MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

Homeotopy groups of rooted tree like non-singular foliations on the plane

Yu. Yu. Soroka

Methods Funct. Anal. Topology, Volume 22, Number 3, 2016, pp. 283-294

Methods of Functional Analysis and Topology
Vol. 22 (2016), no. 3, pp. 283–294

HOMEOTOPY GROUPS OF ROOTED TREE LIKE NON-SINGULAR FOLIATIONS ON THE PLANE

YU. YU. SOROKA

Abstract. Let F be a non-singular foliation on the plane with all leaves being closed subsets, $H^{+}(F)$ be the group of homeomorphisms of the plane which maps leaves onto leaves endowed with compact open topology, and $H^{+}_{0}(F)$ be the identity path component of $H^{+}(F)$. The quotient $\pi_0 H^{+}(F) = H^{+}(F)/H^{+}_{0}(F)$ is an analogue of a mapping class group for foliated homeomorphisms. We will describe the algebraic structure of $\pi_0 H^{+}(F)$ under an assumption that the corresponding space of leaves of F has a structure similar to a rooted tree of finite diameter.

1. Introduction

Non-singular foliations on the plane were studied by W. Kaplan [5, 6] and H. Whitney [17] in the 40–50 years of the XX century. In particular, W. Kaplan in [6] has generalized a theorem of E. Kamke and proved that every non-singular foliation F on the plane admits a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

1) the leaves of f are connected components of level sets $f^{-1}(c)$, $c \in \mathbb{R}$;
2) near each $z \in \mathbb{R}^2$ there are local coordinates (u, v) in which $f(u, v) = u + f(z)$.

This result was further extended to foliations with singularities by W. Boothby [2], and J. Jenkins and M. Morse [4]. Topological classification of different kinds of functions on surfaces was investigated in many papers, see e.g. A. Fomenko and A. Bolsinov [1], A. Oshemkov [9], V. Sharko [14], [15], O. Prishlyak [12], [13], E. Polulyakh and I. Yurchuk [10], E. Polulyakh [11], V. Sharko and Yu. Soroka [16].

W. Kaplan in [5, 6] has also mentioned that a non-singular foliation on the plane is glued of countably many strips along open boundary intervals and such that each strip has a foliation by parallel lines. In a recent paper S. Maksymenko and E. Polulyah [8] studied non-singular foliations F on arbitrary non-compact surfaces Σ glued from strips in a similar way. They proved contractibility of the connected components of groups $H(F)$ of homeomorphisms of Σ mapping leaves onto leaves. Thus the homotopy type of $H(F)$ is determined by the quotient group $\pi_0 H(F) = H(F)/H_0(F)$ of path components of $H(F)$, where $H_0(F)$ is the identity path component of $H(F)$.

In the present paper we compute the groups $\pi_0 H(F)$ for a special class of non-singular foliations on the plane whose spaces of leaves have the structure similar to rooted trees of finite diameter, see Theorem 4.5.

2. Striped surfaces

Let Σ_i be a surface with a foliation F_i, $i = 1, 2$. Then a homeomorphism $h : \Sigma_1 \to \Sigma_2$ will be called *foliated* if it maps leaves of F_1 onto leaves of F_2.

Definition 2.1. A subset $S \subset \mathbb{R}^2$ will be called a *model strip* if the following two conditions hold:

2010 *Mathematics Subject Classification.* Primary 57S05; Secondary 57R30, 55Q05 .
Key words and phrases. Non-singular foliations, homeotopy groups.

1) $\mathbb{R} \times (-1, 1) \subseteq S \subset \mathbb{R} \times [-1, 1]$;
2) $S \cap \mathbb{R} \times \{-1, 1\}$ is a union of open mutually disjoint finite intervals.

Put

$$\partial_{-}S = S \cap (\mathbb{R} \times \{-1\}), \quad \partial_{+}S = S \cap (\mathbb{R} \times \{1\}), \quad \partial S = \partial_{-}S \cup \partial_{+}S.$$

Notice that every model strip has an oriented foliation consisting of horizontal arcs $\mathbb{R} \times t$, $t \in (-1, 1)$, and connected components of ∂S.

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of model strips, and

$$X = \bigcup_{\lambda \in \Lambda} \partial_{-}S_\lambda, \quad Y = \bigcup_{\lambda \in \Lambda} \partial_{+}S_\lambda.$$

By Definition 2.1, X and Y are disjoint unions of open intervals, therefore one can also write

$$X = \bigcup_{\alpha \in A} X_\alpha, \quad Y = \bigcup_{\beta \in B} Y_\beta,$$

where X_α and Y_β are open boundary intervals of those models strips and A and B are some index sets.

We will now glue model strips S_λ by identifying some of the intervals of X_α with some of the intervals of Y_β. In order to make this let us fix any set of indexes C and two injective maps $p : C \to A$ and $q : C \to B$. Notice that for each $\gamma \in C$ there exists a unique preserving orientation affine homeomorphism $\varphi_\gamma : X_{p(\gamma)} \to Y_{q(\gamma)}$. Then the quotient space

$$\Sigma := \bigsqcup_{\lambda \in \Lambda} S_\lambda \Big/ \{X_{p(\gamma)} \overset{\varphi_\gamma}{\sim} Y_{q(\gamma)}\} \tag{2.1}$$

will be called a *striped surface*.

Remark 2.2. A unique preserving orientation affine homeomorphism $\phi : (a, b) \to (c, d)$ is given by $\phi(t) = \frac{c-d}{b-a}(t - a)$.

Remark 2.3. In [8] a wider class of striped surfaces is considered: it is also allowed to identify arbitrary connected components of $\bigsqcup_{\lambda \in \Lambda} \partial S_\lambda$ and the gluing affine homeomorphisms may reverse orientation.

