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THE LIOUVILLE PROPERTY FOR HARMONIC FUNCTIONS ON GROUPS AND HYPERGROUPS

HERBERT HEYER

Dedicated to Professor Satoshi Kawakami on the occasion of his academic retirement

Abstract. A survey is given on the Liouville property of harmonic functions on groups and hypergroups. The discussion of a characterization of that property in terms of the underlying algebraic structures yields interesting open problems.

1. Introduction

It is well known that a bounded analytic function on the complex plane $\mathbb{C}$ is constant. This fact is called the *Liouville property* for analytic functions (see [43]). One can also prove a Liouville property if one replaces analytic functions by harmonic functions and $\mathbb{C}$ by an arbitrary Euclidean space $\mathbb{R}^d$ for $d \geq 1$. More precisely, let f be a real-valued function on an open subset U of $\mathbb{R}^d$. The function f is said to be *harmonic* if it is locally Lebesgue integrable and if it satisfies the averaging property.

Given $x \in U$, $r > 0$ such that $B(x,r) \subset U$,

$$f(x) = \int_{S(x,r)} f(y)\, \sigma_r(x, \mathrm{d}r)\,,$$

where $\sigma_r(x,\cdot)$ denotes the uniform distribution (surface measure) on the sphere $S(x,r)$ of the ball $B(x,r)$ with center x and radius r. Any harmonic function f on U belongs to $\mathrm{C}^\infty(U)$, hence to $\mathrm{C}(U)$, and f is harmonic if and only if $f \in \mathrm{C}^2(U)$ satisfying the Laplace equation $\Delta f = 0$. Applying Harnack's inequality one can easily show the classical *Liouville theorem*: If f is harmonic on $U = \mathbb{R}^d$ and bounded from below and from above, then f is constant.

We note that for U equal to the unit disk $\mathbb{D}$ or U equal to the hyperbolic space $\mathbb{H}^d$ a statement analogous to Liouville's theorem does not hold. The Liouville property has been extended to harmonic functions associated with Markov processes in various kinds of state spaces, for example, in groups and homogeneous spaces, but also in more general algebraic-topological structures such as hypergroups.

During the last 15 years a functional-analytical approach to harmonic functions in L^p-spaces has set the Liouville property into new light. See e.g. [11] by Cho-Ho Chu. The related Bergman spaces have been studied intensively by Cho-Ho Chu and Anthony To-Ming Lau in [14].

In our exposition harmonic functions will be considered within the framework of random walks in a locally compact group. Emphasis will be put on information-theoretic characterizations of the Liouville property for harmonic functions defined with respect to the law of the underlying random walk.

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Interesting problems, in part open ones, show up if the harmonic functions related to a random walk in G behave anti-Liouville, which means that there are unbounded harmonic functions on G. In this situation the Poisson boundary of G introduced by Harry Furstenberg in his work [24] of 1963 is still a rewarding subject of study. An exposition on his seminal contribution had been published informally by Herbert Heyer in [31]. We can only touch upon the construction of the Poisson boundary. Instead we shall apply the theory to a group-theoretical access for harmonic functions on the unit disk $\mathbb{D}$ and their (classical) Poisson representation.

Our setup owes many ideas to the excellent expository articles [17] and [7] of Yves Derriennic and Martine Babillot respectively. In the final section of the present work we shall report on the Liouville property of hypergroups K as it has been layed out by Massoud Amini together with Cho-Ho Chu in [1] and [2] respectively. Although the definition of harmonic functions on K and the standard classes of hypergroups admitting the Liouville property can be provided in analogy to the group case, technical obstacles have to be overcome. After all, to develop a boundary theory in the spirit of Harry Furstenberg for hypergroups remains a challenge for future research.

2. Preliminaries

Although we expect the reader to be familiar with the basics of probability measures on topological groups we shall recall some notation and useful symbols that are used throughout the text.

2.1. **Measures.** Let $(E, \mathfrak{E})$ be a measurable space with a σ-algebra $\mathfrak{E}$ in E. The space of $\mathfrak{E}$-measurable functions on E will be denoted by $\mathrm{M}(E) := \mathrm{M}(E, \mathfrak{E})$, its subspaces of bounded functions by $\mathrm{M}^b(E)$.

We are using the symbols $\mathcal{M}(E) := \mathcal{M}(E, \mathfrak{E})$, $\mathcal{M}^\sigma(E)$ and $\mathcal{M}^1(E)$ for the sets of nonnegative measures, of σ-finite measures and of probability measures on E respectively.

For $x \in E$, ε_x denotes the Dirac measure in x. If $\lambda \in \mathcal{M}^\sigma(E)$, the Lebesgue spaces $\mathrm{L}^p(E, \lambda)$ $(1 \le p \le \infty)$ are the fundamental objects of measure theory. Let τ be a λ-absolutely continuous measure, $\tau \ll \lambda$ in short, then τ has a λ-density φ in the sense that $\tau = \varphi \cdot \lambda$ for $\varphi \in \mathrm{L}^1(E, \lambda)$.

Now, let $(E, \mathfrak{E})$ be the Borel space of a locally compact space E, where $\mathfrak{E}$ stands for the Borel-σ-algebra $\mathfrak{B}(E)$ of E. In this situation measures on E are considered as Borel measures. The set $\mathcal{M}^b(E)$ of bounded (complex) measures on E is a Banach space with respect to the total variation (norm). $\mathcal{M}^b(E)$ carries also the weak topology τ_w defined as the topology associated with the duality between the spaces $\mathrm{C}^b(E)$ of bounded continuous functions on E and $\mathcal{M}^b(E)$. The support of a measure τ on E will be abbreviated by $\operatorname{supp} \tau$.

2.2. **Groups.** Let G be a locally compact group, written multiplicatively with neutral element e. G admits a unique (left invariant) Haar measure $\omega_G \in \mathcal{M}(G)$ with (full) support $\operatorname{supp} \omega_G = G$. In the Banach space $\mathcal{M}^b(G)$ there exists a convolution $*$ between measures and an involution $\sim$ such that it becomes a *-Banach algebra with $\mathrm{L}^1(G, \omega_G)$ as a closed subalgebra.

A measure $\mu \in \mathcal{M}(G)$ is called *adapted* if the closed subgroup $[\operatorname{supp} \mu]$ generated by $\operatorname{supp} \mu$ coincides with G. For any $\mu \in \mathcal{M}(G)$, $\operatorname{supp} \mu$ is σ-compact. If on G there exists an adapted measure, then G is σ-compact.

A measure $\mu \in \mathcal{M}(G)$ is said to be *spread-out* provided there exists a $q > 1$ such that μ^q is singular with respect to ω_G. Clearly, ω_G-absolute continuous measures on G are spread-out. On discrete groups G every $\mu \in \mathcal{M}(G)$ is spread-out. If G is connected, then any spread-out measure is adapted.

2.3. **Special classes of groups.** Given a left invariant Haar measure ω_G, for each $x \in G$ there exists a number $\Delta(x) > 0$ such that

$$\omega_G(f_{x^{-1}}) = \Delta(x)\omega_G(f)$$

whenever $f \in \mathrm{M}(G)$, where $f_{x^{-1}}$ denotes the right translate of f by x^{-1}. The mapping $\Delta\colon G \to \mathbb{R}$ is called the modular function of G.

A locally compact group G is named *unimodular* if $\Delta = 1$. The class of unimodular groups includes all [IN] groups which can be defined as locally compact groups G having a compact invariant neighborhood of e, consequently all central (Z-)group which by definition have a compact quotient $G/Z(G)$ of G by the center $Z(G)$ of G, and therefore all are Abelian and compact groups. *Amenable groups* G are defined by the requirement of existence of a (left) translation invariant positive linear functional of norm 1 (mean) on $\mathrm{L}^\infty(G, \omega_G)$. The class of amenable groups comprises all central groups, but also the solvable groups, hence the nilpotent groups. Finally we would like to recall the notion of groups with growth conditions.

