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ON NEW INVERSE SPECTRAL PROBLEMS FOR WEIGHTED GRAPHS

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Abstract. In this paper, we consider various new inverse spectral problems (ISP) for metric graphs, using maximal eigen values of the adjacency matrix of the graph and its subgraphs as well as the corresponding eigen vectors or some of their components as spectral data. We give examples of spectral data that uniquely determine the metric on the graph. Effective algorithms for solving the considered ISP are given.

1. Introduction

At the present time, spectral theory of matrices is a well developed part of linear algebra. Together with general spectral theory of matrices, spectral properties of special classes of matrices have been well studied. These classes include the class of Jacobi matrices, matrices with nonnegative entries, oscillation matrices, adjacency matrices of metric (weighted) graphs, etc., see, for example, [7]. Due to various applications, a number of inverse spectral problems for various classes of matrices were posed and solved in the literature. Such problems deal with recovering a matrix or its part from its spectrum or spectrum of its submatrices [1, 2]. Such problems also include cases where the values of entries at some fixed places of the sought matrix are known. For example, it is well known that a Jacobi matrix J can be recovered from its spectrum and spectrum of a submatrix obtained by removing from J the last row and the last column, or from the $n \times n$-submatrix $\hat{J}$ obtained from the Jacobi $2n \times 2n$-matrix J by removing n lower rows and n rightmost columns and the complete spectrum of the matrix J [6]. There are many results on problems of finding matrices with a fixed structure (adjacency matrices of graphs with incomplete information on edge weights) from a given spectrum or from a spectrum that satisfies certain conditions on multiplicities of eigen values [7, 3]. However, no complete theory for inverse spectral problems has been developed.

For matrices, an inverse spectral problem in a general setting can be formulated as follows. Suppose we only know elements of a matrix A that are located at certain places, and the rest of the elements are to be found. Let $\mathfrak{A}$ be a set of such matrices, and some functionals $f_1(A), \dots, f_k(A)$ be defined on $\mathfrak{A}$. A collection of values of these functionals for a matrix A is considered to be known and will be called input data for the inverse problem. They will be denoted by $ID(A) = \{f_1(A), \dots, f_k(A)\}$. The inverse problem for a matrix $A \in \mathfrak{A}$ consists of finding all nonnegative elements of the matrix A, or, if such a recovering is not unique, of all matrices $\hat{A}$ from the set $\mathfrak{A}$ such that $ID(\hat{A}) = ID(A)$. An inverse problem for finding a matrix A in the set $\mathfrak{A}$ from the data $ID(A)$ is called well-posed if such a problem has a finite nonempty set of matrix solutions in the set $\mathfrak{A}$.

An example of a well-posed inverse problem is the problem of finding diagonal elements of a matrix A from its spectrum, that is, from a collection of all its eigen values,

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counting multiplicities [3]. An example of an "ill-posed inverse problem" is the problem of recovering all elements of an arbitrary matrix from its spectrum.

Among the functionals $\mathrm{ID}(A)$ that define the initial data for inverse spectral problems for matrices there are certain particular collections of eigen values of the matrix A and eigen values of some submatrices A_τ constructed from the matrix A. Additional functionals could be some components of the eigen vectors corresponding to the eigen values, or other spectral characteristics. In such cases, $\mathrm{ID}(A)$ will be called spectral data and denoted by $\mathrm{SD}(A)$. An inverse problem with initial spectral data $\mathrm{SD}(A)$ will be briefly called ISP.

An important spectral data are the largest eigen values of the matrix or corresponding eigen values of certain submatrices. This the case, first of all, for nonnegative irreducible symmetric matrices. Since the works of Frobenius, its well known that knowledge of a maximal eigen value and the corresponding eigen vector of such a matrix helps to find additional information on elements of the matrix. As far as the adjacency matrix of a graph is concerned, the largest eigen values of the matrix is called index of the graph. This notion is important in the theory of graphs. For oscillation matrices, the corresponding eigen value is directly related to the principle (the least) frequency of oscillation of the mechanical system described by the initial oscillation matrix [4].

In this paper, we consider a number of new formulations for inverse spectral problems, ISP, for weighted graphs such that certain SD uniquely define the weights on all the edges and the vertices. Since an arbitrary connected graph admits an indexing of the vertices such that, as vertices are inductively removed the resulting subgraphs remain connected, the SDV in Definition 5 is defined to be indices of these subgraphs, together with the corresponding eigen vectors.