Let also $p : \bigsqcup_{\lambda \in \Lambda} S_\lambda \to \Sigma$ be the quotient map and $p_\lambda : S_\lambda \to \Sigma$ be the restriction of p to the model strip S_λ. Then the pair (S_λ, p_λ) will be called a *chart* for the strip S_λ.

Since the homeomorphism φ_γ identifies leaves of such foliations, we see that every striped surface has the foliation F consisting of foliations on model strips. This foliation will be called *canonical*.

Moreover, each leaf of the foliation on the model strip is oriented and the gluing preserves orientation. Therefore all leaves of F are oriented.

Special leaves. Let $U \subset \Sigma$ be a subset. Then the union of all leaves of F intersecting U is called the *saturation* of U with respect to F and denoted by $Sat(U)$.

A leaf ω of F will be called *special* if

$$\omega \neq \bigcap_{N(\omega)} \overline{Sat\,(N(\omega))},$$

where $N(\omega)$ runs over all open neighborhoods of ω.

For instance each leaf ω belonging to the interior of a strip is non-special. Moreover, suppose $\omega = X_{p(\gamma)} \sim Y_{q(\gamma)}$ is a leaf such that $\partial_{-}S_\lambda = X_{p(\gamma)}$ and $\partial_{+}S_{\lambda'} = Y_{p(\gamma)}$, see Figure 2.1(a). Then the topological structure of the foliation F near ω is "similar" to the structure of F near "internal" leaves of strips and such a leaf is non-special as well, see [8, Lemma 3.2].

It also follows from that lemma that ω is special if and only if one of the following two conditions hold, see Figure 2.1(b):

1) ω is the image of gluing of leaves $X_{p(\gamma)}$ and $Y_{q(\gamma)}$ such that either $X_{p(\gamma)} \subsetneq \partial_- S_\lambda$ or $Y_{p(\gamma)} \subsetneq \partial_+ S_{\lambda'}$ for some $\gamma \in C$, $\lambda, \lambda' \in \Lambda$;
2) $\omega \subsetneq \partial_- S_\lambda$ or $\omega \subsetneq \partial_+ S_\lambda$ for some $\lambda \in \Lambda$.

(a) Non-special leaf (b) Special leaves

FIGURE 2.1

Reduced striped surfaces. A striped surface Σ will be called *reduced* whenever a leaf ω is special if and only if one of the following conditions holds:

1) ω is an image of gluing of some leaves $X_{p(\gamma)} \sim Y_{q(\gamma)}$ for some $\gamma \in C$;
2) $\omega \subsetneq \partial_- S_\lambda$ or $\omega \subsetneq \partial_+ S_\lambda$ for some $\lambda \in \Lambda$.

Let S be a model strip such that $\partial_- S = (0,1) \times -1$ and $\partial_+ S = (0,1) \times 1$. Let also $\phi : \partial_- S \to \partial_+ S$ be a homeomorphism defined by $\phi(t,-1) = (t,1)$, $t \in (0,1)$, and $\mathcal{C} = S/\phi$ be the quotient space obtained by identifying each $x \in \partial_- S$ with $\phi(x) \in \partial_+ S$.

Then $\mathcal{C}$ is a striped surface homeomorphic with a cylinder, and its canonical foliation has no special leaves.

It follows from [8, Theorem 3.7] that every striped surface (in the sense of (2.1), see Remark 2.3) is foliated homeomorphic either to $\mathcal{C}$ or to a reduced surface.

Graph of a striped surface. For a reduced striped surface Σ not foliated homeomorphic with $\mathcal{C}$ define an oriented graph $\Gamma(\Sigma)$ whose vertexes are strips and whose edges are special leaves. More precisely: if $\omega = X_{p(\gamma)} \sim Y_{q(\gamma)}$ is a special leaf of F, $X_{p(\gamma)} \subset \partial_- S_{\lambda_0}$, and $Y_{q(\gamma)} \subset \partial_+ S_{\lambda_1}$, then we assume that ω is an *edge* between *vertices* S_{λ_0} and S_{λ_1} oriented from S_{λ_1} to S_{λ_0}.

If $\lambda_0 = \lambda_1$, then ω correspond to a loop in $\Gamma(\Sigma)$ at $S_{\lambda_0} = S_{\lambda_1}$ being a vertex of $\Gamma(\Sigma)$.

Since a model strip may have infinitely many boundary components, we see that a graph $\Gamma(\Sigma)$ can be not locally finite. On the other hand, it can have a finite diameter $\operatorname{diam}\Gamma(\Sigma)$, being the minimal non-negative integer d such that every two vertices v_1 and v_2 are connected in $\Gamma(\Sigma)$ by a path consisting at most d edges.

Admissible striped surfaces. Recall that a family $\mathcal{V} = \{V_i\}_{i\in\Lambda}$ of subsets in a topological space X is called *locally finite* whenever for each $x \in X$ there exists an open neighborhood intersecting only finitely many elements from $\mathcal{V}$.

It is well known and is easy to see that *a union of a locally finite family of closed subsets is closed*, e.g. [7, Chapter 1, § 5.VIII].

Definition 2.4. A model strip S will be called *admissible* if the closures of intervals in $\partial_- S$ and $\partial_+ S$ are mutually disjoint and constitute a locally finite family in $\mathbb{R}^2$.

Example 2.5. A model strip with

$$\partial_+ S = \bigcup_{n\in\mathbb{Z}\setminus\{-1,0\}} \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right) \times 1$$

is not admissible, since condition 2) of Definition 2.1 fails.