From now on we assume G to be 2nd countable which implies that there is a (left) invariant distance on G. Consequently balls $B(x, r)$ with center $x \in G$ and radius $r > 0$ can be formed. Now, the growth rate of G is given as

$$c := \limsup_{n\to\infty} \frac{1}{n} \log \omega_G\left(B(e, n)\right) .$$

A group G has *subexponential growth* if $c = 0$ and exponential growth if $c > 0$. It is known that groups with subexponential growth are amenable.

3. Mutual information

In this and the following sections we repeat some of the material exposed in a previous paper [32] except for the generalization beyond random walks. Let $(E, \mathfrak{E})$ be an arbitrary measurable space with $\mu, \nu \in M^1(E)$. The *relative entropy* between μ and ν is given by

$$H\left(\mu \parallel \nu\right) := \sup_{\mathfrak{F}} \sum_{F\in\mathfrak{F}} \left(\log \frac{\mu(F)}{\nu(F)}\right) \mu(F)\,,$$

where $\mathfrak{F}$ denotes the partition of finite sets $F \in \mathfrak{E}$ with $\nu(F) > 0$. Obviously

$$H\left(\mu \parallel \nu\right) \geq 0, \quad \text{possibly} \quad +\infty$$

and

$$H\left(\mu \parallel \nu\right) = 0 \quad \text{if and only if} \quad \mu = \nu\,.$$

3.1. **Theorem - M. S. Pinsker** [39]**.**

(i) $H\left(\mu \parallel \nu\right) < \infty$ implies $\mu \ll \nu$.

(ii) If $\mu \ll \nu$, then

$$H\left(\mu \parallel \nu\right) = \int_E \left(\log \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right) \mathrm{d}\mu\,,$$

where the integral may be $+\infty$.

In the case (i) $H\left(\mu \parallel \nu\right)$ is called the *Kullback-Leibler information* between μ and ν.

3.2. **Special case.** If there exists $\lambda \in \mathcal{M}^\sigma(E)$ such that $\{\mu, \nu\} \ll \lambda$, i.e., $\mu = f \cdot \lambda$ and $\nu = g \cdot \lambda$ with $f, g \in \mathrm{L}^1(E, \lambda)$, then

$$H\,(f \parallel g) := H(f \cdot \lambda, g \cdot \lambda) = \int_E f \left(\log \frac{f}{g}\right) \mathrm{d}\lambda\,.$$

The notation

$$H(f) := -H\,(f \parallel \mathbb{1}_E)$$

is in accordance with the definition of Claude Shannon's differential entropy of a Lebesgue density f on $E = \mathbb{R}$.

Now let X, Y be random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with values in arbitrary measurable spaces. Here $(\Omega, \mathfrak{A})$ is a measurable space and $\mathbb{P} \in \mathcal{M}^1(\Omega)$.

The *mutual information* between X and Y is given by

$$I(X, Y) := H\left(\mathbb{P}_{(X,Y)} \parallel \mathbb{P}_X \otimes \mathbb{P}_Y\right)\,,$$

where $\mathbb{P}_Z$ denotes the probability distribution of a random variable Z on $(\Omega, \mathfrak{A}, \mathbb{P})$ and $\otimes$ signifies the (tensor) product measure.

3.3. **Properties of the mutual information.**

3.3.1 $I(X, Y) \geq 0$.

3.3.2 $I(X, Y) = 0$ if and only if X, Y are (stochastically) independent (with respect to $\mathbb{P}$).

3.3.3 Let φ be any measurable function on the range (measure space) of Y. Then

$$I(X, \varphi \circ Y) \leq I(X, Y)\,.$$

3.3.4 For a sequence $(Y_k)_{k \geq 1}$ of random variables Y_k on $(\Omega, \mathfrak{A}, \mathbb{P})$ there exists

$$\lim_{n \to \infty} \uparrow I(X, (Y_1, \ldots, Y_n)) = I(X, (Y_1, \ldots, Y_n, \ldots))\,.$$

3.3.5 Suppose that there exists an $n \geq 1$ such that

$$I(X, (Y_n, Y_{n+1}, \ldots)) < \infty\,.$$

Then

$$\lim_{n \to \infty} \downarrow I(X, (Y_n, Y_{n+1}, \ldots)) \text{ exists }\;;$$

it will be denoted by $I(X, \mathfrak{S})$ where

$$\mathfrak{S} := \bigcap_{n \geq 1} \sigma(Y_n, Y_{n+1}, \ldots)$$

denotes the *tail*- or *asymptotic σ-algebra* of $(Y_k)_{k \geq 1}$.

Let X, Y, Z be random variables on $(\Omega, \mathfrak{A}, \mathbb{P})$ taking their values in a measurable space with a separable σ-algebra such that the conditional probabilities $\mathbb{P}((X, Y) \mid Z = z)$, $\mathbb{P}(X \mid Z = z)$ and $\mathbb{P}(Y \mid Z = z)$ are regular. Then the *conditional mutual information* of X and Y under Z is defined by

$$\mathbb{E}I((X, Y) \mid Z) = \int_\Omega I((X, Y) \mid Z = z)\, \mathbb{P}_Z(\mathrm{d}z)\,,$$

where

$$I((X, Y) \mid Z = z) := H\left(\mathbb{P}((X, Y) \mid Z = z) \parallel \mathbb{P}(X \mid Z = z) \otimes \mathbb{P}(Y \mid Z = z)\right)\,.$$

One has *Kolmogorov's formula*

$$I((X, Z), Y) = I(Y, Z) + \mathbb{E}I((X, Y) \mid Z)\,.$$

Given random variables Y_1, Y_2, Y_3 on $(\Omega, \mathfrak{A}, \mathbb{P})$ the sequence $\{Y_1, Y_2, Y_3\}$ forms a Markov chain if and only if

$$\mathbb{E}\left(I(Y_1, Y_2) \mid Y_3)\right) = 0\,.$$

Consequently, for a Markov chain $(Y_n)_{n\geq 1}$ in a separable measurable space $(E, \mathfrak{E})$ one has

$$I(Y_1, Y_n) = I(Y_1, (Y_n, Y_{n+1}, \dots))\,,$$

the sequence $(I(Y_1, Y_n))_{n\geq 1}$ decreases, and

$$\lim_{n\to\infty} I(Y_1, Y_n) = I(Y, \mathfrak{S})$$

provided

$$I(Y_1, Y_n) < \infty \quad \text{for all} \quad n \geq 1\,.$$

Further properties of the asymptotic σ-algebra of Markov chains in groups are contained in the work of Y. Derriennic in [19] and of W. Jaworski in [33].

4. Asymptotic entropy

Let $(Y_n)_{n\geq 0}$ be a stationary Markov chain in a measurable space $(E, \mathfrak{E})$ with transition function P. If (Y_n) starts at $Y_0 = y \in E$, the number

$$h(P, y) := I_y(Y, \mathfrak{S})$$

is called the *asymptotic entropy* of (Y_n). In other words, $h(P, y)$ is the mutual information between the Markov chain (Y_n) at time 1 and its asymptotic σ-algebra $\mathfrak{S}$, with respect to the distribution $\mathbb{P}^y$ of (Y_n) starting at $Y_0 = y$.

4.1. **Observation.** If $\mathfrak{S}$ is trivial $[\mathbb{P}^y]$, then $h(P, y) = 0$.

In general $h(P, y)$ depends on y. We now assume that (Y_n) is *spatially homogeneous* in the sense of the following requirements. There exists a family $\mathcal{T}$ of bijective, bimeasurable transformations T on $(E, \mathfrak{E})$ such that

a) $\mathcal{T}$ commutes with the transition function P of (Y_n), i.e.,

$$P(Ty, TB) = P(y, B)$$

or

$$\mathbb{E}(Y_{n+1} \in TB \mid Y_n = Ty) = \mathbb{E}(Y_{n+1} \in B \mid Y_n = y) = \mathbb{E}(Y_1 \in B \mid Y_n = y)$$

for all $y \in E$, $B \in \mathcal{E}$.

b) $\mathcal{T}$ acts transitively on E which says that for $x, y \in E$ there exists a $T \in \mathcal{T}$ satisfying

$$Tx = y\,.$$

Under the condition of spatial homogeneity

$$h(P, y) = \text{constant}\,.$$

We summarize in

4.2. **Theorem.** If (Y_n) is a spatially homogeneous (stationary) Markov chain in $(E, \mathfrak{E})$ with transition function P, then its asymptotic entropy

$$h(P) = I(Y, \mathfrak{S})$$

is independent of the starting point (or of the initial distribution).