For graphs that are trees, we consider several definitions of SD (Definition 8, Remarks 5, 6) that lead to uniqueness of solution of the IP (Theorem 3).

If a connected graph is not a tree, then an inductive removal of vertices will lead to a subgraph that will be a tree. Such a subgraph will be called a skeleton of the initial graph. For such a case, Definition 9 gives SD in the form of SDT for a skeleton of the graph, together with indices of the graph and indices of the subgraphs obtained in the course of removal of the vertices. Such SD imply uniqueness of the IP (Theorem 4).

In fact, the considered ISP for weighted graphs deal with their adjacency matrices that are irreducible symmetric matrices with nonnegative entries off the main diagonal. Each such a matrix can be regarded as an adjacency matrix of a weighed connected graph. Hence the obtained results can be regarded as new formulations of ISP for such matrices.

2. Some facts from the theory of graphs and nonnegative matrices

Throughout this paper, by a graph $G = G(V, E)$ we mean a simple undirected graph (no loops, no multiple edges) with a nonempty set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and a set of edges $E = \{e_1, e_2, \ldots, e_m\}$, where each edge $e \in E$ corresponds to two distinct vertices $v_1(e), v_2(e)$ that connects them. Hence, an edge e, in an undirected graph, can be identified with an unordered pair of vertices, (v_1, v_2), connected by the edge. A graph $G_1 = G(V_1, G_1)$ is called a subgraph of the graph $G = G(V, E)$, if $V_1 \subset V$ and $E_1 \subset E$. By subgraph $G(V_1)$ on vertices $V_1 \subset V$, we mean a vertex-induced subgraph of G that contains all edges of G with both ends in V_1. By a subgraph $G(E_1)$ on edges $E_1 \subset E$, we mean an edge-induced subgraph of G that contains all the edges from E_1 with their endpoints and no other vertices. Recall that a vertex v is called a leaf vertex, if its degree equals one. Hence, for example, $G(V, E \setminus \{e\}) \neq G(E \setminus \{e\})$, if the edge e comes from a leaf vertex.

A graph $G(V, E)$ is called weighted if the is a real-valued function w defined on the sets V and E such that to each edge $e \in E$ there corresponds a positive number $w(e)$, a wight of the edge e, end to every vertex $v \in V$ there corresponds a real number $w(v)$, a weight of the vertex v. A weighted graph will be denoted by $G(V, E, w)$. If the weight of each vertex is zero, then the weighted graph is call a metric graph, and the number $w(e)$ is a weight of the edge e, or its length.

If all vertices of a graph are indexed with natural numbers $1, 2, \dots, n$, then, to such a graph, one assigns a square symmetric matrix $A(G) = \|a_{ij}\|_{i,j=1}^{n}$, called an adjacency matrix of the graph G, as follows: $a_{ij} = w(e_{i,j})$, where the edge $e_{i,j}$ connects the vertex indexed with i and the vertex indexed with j, if $i \neq j$. If there is no such an edge, then $a_{ij} = 0$. Diagonal elements are defined by $a_{ii} = w(v_i)$, that is, the weights are weights of the vertices with the corresponding indices.

The largest eigen value of the adjacency matrix of a graph $G(V, E, w)$ is called an index of the graph and is denoted by $\operatorname{ind} G$. It is clear that a change of indexing of the vertices of a weighted graph leads to a certain permutation of rows and corresponding columns of the adjacency matrix. Of course, the index of the graph does not depend on a particular indexing of vertices of a weighted graph.

A sequence of edges $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$ is called a path that connects the vertex v_1 with the vertex v_k. A path is a closed, if $v_k = v_1$. A path is called a chain, if all its edges are distinct, and it is a simple chain, if all its vertices are distinct. A closed simple chain is called a simple cycle. A graph G is called connected, if each pair of vertices can be connected with a path.

A graph is connected if and only if its adjacency matrix is irreducible. The length of a path (a chain, a simple chain, respectively) is given by the number of edges needed to be passed. A connected graph with no cycles is called a tree. A subgraph $G(V, E_1)$, which is a tree, is called a skeleton of the graph $G(V, E)$. The initial graph $G(V, E)$ can be obtained from a skeleton by supplementing it with edges in $E \setminus E_1$. Each graph has a skeleton.

Definition 1. A vertex v of a simple connected graph $G(V, E)$ is called *admissible* for removal, if the subgraph $G(V \setminus \{v\})$ is connected. An edge e is called *admissible* for removal, if the subgraph $G(E \setminus \{e\})$ is connected.

