tau(\lambda)$ is defined by (11), $\|\cdot\|_{\mathfrak{B}}$ is the norm in $\mathfrak{B}$.

Proof. Show first that the spaces $C_{\{m_n\}}(A)$ and $C_{(m_n)}(A)$ considered as subspaces of $\mathfrak{H}$ coincide with the corresponding subspaces $C^k_{\{m_n\}}(A)$ and $C^k_{(m_n)}(A)$ constructed in the Hilbert space $\mathfrak{H}^k$ from the restriction $A \restriction \mathfrak{H}^k$ which is a nonnegative self-adjoint operator in $\mathfrak{H}^k$.

So, let $f \in C_{\{m_n\}}(A)$. Then

$$\exists\alpha > 0,\ \exists c > 0 : \left\|A^i f\right\|_{\mathfrak{H}^k} = \left(\left\|A^i f\right\|^2 + \left\|A^{i+k} f\right\|^2\right)^{1/2}$$

$$\le c\left(\alpha^{2i} m_i^2 + \alpha^{2(i+k)} m_{k+i}^2\right)^{1/2} \le \widetilde{c}\left(\alpha h^k\right)^i m_i, \tag{20}$$

i.e., $C_{\{m_n\}}(A) \subseteq C^k_{\{m_n\}}(A)$. From (19) we also have the embedding $C_{(m_n)}(A) \subseteq C^k_{(m_n)}(A)$. The inverse embeddings are consequences of the estimate $\left\|A^i f\right\|_{\mathfrak{B}} \le \left\|A^i f\right\|_{\mathfrak{H}^k}$. Thus we have

$$C_{\{m_n\}}(A) = C^k_{\{m_n\}}(A), \quad C_{(m_n)}(A) = C^k_{(m_n)}(A). \tag{21}$$

It is also evident that $C^\infty_k(A) = C^\infty(A)$.

Since for a vector $g \in C_{\{1\}}(A)$ $(m_n \equiv 1)$ of type $\sigma(g, A) \le k$, the inequality

$$\left\|A^i f\right\|_{\mathfrak{H}^k} = \left(\left\|A^i g\right\|^2 + \left\|A^{i+k} g\right\|^2\right)^{1/2} \le c\left(\alpha^{2i} + \alpha^{2(i+k)}\right)^{1/2} \le \widetilde{c}\alpha^i, \quad \widetilde{c} = \left(1 + \alpha^{2k}\right)^{1/2},$$

is valid, the space $C_{\{1\}}(A)$ coincides with $C^k_{\{1\}}(A)$. Moreover, the type of g for the operator A in the space $C_{\{1\}}(A)$ is the same as the one for $A \upharpoonright \mathfrak{H}^k$ in the space $C^k_{\{1\}}(A)$.

Denote by $\widetilde{A}$ the closure of A in the space $\mathfrak{H}^{-k}$. It is easy to make sure that $\widetilde{A}$ is a nonnegative self-adjoint operator in $\mathfrak{H}^{-k}$ and if the space $\mathfrak{H}$ is considered as a subspace of $\mathfrak{H}^{-k}$, then we arrive at the previous situation. For this reason,

$$C_{\{m_n\}}(A) = C^{-k}_{\{m_n\}}(A), \quad C_{(m_n)}(A) = C^{-k}_{(m_n)}(A), \tag{22}$$

and

$$\forall g \in C_{\{1\}}(A) = C^k_{\{1\}}(A) : \sigma(g, A) = \sigma(g, \widetilde{A}).$$

Taking into account that the restriction (extension) of the semigroup $\left\{e^{-At}\right\}_{t\in\mathbb{R}_+}$ to the space $\mathfrak{H}^{k_1}$, $k_1 > 0$, (to $\mathfrak{H}^{-k_2}$, $k_2 > 0$) is an analytic contractive C_0-semigroup in $\mathfrak{H}^{k_1}$ (in $\mathfrak{H}^{-k_2}$), the embeddings $C^k_{\{m_n\}}(A) \subseteq \mathfrak{B}$, $C_{(m_n)}(A) \subseteq \mathfrak{B}$ and the chain (19), we obtain for $y(t) = e^{-At}f$, $f \in C_{\{m_n\}}(A)$ and $y_0(t) = e^{-At}g$, $g \in C_{\{1\}}(A)$ that