Moreover,

$$\mathfrak{S} \text{ is trivial } [\mathbb{P}^y] \quad \text{if and only if} \quad h(P) = 0\,.$$

The proof of this equivalence given by Yves Derriennic in [17] for random walks in a locally compact (second countable) group extends to Markov chains in general measurable spaces $(E, \mathfrak{E})$ (with separable σ-algebra $\mathfrak{E}$).

4.3. **The case of a random walk.** Let $(S_n)_{n\geq 1}$ be a (right) random walk in a locally compact (second countable) group G. For each $n \geq 1$

$$S_n := \prod_{k=1}^{n} X_k \,,$$

where $(X_k)_{k\geq 1}$ is a sequence of independent, identically distributed random variables in G with

$$\mathbb{P}_{X_k} = \mu \in \mathcal{M}^1(G)$$

for all $k \geq 1$. We note that for any $y \in G$, $(yS_n)_{n\geq 1}$ is a stationary spatially homogeneous Markov chain in $(G, \mathfrak{B}(G))$ with starting point y and transition function P given by

$$P(x, B) := \mu(x^{-1}B)$$

whenever $x \in G$, $B \in \mathfrak{B}(G)$. Without loss of generality we may choose $y = e$. The notation

$$h(\mu) := h(P) = I(S_1, \mathfrak{S})$$

suggests the dependence of the asymptotic entropy only on μ; $\mathfrak{S}$ stands for the asymptotic σ-algebra of (S_n).

Associated with a right random walk (S_n) is the *adjoint* left *random walk* (S_n^{-1}) given by

$$S_n^{-1} = X_n^{-1} \cdot \ldots \cdot X_1^{-1}$$

with defining adjoint measure $\mu^{\sim}$. Now let $\mu \ll \omega_G$ in the sense that $\mu = \varphi \cdot \omega_G$ with $\varphi \in \mathrm{L}^1(G, \omega_G)$. For each $n \geq 1$

$$\mu^n = \varphi_n \cdot \omega_G$$

with

$$\varphi_n(y) = \int_G \varphi(x)\varphi_{n-1}(x^{-1}y)\, \omega_G(\mathrm{d}x)\,,$$

and

$$(\mu^n)^{\sim} = \psi_n \cdot \omega_G$$

with

$$\psi_n(y) = \varphi_n(y^{-1})\Delta(y)^{-1}$$

whenever $y \in G$. Here Δ denotes the modular function of G. By the way, the random walks (S_n) and (S_n^{-1}) are defined by the same measure $\mu \in \mathcal{M}^1(G)$.

4.3.1. *Proposition.* Suppose that

$$\int_G \log \Delta(x)\, \mu(\mathrm{d}x) = \mathbb{E}\left(\log \Delta \circ X_1\right) < \infty\,.$$

Then

$$H(\mu^n) = H(\varphi_n) < \infty \quad \text{if and only if} \quad H((\mu^n)^{\sim}) = H(\psi_n) < \infty\,,$$

and under this condition

$$H(\psi_n) = H(\varphi_n) - n\mathbb{E}\left(\log \Delta \circ X_1\right)\,.$$

We keep the assumption of the proposition and deduce

4.3.2. *Theorem.*

$$h(\mu) \geq \mathbb{E}\left(\log \Delta \circ X_1\right)$$

and

$$h(\mu^{\sim}) = h(\mu) - \mathbb{E}\left(\log \Delta \circ X_1\right) .$$

Moreover, if G is unimodular and

$$H(\varphi_n) < \infty ,$$

then

$$h(\mu) = h(\mu^{\sim}) .$$

5. The Liouville property

We are returning to the situation of a general spatially homogeneous stationary Markov chain $(Y_n)_{n\geq 0}$ in a measurable space $(E, \mathfrak{E})$ with transition function P. (Y_n) can be viewed as a canonical Markov chain on the path space

$$\left(E^{\mathbb{Z}_+}, \mathfrak{E}^{\otimes \mathbb{Z}_+} \, \mathbb{P}^y\right) ,$$

where the probability measure $\mathbb{P}^y$ on $\left(E^{\mathbb{Z}_+}, \mathfrak{E}^{\otimes \mathbb{Z}_+}\right)$ is constructed for every starting point $y \in E$.

The *Poisson equation* associated with (Y_n) reads as

$$(I - P)f = g, \tag{PE}$$

where $f, g \in \mathrm{M}(E)$.

5.1. **Definition.** A function $f \in \mathrm{M}(E)$ is said to be *harmonic* on E with respect to (Y_n) or P if f is a solution of (PE) for $g = 0$.

It is easily observed that the set $\mathrm{H}(E)$ of harmonic functions on E is a vector space, and its subset $\mathrm{H}^b(E)$ of bounded harmonic functions (on E) is a Banach space with respect to the sup-norm in $\mathrm{M}^b(E)$. On the path space of (Y_n) the shift operator θ is defined by

$$\theta\left((x_0, x_1, \dots, x_n, \dots)\right) := (x_1, x_2, \dots, x_{n+1}, \dots)$$

for all $(x_0, x_1, \dots, x_n, \dots) \in E^{\mathbb{Z}_+}$. Obviously θ is a measurable mapping on $E^{\mathbb{Z}_+}$, and

$$Y_n \circ \theta = Y_{n+1}$$

for the coordinate mappings Y_n on $(E^{\mathbb{Z}_+}, \mathfrak{E}^{\otimes \mathbb{Z}_+})$, $n \geq 0$.

Next we introduce the *invariance σ-algebra*

$$\mathfrak{I} := \left\{A \in \mathfrak{E}^{\otimes \mathbb{Z}_+} \, : \theta^{-1} A = A\right\}$$

of the Markov chain (Y_n), for which obviously the inclusion

$$\mathfrak{I} \subset \mathfrak{S} = \bigcap_{n\geq 0} \theta^{-n} \mathfrak{E}^{\otimes \mathbb{Z}_+} \; = \bigcap_{n\geq 0} \sigma\left(\{Y_m \colon m \geq n\}\right)$$

holds.

A random variable Z on $(E^{Z_+}, \mathfrak{E}^{\otimes \mathbb{Z}}, \mathbb{P}^y)$ (with values in $(E, \mathfrak{E})$) is said to be *invariant* if it is $\mathfrak{I}$-measurable.

Now we can phrase the *probabilistic version of the Liouville property* for the Markov chain (Y_n).

First of all we need an important proposition

5.2. **Proposition.** There is a one-to-one and onto correspondence between $\mathbb{P}^y$-equivalence classes of bounded invariant random variables Z on $(E^{\mathbb{Z}_+}, \mathfrak{E}^{\otimes \mathbb{Z}_+}, \mathbb{P}^y)$ and bounded harmonic functions f on E given by

$$f(y) = \mathbb{E}^y(Z)(= \mathbb{E}^{\mathbb{P}^y}(Z)) .$$

Moreover,

$$\lim_{n\to\infty} f \circ Y_n = Z[\mathbb{P}^y]$$

whenever $y \in E$.