Proposition 1. *Let $G(V, E)$ be a connected graph.*

1) *An edge $e \in E$ is admissible for removal if and only if one of the vertices connected with this edge is a leaf vertex or else the graph $G(V, E)$ has a cycle that contains the edge e.*
2) *A vertex $v \in V$ is admissible for removal if and only if this is a leaf vertex of the graph $G(V, E)$ or else each edge that comes out of the vertex v is a part of a cycle of the graph $G(V, E)$.*

Proof. The proof directly follows from Definition 1. □

Remark 1. In a connected graph, admissible for removal are only leaf vertices and edges that connect such leaf vertices.

Proposition 2. *Every simple connected graph $G(V, E)$ that has more than two vertices always has at least two vertices admissible for removal, and at least two edges admissible for removal.*

Proof. If a connected graph $G(V, E)$ has a cycle, then each edge of the cycle is admissible for removal by Proposition 1. If the graph has no cycles, then it is a tree with more than two vertices, which means that there are at least two leaf vertices. By Remark 1, the graph has at least two edges admissible for removal.

Consider now removal of vertices. Let $L = (v_1, v_2, \dots, v_k)$ be a simple chain of maximal length, with $k \geq 3$, of a connected graph $G(V, E)$ with n vertices. Let v_1 be an initial vertex of the chain L, and v_k be a terminal vertex. Then v_1 and v_k are vertices of $G(V,E)$ admissible for removal. Indeed, either v_1 is a leaf vertex in $G(V,E)$, hence it is admissible for removal, or there are vertices $v_{i_1},\dots,v_{i_s}\in V$ such that $i_j>2$ and $(v_1,v_{i_j})\in E$ for all $j=1,\dots,s$. Since L is a simple chain of maximal length, $i_j\in\{3,\dots,k\}$ for every $j=1,\dots,s$, that is, (v_1,v_{i_j}) is a part of a cycle in $G(V,E)$ for each value of j. By Proposition 1, the vertex v_1 is admissible for removal in this case. A similar reasoning shows that the vertex v_k is also admissible for removal. □

Definition 2. Let G be a simple connected graph with a finite number n of vertices and m edges. A vertices indexing $v_1,\dots,v_n$ (resp., edge indexing $e_1,\dots,e_m$) is called admissible if the subgraphs $G(v_1,\dots,v_k)$ for all $k=1,\dots,n$ (resp., the subgraphs $G(e_1,\dots,e_s)$ for all $s=1,\dots,m$) are connected.

By the definition and Proposition 2, each simple connected graph has an admissible vertex indexing and an admissible edge indexing. Moreover, we have the following.

Proposition 3. *Let $G(V,E)$ be a simple connected graph with a finite number n of vertices and m edges. Then, for a fixed vertex $v\in V$ there exists an admissible vertex indexing $v_1,\dots,v_n$ such that $v_1=v$. There also exists an admissible edge indexing $e_1,\dots,e_m$ such that the edge e_1 coincides with a chosen edge $e\in E$.*

Proof. Let $v_1,\dots,v_n$ be a vertex indexing, and $G_k=G(v_1,\dots,v_k)$, $k=1,\dots,n$. By Proposition 2 there are two vertices in G_k admissible for removal, if $k\ge 2$. Without loss of generality, we can assume that v_k is a vertex in G_k admissible for removal and, moreover, we assume that the chosen vertex is different from v, since there are at least two vertices that can be chosen. By the assumption, such an indexing exists, is admissible, and $v_1=v$.

A similar reasoning with a use of Proposition 3 shows that there is an admissible edge indexing. □

Definition 3. Let a connected graph $G(V,E)$ be a tree with n vertices. An admissible vertex indexing $v_1,\dots,v_n$ will be called matched with an admissible edge indexing $e_1,\dots,e_{n-1}$ (and vice versa) if the subgraphs $G(v_1,\dots,v_k)=G(e_1,\dots,e_{k-1})$ coincide for $k=2,\dots,n$.

Remark 1 evidently implies the following.

Proposition 4. *In a tree, an admissible vertex indexing is matched with an admissible edge indexing if and only if for each k the edge e_{k-1} connects the vertex v_k.*

The problem of choosing admissible indexing on a graph is naturally related to a skeleton tree of the graph. Recall that a *skeleton* of a graph G is a graph $S(G)$ that contains all vertices of the graph G. Here, if $S(G)$ is a tree, then $S(G)$ is called a *skeleton tree* of G. Of course, each undirected connected graph $G=G(V,E)$ has a skeleton tree. The following proposition directly follows from the definition of an admissible edge indexing.