It will be convenient to use the following notation:

$$[0] = \varnothing, \quad [n] = \{1, 2, \dots, n\}, \quad -\mathbb{N} = \{-1, -2, \dots\}.$$

Let also $J_i = (i, i + 0.5)$, $i \in \mathbb{Z}$, and for a subset $\Delta \subset \mathbb{Z}$ denote

$$A_\Delta = \bigcup_{i\in\Delta} J_i.$$

In particular, consider the following collections of mutually disjoint open intervals:

$$A_{[n]} = \bigcup_{i=1}^{n} (i, i + 0.5), \quad n = 0, 1, \dots, \qquad A_{\mathbb{N}} = \bigcup_{i\in\mathbb{N}} (i, i + 0.5),$$

$$A_{-\mathbb{N}} = \bigcup_{-i\in\mathbb{N}} (i, i + 0.5), \qquad A_{\mathbb{Z}} = \bigcup_{i\in\mathbb{Z}} (i, i + 0.5),$$

which will be called *standard*. The following easy lemma is left for the reader.

Lemma 2.6. *Let S be an admissible model strip. Then there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ preserving each line $\mathbb{R} \times t$, $t \in (-1, 1)$, with its orientation, and such that $h(S)$ is a model strip with $\partial_- h(S) = A_\alpha \times \{-1\}$ and $\partial_+ h(S) = A_\beta \times \{1\}$, where A_α and A_β are standard collections of intervals, i.e. $\alpha, \beta \in \{[0], [1], \dots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}$, see Figure 2.2. Moreover, α and β do not depend on a particular choice of such h.*

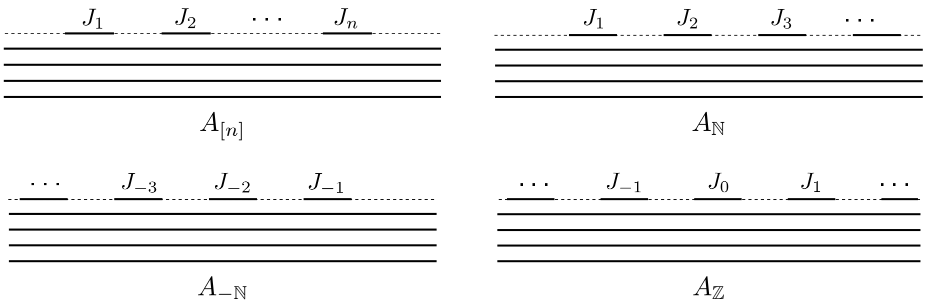

FIGURE 2.2. Types of $\partial_+ S$

Thus for an admissible model strip S its foliated topological type is determined by the ordinal type of collections of boundary intervals in $\partial_- S$ and $\partial_+ S$.

3. WREATH PRODUCTS

Let H and S be two groups. Denote by $Map(H, S)$ the group of all *maps* (not necessarily homomorphisms) $\varphi : H \to S$ with respect to the point-wise multiplication. Then the group H acts on $Map(H, S)$ by the following rule: the result of the action of $\varphi \in Map(H, S)$ on $h \in H$ is the composition map:

$$\varphi \circ h : H \longrightarrow H \longrightarrow S.$$

The semidirect product $Map(H, S) \rtimes H$ corresponding to this action will be denoted by $S \wr H$ and called the *wreath product* of S and H. Thus

$$S \wr H = Map(H, S) \rtimes H$$

is the Cartesian product $Map(H, S) \times H$ with the multiplication given by the formula

$$(\varphi_1, h_1) \cdot (\varphi_2, h_2) = \big((\varphi_1 \circ h_2) \cdot \varphi_2, h_1 \cdot h_2\big)$$

for $(\varphi_1, h_1), (\varphi_2, h_2) \in Map(H, S) \rtimes H$.

Let $\varepsilon : H \to S$ be the constant map into the unit of S. Then the pair $(\varepsilon, \mathrm{id}_H)$ is the unit element of $S \wr H$. Moreover, if $(\varphi, h) \in S \wr H$ and $\varphi^{-1} \in Map(H, S)$ is the point-wise inverse of φ, then $(\varphi^{-1} \circ h^{-1}, h^{-1})$ is the inverse of (φ, h) in $S \wr H$.

We also have the following exact sequence:

$$1 \to Map(H, S) \xrightarrow{i} S \wr H \xrightarrow{\pi} H \to 1,$$

where $i(\varphi) = (\varphi, e)$, e is the unit of H, and $\pi(\varphi, h) = h$. Moreover, π admits a section $s : H \to S \wr H$ defined by $s(h) = (\varepsilon, h)$.

4. Main result

Homeotopy group of a canonical foliation. Let Σ be striped surface with a canonical foliation F. Denote by $H(F)$ the groups of all foliated homeomorphisms $h : \Sigma \to \Sigma$, i.e. homeomorphisms mapping leaves of F onto leaves. We will endow $H(F)$ with the corresponding compact open topology.

Recall that all leaves of F are oriented. Then we denote by $H^+(F)$ the subgroup of $H(F)$ consisting of homeomorphisms $h : \Sigma \to \Sigma$ such that for each leaf ω the restriction map $h : \omega \to h(\omega)$ is orientation preserving.

Let $H_0^+(F)$ be the identity path component of $H^+(F)$. It consists of all $h \in H^+(F)$ isotopic to id_Σ in $H^+(F)$. Then $H_0^+(F)$ is a normal subgroup of $H^+(F)$, and the corresponding quotient

$$\pi_0 H^+(F) = H^+(F)/H_0^+(F)$$

will be called the *homeotopy* group of F.

Class $\mathfrak{F}$. Denote by $\mathfrak{F}$ the class of striped surfaces

$$\Sigma = \bigsqcup_{\lambda \in \Lambda} S_\lambda \big/ \{X_{p(\gamma)} \overset{\varphi_\gamma}{\sim} Y_{q(\gamma)}\}$$

satisfying the following conditions:

1) each S_λ, $\lambda \in \Lambda$, is admissible,
$$\partial_- S_\lambda = J_1 \times \{-1\} = A_{[1]} \times \{-1\}, \quad \partial_+ S_\lambda = A_{\Delta_\lambda} \times \{1\},$$
where Δ_λ coincides with one of the standard collections $A_{[n]}$, $A_{\mathbb{N}}$, $A_{-\mathbb{N}}$, or $A_{\mathbb{Z}}$;
2) the graph $\Gamma(\Sigma)$ is connected and has a finite diameter and no cycles.