For any $A \in \mathfrak{E}^{\otimes \mathbb{Z}_+}$ one introduces the set

$$R(A) := \{(x_0, x_1, \ldots, x_n, \ldots) \in E^{\mathbb{Z}_+} \colon \lim_{n\to\infty} 1\!\!1_A \circ Y_n ((x_0, x_1, \ldots, x_n, \ldots)) = 1\}$$

of all paths of the Markov chain (Y_n) which hit A infinitely often. $R(A) \in \mathfrak{I}$, and

$$h_A := \mathbb{P}^{\cdot} (R(A)) \in \mathrm{H}^b(E) .$$

Here $h_A(y)$ is the probability that the Markov chain (Y_n) with start in y hits A infinitely many times. We say that $A \in E^{\mathbb{Z}_+}$ is *transient* if $h_A = 0$ and *recurrent* if $h_A = 1$.

5.3. **Definition.** The state space $(E, \mathfrak{E})$ of the Markov chain (Y_n) with transition function P and start in $y \in E$ is said to admit the Liouville property (LP) if $\mathrm{H}^b(E)$ contains only constant functions.

5.4. **Theorem.** The following statements are equivalent.

(i) (LP) holds.
(ii) $\mathfrak{I}$ is trivial $[\mathbb{P}^y]$.
(iii) Each $A \in \mathfrak{E}^{\otimes \mathbb{Z}_+}$ is either transient or recurrent.

The equivalence (i) $\Longleftrightarrow$ (ii) follows from Proposition Proposition 5.2. The equivalence (ii) $\Longleftrightarrow$ (iii) has been proved in the book [41] by A. Revuz.

5.5. Now we are returning to the case of a random walk. As in Subsection 4.3 let (S_n) be a random walk in a locally compact and 2$^{\text{nd}}$ countable group G with defining measure $\mu \in \mathcal{M}^1(G)$. The corresponding canonical process is

$$\left(G^{\mathbb{N}}, \mathfrak{B}(G)^{\otimes \mathbb{N}}, \mathbb{P}^e\right) ,$$

where $\mathbb{P}^e := \gamma\left(\mu^{\otimes \mathbb{N}}\right)$, with a mapping

$$\gamma \colon G^{\mathbb{N}} \to G^{\mathbb{N}}$$

given by

$$\gamma(x_1, x_2, \ldots, x_n, \ldots) := (s_1 = x_1, s_2 = x_1 \cdot x_2, \ldots, s_n = x_1 \cdot \cdots \cdot x_n, \ldots)$$

for all $(x_1, x_2, \ldots, x_n, \ldots) \in G^{\mathbb{N}}$.

The shift operator θ takes the form

$$\theta(s_1, s_2, \ldots, s_n, \ldots) := (s_2, s_3, \ldots, s_{n+1}, \ldots)$$

for all $(s_1, s_2, \ldots, s_n, \ldots) \in G^{\mathbb{N}}$. $\mathfrak{I}$ denotes the corresponding invariance σ-algebra, Applying his 0-2-law [16] Yves Derriennic showed

5.5.1. *Theorem.*

$$\mathfrak{I} = \mathfrak{S}[\mathbb{P}^e]\,.$$

Since the random walk (S_n) is spatially homogeneous, its transition function P has the form

$$P(x,B) = \mathbb{P}^e\left(S_{n+1} \in B \mid S_n = x\right) = \varepsilon_x * \mu(B)$$

for all $(x,B) \in G \times \mathfrak{B}(G)$.

In this case the harmonic functions with respect to (Y_n) (or P) will be called μ-*harmonic.*

The sets of bounded or bounded continuous μ-harmonic functions on G will be abbreviated by $\mathrm{H}^b_\mu(G)$ and $\mathrm{H}^0_\mu(G)$ respectively.

We note that if μ is spread out,

$$\mathrm{H}^b_\mu(G) \subset \mathrm{H}^0_\mu(G)\,.$$

Clearly, $f \in \mathrm{H}^b_\mu(G)$ if and only if it satisfies the *mean value property*

$$f(x) = \int_G f(xy)\,\mu(\mathrm{d}y)\,(= Pf(x)) \ \ (\mathrm{MVP})\,,$$

whenever $x \in G$.

Within this setting Definition 5.3 reads as follows

5.5.2. *Definition.* The state space $(G, \mathfrak{B}(G))$ of the random walk (S_n) with defining measure $\mu \in \mathcal{M}^1(G)$ is said to admit the *Liouville property* (LP) or (LP^0) provided $\mathrm{H}^b_\mu(G)$ or $\mathrm{H}^0_\mu(G)$ contain only constant functions respectively.

The symmetry properties for random walks mentioned in subsection 4.3 lead to the equality

$$h(\mu) = h(\mu^\sim)$$

under the assumptions that G is unimodular and $\mu << \omega_G$. From this equality follow the one-to-one correspondences

$$\mathrm{H}^b_\mu(G) \leftrightarrow \mathrm{H}^b_{\mu^\sim}(G)$$

and

$$\mathrm{H}^0_\mu(G) \leftrightarrow \mathrm{H}^0_{\mu^\sim}(G)$$

as well as those between the related Liouville properties (LP) and (LP^0).

The following result is useful for the determination of Liouville properties depending on G and μ.

5.5.3. *Theorem.* The subsequent statements are equivalent for spread-out measures $\mu \in \mathcal{M}^1(G)$.

(i) (LP) holds,
(ii) (LP^0) holds,
(iii) $\mathfrak{I}$ is trivial $[\mathbb{P}^e]$,
(iv) $\mathfrak{S}$ is trivial $[\mathbb{P}^e]$,
(v) $h(\mu) = 0$.

The equivalence (i) $\Longleftrightarrow$ (ii) depends on the inclusion $\mathrm{H}^b_\mu(G) \subset \mathrm{H}^0_\mu(G)$. (i) $\Longleftrightarrow$ (iii) follows from Theorem 5.4, (i) $\Longleftrightarrow$ (iv) has been quoted in Theorem 5.5.1, and (iv) $\Longleftrightarrow$ (v) is the contents of Theorem 4.2.

It should be noted that μ-harmonic functions on a locally compact group G can be introduced without reference to a random walk (S_n), by the mean value property (MVP). There is, however, the canonical construction of a random walk for a given measure μ an the transition function induced by μ.

5.5.4. *Remark.* Necessary as well as sufficient conditions for the validity of (LP^0) depending on special choices of the measure $\mu \in \mathcal{M}^1(G)$ defining a random walk (S_n) in G can be proved independently of the previous theorem (Y. Derriennic [17]).

(1) Suppose that μ is adapted. Then the fact that $\mathfrak{I}$ is trivial $[\mathbb{P}^e]$ implies (LP^0).
(2) If μ is spread-out, then (LP^0) implies that $\mathfrak{I}$ is trivial $[\mathbb{P}^e]$.

5.5.5. *Remark.* In confirming the validity of (LP) and (LP^0) for a random walk (S_n) in a locally compact second countable group G and its defining measure $\mu = \varphi \cdot \omega_G$ with $\varphi \in \mathrm{L}^1(G, \omega_G)$ the following representation of the asymptotic entropy of (S_n) is successfully applied.

Suppose that for all $n \geq 1$

$$H(\mu^n) =: H(\varphi_n) = -\int_G \varphi_n(x) \log \varphi_n(x)\, \omega_G(\mathrm{d}x) < \infty\,.$$

Then

$$h(\mu) = \lim_{n\to\infty} (H(\varphi^n) - H(\varphi_{n-1})) = \frac{1}{n} H(\varphi_n)\,.$$

5.5.6. *Selected results on the Liouville property.* of a random walk in G with defining measure $\mu \in \mathcal{M}^1(G)$.
Classifying groups G with.

$$h(\mu) = 0 \qquad (*)$$

for some $\mu \in \mathcal{M}^1(G)$. R. Azencott ([6]), V. A. Kaimanovich ([35]) and A. M. Vershik ([37]).

There exists an adapted μ with $(*)$ if and only if G is amenable. See V. A. Kaimanovich ([36]), J. Rosenblatt ([42]).
Classes of groups satisfying (LP).

