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L-DUNFORD-PETTIS PROPERTY IN BANACH SPACES

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Abstract. In this paper, we introduce and study the concept of L-Dunford-Pettis sets and L-Dunford-Pettis property in Banach spaces. Next, we give a characterization of the L-Dunford-Pettis property with respect to some well-known geometric properties of Banach spaces. Finally, some complementability of operators on Banach spaces with the L-Dunford-Pettis property are also investigated.

1. Introduction and notation

A norm bounded subset A of a Banach space X is called Dunford-Pettis (DP for short) if every weakly null sequence (f_n) in X' converge uniformly to zero on A, that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |f_n(x)| = 0$.

An operator T between two Banach spaces X and Y is completely continuous if T maps weakly null sequences into norm null ones.

Recall from [11], that an operator $T : X \to Y$ between two Banach spaces is Dunford-Pettis completely continuous (abb. $DPcc$) if it carries a weakly null sequence, which is a DP set in X to norm null ones in Y. It is clear that every completely continuous operator is DPcc. Also every weakly compact operator is DPcc (see Corollary 1.1 of [11]).

A Banach space X has:

- a relatively compact Dunford-Pettis property (DPrcP for short) if every Dunford-Pettis set in X is relatively compact [5]. For example, every Schur spaces have the DPrcP.
- a Grothendieck property (or a Banach space X is a Grothendieck space) if weak* and weak convergence of sequences in X' coincide. For example, each reflexive space is a Grothendieck space.
- a Dunford-Pettis property (DP property for short) if every weakly compact operator T from X into another Banach space Y is completely continuous, equivalently, if every relatively weakly compact subset of X is DP.
- a reciprocal Dunford-Pettis property (RDP property for short) if every completely continuous operator on X is weakly compact.

A subspace X_1 of a Banach space X is complemented if there exists a projection P from X to X_1 (see page 9 of [2]).

Recall from [1], that a Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$.

We denote by c_0, ℓ^1, and ℓ^∞ the Banach spaces of all sequences converging to zero, all absolutely summable sequences, and all bounded sequences, respectively.

Let us recall that a norm bounded subset A of a Banach space X' is called L-set if every weakly null sequence (x_n) in X converge uniformly to zero on A, that is,

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$\lim_{n\to\infty} \sup_{f\in A} |f(x_n)| = 0$. Note also that a Banach space X has the RDP property if and only if every L-set in X' is relatively weakly compact.

In his paper, G. Emmanuelle in [4] used the concept of L-set to characterize Banach spaces not containing ℓ^1, and gave several consequences concerning Dunford-Pettis sets. Later, the idea of L-set is also used to establish a dual characterization of the Dunford-Pettis property [6].

The aim of this paper is to introduce and study the notion of L-Dunford-Pettis set in a Banach space, which is related to the Dunford-Pettis set (Definition 2.1), and note that every L-set in a topological dual of a Banach space is L-Dunford-Pettis set (Proposition 2.3). After that, we introduce the L-Dunford-Pettis property in Banach space which is shared by those Banach spaces whose L-Dunford-Pettis subsets of his topological dual are relatively weakly compact (Definition 2.6). Next, we obtain some important consequences. More precisely, a characterizations of L-Dunford-Pettis property in Banach spaces in terms of DPcc and weakly compact operators (Theorem 2.7), the relation between L-Dunford-Pettis property with DP and Grothendieck properties (Theorem 2.8), a new characterizations of Banach space with DPrcP (resp, reflexive Banach space) (Theorem 2.5) (resp, Corollary 2.10). Finally, we investigate the complementability of the class of weakly compact operators from X into ℓ^∞ in the class of DPcc from X into ℓ^∞ (Theorem 2.13 and Corollary 2.14).

The notations and terminologies are standard. We use the symbols X, Y for arbitrary Banach spaces. We denoted the closed unit ball of X by B_X, the topological dual of X by X' and $T' : Y' \to X'$ refers to the adjoint of a bounded linear operator $T : X \to Y$. We refer the reader for undefined terminologies to the references [1, 8, 9].

2. Main results

Definition 2.1. Let X be a Banach space. A norm bounded subset A of the dual space X' is called an L-Dunford-Pettis set, if every weakly null sequence (x_n), which is a DP set in X converges uniformly to zero on A, that is, $\lim_{n\to\infty} \sup_{f\in A} |f(x_n)| = 0$.

For a proof of the next Proposition, we need the following Lemma which is just Lemma 1.3 of [11].