In particular, if $\Sigma \in \mathfrak{F}$, then each model strip S_λ of Σ regarded as a vertex of $\Gamma(\Sigma)$ has at most one incoming edge and at most countably many outcoming edges linearly ordered with respect to Δ_λ.

Since $\Gamma(\Sigma)$ is connected and has a finite diameter and no cycles, it follows that there exists a unique vertex having no incoming edges. We will call this vertex a *root* and the corresponding strip a *root* strip.

Thus every surface $\Sigma \in \mathfrak{F}$ of diameter d can be represented as follows, see Figure 4.1:

$$\Sigma \;=\; S \underset{\partial_+ S}{\cup} \Big(\bigcup_{i \in \Delta} \Sigma_i \Big), \tag{4.1}$$

where

- S is a root strip of Σ,
$$\partial_- S = J_1 \times \{-1\}, \quad \partial_+ S = \underset{i \in \Delta}{\cup} J_i \times \{1\},$$
where $\Delta \in \{[0], [1], \dots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}$.
- Σ_i is either empty or it is a striped surface belonging to $\mathfrak{F}$ and its graph $\Gamma(\Sigma_i)$ has diameter less than d.

- Suppose Σ_i is non-empty and let S_i be the root strip of Σ_i. Then $\partial_- S_i = J_1 \times \{-1\}$ is glued to the boundary interval $J_i \times \{1\}$ of $\partial_+ S$ by the homeomorphism

$$\varphi : J_1 \equiv (1, 1.5) \longrightarrow J_i \equiv (i, i+0.5), \quad \varphi(t) = t + i - 1.$$

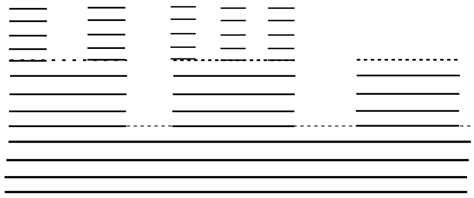

FIGURE 4.1. A striped surface $\Sigma \in \mathfrak{F}$ whose graph $\Gamma(\Sigma)$ has diameter 3

Obviously, $\Sigma \in \mathfrak{F}$ is a connected and simply connected non-compact surface. Therefore it follows from [3] that the interior of Σ is homeomorphic to $\mathbb{R}^2$.

The class of homeotopy groups of foliations on striped surfaces which belongs to the class $\mathfrak{F}$ will be denoted by $\mathcal{P}$, i.e.

$$\mathcal{P} = \{\pi_0 H^+(F) \mid F \text{ is a canonical foliation of some striped surface } \Sigma \in \mathfrak{F}\}.$$

We will also define another class of groups $\mathcal{G}$.

Definition 4.1. Let $\mathcal{G}$ be the minimal class of groups satisfying the following conditions:

1) $\{1\} \in \mathcal{G}$;
2) if $A_i \in \mathcal{G}$ for $i \in \mathbb{N}$, then $\prod_{i\in\mathbb{N}} A_i \in \mathcal{G}$;
3) if $A \in \mathcal{G}$, then $A \wr \mathbb{Z} \in \mathcal{G}$.

Lemma 4.2. *A group G belongs to $\mathcal{G}$ if and only if it can be obtained from the unit group $\{1\}$ by a composition of finitely many operations of the following types:*

(a) *countable direct products;*
(b) *wreath product with the group $\mathbb{Z}$.*

Proof. Let $\mathcal{G}_0$ be the class of groups G which can be obtained from the unit group $\{1\}$ by a composition of finitely many operations of types (a) and (b). Then any class of groups satisfying conditions 1)–3) of Definition 4.1 contains $\mathcal{G}_0$, whence $\mathcal{G}_0 \subset \mathcal{G}$. On the other hand, $\mathcal{G}_0$ also satisfies conditions 1)–3) of Definition 4.1, whence $\mathcal{G} \subset \mathcal{G}_0$ as well. $\square$

Every representation $\xi(G)$ of G as a composition of operations (a) and (b) will be called *a representation of G in the class $\mathcal{G}$*. Such a representation is not unique. For example,

$$\mathbb{Z} \;\cong\; \{1\} \wr \mathbb{Z} \;\cong\; 1 \times (1 \wr \mathbb{Z}) \;\cong\; (1 \times 1 \times 1) \wr \mathbb{Z}. \tag{4.2}$$

Definition 4.3. The *height* of a representation $\xi(G)$ of G in the class $\mathcal{G}$ is a non-negative integer defined inductively as follows:

1) $h(\{1\}) = 0$;
2) $h(\xi(G) \wr \mathbb{Z}) = 1 + h(\xi(G))$;
3) $h\left(\prod_{i\in\Lambda} \xi(A_i)\right) = 1 + \max_i \{h(\xi(A_i))\}$.

Example 4.4. Below are examples of representations of groups $\{1\}$, $\mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$ in the class $\mathcal{G}$ and their heights:

$$h(\{1\}) = 0, \qquad h(\{1\} \times \{1\}) = 1,$$
$$h(\{1\} \wr \mathbb{Z}) = 1, \qquad h((\{1\} \times \{1\}) \wr \mathbb{Z}) = 2,$$
$$h\big((\{1\} \wr \mathbb{Z}) \times (\{1\} \wr \mathbb{Z}) \big) = 2, \qquad h\big(((\{1\} \times \{1\}) \wr \mathbb{Z}) \times (\{1\} \wr \mathbb{Z}) \big) = 3.$$

Let $\mathcal{G}' \subset \mathcal{G}$ be a subclass of $\mathcal{G}$ consisting of groups admitting a representation of finite height in $\mathcal{G}$. The aim of the present paper is to prove the following theorem:

Theorem 4.5. *Classes $\mathcal{P}$ and $\mathcal{G}'$ coincide.*

In other words, a group G is isomorphic with a homeotopy group $H^+(F)$ of some striped surface $\Sigma \in \mathfrak{F}$ with a canonical foliation F if and only if G can be obtained from the unit group $\{1\}$ by a composition of finitely many operations of types (a) and (b) of Lemma 4.2.