$$\left\|e^{-At}f - e^{-As}g\right\|_{\mathfrak{B}} \le \left\|e^{-At}f - e^{-As}g\right\|_{\mathfrak{H}^{k_1}} \le \|f - g\|_{\mathfrak{H}^{k_1}},$$

whence

$$\mathcal{E}_r(y, \mathfrak{B}) = \inf_{y_0 \in S_0 : \sigma(y_0) \le r} \sup s \in \mathbb{R}_+ \left\|e^{-As}f - e^{-As}g\right\|_{\mathfrak{B}} \le \|f - g\|_{\mathfrak{H}^{k_1}},$$

that is,

$$\mathcal{E}_r(y, \mathfrak{B}) \le \mathcal{E}_r(f, A \upharpoonright \mathfrak{H}^{k_1}). \tag{23}$$

From (19) it follows that

$$\forall t \in \mathbb{R}_+ : \left\|e^{-At}f - e^{-At}g\right\|_{\mathfrak{H}^{-k_2}} \le \left\|e^{-At}f - e^{-At}g\right\|_{\mathfrak{B}}.$$

This implies that

$$\|f - g\|_{\mathfrak{H}^{-k_2}} = \sup_{t\in\mathbb{R}_+} \left\|e^{-At}f - e^{-At}g\right\|_{\mathfrak{H}^{-k_2}} \le \sup_{t\in\mathbb{R}_+} \left\|e^{-At}f - e^{-At}g\right\|_{\mathfrak{B}}$$

and, hence,

$$\inf_{g\in C_{\{1\}}(A):\sigma(g)\le r} \|f - g\|_{\mathfrak{H}^{-k_2}} \le \inf_{y_0\in S_0:\sigma(y_0)\le r} \sup_{t\in\mathbb{R}_+} \|y(t) - y_0(t)\|_{\mathfrak{B}}.$$

Thus,

$$\mathcal{E}_r(f, \widetilde{A}) \le \mathcal{E}_r(y, \mathfrak{B}). \tag{24}$$

Inequalities (23) and (24), with regard to (19), (21), (22) and Theorem 5, complete the proof of Theorem 7. □

5. Let A be a self-adjoint operator in $\mathfrak{H}$ whose spectrum is discrete. Assume that its eigenvalues $\lambda_k = \lambda_k(A)$, $k \in \mathbb{N}$, satisfy the condition $\sum_{k=1}^{\infty} \lambda_k^{-p} < \infty$ with some $p > 0$. Suppose also that λ_k are enumerated in ascending order and each one is counted according to its multiplicity and denote by $\{e_n\}_{n\in\mathbb{N}}$ the orthonormal basis in $\mathfrak{H}$ consisting of eigenvectors of A. Then the spectral function $E(\lambda)$ of the operator A has the form

$$E(\lambda)f = \sum_{\lambda_k \le \lambda} f_k e_k,$$

where $f_k = (f_k, e_k)$ are the Fourier coefficients of f, and

$$\mathcal{E}_r(f, A) = \sum_{\lambda_k > r} f_k e_k.$$

As it has been shown in [6], the following assertion holds true.

Proposition 1. *The following equivalence relations are valid:*

$$\begin{aligned} f \in C^{\infty}(A) &\iff \forall \alpha > 0,\ \exists c = c(\alpha) > 0 : |f_k| \le c\lambda_k^{-\alpha}, \\ f \in C_{\{1\}}(A) &\iff \exists n_0 \in \mathbb{N} : f_k = 0 \text{ as } k \ge n_0, \\ f \in C_{\{m_n\}}(A) &\iff \exists \alpha > 0,\ \exists c > 0 : |f_k| \le c\tau^{-1}(\alpha\lambda_k), \\ f \in C_{(m_n)}(A) &\iff \forall \alpha > 0,\ \exists c = c(\alpha) > 0 : |f_k| \le c\tau^{-1}(\alpha\lambda_k). \end{aligned}$$

(The function $\tau(\lambda)$ was defined in (11)).

Let now $y(t)$ be a weak solution of (1). Then

$$y(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} f_k e_k, \quad \sum_{k=1}^{\infty} |f_k|^2 < \infty. \tag{25}$$

The solution $y(t)$ is an entire vector-valued function of exponential type ($y \in S_0$) if and only if

$$\exists n_0 \in \mathbb{N} : f_k = 0 \text{ as } k \ge n_0.$$

Proposition 1, Theorems 3, 5 and (13) imply the following.