Proposition 5. *Let $T(G)=G(V,E_1)$ be a skeleton tree of a graph $G(V,E)$, $\{e_1,\dots,e_{n-1}\}=E_1$, and $\{f_1,\dots,f_m\}=E\setminus E_1$. If $e_1,\dots,e_{n-1}$ is an admissible edge indexing of $T(G)$, then any edge indexing of $E\setminus E_1$ with numbers from n to $n+m-1$ forms an admissible edge indexing of the graph $G(V,E)$.*

In order to consider the procedure of extracting an edge (or a vertex) from a graph G, let us look at the action it produces on the adjacency matrix $A(G)$. This matrix has real entries, and adding a scalar matrix αI to it, with a sufficiently large $\alpha>0$, transforms it to a matrix with nonnegative elements, which permits to apply a well developed theory of nonnegative matrices. Let us recall the needed results.

By the Peron-Frobenius theorem, the module greatest eigen value of a nonnegative irreducible matrix A is a positive number $\lambda_{\max}(A)$, it has multiplicity one, the corresponding eigen vector $\phi_{\lambda_{\max}}$ can be chosen to have positive coordinates. Moreover, if there is another matrix C with nonnegative off-diagonal elements such that the difference $A-C$ is a nonzero nonnegative matrix, then $\lambda_{\max}(A)$ is greater than any real eigen value of the matrix C, see [5, Theorem 6', p. 355]. This gives the following.

Proposition 6. *Let A, A_1, A_2 be matrices with real elements and nonnegative off-diagonal elements. If A is irreducible and the differences A_2-A and $A-A_1$ are matrices with nonnegative elements, that is,*

$$A_1 \leqslant A \leqslant A_2, \tag{1}$$

then

$$\lambda_{\max}(A_1) \le \lambda_{\max}(A) \le \lambda_{\max}(A_2), \tag{2}$$

where $\lambda_{\max}(X)$ is the greatest real eigen value of the matrix X. With this, if $A_i \neq A$, then the corresponding inequality in (2) is strict.

This shows that reducing (or making zero) the weights of edges or vertices reduces the value of greatest eigen value of the corresponding adjacency matrix. This monotone dependence permits to recover the value of a weight of the graph from its index. We will show this by using properties of cofactors of nonnegative matrices, see [5, the proof of the Frobenius Theorem, p. 342].

Proposition 7. *Let A be an irreducible matrix with nonnegative elements and $\lambda_{\max}(A)$ its greatest real eigen value. Then any cofactor of the matrix $\lambda I - A$ is positive for $\lambda > \lambda_{\max}(A)$.*

Proposition 8. *Let A be an irreducible matrix with nonnegative entries. If there is only one unknown element a_{ij} in a matrix A but the greatest real eigen value λ is known, then a_{ij} is uniquely expressed in terms of the known elements of A and λ.*

Proof. The determinant of the matrix $\lambda I_n - A$ equals zero by construction. Let us expand it in the ith row. Let B_{kl} be a cofactor of the element $\lambda\delta_{kl} - a_{kl}$ in the matrix $\lambda I - A$, where δ_{kl} is Kronecker's symbol. Then

$$\begin{aligned} B_{i1}(-a_{i1}) + \cdots + B_{i\ i-1}(-a_{i\ i-1}) + B_{ii}(\lambda - a_{ii}) \\ + B_{i\ i+1}(-a_{i\ i+1}) + \cdots + B_{in}(-a_{in}) = 0. \end{aligned} \tag{3}$$

By Proposition 7, $B_{ij} \neq 0$. Hence, it follows from (3) that a_{ij} can be uniquely expressed in terms of all other elements of A and the number λ. ☐

Remark 2. Proposition 8 is also true if the matrix A, in addition, is symmetric. A nonnegative element $a_{ij} = a_{ji}$ can be uniquely found from the equation $\det(\lambda I - A) = 0$, which, for $i \neq j$, can be reduced to a quadratic equation that has a unique nonnegative solution.

3. Inverse spectral problems for graphs

Spectral theory of weighted graphs is a spectral theory of their adjacency matrices. Since an adjacency matrix is a symmetric matrix with nonnegative off-diagonal elements, the theory of Perron-Frobenius, which is developed to great details, gives a number of spectral characteristics of such matrices [7, 5, 4]. As we have already mentioned, an index of a graph G is the greatest eigen value of its adjacency matrix $A(G)$. By spectrum of a graph G, we understand spectrum of its adjacency matrix, $\sigma(G) = \sigma(A(G))$, that is, the set of all eigen values of the adjacency matrix.