(1) G. Choquet and J. Deny ([10])
If G is Abelian, then (LP) holds if and only if μ is adapted.
(2) Y. Kawada and K. Ito ([38]), E. B. Dynkin and M. B. Maljutov ([20])
If G is compact or nilpotent and μ adapted and $\mu << \omega_G$, then (LP), hence (LP^0) holds.
(3) A. Raugi ([40]), B. E. Johnson ([34])
If G is nilpotent of degree > 2 and μ adapted (not necessarily spread out), then (LP^0) holds.
(4) S. Glasner ([27]), C.-H. Chu and A. T-M. Lau ([13])
If G is a central group, $\mu \in \mathcal{M}^1(G)$ adapted, then (LP^0) holds.

Exhibiting measures $\mu \in \mathcal{M}^1(G)$ satisfying $(*)$.

(1) A. Avez ([4],[5]), see also ([3])
Let G be non-exponential, $\mu = \varphi \cdot \omega_G$ with bounded φ having compact support. Then $(*)$, hence (LP) holds.
(2) Y. Guivarc'h ([29],[28])
Let G be amenable, connected Lie and $\mu \in \mathcal{M}^1(G)$ be centered of the form $\mu = \varphi \cdot \omega_G$, where φ is continuous with compact support. Then $(*)$ hence (LP) holds.
(3) C.-H. Chu and C.-W. Leung ([15])
If G is an almost connected [IN] group and μ is adapted, then (LP) holds.

5.6. Remark. In view of the discussion of the non-Liouville behavior of random walks in G which will follow in the next sections we comment briefly on two groups for which only a very special choice of $\mu \in \mathcal{M}^1(G)$ yields the condition $(*)$.

Let S_∞ denote the group of permutations of finite order of a countably infinite set. There exists a symmetric $\mu \in \mathcal{M}^1(S_\infty)$ satisfying $0 < H(\mu) < \infty$. Since every element of S_∞ is of finite order, for each $\mu \in \mathcal{M}^1(S\infty)$ of finite support one has $h(\mu) = 0$. On the other hand the group $\mathbb{F}_0(\mathbb{Z}^k) := \mathbb{F}_0(\mathbb{Z}^k, \mathbb{Z}/\mathbb{Z}^2)$ of finite configurations of $\mathbb{Z}^3$ admits a measure $\mu \in \mathcal{M}^1(\mathbb{F}_0(\mathbb{Z}^3))$ with $h(\mu) = 0$, while for a general adapted μ of finite support $h(\mu) > 0$.

The groups $\mathbb{F}_0(\mathbb{Z}^k)$ for $k \geq 1$ are solvable of degree 2 and of exponential growth (not nilpotent). A general result due to V. A. Kaimanovich and A. M. Vershik ([37]) exhibits a necessary and sufficient condition for some specified measure μ on $\mathbb{F}_0(\mathbb{Z}^k)$ such that (LP) holds.

More precisely, this condition says that the component random walk on $\mathbb{Z}^k$ of the random walk defined by μ on $\mathbb{F}_0(\mathbb{Z}^k)$ viewed as a semidirect product is recurrent. This property in turn can be verified for symmetric μ in the cases $k = 1$ and $k = 2$. For detailed arguments see also Y. Derriennic ([17]).

6. The Poisson boundary

We start with a few technical preparations. A topological space M is called a (topological) *G-space* if there is a locally compact group G acting on M in the sense that

$$(x, m) \mapsto xm$$

from $G \times M$ in to M is continuous and satisfies the conditions $em = m$ and

$$(xy)m = x(ym)$$

whenever $x, y \in G$ and $x \in M$.

At a later stage of our exposition we shall also consider (measurable) G-spaces M furnished with a σ-algebra $\mathfrak{M}$ and an $\mathfrak{M}$-measurable action of (a locally compact) group G.

Special G-spaces are the *homogeneous spaces* M of the form G/H, where H is a closed subgroup of G. In this case G acts transitively on M.

Further special G-spaces are the *symmetric spaces* of a semisimple Lie group G which have the form $M = G/K$, where K is a maximal compact subgroup of G. For some spaces G acts on M by isometries, i.e., G/K is a metric space, and G preserves the metric. Examples of G-spaces, in particular of (Riemannian) symmetric spaces are the spaces arising from groups $SL(r, \mathbb{R})$ with $r \geq 2$.

On measurable G-spaces a convolution of measures $\mu \in \mathcal{M}^1(G)$ and $\nu \in \mathcal{M}^1(G)$ can be introduced as a measure $\rho \in \mathcal{M}^1(M)$ by

$$\int f(m)\,\rho(\mathrm{d}m) := \int_G \left(\int_M f(xm)\,\nu(\mathrm{d}m) \right) \mu(\mathrm{d}x)$$

for all $f \in \mathrm{M}(M, \mathfrak{M})$.

Let M and M' be two G-spaces. A mapping $\Phi \colon M \to M'$ is called *equivariant* if

$$\Phi(xm) = x\,\Phi(m)$$

for all $x \in G$, $m \in M$. For measures $\mu \in \mathcal{M}^1(G)$ and $\nu \in \mathcal{M}^1(M)$ one has

$$\Phi(\mu * \nu) = \mu * \Phi(\nu)\,.$$

Given a measure $\mu \in \mathcal{M}^1(G)$ we have the notion of a *μ-invariant measure* $\nu \in \mathcal{M}^1(M)$ given by

$$\mu * \nu = \nu\,.$$

μ-invariant measures will be of importance in connection with μ-boundaries of groups G.

As in the previous section we are considering a random walk (S_n) in a second countable locally compact group G with defining measure $\mu \in \mathcal{M}^1(G)$ and start in e. (S_n) is canonically realized on the probability space

$$(\Omega, \mathfrak{A}, \mathbb{P}^e) := (G^{\mathbb{N}}, \mathfrak{B}(G)^{\otimes\mathbb{N}}, \mathbb{P}^e)\,,$$

where $\mathbb{P}^e$ denotes the law of (S_n).

Since G acts on $G^{\mathbb{N}}$, a convolution $\rho * \sigma$ of measures $\rho \in \mathcal{M}^1(G)$ and $\sigma \in \mathcal{M}^1(G^{\mathbb{N}})$ is defined as a measure in $\mathcal{M}^1(G^{\mathbb{N}})$.

Clearly,

$$\theta\mathbb{P}^e = \mu * \mathbb{P}^e\,,$$

where θ denotes the shift operator on $\Omega := G^{\mathbb{N}}$.

Now one can compute for each $n \geq 1$ the mutual information

$$\begin{aligned} I(S_1, S_{n+1}) &= \int_G \left(\int_G \log \frac{\mathrm{d}(\varepsilon_x * \mu^n)}{\mathrm{d}\mu^{n+1}}(y)\, (\varepsilon_x * \mu^n)(\mathrm{d}y) \right) \mu(\mathrm{d}x) \\ &= \int_G \left(\int_\Omega \log \frac{\mathrm{d}(\varepsilon_x * \mathbb{P}^e)}{\mathrm{d}\theta\mathbb{P}^e}\Big|_{\sigma(\{S_n\colon n\geq 1\})} (\omega)\, (\varepsilon_x * \mathbb{P}^e)(\mathrm{d}\omega) \right) \mu(\mathrm{d}x) \end{aligned}$$

In fact, the computation relies on the joint probabilities

$$\begin{aligned} \mathbb{P}^e\left(S_1 \in B_1, S_{n+1} \in B_{n+1}\right) &= \int_{B_1} (\varepsilon_x * \mu^n)\,(B_{n+1})\, \mu(\mathrm{d}x) \\ &= \int_{B_1} (\varepsilon_x * \mathbb{P}^e)\,(S_n \in B_{n+1})\, \mu(\mathrm{d}x)\,, \end{aligned}$$

$B_n \in \mathfrak{B}(G)$, $n \geq 2$ and applies favorably to a measure-theoretic result on the Markov kernel

$$(x, \cdot) \mapsto \varepsilon_x * \mathbb{P}^e(\cdot)$$

from G to Ω.