Lemma 2.2. *A sequence (x_n) in X is DP if and only if $f_n(x_n) \to 0$ as $n \to \infty$ for every weakly null sequence (f_n) in X'.*

The following Proposition gives some additional properties of L-Dunford-Pettis sets in a topological dual Banach space.

Proposition 2.3. *Let X be a Banach space. Then*

(1) *every subset of an L-Dunford-Pettis set in X' is L-Dunford-Pettis,*
(2) *every L-set in X' is L-Dunford-Pettis,*
(3) *relatively weakly compact subset of X' is L-Dunford-Pettis,*
(4) *absolutely closed convex hull of an L-Dunford-Pettis set in X' is L-Dunford-Pettis.*

Proof. (1) and (2) are obvious.

(3) Suppose $A \subset X'$ is relatively weakly compact but it is not an L-Dunford-Pettis set. Then, there exists a weakly null sequence (x_n), which is a DP set in X, a sequence (f_n) in A and an $\epsilon > 0$ such that $|f_n(x_n)| > \epsilon$ for all integer n. As A is relatively weakly compact, there exists a subsequence (g_n) of (f_n) that converges weakly to an element g in X'. But from

$$|g_n(x_n)| \le |(g_n - g)(x_n)| + |g(x_n)|$$

and Lemma 2.2, we obtain that $|g_n(x_n)| \to 0$ as $n \to \infty$. This is a contradiction.

(4) Let A be a L-Dunford-Pettis set in X', and (x_n) be a weakly null sequence, which is a DP set in X. Since

$$\sup_{f \in a\overline{co}(A)} |f(x_n)| = \sup_{f \in A} |f(x_n)|$$

for each n, where $a\overline{co}(A) = \overline{\{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \forall i, \sum_{i=1}^{n} |\lambda_i| \leq 1\}}$ is the absolutely closed convex hull of A (see [1, pp. 148, 151]), then it is clear that $a\overline{co}(A)$ is L-Dunford-Pettis set in X'. □

We need the following Lemma which is just Lemma 1.2 of [11].

Lemma 2.4. *A Banach space X has the DPrcP if and only if any weakly null sequence, which is a DP set in X is norm null.*

From Lemma 2.4, we obtain the following characterization of DPrcP in a Banach space in terms of an L-Dunford-Pettis set of his topological dual.

Theorem 2.5. *A Banach space X has the DPrcP if and only if every bounded subset of X' is an L-Dunford-Pettis set.*

Proof. ($\Leftarrow$) Let (x_n) be a weakly null sequence, which is a DP set in X. As

$$\|x_n\| = \sup_{f \in B_{X'}} |f(x_n)|$$

for each n, and by our hypothesis, we see that $\|x_n\| \to 0$ as $n \to \infty$. By Lemma 2.4 we deduce that X has the DPrcP.

($\Rightarrow$) Assume by way of contradiction that there exist a bounded subset A, which is not an L-Dunford-Pettis set of X'. Then, there exists a weakly null sequence (x_n), which is a Dunford-Pettis set of X such that $\sup_{f \in A} |f(x_n)| > \epsilon > 0$ for some $\epsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \epsilon$.

On the other hand, since $(f_n) \subset A$, there exist some $M > 0$ such that $\|f_n\|_{X'} \leq M$ for all n. Thus,

$$|f_n(x_n)| \leq M \|x_n\|$$

for each n, then by our hypothesis and Lemma 2.4, we have $|f_n(x_n)| \to 0$ as $n \to \infty$, which is impossible. This completes the proof. □

Remark 1. Note by Proposition 2.3 assertion (3) that every relatively weakly compact subset of a topological dual Banach space is L-Dunford-Pettis. The converse is not true in general. In fact, the closed unit ball B_{ℓ^∞} of ℓ^∞ is L-Dunford-Pettis set (see Theorem 2.5), but it is not relatively weakly compact.

We make the following definition.

Definition 2.6. A Banach space X has the L-Dunford-Pettis property, if every L-Dunford-Pettis set in X' is relatively weakly compact.

As is known a DPcc operator is not weakly compact in general. For example, the identity operator $Id_{\ell^1} : \ell^1 \to \ell^1$ is DPcc, but it is not weakly compact.

In the following Theorem, we give a characterizations of L-Dunford-Pettis property of Banach space in terms of DPcc and weakly compact operators.