5. Preliminaries

Let Σ be a striped surface belonging to $\mathfrak{F}$ presented in the form (4.1), and S be the root strip of Σ. We will use coordinates (x, y) from the chart for S, so we can assume that $\partial_+ S = \cup_{i\in\Delta} J_i \times \{1\}$.

Notice that if $h \in H^+(F)$, then $h(S) = S$, whence there exists a unique number $\eta(h) \in \mathbb{Z}$ such that in the chart for S we have that

$$h(J_i \times \{1\}) = J_{i+\eta(h)} \times \{1\}$$

for all $i \in \Delta$. One can easily check that the correspondence $h \mapsto \eta(h)$ is a homomorphism

$$\eta : H^+(F) \to \mathbb{Z}. \tag{5.1}$$

Obviously, η can be a non-zero homomorphism only when $\Delta = \mathbb{Z}$.

Consider the following two subgroups of $H^+(F)$:

$$Q_S = \left\{ h \in H^+(F) \mid h(\omega) = \omega, \text{for each leaf } \omega \text{ of } F \subset S \right\},$$
$$H^+(F, S) = \left\{ h \in H^+(F) \mid h|_S = \mathrm{id}|_S \right\}.$$

It is evident that

$$H^+(F, S) \ \subset \ Q_S \ \subset \ker(\eta). \tag{5.2}$$

Lemma 5.1. *Embeddings* (5.2) *are homotopy equivalences.*

Proof. First we will construct a deformation of $\ker(\eta)$ into Q_S. Let $h \in \ker(\eta)$. Since $h(S) = S$, it follows that h interchanges leaves of F. In the coordinates (x, y) in the chart for S these leaves are the lines $y = \mathrm{const}$, whence

$$h(x, y) = (\alpha(x, y), \beta(y)),$$

where $\alpha : S \to \mathbb{R}$ and $\beta : [-1, 1] \to [-1, 1]$ are continuous functions such that for each $y \in (0, 1)$ the correspondence $x \mapsto \alpha(x, y)$ is a preserving orientation homeomorphism $\mathbb{R} \to \mathbb{R}$.

Then $h \in Q_S$ iff $\beta(y) = y$ for all $y \in [0, 1]$. Define the map $H : \ker(\eta) \times [0, 1] \to \ker(\eta)$ by the formula

$$H(h, t)(z) = \begin{cases} (\alpha(x, y), (1 - t)\beta(y) + ty), & z = (x, y) \in S, \\ z, & z \in \Sigma \setminus S. \end{cases}$$

One can easily check that $H_0 = \mathrm{id}_{\ker(\eta)}$, $H_t(Q_S) \subset Q_S$ for all $t \in [0, 1]$, and $H(h, 1) \in Q_S$. Hence H is a deformation of $\ker(\eta)$ into Q_S, and so the inclusion $Q_S \subset \ker(\eta)$ is a homotopy equivalence.

Similarly, let $h \in Q_S$, so

$$h(x,y) = (\alpha(x,y), y)$$

for all $(x,y) \in S$. Notice that $h \in H^+(F,S)$ iff $\alpha(x,y) = x$ and $\beta(y) = y$ for all $(x,y) \in S$.

Let

$$h(x,y) = (\alpha_i(x,y), \beta_i(y))$$

be the restriction of h onto root strip S_i of Σ_i in the corresponding chart of S_i. Since $\partial_- S_i = J_1 \times \{-1\}$, we see that if $h \in H^+(F,S)$, then $\alpha_i(x,-1) = x$ for all $x \in J_1$ and $i \in \Delta$.

Fix a continuous function $\varepsilon : [-1,1] \to [0,1]$ such that

$$\varepsilon(y) = \begin{cases} 0, & y \in (-1,-0.8), \\ 1, & y \in (0,1) \end{cases}$$

and define the following homotopy $G : Q_S \times [0,1] \to Q_S$ by

$$G(h,t)(z) = \begin{cases} ((1-t)\alpha(x,y) + tx, y), & z = (x,y) \in S, \\ \big((1-t\varepsilon(y))\alpha_i(x,y) + t\varepsilon(y)x, \beta(y)\big), & z = (x,y) \in S_i, \\ z & z \notin S \cup \big(\cup_{i\in\Delta} S_i\big). \end{cases}$$

Since $\partial_- S_i$ is glued to the boundary component $J_i \times \{1\}$ by an affine homeomorphism, and the formulas for G are affine for each fixed t and y, it follows that those formulas agree on $J_i \times \{1\}$ and $\partial_- S_i$, c.f. [8]. This implies that G is a continuous map.

Moreover, one can easily check that $G_0 = \mathrm{id}_{Q_S}$, $G_t(H^+(F,S)) \subset H^+(F,S)$ for all $t \in [0,1]$, and $G_1(Q_S) \subset H^+(F,S)$. Hence G is a deformation of Q_S into $H^+(F,S)$, and therefore the inclusion $H^+(F,S) \subset Q_S$ is a homotopy equivalence as well. □

Suppose Σ_i is non-empty for some $i \in \Delta$. Let F_i be the canonical foliation on Σ_i and S_i be the root strip of Σ_i. We will denote by $H^+(F_i, \partial_- S_i)$ the subgroup of $H^+(F_i)$ consisting of homeomorphisms fixed on $\partial_- S_i$.

If $\Sigma_i = \varnothing$, then we will assume that $H^+(F_i, \partial_- S_i) = \{1\}$.