Theorem 8. *The following equivalence relations take place:*

$$\begin{aligned} y \in C^n(\mathbb{R}_+, \mathfrak{H}) &\iff \mathcal{E}_{\lambda_k}(y) = o\left(\lambda_{k+1}^{-n}\right) \quad (n \to \infty), \\ y \in C^{\infty}(\mathbb{R}_+, \mathfrak{H}) &\iff \forall \alpha > 0 : \mathcal{E}_{\lambda_k}(y) = O\left(\lambda_{k+1}^{-\alpha}\right) \quad (n \to \infty), \\ y \in C_{\{m_n\}} &\iff \exists \alpha > 0 : \mathcal{E}_{\lambda_k}(y) = O\left(\tau^{-1}(\alpha\lambda_{k+1})\right) \quad (n \to \infty), \\ y \in C_{(m_n)} &\iff \forall \alpha > 0 : \mathcal{E}_{\lambda_k}(y) = O\left(\tau^{-1}(\alpha\lambda_{k+1})\right) \quad (n \to \infty). \end{aligned}$$

6. Put $\mathfrak{H} = L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^q$ with piecewise smooth boundary $\partial\Omega$, and denote by B' the operator generated in $L_2(\Omega)$ by the differential expression

$$(\mathcal{L}u)(x) = -\sum_{i=1}^{q} \sum_{k=1}^{q} \frac{\partial}{\partial x_i} \left(a_{ik}(x) \frac{\partial u(x)}{\partial x_k} \right) + c(x)u(x), \tag{26}$$

on

$$\mathcal{D}(B') = \left\{ u \in C^2(\overline{\Omega}) \middle| u \restriction_{\partial\Omega} = 0 \right\}. \tag{27}$$

It is assumed that $a_{ik}(x), c(x) \in C^{\infty}(\overline{\Omega}),\ c(x) \ge 0$. Suppose also the expression (26) to be of elliptic type in $\overline{\Omega}$. In this case all the eigenvalues $\mu_i(x),\ i = 1, \dots q$, of the matrix $\|a_{ik}(x)\|_{i,k=1}^{q},\ x \in \overline{\Omega}$, have the same sign; without loss of generality we may assume $\mu_i(x) > 0,\ x \in \overline{\Omega}$.

It is not hard to make sure that B' is a positive definite Hermitian operator with dense domain in $L_2(\Omega)$. So, B' admits a closure to a positive definite selfadjoint operator B on $L_2(\Omega)$. We shall call B the operator generated by (26), (27). The spectrum of B is discrete, and for its eigenvalues, $\lambda_1(B) < \lambda_2(B) < \cdots < \lambda_n(B) < \dots$, the estimate

$$c_1 n^{2/q} \le \lambda_n(B) \le c_2 n^{2/q}, \quad 0 < c_i = \text{const}, \quad i = 1, 2. \tag{28}$$

is valid (see [8]). Denote by $e_n(x),\ n \in \mathbb{N}$, the orthonormal basis in $L_2(\Omega)$, consisting of eigenfunctions of B.

In the case where Ω is a q-dimensional cube, $0 < x_k < a,\ k = 1, \dots, q,\ a > 0$, and $\mathcal{L} = -\sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2}$, the following formulas for the eigenvalues $\lambda_{n_1 \dots n_q},\ n_k \in \mathbb{N}$, and eigenfunctions $e_{n_1 \dots n_q}(x)$ of the operator B hold:

$$\lambda_{n_1 \dots n_q} = \frac{\pi^2}{a^2} \sum_{k=1}^{q} n_k^2; \quad e_{n_1 \dots n_q}(x) = \left(\frac{2}{a}\right)^{q/2} \prod_{k=1}^{q} \sin \frac{\pi}{a} x_k.$$