6.1. Lemma Y. Derriennic [17]**.** Let $(E, \mathfrak{E}, \nu)$ be a probability space and N a Markov kernel from $(E, \mathfrak{E})$ to a separable measurable space $(F, \mathfrak{F})$. Putting

$$\nu' := N\nu$$

and

$$\rho := \nu \otimes N\,,$$

we have $\rho << \nu \otimes \nu'$ if and only if for $\nu \otimes \nu'$-a.a. $x \in E$, $N(x, \cdot) << \nu'$, hence

$$\frac{\mathrm{d}\rho}{\mathrm{d}(\nu \otimes \nu')}(x, y) = \frac{\mathrm{d}N(x, \cdot)}{\mathrm{d}\nu'}(y)$$

for $\nu \otimes \nu'$ -a.a. $(x, y) \in E \times F$.

Moreover,

$$H(\rho \parallel \nu \otimes \nu') < \infty$$

if and only if

$$\int_E \left(\int_F \log \frac{\mathrm{d}N(x, \cdot)}{\mathrm{d}\nu'}(y)\, N(x, \cdot)(\mathrm{d}y) \right) \nu(\mathrm{d}x)$$

exists, and in this case both terms coincide.

In a next step we replace the separable measurable space $(\Omega, \sigma(\{S_n\colon n \geq 1\}))$ by the measurable space $(\Omega, \mathfrak{J})$ not ignoring that in general this space is not separable. Fortunately, however, there exists a separable subspace $(\Omega, \mathfrak{J}')$ with $\mathfrak{J}' = \mathfrak{J}[\mathbb{P}^e]$.

Now

$$h(\mu) = I(S, \mathfrak{J}) = I(S, \mathfrak{J}')$$

can be computed with the help of the joint probabilities

$$\mathbb{P}^e\left((S_1 \in B_1) \cap J\right) = \int_{B_1} \varepsilon_x * \mathbb{P}^e(J)\, \mu(\mathrm{d}x)\,,$$

$B_1 \in \mathfrak{B}(G)$, $J \in \mathfrak{J}$, together with another application of the lemma.

We arrive at the

6.2. Theorem.

$$h(\mu) = \int_G \left(\int_\Omega \log \frac{\mathrm{d}(\varepsilon_x * \mathbb{P}^e)}{\mathrm{d}\theta\mathbb{P}^e}\bigg|_{\mathfrak{J}'} (\omega)\, (\varepsilon_x * \mathbb{P}^e)(\mathrm{d}\omega) \right) \mu(\mathrm{d}x)\,.$$

If, in addition, $\mathfrak{J}'$ is stable with respect to the action of G, then

$$h(\mu) = \int_G \left(\int_\Omega \log \frac{\mathrm{d}(\varepsilon_{x^{-1}} * \mathbb{P}^e)}{\mathrm{d}\mathbb{P}^e}\bigg|_{\mathfrak{J}'} (\omega)\, \mathbb{P}^e(\mathrm{d}\omega) \right) \mu(\mathrm{d}x)\,.$$

For a motivation of the integral term see [23] by H. Furstenberg.

Meanwhile we are on the way to discuss the anti-Liouville property, i.e., the case that $h(\mu) > 0$ or that the space $\mathrm{H}^b_\mu(G)$ is *trivial* in the sense that it contains non-constant functions.

Here the notion of a Poisson boundary will be essential. We are going to present two approaches to the boundary theory for random walks in G with defining measure $\mu \in \mathcal{M}^1(G)$.

6.3. Furstenberg's approach to the topological Poisson boundary. Let M be a topological G-space and $\nu \in \mathcal{M}^1(M)$ a μ-invariant measure. For any function $\varphi \in \mathrm{M}^b(M, \mathfrak{B}(M))$ the mapping

$$x \mapsto f(x) := \int_M \varphi(xz)\, \nu(\mathrm{d}z) = \int_M \varphi(z)\, \varepsilon_x * \nu(\mathrm{d}z)$$

from G into $\mathbb{R}$ belongs to $\mathrm{H}^b_\mu(G)$.

For a converse of this statement one has the profound

6.3.1. *Theorem - H. Furstenberg* [26]. Let $\mu \in \mathcal{M}^1(G)$. There exist a G-space $(B, \mathfrak{B}(G))$ and a μ-invariant measure $\nu \in \mathcal{M}^1(M)$ such that every function $f \in \mathrm{H}^b_\mu(G)$ admits a *Poisson representation* of the form

$$f(x) = \int_M \varphi(z)\, \varepsilon_x * \nu(\mathrm{d}z) \qquad \text{(PR)}$$

for all $x \in G$, where $\varphi \in \mathrm{M}^b(B, \mathfrak{B}(G))$ is suitably chosen.

The G-space $(B, \mathfrak{B}(G))$ is said to be the *Poisson space* $P(G, \mu)$ corresponding to the pair (G, μ) (or just of G). We also apply the notation (B, ν) for $P(G, \mu)$ in order to specify the underlying μ-invariant measure ν.

In fact, (B, ν) is a μ-boundary and hence called the *Poisson boundary* of G as we shall see below.

Starting with a random walk (S_n) in G with defining measure $\mu \in \mathcal{M}^1(G)$ it can be shown that given a G-space $(M, \mathfrak{B}(M))$ and a μ-invariant measure $\nu \in \mathcal{M}^1(M)$ the sequence

$$\left(\varepsilon_{S_n(\omega)} * \nu\right)_{n \geq 1}$$

τ_w-converges for $\mathbb{P}^e$-a.a. $\omega \in \Omega = G^{\mathbb{N}}$.

If there exists an $m \in M$ such that

$$\varepsilon_{S_n(\omega)} * \nu \to \varepsilon_m \quad (\tau_w)$$

for $\mathbb{P}^e$-a.a. $\omega \in \Omega$, as $n \to \infty$, then (M, ν) is called a *μ-boundary* of G.

6.3.2. *Theorem.* Every μ-boundary (M, ν) is an equivariant image of the Poisson boundary $P(G, \mu)$.

In other words, if

$$P(G, \mu) = (B, \nu_0)$$

for some μ-invariant measure $\nu_0 \in \mathcal{M}^1(B)$, then there exists an equivariant measurable mapping $\Phi\colon B \to M$ such that

$$\Phi(\nu_0) = \nu\,.$$

6.3.3. *Special case.* Let G be a semi-simple Lie group (with finite center) and $\mu \in \mathcal{M}^1(G)$ with $\mu << \omega_G$. H. Furstenberg showed in [24] that the Poisson boundary

$$(B, \nu) = P(G, \mu)$$

of G is a compact homogeneous (G-)space.

In fact, (B, ν) is necessarily one of finitely many covering spaces of the homogeneous space $B(G)$ arising from the Iwasawa decomposition $G = KAN$ of G, namely

$$B(G) = G/T\,,$$

where T is the normalizer in G of AN.

Since $KT = G$, K acts transitively on $B(G)$. Suppose that μ is *spherical* in the sense that it is right and left K-invariant, then

$$P(G, \mu) = (B(G), \omega_B)\,,$$

where ω_B denotes the unique K-invariant measure in $\mathcal{M}^1(B(G))$.

From Theorem 6.3.1 follows that all spherical measures in $\mathcal{M}^1(G)$ lead to the same class $\mathrm{H}^b_\mu(G)$. Thus the μ-harmonic functions on G are independent of μ, but still depend on the choice of the maximal compact subgroup K of G.

6.4. Derriennic's approach to the measurable Poisson boundary. It gives some insight into the construction of the Poisson boundary $P(G, \mu)$ which is the content of Theorem 6.3.1.

Let $(M, \mathfrak{M})$ be a (measurable) G-space with a σ-algebra $\mathfrak{M}$ having a countable basis, and let $\nu \in \mathcal{M}^1(M)$ be μ-invariant.

For any $f \in \mathcal{M}^b(M, \mathfrak{M})$ we define the function $g = Rf \in \mathrm{H}^b_\mu(G)$ by

$$g(x) := \int_M f(xz)\,\nu(\mathrm{d}z) = \int_M f(z)\,\varepsilon_x * \nu(\mathrm{d}z)$$

for all $x \in G$. Compare with the Poisson representation (PR) in Section 6.3.