Theorem 2.7. *Let X be a Banach space, then the following assertions are equivalent:*

(1) *X has the L-Dunford-Pettis property,*
(2) *for each Banach space Y, every DPcc operator from X into Y is weakly compact,*
(3) *every DPcc operator from X into ℓ^∞ is weakly compact.*

Proof. (1) ⇒ (2) Suppose that X has the L-Dunford-Pettis property and $T : X \to Y$ is DPcc operator. Thus $T'(B_{Y'})$ is an L-Dunford-Pettis set in X'. So by hypothesis, it is relatively weakly compact and T is a weakly compact operator.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) If X does not have the L-Dunford-Pettis property, there exists an L-Dunford-Pettis subset A of X' that is not relatively weakly compact. So there is a sequence $(f_n) \subseteq A$ with no weakly convergent subsequence. Now, we show that the operator $T : X \to \ell^\infty$ defined by $T(x) = (f_n(x))$ for all $x \in X$ is DPcc but it is not weakly compact. As $(f_n) \subseteq A$ is L-Dunford-Pettis set, for every weakly null sequence (x_m), which is a DP set in X we have

$$\|T(x_m)\| = \sup_n |f_n(x_m)| \to 0, \text{ as } m \to \infty,$$

so T is a Dunford-Pettis completely continuous operator. We have $T'((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$ for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)'$. If e'_n is the usual basis element in ℓ^1 then $T'(e'_n) = f_n$, for all $n \in N$. Thus, T' is not a weakly compact operator and neither is T. This finishes the proof. □

Theorem 2.8. *Let E be a Banach lattice.*
If E has both properties of DP and Grothendieck, then it has the L-Dunford-Pettis property.

Proof. Suppose that $T : E \to Y$ is DPcc operator. As E has the DP property, it follows from Theorem 1.5 [11] that T is completely continuous.

On the other hand, ℓ^1 is not a Grothendieck space and Grothendieck property is carried by complemented subspaces. Hence the Grothendieck space E does not have any complemented copy of ℓ^1. By [10], E has the RDP property and so the completely continuous operator T is weakly compact. From Theorem 2.7 we deduce that E has the L-Dunford-Pettis property. □

Remark 2. Since ℓ^∞ has the Grothendieck and DP properties, it has the L-Dunford-Pettis property.

Let us recall that K is an infinite compact Hausdorff space if it is a compact Hausdorff space, which contains infinitely many points.

For an infinite compact Hausdorff space K, we have the following result for the Banach space $C(K)$ of all continuous functions on K with supremum norm.

Corollary 2.9. *If $C(K)$ contains no complemented copy of c_0, then it has L-Dunford-Pettis property.*

Proof. Since $C(K)$ contains no complemented copy of c_0, it is a Grothendieck space [3]. On the other hand, $C(K)$ be a Banach lattice with the DP property, and by Theorem 2.8 we deduce that $C(K)$ has L-Dunford-Pettis property. □

Corollary 2.10. *A DPrc space has the L-Dunford-Pettis property if and only if it is reflexive.*

Proof. (⇒) If a Banach space X has the DPrcP, then by Theorem 1.3 of [11], the identity operator Id_X on X is DPcc. As X has the L-Dunford-Pettis property, it follows from Theorem 2.7 that Id_X is weakly compact, and hence X is reflexive.

(⇐) Obvious. □

Remark 3. Note that the Banach space ℓ^1 is not reflexive and has the DPrcP, then from Corollary 2.10, we conclude that ℓ^1 does not have the L-Dunford-Pettis property.

Theorem 2.11. *If a Banach space X has the L-Dunford-Pettis property, then every complemented subspace of X has the L-Dunford-Pettis property.*

Proof. Consider a complemented subspace X_1 of X and a projection map $P : X \to X_1$. Suppose $T : X_1 \to \ell^\infty$ is DPcc operator, then $TP : X \to \ell^\infty$ is also DPcc. Since X has L-Dunford-Pettis, by Theorem 2.7, TP is weakly compact. Hence T is weakly compact, also from Theorem 2.7 we conclude that X_1 has L-Dunford-Pettis, and this completes the proof. □

Let X be a Banach space. We denote by $L(X, \ell^\infty)$ the class of all bounded linear operators from X into ℓ^∞, by $W(X, \ell^\infty)$ the class of all weakly compact operators from X into ℓ^∞, and by $DPcc(X, \ell^\infty)$ the class of all Dunford-Pettis completely continuous operators from X into ℓ^∞.

Recall that Bahreini [2] investigated the complementability of $W(X, \ell^\infty)$ in $L(X, \ell^\infty)$, and she proved that if X is not a reflexive Banach space, then $W(X, \ell^\infty)$ is not complemented in $L(X, \ell^\infty)$. In the next Theorem, we establish the complementability of $W(X, \ell^\infty)$ in $DPcc(X, \ell^\infty)$.