Definition 4. A spectral pair of a graph G is the pair $(\operatorname{ind} G, \phi)$, where ϕ is an eigen vector of the adjacency matrix $A(G)$ of the graph G corresponding to the greatest eigen value $\lambda_{\max}(A) = \operatorname{ind} G$. The vector ϕ satisfies a special normalization condition, the last coordinate equals one, $\phi = (\phi_1, \dots, \phi_{n-1}, 1)$.

Lemma 1. *Let $G(V, E, \tilde{w})$ be a weighted tree with a vertex admissible indexing $v_1, \dots, v_n$. Let $\tilde{w}(v_j) = 0$ for $j = 2, \dots, n$, and let a spectral pair $S(G) = (\operatorname{ind} G, \phi)$ of the graph G be given. Then all weights of the edges and the weight of the vertex v_1 are uniquely determined.*

Proof. Let $A = \|a_{ij}\|_{i,j=1}^n$ be the adjacency matrix of the graph G. All the diagonal elements a_{ii}, save for one, are equal to zero, $a_{ii} = 0$, $i = 2, \dots, n$. By the Frobenius theorem, all components of the eigen vector ϕ of the irreducible matrix A are positive. Denoting $\lambda = \operatorname{ind} G$ we have

$$A\phi = \lambda\phi. \tag{4}$$

Consider the last component in the vector identity (4). There is only one element in the last row of the matrix $A = \|a_{ij}\|_{i,j=1}^n$ which is nonzero. This is a_{nk}, where k in the number of the vertex that is connected to the vertex v_n with an edge. This implies that $a_{nk}\phi_k = \lambda\phi_n$, which yields $a_{nk} = \lambda\phi_k^{-1}$, since $\phi_n = 1$. Consider, in the vector identity (4), the subsequent components; first, with the index $n-1$, then with the index $n-2$, etc., up to component with the index s, $s > 1$. Since there is only one edge that can connect the vertices v_s and v_r, where r is less than s, for otherwise G would have a cycle, which is impossible for a tree, the inductive computation for $n-1, n-2, \dots, 2$ using (4) permits to uniquely calculate all the elements a_{ij} of the matrix A except for the element a_{11}. However, the equality of the first components in (4) gives $a_{11}\phi_1 + \sum_{j=2}^n a_{1j}\phi_j = \lambda\phi_1$, which uniquely defines a_{11}. □

Actually, Lemma 1 is a result of the inverse spectral problem for a tree. Inverse spectral problems for weighted graphs consist in recovering or finding weights of edges and vertices from certain spectral information that will be called spectral data, (SD), which is the initial data for the ISP.

Let us list several important general conditions on the SD.

1) SD should have a clear physical meaning and be experimentally verifiable.
2) The number of numeric parameters in SD must coincide with the number of weights to be found.
3) A solution of the IP with a given SD must be unique or the number of solutions must be finite, that is, such an ISP must be a well-posed problem.

Definition 5. Let $G(V, E, w)$ be a simple connected weighted graph with an admissible vertex indexing $v_1, \dots, v_n$. A collection of spectral pairs for the subgraphs $G(v_1, \dots, v_k)$, $k = 1, \dots, n$, will be called graph spectral data corresponding to the admissible vertex indexing. It will be denoted by $\mathrm{SDV}(G)$,

$$\mathrm{SDV}(G) = \{S(G(v_1, \dots, v_k)) \mid k = 1, \dots, n\} = \{(\operatorname{ind} G(v_1, \dots, v_k), \phi^{(k)}) \mid k = 1, \dots, n\}, \tag{5}$$

where $\phi^{(k)}$ is an eigen vector of the adjacency matrix of the subgraph $G(v_1, \dots, v_k)$ corresponding to the eigen value $\operatorname{ind} G(v_1, \dots, v_k)$. The problem of finding the weights of a connected graph G from $\mathrm{SDV}(G)$ will be denoted by ISPV.