Let $y(t) = u(t, x) \in C(\mathbb{R}_+, L_2(\Omega))$ be a weak solution of the problem

$$\left(\frac{\partial}{\partial t} - \sum_{k=1}^{q} \sum_{i=1}^{q} \frac{\partial}{\partial x_k} \left(a_{ki}(x) \frac{\partial}{\partial x_i} \right) + c(x) \right) u(t, x) = 0, \tag{29}$$

$$\forall t > 0,\ \forall x \in \partial\Omega : u(t,x) = 0, \tag{30}$$

where the conditions on $a_{ki}(x)$ and $c(x)$ are the same as before. Then $u(t,x)$ admits a representation of form (25). Set

$$L_2^n = \mathcal{D}(B^n), \quad \|f\|_{L_2^n} = \left(\|f\|^2 + \|B^n f\|^2\right)^{1/2},$$

where B is an operator generated in the space $L_2(\Omega)$ by expression (26) and boundary value condition (27). The space L_2^n is continuously and densely embedded into $L_2(\Omega)$. Denote by L_2^{-n} the negative space corresponding to the positive one $L_2^n \subset L_2(\Omega)$. In the case where $\mathcal{L} = -\sum_{k=1}^{q} \frac{\partial^2}{\partial x_k^2}$, L_2^n is none other than the well-known Sobolev space $\overset{\circ}{W}{}_2^{2n}(\Omega)$.

Using estimate (28) for $\lambda_k(B)$ and Theorem 8, we obtain in a way analogous to that used in the proof of Theorem 7 the following assertion.

Theorem 9. *Let $\mathfrak{B}$ be a Banach space and let*

$$L_2^{n_1} \subseteq \mathfrak{B} \subseteq L_2^{-n_2}, \quad n_1, n_2 \in \mathbb{N},$$

be a chain of continuously and densely embedded into each other spaces. Suppose also the sequence $\{m_n\}_{n=1}^{\infty}$ to satisfy (10) and (12). Then

$$\begin{aligned} y(t) = u(t,x) \in C^\infty(\mathbb{R}_+, L_2(\Omega)) \quad &\Longleftrightarrow \quad \alpha > 0 : \mathcal{E}_{\lambda_k}^{\mathfrak{B}}(y) = O\left(\tfrac{1}{(k+1)^\alpha}\right), \\ y(t) \in C_{\{m_n\}} \quad &\Longleftrightarrow \quad \exists \alpha > 0 : \mathcal{E}_{\lambda_k}^{\mathfrak{B}}(y) = O\left(\tau^{-1}\left(\alpha(k+1)^{2/q}\right)\right), \\ y(t) \in C_{(m_n} \quad &\Longleftrightarrow \quad \forall \alpha > 0 : \mathcal{E}_{\lambda_k}^{\mathfrak{B}}(y) = O\left(\tau^{-1}\left(\alpha(k+1)^{2/q}\right)\right), \end{aligned}$$

where

$$\mathcal{E}_{\lambda_k}^{\mathfrak{B}}(y) = \inf_{y_0 \in S_0 : \sigma(y_0) \le \lambda_k} \sup_{s \in \mathbb{R}_+} \|y(s) - y_0(s)\|_{\mathfrak{B}}.$$

It is relevant to remark that in the case where $a_{ki}(x) = \delta_{ki}$ and $c(x) \equiv 0$, by virtue of the embedding theorems for Sobolev spaces, one can take the space $C(\overline{\Omega})$ of continuous in $\overline{\Omega}$ functions or $L_p(\Omega)$, $1 \le p < \infty$, as $\mathfrak{B}$ and consider not only the Dirichlet but some other boundary value problems, in particular, the Neumann problem.

References

1. J. M. Ball, *Continuity properties of nonlinear semigroups*, J. Functional Analysis **17** (1974), no. 1, 91–103.
2. Yu. M. Berezanskiĭ, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Transl. Math. Monographs, vol. 63, American Mathematical Society, Providence, RI, 1986.
3. M. L. Gorbachuk, *On analytic solutions of operator-differential equations*, Ukrainian Math. J. **52** (2000), no. 5, 680–693.
4. M. L. Gorbachuk and V. I. Gorbachuk, *Operator approach to approximation problems*, St. Petersburg Math. J. **9** (1998), no. 6, 1097–1110.
5. M. L. Gorbachuk, Ya. I. Grushka, and S. M. Torba, *Direct and inverse theorems in the theory of approximations by the Ritz method*, Ukrainian Math. J. **57** (2005), no. 5, 751–764.
6. V. I. Gorbachuk, *On summability of expansions in eigenfunctions of self-adjoint operators*, Soviet Math. Dokl. **35** (1987), no. 1, 11–15.
7. N. P. Kupcov, *Direct and inverse theorems of approximation theory and semigroups of operators*, Uspehi Mat. Nauk **23** (1968), no. 4, 117–178. (Russian)
8. S. G. Mikhlin, *Linear Partial Differential Equations*, Vysshaya Shkola, Moscow, 1977. (Russian)

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