Lemma 5.2. *We have an isomorphism*

$$\pi_0 \ker(\eta) \cong \prod_{i\in\Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

Proof. Evidently, we have a canonical isomorphism

$$\alpha : H^+(F,S) \cong \prod_{i\in\Delta} H^+(F_i, \partial_- S_i), \quad \alpha(h) = (h|_{\Sigma_i})_{i\in\Delta}.$$

Then from Lemma 5.1 we get the following sequence of isomorphisms:

$$\pi_0 \ker(\eta) \cong \pi_0 H^+(F,S) \cong \pi_0 \prod_{i\in\Delta} H^+(F_i, \partial_- S_i) = \prod_{i\in\Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

Lemma is proved. □

Theorem 5.3. 1) *If η is zero homomorphism, then the group $\pi_0 H^+(F)$ is isomorphic to $\prod\limits_{i\in\Delta} \pi_0 H^+(F_i, \partial_- S_i)$.*

2) *Suppose the image of η is $k\mathbb{Z}$ for some $k \geq 1$, so $\Delta = \mathbb{Z}$. Then the group $\pi_0 H^+(F)$ is isomorphic to $\left(\prod\limits_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i)\right) \wr \mathbb{Z}$.*

Proof. 1) The assumption that η is zero homomorphism means that $H^+(F) = \ker(\eta)$, whence we get from Lemma 5.2 that

$$\pi_0 H^+(F) \cong \prod_{i\in\Delta} \pi_0 H^+(F_i, \partial_- S_i).$$

2) Suppose $\operatorname{Im}\eta = k\mathbb{Z}$. Then we have an *epimorphism* $\widehat{\eta} : H^+(F) \to \mathbb{Z}$ defined by $\widehat{\eta}(h) = \eta(h)/k$ and such that

$$h(\Sigma_r) = \Sigma_{r+k\cdot\widehat{\eta}(h)}, \quad r = 0, 1, \ldots, k-1.$$

Let

$$X = \bigcup_{i=0}^{k-1} \Sigma_i, \quad \partial_- X = \bigcup_{i=0}^{k-1} \partial_- S_i,$$

and F_X be the oriented foliation on X induced by F. Denote by $H^+(F_X, \partial_- X)$ the group of homeomorphisms of X fixed on $\partial_- X$ and mapping leaves of F_X onto leaves and preserving their orientation. Then we have a natural isomorphism

$$\prod_{i=0}^{k-1} H^+(F_i, \partial_- S_i) \ \cong\ H^+(F_X, \partial_- X)$$

which yields an isomorphism

$$\prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \ \cong\ \pi_0 H^+(F_X, \partial_- X).$$

Therefore for the proof of Theorem 5.3 we should construct an isomorphism

$$\beta : \pi_0 H^+(F) \ \longrightarrow\ \pi_0 H^+(F_X, \partial_- X) \wr \mathbb{Z} \ \equiv\ Map\Big(\mathbb{Z}, \pi_0 H^+(F_X, \partial_- X)\Big) \rtimes \mathbb{Z}.$$

Fix any $g \in H^+(F)$ with $\widehat{\eta}(g) = 1$. Then

$$g^{-\widehat{\eta}(h)} \circ h\,(\Sigma_i) = \Sigma_i,$$

for all $h \in H^+(F)$ and $i \in \mathbb{Z}$, whence $g^{-\widehat{\eta}(h)} \circ h \in \ker(\eta)$. Thus we get a well-defined function

$$\varphi_h : \mathbb{Z} \to \pi_0 H^+(F_X, \partial_- X), \quad \varphi_h(j) = \left[g^{-j-\widehat{\eta}(h)} \circ h \circ g^j\big|_X \right].$$

Define the following map:

$$\beta : \pi_0 H^+(F) \longrightarrow \pi_0 H^+(F_X, \partial_- X)$$

by the formula

$$\beta(h) = (\varphi_h, \widehat{\eta}(h)), \quad h \in \pi_0 H^+(F).$$

We claim that β is an isomorphism. First notice that the composition operation in $H^+(F_X, \partial_- X) \wr \mathbb{Z}$ is given by the following rule:

$$(\varphi_{h_1}, n) \cdot (\varphi_{h_2}, m) = (\varphi_{h_1}^m \cdot \varphi_{h_2}, n+m),$$

where $\varphi_h^m(j) = \varphi_h(j+m)$.

Proof that β is a homomorphism. Let $h_1, h_2 \in H^+(F)$. Then

$$\begin{aligned}
\beta(h_1) \circ \beta(h_2) &= \big(\varphi_{h_1},\ \widehat{\eta}(h_1)\big) \cdot \big(\varphi_{h_2},\ \widehat{\eta}(h_2)\big) \\
&= \big(\varphi_{h_1}^{\widehat{\eta}(h_2)} \cdot \varphi_{h_2},\ \widehat{\eta}(h_1) + \widehat{\eta}(h_2)\big) \\
&= \Big(\big[g^{-j-\widehat{\eta}(h_1)-\widehat{\eta}(h_2)} \circ h_1 \circ g^{j+\widehat{\eta}(h_2)} \circ g^{-j-\widehat{\eta}(h_2)} \circ h_2 \circ g^j|_X\big],\ \widehat{\eta}(h_1 \circ h_2)\Big) \\
&= \Big(\big[g^{-j-\widehat{\eta}(h_1\circ h_2)} \circ h_1 \circ h_2 \circ g^j|_X\big],\ \widehat{\eta}(h_1 \circ h_2)\Big) \\
&= \big(\varphi_{h_1\circ h_2},\ \widehat{\eta}(h_1 \circ h_2)\big) = \beta(h_1 \circ h_2).
\end{aligned}$$

Proof that β is injective. Let $h \in H^+(F)$ be such that $[h] \in \ker\beta$. We should prove that h is isotopic in $H^+(F)$ to id_Σ.