The operator $R\colon \mathrm{M}^b(M, \mathfrak{M}) \to \mathrm{H}^b_\mu(G)$ is a linear contraction with respect to the sup-norm.

R is said to be injective if $Rf = 0$ if and only if $f = 0$ $[\varepsilon_x * \nu]$ for each $x \in G$.

In $\mathrm{H}^b_\mu(G)$ one introduces a multiplication

$$(g, g') \mapsto g \odot g'$$

by

$$g \odot g' := \lim_{n\to\infty} \int_G g(xy) g'(xy)\,\mu^n(\mathrm{d}y) = \int_\Omega \lim_{n\to\infty} g(xS_n) g'(xS_n)\,\mathrm{d}\mathbb{P}^e$$

for all $g, g' \in \mathrm{H}^b_\mu(G)$, $x \in G$.

With this multiplication $\mathrm{H}^b_\mu(G)$ becomes an algebra isomorphic to the algebra of invariant G-valued random variables on $(G^{\mathbb{N}}, \mathfrak{B}(G)^{\otimes\mathbb{N}}, \mathbb{P}^e)$ considered modulo $\varepsilon_x * \mathbb{P}^e$ for all $x \in G$.

Now, (M, ν) is called a *μ-boundary* if R is injective and multiplicative with respect to the multiplication $\odot$, and a *Poisson boundary* provided R is surjective.

This definition appears to be consistent with that given in Section 6.3.

From Theorem 6.2 one obtains the following useful application.

6.4.1. *Theorem - Y. Derriennic* [17]. For a μ-boundary (M, ν) of G one has the inequality

$$h(\mu) \geq - \int_G \left(\int_M \log \frac{\mathrm{d}(\varepsilon_{x^{-1}} * \nu)}{\mathrm{d}\nu}(z)\, \nu(\mathrm{d}z) \right) \mu(\mathrm{d}x)$$

and equality between the two terms if and only if (M, ν) is the Poisson boundary $P(G, \mu)$ of G provided $h(\mu) < \infty$.

Referring to Theorem 5.5.3 we can rephrase equivalence (i) $\Longleftrightarrow$ (v) by saying that $h(\mu) = 0$ if and only if $P(G, \mu)$ is trivial in the sense that $\mathrm{H}^b_\mu(G)$ consists only of constant functions, i.e., (LP) holds.

7. Examples of Poisson boundaries

Most of the following examples will uncover nontrivial Poisson spaces. It should be noted that the description of nontrivial Poisson spaces can be quite involved (see f.e. A. Erschler's work [22]).

7.1. For the groups G listed in subsection 5.5.6 the Poisson boundary $P(G, \mu)$ is trivial at least for adapted and spread-out measures $\mu \in \mathcal{M}^1(G)$. (See Theorem 5.5.3 together with the note at the end of the previous section.)

7.2. Let G be the affine group $Aff(\mathbb{Z}(\frac{1}{2}))$ of the dyadic rational line, i.e.,

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a = 2^n, b = \frac{k}{2^l} \text{ with } k, l, n \in \mathbb{Z} \right\}$$

generated by the matrices

$$\alpha := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

together with the relation

$$\beta^2 \alpha = \alpha \beta\,.$$

G is solvable of length 2 and has exponential growth.

7.2.1. Choosing $\mu \in \mathcal{M}^1(G)$ with

$$\mu(\alpha) = \mu(\alpha^{-1}) := \frac{1}{2}p$$

and

$$\mu(\beta) = \mu(\beta^{-1}) := \frac{1}{2}q,$$

where $p, q > 0$, $p + q = 1$, one obtains $h(\mu) = 0$, which implies that $P(G, \mu)$ is trivial.

7.2.2. If one takes

$$\mu(\alpha) := p \text{ and } \mu(\beta) := q$$

with $p, q > 0$, $p + q = 1$, then $h(\mu) > 0$, i.e., $P(G, \mu)$ is nontrivial.

7.3. The connected real affine group $G = Aff(\mathbb{R})$ presents a behavior contrasting that of example 7.2.

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} =: (a(g), b(g)) = (a, b) \colon a > 0, b \in \mathbb{R} \right\}$$

is solvable, of exponential growth, and non-unimodular. Clearly,

$$\Delta(g) = \frac{1}{a(g)}$$

for all $g = (a(g), b(g)) \in G$, and

$$\mathrm{d}\omega_G(a, b) = \frac{1}{a^2} \mathrm{d}a \mathrm{d}b \, .$$

Let $\mu \in \mathcal{M}^1(G)$ be a measure with bounded ω_G-density and compact support.

We first note that any function in $\mathrm{H}^0_\mu(G)$ is constant, i.e., (LP^0) holds provided

$$\int_G \log a(g) \, \omega_G(\mathrm{d}g) \geq 0 \, .$$

Moreover,

$$h(\mu^\sim) = h(\mu) - \int_G \log \Delta(g) \, \mu(\mathrm{d}g)$$

and

$$h(\mu) = \max \left(0, - \int_G \log a(g) \, \mu(\mathrm{d}g) \right) .$$

Thus there exists a non constant function in $\mathrm{H}^0_\mu(G)$, provided

$$\int_G \log a(g) \, \mu(\mathrm{d}g) < 0 \, , \qquad (*)$$

hence $P(G, \mu)$ is nontrivial.

We also know that under the condition (*) there exists a unique μ-invariant measure $\nu \in \mathcal{M}^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \left(\int_G \varphi(gx) \, \mu(\mathrm{d}g) \right) \nu(\mathrm{d}x) = \int_{\mathbb{R}} \varphi(x) \, \nu(\mathrm{d}x)$$

for all bounded continuous functions φ on $\mathbb{R}$. Given any such function φ on $\mathbb{R}$, the mapping

$$g \mapsto f(g) := \int_{\mathbb{R}} \varphi(gx) \, \nu(\mathrm{d}x) \qquad (**)$$

belongs to $\mathrm{H}^0_\mu(G)$. Conversely, if (*) is fulfilled, then any $f \in \mathrm{H}^b_\mu(G)$ is of the form (**) for some $\varphi \in \mathrm{M}^b(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ (and the unique μ-invariant measure $\nu \in \mathcal{M}^1(\mathbb{R})$). In other words $P(G, \mu) = (\mathbb{R}, \nu)$. This statement, due to L. Elie in [21], can be deduced from Theorem 6.4.1 as indicated in [18] by Y. Derriennic. In fact, it suffices to show that

$$h(\mu) = \int_G \left(\int_{\mathbb{R}} \log \frac{\mathrm{d}(\varepsilon_{g^{-1}} * \nu)}{\mathrm{d}\nu}(x) \, \nu(\mathrm{d}x) \right) \mu(\mathrm{d}g),$$

which appears to be identical with

$$- \int_G \log a(g) \, \mu(\mathrm{d}g) \, .$$

For the last computation one applies that $\mu << \omega_G$ implies $\nu = \psi \cdot \lambda$ for $\psi \in \mathrm{L}^1(\mathbb{R}, \lambda)$, where λ denotes the Lebesgue measure of $\mathbb{R}$. Then

$$\frac{\mathrm{d}(\varepsilon_g * \nu)}{\mathrm{d}\nu}(x) = \frac{1}{a(g)} \frac{\psi\left(\frac{x - b(g)}{a(g)}\right)}{\psi(x)}$$

for all $g \in G$, $x \in \mathbb{R}$.