Theorem 1. *If G is a simple weighted graph, then the problem ISPV has a unique solution.*

Proof. Consider the subgraph $G(v_1)$. This is a one-vertex graph with $w(v_1) = \operatorname{ind} G(v_1)$. For the subgraph $G(v_1, v_2)$, the adjacency matrix is symmetric and can be written as $A_2 =$

$\|a_{ij}\|_{i,j=1}^2$, where a_{11} has already been found but the elements $a_{12} = a_{21}$ and a_{22} are still unknown. They can be found from the spectral pair $S(G(v_1, v_2)) = (\lambda_2, \phi^{(2)})$. Indeed, $A_2\phi^{(2)} = \lambda_2\phi^{(2)}$, which leads, using $\phi_2^{(2)} = 1$, to the linear system

$$\begin{cases} a_{11}\phi_1^{(2)} + a_{12} &= \lambda_2\phi_1^{(2)}, \\ a_{12}\phi_1^{(2)} + a_{22} &= \lambda_2. \end{cases}$$

Since $\phi_1^{(2)}$ is not zero, the unknowns a_{12} and a_{22} can be uniquely determined.

Suppose now that all elements of the adjacency matrix A_{s-1} of the subgraph $G(v_1, \dots, v_{s-1})$ have been found, and show that the spectral pair $S(G(v_1, \dots, v_s)) = (\lambda_s, \phi^{(s)})$ uniquely determines all elements of the adjacency matrix A_s of the graph $G(v_1, \dots, v_s)$. Indeed, only the elements $a_{1s} = a_{s1}, a_{2s} = a_{s2}, \dots, a_{ss}$ of the matrix A_s are unknown. These elements can uniquely be found from the equation

$$A_s\phi^{(s)} = \lambda_s\phi^{(s)} \tag{6}$$

in terms of λ_s and the eigen vector $\phi^{(s)}$ that define a spectral pair of the subgraph $S(G(v_1, \dots, v_s))$. Indeed, write (6) as the system

$$(A_{s-1} - \lambda_s I_{s-1})\operatorname{col}(\phi_1^{(s)}, \dots, \phi_{s-1}^{(s)}) + \operatorname{col}(a_{1s}, \dots, a_{s-1\ s}) = 0 \tag{7}$$

and one equation written separately,

$$\sum_{j=1}^{s-1} a_{js}\phi_j^{(s)} + a_{ss} = \lambda_s. \tag{8}$$

Using (7) we can find all $a_{1s}, \dots, a_{s-1\ s}$, since $A_{s-1} - \lambda_s I_{s-1}$ is invertible by Proposition 7. Then (8) permits to uniquely find the unknown element a_{ss}. Hence, the elements of the matrix A_s can uniquely be found from A_{s-1} and the spectral pair $S(G(v_1, \dots, v_s))$.

By using inductively the spectral pairs in ISPV, we can continue this process until $s = n$. This gives all elements of the adjacency matrix of the weighted graph $G(V, E, w)$, hence the weight function w on the graph. □

Remark 3. For a connected graph $G(V, E, w)$ that has n vertices, the number of parameters in SDV equals $n(n+1)/2$. The number of weights in the graph $G(V, E, w)$ is $n+m$, where m is the number of edges. For a complete graph where each pair of vertices is connected with an edge, we have $m = C_n^2 = n(n-1)/2$. Thus, the number of parameters in SDV coincides with the number of the sought weights. If a graph is not complete, then its adjacency matrix has a number of zeros, and then $n + m < n(n+1)/2$. In such a case, the SDV is overdetermined. A close look at the proof of Theorem 1 shows that, in the algorithm for solving the SDV in the case where the adjacency matrix has zeros as entries, a number of components of the eigen vectors in SDV are not used and thus such parameters can be dropped in the definition of the SD.

Definition 6. Let $G(V, E, w)$ be a simple connected graph, and $v \in V$ be a vertex admissible for removal. A truncated spectral pair for a graph G with a fixed vertex v is a pair $S(\operatorname{ind} G, \phi)$ such that the only given components of the specially normalized eigen vector ϕ are $\phi_{i_1}, \dots, \phi_{i_k}$, where the numbers $i_1, \dots, i_k$ are indexes of all vertices of the graph which are connected with the vertex v. We will briefly denote this spectral pair by

$$S_v(G) = (\operatorname{ind} G, \phi_{i_1}, \dots, \phi_{i_k}).$$

Definition 7. Let $G(V, E, w)$ be a simple connected weighted graph with an admissible vertex indexing $v_1, \dots, v_n$. By minimal spectral data minSDV(G) connected with an admissible sequence of vertices we call the set of truncated spectral pairs of all subgraphs $G(v_1, \dots, v_k)$, $k = 1, \dots, n$.