The assumption $[h] \in \ker\beta$ means that $\beta(h) = (\varphi_h, \widehat{\eta}(h)) = (\varepsilon, 0)$, where $\varepsilon : \mathbb{Z} \to [\mathrm{id}_X]$ is the constant map into the unit of $\pi_0 H^+(F_X, \partial_- X)$. In particular, since $\eta(h) = 0$, we get from Lemma 5.1 that h is isotopic in $H^+(F)$ to a homeomorphism fixed on S. Therefore we can assume that h itself is fixed on S, that is $h \in H^+(F, S)$. Then

$$\varphi_h(j) = \left[g^{-j} \circ h \circ g^j|_X\right] = \varepsilon(j) = [\mathrm{id}_X] \in \pi_0 H^+(F_X, \partial_- X) \tag{5.3}$$

for each $j \in \mathbb{Z}$. In other words, $g^{-j} \circ h \circ g^j|_X$ is isotopic to id_X relatively $\partial_- X$.

It suffices to prove that for each $i \in \mathbb{Z}$ the restriction $h|_{\Sigma_i}$ is isotopic in $H^+(F_i, \partial_- S_i)$ to id_{Σ_i} relatively to $\partial_- S_i$.

Write $i = r + jk$ for a unique $r \in \{0, k-1\}$. Then we have the following commutative diagram:

Therefore, we get from (5.3) that $[h|_{\Sigma_i}] = [\mathrm{id}_{\Sigma_i}] \in H^+(F_i, \partial_- S_i)$. Hence h is isotopic ot id_Σ in $H^+(F)$.

Proof that β is surjective. Let $(\varphi, n) \in \pi_0 H^+(F_X, \partial_- X) \wr \mathbb{Z}$. For each $j \in \mathbb{Z}$ fix a homeomorphism $h_j \in H^+(F_X, \partial_- X)$ such that $[h_j] = \phi(j) \in \pi_0 H^+(F_X, \partial_- X)$. Now define the following homeomorphism $\widehat{h}$ of Σ by the formula:

$$\widehat{h} = \begin{cases} \mathrm{id}_S, & \text{on } S, \\ [g^j \circ h_j \circ g^{-j}] & \text{on } g^j(X) \end{cases}$$

and put $h = g^n \circ \widehat{h}$. Then it is easy to see that $\beta([h]) = (\phi, n)$, whence β is surjective. Thus β is an isomorphism. □

6. Proof of Theorem 4.5

We should prove that $\mathcal{P} = \mathcal{G}'$.

1. First we will show that $\mathcal{G}' \subset \mathcal{P}$.

Let $G \in \mathcal{G}'$, so G has a representation $\xi(G)$ in the class $\mathcal{G}$ of finite height $k = h(\xi(G))$. We have to show that there exists a striped surface $\Sigma \in \mathfrak{F}$ with canonical foliation F such that $G \cong \pi_0 H^+(F)$.

If $k = h(\xi(G)) = 0$, then G is the unit group $\{1\}$ and $\xi(G) = \{1\}$. Let S be an admissible model strip with $\partial_- S = A_{[1]} \times \{-1\}$ and $\partial_+ S = \varnothing$. Then $S \in \mathfrak{F}$. Let also F be the canonical foliation on S. Then

$$\pi_0 H^+(F) = \{1\} = G,$$

i.e. $G \in \mathcal{P}$.

Suppose that we have established our statement for all k being less than some $\bar{k} > 0$. Let us prove it for $k = \bar{k}$. It follows from Definition 4.3 that either

(i) $\xi(G) = \prod\limits_{i \in \mathbb{N}} A_i$ where each group A_i has a representation $\xi(A_i)$ in the class $\mathcal{G}$ of height $h(\xi(A_i)) < k$, or

(ii) $\xi(G) = A \wr \mathbb{Z}$, and A has a representation $\xi(A)$ in the class $\mathcal{G}$ of height $h(\xi(A)) < k$.

In the case (i) due to the inductive assumption for each $i \in \mathbb{N}$ there exists a striped surface $\Sigma_i \in \mathfrak{F}$ with foliations F_i such that $A_i = \pi_0 H^+(F_i)$.

Let S be an admissible model strip with $\partial_- S = A_{[1]} \times \{-1\}$ and $\partial_+ S = A_{\mathbb{N}} \times \{1\}$, and S_i be the root strip of Σ_i, $i \in \mathbb{N}$. Define the striped surface

$$\Sigma \;=\; S \bigcup_{\partial_+ S} \left(\bigcup_{i \in \mathbb{N}} \Sigma_i\right)$$

obtained by identifying $\partial_- S_i \subset \Sigma_i$ with $J_i \times \{1\} \subset \partial_+ S$. Then by Theorem 5.3 η is a trivial homomorphism, and $\pi_0 H^+(F) \cong \prod_{i \in \mathbb{N}} \pi_0 H^+(F_i) \cong \prod\limits_{i \in \mathbb{N}} A_i \cong G$. So $G \in \mathcal{P}$.

In the case (ii) again by inductive assumption there exists a striped surface $\widehat{\Sigma} \in \mathfrak{F}$ with a canonical foliation $\widehat{F}$ such that $A = \pi_0 H^+(\widehat{F})$.

Take countably many copies $\widehat{\Sigma}_i$, $i \in \mathbb{Z}$, of $\widehat{\Sigma}$. Let $\widehat{S}_i$ be the root strip of $\widehat{\Sigma}_i$ and $\widehat{F}_i$ be the canonical foliation on $\widehat{\Sigma}_i$.

Let also S be an admissible model strip with $\partial_- S = A_{[1]} \times \{-1\}$ and $\partial_+ S = A_{\mathbb{Z}} \times \{1\}$. Define the following striped surface:

$$\Sigma \;=\; S \mathop{\cup}_{\partial_+ S} \Bigl(\mathop{\cup}_{i\in\mathbb{N}} \widehat{\Sigma}_i \Bigr).$$

Obtained by gluing each $\widehat{\Sigma}_i$ to S by identifying $\partial_- \widehat{S}_i \subset \widehat{\Sigma}_i$ with $J_i \times \{1\} \subset \partial_+ S$, $i \in \mathbb{Z}$.