We need the following lemma of [7] .

Lemma 2.12. *Let X be a separable Banach space, and $\phi : \ell^\infty \to L(X, \ell^\infty)$ is a bounded linear operator with $\phi(e_n) = 0$ for all n, where e_n is the usual basis element in c_0. Then there is an infinite subset M of N such that for each $\alpha \in \ell^\infty(M)$, $\phi(\alpha) = 0$, where $\ell^\infty(M)$ is the set of all $\alpha = (\alpha_n) \in \ell^\infty$ with $\alpha_n = 0$ for each $n \notin M$.*

Theorem 2.13. *If X does not have the L-Dunford-Pettis property, then $W(X, \ell^\infty)$ is not complemented in $DPcc(X, \ell^\infty)$.*

Proof. Consider a subset A of X' that is L-Dunford-Pettis but it is not relatively weakly compact. So there is a sequence (f_n) in A such that has no weakly convergent subsequence. Hence $S : X \to \ell^\infty$ defined by $S(x) = (f_n(x))$ is an DPcc operator but it is not weakly compact. Choose a bounded sequence (x_n) in B_X such that $(S(x_n))$ has no weakly convergent subsequence. Let $X_1 = \langle x_n \rangle$, the closed linear span of the sequence (x_n) in X. It follows that X_1 is a separable subspace of X such that S/X_1 is not a weakly compact operator. If $g_n = f_n/X_1$, we have $(g_n) \subseteq X_1'$ is bounded and has no weakly convergent subsequence.

Now define the operator $T : \ell^\infty \to DPcc(X, \ell^\infty)$ by $T(\alpha)(x) = (\alpha_n f_n(x))$, where $x \in X$ and $\alpha = (\alpha_n) \in \ell^\infty$. Then

$$\|T(\alpha)(x)\| = \sup_n |\alpha_n f_n(x)| \leq \|\alpha\| \, . \, \|f_n\| \, . \, \|x\| < \infty.$$

We claim that $T(\alpha) \in DPcc(X, \ell^\infty)$ for each $\alpha = (\alpha_n) \in \ell^\infty$.

Let $\alpha = (\alpha_n) \in \ell^\infty$ and let (x_m) be a weakly null sequence, which is a DP set in X. As (f_n) is L-Dunford-Pettis set $\sup_n |f_n(x_m)| \to 0$ as $m \to \infty$. So we have

$$\|T(\alpha)(x_m)\| = \sup_n |\alpha_n f_n(x_m)| \leq \|\alpha\| \, . \sup_n |f_n(x_m)| \to 0,$$

as $m \to \infty$. Then this finishes the proof that T is a well-defined operator from ℓ^∞ into $DPcc(X, \ell^\infty)$.

Let $R : DPcc(X, \ell^\infty) \to DPcc(X_1, \ell^\infty)$ be the restriction map and define

$$\phi : \ell^\infty \to DPcc(X_1, \ell^\infty) \quad \text{by} \quad \phi = RT.$$

Now suppose that $W(X, \ell^\infty)$ is complemented in $DPcc(X, \ell^\infty)$ and

$$P : DPcc(X, \ell^\infty) \to W(X, \ell^\infty)$$

is a projection. Define $\psi : \ell^\infty \to W(X_1, \ell^\infty)$ by $\psi = RPT$. Note that as $T(e_n)$ is a one rank operator, we have $T(e_n) \in W(X, \ell^\infty)$. Hence

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n)$$

for all $n \in N$. From Lemma 2.12, there is an infinite set $M \subseteq N$ such that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in \ell^\infty(M)$. Thus $\phi(\chi_M)$ is a weakly compact operator. On the other hand, if e'_n is the usual basis element of ℓ^1, for each $x \in X_1$ and each $n \in M$, we have

$$(\phi(\chi_M))'(e'_n)(x) = f_n(x).$$

Therefore $(\phi(\chi_M))'(e'_n) = f_n/X_1 = g_n$ for all $n \in M$. Thus $(\phi(\chi_M))'$ is not a weakly compact operator and neither is $\phi(\chi_M)$. This contradiction ends the proof. □

As a consequence of Theorem 2.7 and Theorem 2.13, we obtain the following result.

Corollary 2.14. *Let X be a Banach space. Then the following assertions are equivalent:*

(1) *X has the L-Dunford-Pettis property,*
(2) $W(X, \ell^\infty) = DPcc(X, \ell^\infty)$,
(3) *$W(X, \ell^\infty)$ is complemented in $DPcc(X, \ell^\infty)$.*

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