Theorem 2. *The inverse spectral problem of recovering all weights from minSDV(G) for a simple connected graph with an admissible vertex indexing has a unique solution.*

Proof. The adjacency matrix A_1 of the graph $G(v_1)$ is the number $\operatorname{ind} G(v_1)$. Let $s > 1$, $S_{v_s}(G(v_1, \dots, v_s)) = (\lambda_s, \phi_{i_1}, \dots, \phi_{i_k})$ be a truncated spectral pair for the graph $G(v_1, \dots, v_s)$, and assume that all weights of the graph $G(v_1, \dots, v_{s-1})$ are known, and denote its adjacency matrix by A_{s-1}. The proof of the theorem relies on the proof of Theorem 1, and we will use the same notations. The identities (7) and (8) hold true. Let $j_1, \dots, j_{s-k}$ be an increasing sequence of vertex indexes that are not connected to v_s in $G(v_1, \dots, v_s)$. Since $a_{j_l\ s} = 0$, $l = 1, \dots, s-k$, it follows from (7) that

$$\tilde{A}\mathrm{col}(\phi^{(s)}_{j_1}, \dots, \phi^{(s)}_{j_{s-k}}) + \tilde{B}\mathrm{col}(\phi^{(s)}_{i_1}, \dots, \phi^{(s)}_{i_k}) = 0, \tag{9}$$

where $\tilde{A}$ is a submatrix of the matrix $A_{s-1} - \lambda_s I_{s-1}$, which is formed by elements of the matrix $A_{s-1} - \lambda_s I_{s-1}$ with the indexes $j_1, \dots, j_{s-k}$, and $\tilde{B}$ is a submatrix of the matrix $A_{s-1} - \lambda_s I_{s-1}$ formed by elements in columns with the indexes $i_1, \dots, i_s$ and rows with the indexes $j_1, \dots, j_{s-k}$. By Proposition 7, $\tilde{A}$ is invertible and, hence, $\phi^{(s)}_{j_1}, \dots, \phi^{(s)}_{j_{s-k}}$ can be uniquely expressed in terms of elements of the matrix A_{s-1}, $\phi^{(s)}_{i_1}, \dots, \phi^{(s)}_{i_k}$, and λ_s. This permits to recover the entire eigen vector $(\phi^{(s)}_1, \dots, \phi^{(s)}_{s-1}, 1)$, and now Theorem 2 follows directly from Theorem 1. □

Definition 8. Let $T(V, E, w)$ be a connected tree with an admissible vertex indexing $v_1, \dots, v_n$, and $T(V, E, \tilde{w})$ be the same graph with another weight $\tilde{w}$ that coincides with w on all edges $e \in E$ and on the vertex v_1: $\tilde{w}(v_1) = w(v_1)$, however, $\tilde{w}(v_j) = 0$ for each $j = 2, \dots, n$. Let $(\operatorname{ind} T, \phi)$ be a spectral pair for the graph $T(V, E, w)$ and $(\operatorname{ind} \tilde{T}, \tilde{\phi})$ be a spectral pair for $T(V, E, \tilde{w})$. Spectral data SDT for the tree $T(V, E, w)$ will be a collection of the vectors ϕ and $\tilde{\phi}$, together with the index $\operatorname{ind} T$ or $\operatorname{ind} \tilde{T}$,

$$\mathrm{SDT} = \{\phi, \tilde{\phi}, \lambda\}, \quad \lambda = \operatorname{ind} T \quad \lambda = \operatorname{ind} \tilde{T}. \tag{10}$$

Theorem 3. *The weights of a connected weighted graph $T(V, E, w)$ with an admissible vertex indexing are uniquely recovered from the spectral data SDT of the form (10).*

Proof. Let $\mathrm{SDT} = \{\phi, \tilde{\phi}, \operatorname{ind} \tilde{T}\}$. By Lemma 1 all the associated weights $\tilde{w}$ are uniquely determined, hence the same is true for all the elements of the adjacency matrix $\tilde{A}(\tilde{T})$. Since the adjacency matrix $A(T)$ coincides with $\tilde{A}(\tilde{T})$ except for the diagonal elements a_{jj}, $j = 2, \dots, n$, the identity

$$A\phi = \operatorname{ind} T \phi \tag{11}$$

for the first components

$$\sum_{i=1}^{n} a_{1i}\phi_i = \sum_{i=1}^{n} \tilde{a}_{1i}\phi_i = \operatorname{ind} T \phi_1 \tag{12}$$

gives the index of $\operatorname{ind} T$. The unknown diagonal elements a_{jj}, $j \geq 2$, can be found from the corresponding jth equation of system (11),

$$\sum_{i=1}^{n} a_{ji}\phi_i = a_{jj}\phi_j + \sum_{i=1}^{n} \tilde{a}_{ji}\phi_i = \operatorname{ind} T \phi_j. \tag{13}$$