Then for every pair $i, j \in \mathbb{Z}$ there exists $h \in H^+(F)$ such that $h(\widehat{\Sigma}_i) = \widehat{\Sigma}_j$, whence the homomorphism η, see (5.1) is surjective. Hence by Theorem 5.3

$$\pi_0 H^+(F) \cong \pi_0 H^+(\widehat{F}) \wr \mathbb{Z} \cong A \wr \mathbb{Z} \cong G.$$

Thus, $G \in \mathcal{P}$ and so $\mathcal{G}' \subset \mathcal{P}$.

2. Conversely, let us show that $\mathcal{P} \subset \mathcal{G}'$.

Let $\Sigma \in \mathfrak{F}$ be a striped surface presented in the form (2.1) with canonical foliation F and such $\operatorname{diam} \Gamma(\Sigma) = k$. We should prove that $\pi_0 H^+(F)$ has a finite presentation in the class $\mathcal{G}$, which means that $\pi_0 H^+(F) \in \mathcal{G}'$.

If $k = 0$, then Σ is an admissible model strip with

$$\partial_- \Sigma = A_{[1]} \times \{-1\}, \quad \partial_+ \Sigma = A_\alpha, \quad \alpha \in \{[0], [1], \ldots, \mathbb{N}, -\mathbb{N}, \mathbb{Z}\}.$$

Then it easily follows from Theorem 5.3 that $\pi_0 H^+(F) \cong \mathbb{Z} \cong \{1\} \wr \mathbb{Z}$ if $\alpha = \mathbb{Z}$, and $\pi_0 H^+(F) \cong \{1\}$ otherwise. In both cases $\pi_0 H^+(F) \in \mathcal{G}$.

Suppose that we have established our statement for all k being less than some $\bar{k} > 0$. We should prove it for $k = \bar{k}$. Let

$$\Sigma \;=\; S \mathop{\cup}_{\partial_+ S} \Bigl(\bigcup_{i\in\Delta} \Sigma_i \Bigr) \;\in\; \mathfrak{F}$$

be such that $\Gamma(\Sigma)$ has diameter k. Then $\Gamma(\Sigma_i)$ has diameter less than k, and so by inductive assumption $\pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G}$. Moreover, according to Theorem 5.3 we have that

(i) if $\operatorname{image}(\eta) = 0$, then $\pi_0 H^+(F) \cong \prod_{i\in\Delta} \pi_0 H^+(F_i, \partial_- S_i) \in \mathcal{G}$,

(ii) if $\operatorname{image}(\eta) = k\mathbb{Z}$, then $\pi_0 H^+(F) \cong \Bigl(\prod_{i=0}^{k-1} \pi_0 H^+(F_i, \partial_- S_i) \Bigr) \wr \mathbb{Z} \in \mathcal{G}$.

Thus $\mathcal{P} \subset \mathcal{G}'$, and so $\mathcal{P} = \mathcal{G}'$. Theorem 5.3 completed.

References

1. A. V. Bolsinov and A. T. Fomenko, *Vvedenie v topologiyu integriruemykh gamiltonovykh sistem (introduction to the topology of integrable hamiltonian systems)*, Nauka, Moscow, 1997 (Russian).
2. William M. Boothby, *The topology of regular curve families with multiple saddle points*, Amer. J. Math. **73** (1951), 405–438.
3. D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.
4. James A. Jenkins and Marston Morse, *Contour equivalent pseudoharmonic functions and pseudoconjugates*, Amer. J. Math. **74** (1952), 23–51.
5. Wilfred Kaplan, *Regular curve-families filling the plane, I*, Duke Math. J. **7** (1940), 154–185.
6. Wilfred Kaplan, *Regular curve-families filling the plane, II*, Duke Math J. **8** (1941), 11–46.
7. K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York-London; Panstwowe Wydawnictwo Naukowe, Warsaw, 1966.
8. Sergiy Maksymenko and Eugene Polulyakh, *Foliations with non-compact leaves on surfaces*, Proceedings of Geometric Center **8** (2015), no. 3–4, 17–30, arXiv:1512.07809.

9. A. A. Oshemkov, *Morse functions on two-dimensional surfaces. Coding of singularities*, Proc. Steklov Inst. Math. **205** (1995), no. 4, 119–127.
10. E. Polulyakh and I. Yurchuk, *On the pseudo-harmonic functions defined on a disk*, vol. 80, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 2009 (Ukrainian).
11. E. A. Polulyakh, *Kronrod-Reeb graphs of functions on noncompact two-dimensional surfaces. I*, Ukrainian Math. J. **67** (2015), no. 3, 431–454.
12. A. O. Prishlyak, *Conjugacy of morse functions on surfaces with values on the line and circle*, Ukrainian Math. J. **52** (2000), no. 10, 1623–1627.
13. A. O. Prishlyak, *Morse functions with finite number of singularities on a plane*, Methods Funct. Anal. Topology **8** (2002), no. 1, 75–78.
14. V. V. Sharko, *Smooth and topological equivalence of functions on surfaces*, Ukrainian Math. J. **55** (2003), no. 5, 832–846.
15. V. V. Sharko, *Smooth functions on non-compact surfaces*, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. **3** (2006), no. 3, 443–473, `arXiv:0709.2511`.
16. V. V. Sharko and Yu. Yu. Soroka, *Topological equivalence to a projection*, Methods Funct. Anal. Topology **21** (2015), no. 1, 3–5.
17. Hassler Whitney, *Regular families of curves*, Ann. of Math. (2) **34** (1933), no. 2, 244–270.

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail address: `sorokayulya15@gmail.com`

Received 31/03/2016