If $\mathrm{SDT} = \{\phi, \tilde{\phi}, \operatorname{ind} T\}$, then, as in the proof of Lemma 1, $a_{ij} = \rho_{ij} \operatorname{ind} \tilde{T}$, $i \neq j$, or $i = j = 1$, where ρ_{ij} can be explicitly expressed in terms of components of the vector $\tilde{\phi}$. But then (12) gives the identity

$$\operatorname{ind} \tilde{T} \sum_{i=1}^{n} \rho_{1i}\phi_i = \operatorname{ind} T \phi_1.$$

This identity, if the vectors ϕ and $\tilde{\phi}$ are given, permits to uniquely express $\operatorname{ind}\tilde{T}$ in terms of $\operatorname{ind}T$. Hence, in the case where $\mathrm{SDT} = \{\phi, \tilde{\phi}, \operatorname{ind}T\}$, the proof directly follows from the case considered above, $\mathrm{SDT} = \{\phi, \tilde{\phi}, \operatorname{ind}\tilde{T}\}$. □

Remark 4. It is easy to see that the number of SDT in Theorem 3 equals $2n-1$ and coincides with the number of the weights, since the number of edges in a tree is $m = n-1$, that is, the SDT do not make overdetermined data.

Remark 5. Theorem 2 on uniqueness of a solution of the ISPV can also be applied to connected trees $T(V,E,w)$. In such a case, the SDV differ from the SDT considered in Theorem 2. A truncated spectral pair for a subgraph $T(v_1,\dots,v_k)$ consists of $\operatorname{ind}T(v_1,\dots,v_k)$ and the value ϕ_k of one of the components of the normalized eigen vector ϕ. This component is defined by the index $s(k) < k$ of the vertex connected to the vertex v_k with an edge, $S_{v_k}(T(v_1,\dots,v_k)) = \{\operatorname{ind}T(v_1,\dots,v_k), s(k), \phi_s\}$. A collection of all such quantities for $k = 1,2,\dots,n$ makes spectral data SDTV for the tree. By Theorem 2, ISPTV has a unique solution.

Remark 6. For trees, one can also consider spectral data other than SDT and SDTV. In particular, for a connected weighted tree $T(V,E,w)$ with an admissible vertex indexing $v_1,\dots,v_n$, spectral data can be taken as the collection $\operatorname{ind}T(v_1,\dots,v_k)$, $k = 1,\dots,n$, together with the collection $\operatorname{ind}T(v_1,\dots,v_k \mid w(v_k) = 0)$ for the subgraphs with the weight $w(v_k) = 0$ modified only in a single vertex v_k. Such spectral data will be denoted by SDTW. It is easy to see that SDTW uniquely define weights for the tree $T(V,E,w)$.

Definition 9. Let $G(V,E,w)$ be a simple weighted graph and $T(V,E_1,w)$ be its skeleton tree. Let n be the number of vertices in V, $n-1$ the number of edges in E_1, and $e_n,\dots,e_m$ all edges in $E \setminus E_1$. By spectral data SDVT for the graph G with a skeleton T, we call the union SDT of spectral data for the skeleton T and a sequence of indices of the subgraphs $G(V, E_1 \cup \{e_n,\dots,e_s\}, w)$, where s assumes the values from n to m,

$$\mathrm{SDVT}(G,T) = \{\mathrm{SDT}, \operatorname{ind}G(V, E_1 \cup \{e_n,\dots,e_s\}, w), s = n,\dots,m\}. \tag{14}$$

Theorem 4. *The inverse spectral problem for determining weights of a weighted graph G with a skeleton tree T from the SD in the form of (14) has a unique solution.*

Proof. By Theorem 3, the SDT permits to recover all weights of the vertices in V and all weights of the edges that belong to the skeleton tree T. By Remark 2, one can uniquely find the weight $w(e_n)$ of the edge e_n from $\operatorname{ind}G(V, E_1 \cup \{e_n,\}, w)$. Now, using the value of $w(e_n)$ and taking into account Remark 2 we can find the weight of the edge e_{n+1}. Continuing this process, we get values of $w(e_s)$ for all $s = n,\dots,m$. □

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