7.4. For the free group $G := \mathbb{F}_2$ of two generators a, b and the measure

$$\mu := \frac{1}{4}(\varepsilon_a + \varepsilon_b + \varepsilon_{a^{-1}} + \varepsilon_{b^{-1}}) \in \mathcal{M}^1(\mathbb{F}_2)$$

one can compute

$$h(\mu) = \frac{1}{2}\log 3\,.$$

In order to describe the Poisson boundary of G one chooses M to be the space of reduced infinite words a, b, a^{-1}, b^{-1} and $\nu \in \mathcal{M}^1(M)$ to be the law of the Markov chain with 4 states whose transition probabilities are given by the following diagram

	a	b	a^{-1}	b^{-1}
a	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
b	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
a^{-1}	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
b^{-1}	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$

its initial distribution being the uniform distribution charging each state with $\frac{1}{4}$. Then

$$\frac{\mathrm{d}(\varepsilon_a * \nu)}{\mathrm{d}\nu}(m) = \begin{cases} 3 & \text{for } m \in \{m_0 = a \text{ or } b \text{ or } b^{-1}\}, \\ \frac{1}{3} & \text{for } m \in \{m_0 = a^{-1}\} \end{cases}$$

implies

$$\int_M \log \frac{\mathrm{d}(\varepsilon_a * \nu)}{\mathrm{d}\nu}(m)\,\nu(\mathrm{d}m) = \frac{3}{4}\log 3 - \frac{1}{4}\log 3\,,$$

hence

$$\int_G \left(\int_M \frac{\mathrm{d}(\varepsilon_{x^{-1}} * \nu)}{\mathrm{d}\nu}(m)\,\nu(\mathrm{d}m) \right) \mu(\mathrm{d}x) = \frac{1}{2}\log 3\,.$$

Theorem 6.4.1 implies that $P(G, \mu) = (M, \nu)$.

The first proof of this fact is due to E .B. Dynkin and M. B. Maljutov [20].

7.5. Let $G := SL(d, \mathbb{R})$ with a decomposition KS, where K denotes the subgroup of G consisting of orthogonal $(d \times d)$-matrices and S the subgroup of upper triangular $(d \times d)$-matrices with negative terms along the diagonal. $K \cap S$ is nontrivial, though finite.

As in subsection 6.3.3 we assume $\mu \in \mathcal{M}^1(G)$ to be a spherical measure with $\mu << \omega_G$. Let $B := G/S$. Since K acts transitively on the G-space B, there exists a unique K-invariant measure ω_B which is the unique μ-invariant measure in $\mathcal{M}^1(B)$.

It turns out that (B, ω_B) is the Poisson boundary $P(G, \mu)$ of G.

7.5.1. Now let $d = 2$. Since B can be identified with the projective space P^1, (P^1, ω_B) is also a μ-boundary of G (for spherical μ).

In order to study harmonic functions on G it is useful to consider the symmetric space G/K. G acts on G/K by isometries, i.e., G/K is a metric space and G preserves this metric.

It is known that G/K can be identified with the open unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}\,.$$

Further identifications are

$$G/S \simeq \partial\mathbb{D} := \{z \in \mathbb{C} \colon |z| = 1\} \simeq B \simeq \mathbb{P}^1\,.$$

We turn to the (μ-)harmonic functions f on G that depend only on the cosets of K.

For all $x \in G$ and $k \in K$

$$f(xk) = \int_G f(xky)\,\mu(\mathrm{d}y) = \int_G f(xz)\,\varepsilon_k * \mu(\mathrm{d}z) = \int_G f(xz)\,\mu(\mathrm{d}z) = f(x)\,,$$

since μ is spherical and $\varepsilon_k * \mu = \mu$. To f there is associated a function $\overline{f}$ on G/K, which is also called harmonic and where the Poisson representation (PR) in the sense of Theorem 6.3.1 is given by

$$\overline{f}(x(a))(= \overline{f}(xk)) = f(x) = \int_B \hat{f}(x\xi)\,\omega_B(\mathrm{d}\xi) = \int_B \hat{f}(\xi)\frac{\mathrm{d}(\varepsilon_x * \omega_B)}{\mathrm{d}\omega_B}(\xi)\,\omega_B(\mathrm{d}\xi)$$

with some bounded function $\hat{f}$ on B and

$$\frac{\mathrm{d}(\varepsilon_x * \omega_B)}{\mathrm{d}\omega_B}(\xi) = P(x(a), \xi)$$

for all $x \in G$, $\xi \in B$, where $P(z, \xi)$ denotes the classical Poisson kernel of the harmonic function $\overline{f}$ on $\mathbb{D}$. The formula

$$\overline{f}(z) = \int_B \hat{f}(\xi) P(z, \xi)\,\omega_B(\mathrm{d}\xi)\,,$$

valid for all $z \in \mathbb{D}$, implies that the harmonic functions on the group $SL(2, \mathbb{R})$ coincide with those on the disk $\mathbb{D}$.

7.5.2. *Remark.* In analogy to $SL(2, \mathbb{R})$ H. Furstenberg discusses in [25] the case $SL(3, \mathbb{R})$. Moreover, he identifies the Poisson boundary of discrete subgroups of $SL(d, \mathbb{R})$ for arbitrary $d \geq 2$.

8. Extension to hypergroups

Hypergroups are locally compact spaces K on which the bounded measures convolve as in the case of a locally compact group. More precisely, on K there exists a convolution $*$ such that $(\mathcal{M}^b(K), *)$ becomes a Banach algebra with the following properties

HG1 The mapping

$$(\mu, \nu) \mapsto \mu * \nu$$

from $\mathcal{M}^b(K) \times \mathcal{M}^b(K)$ into $\mathcal{M}^b(K)$ is τ_w-continuous.

HG2 For $x, y \in K$ the convolution $\varepsilon_x * \varepsilon_y$ belongs to $\mathcal{M}^1(K)$ and has compact support.

HG3 There exists a unit element $e \in K$ with

$$\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$$

for all $x \in K$, and an involution

$$\mu \mapsto \mu^-$$

such that

$$\varepsilon_{x^-} * \varepsilon_{y^-} = (\varepsilon_x * \varepsilon_y)^-$$

with the additional property

$$e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y) \text{ if and only if } x = y^-$$

whenever $x, y \in K$.

HG4 The mapping

$$(x, y) \mapsto \operatorname{supp}(\varepsilon_x * \varepsilon_y)$$

from $K \times K$ into the space of compact subsets of K furnished with the Michael topology is continuous.

A hypergroup $K(= (K, *))$ is said to be commutative if

$$\mu * \nu = \nu * \mu$$

for all $\mu, \nu \in \mathcal{M}^b(K)$. In this case $(M^b(K), *, -)$ is a commutative Banach *-algebra.

Clearly, locally compact groups are hypergroups. By τ_w-continuous linear extension the convolution $*$ of a hypergroup K is uniquely determined by the convolution of Dirac measures. More precisely, given $\mu, \nu \in \mathcal{M}^b(K)$,

$$\mu * \nu(f) = \int_G \left(\int_G f(x * y)\, \mu(\mathrm{d}x) \right) \nu(\mathrm{d}y)\,,$$

where

$$f(x * y) := \varepsilon_x * \varepsilon_y(f) = \int_G f\, \mathrm{d}(\varepsilon_x * \varepsilon_y)$$

whenever f belongs to the space $\mathrm{C}_0(K)$ of continuous functions f on K vanishing at infinity.

There are various constructions of hypergroup structures, f.e., on $K = \mathbb{Z}_+$ (via polynomials) and on $K = \mathbb{R}_+$ (via special functions), but also extension procedures to obtain new hypergroups from known ones.

For detailed knowledge on hypergroups the reader is referred to the book [9] by W. R. Bloom and H. Heyer. In the seminal monograph [8] by Yu. M. Berezansky and A. A. Kalyuzhnyi harmonic analysis has been developed in a slightly different axiomatic setting.

To recall a few basic notions and some useful notation seems to be in order.

A *subhypergroup* of a hypergroup K is a subset H of K satisfying the condition $H^- = H$ and $H * H \subset H$, where for arbitrary sets A, B in K

$$A * B := \bigcup \{\mathrm{supp}(\varepsilon_a * \varepsilon_b)\colon a \in A, b \in B\}\,.$$

A subhypergroup H of K is said to be *supernormal* if

$$\{x\} * H * \{x^-\} \subset H$$

for each $x \in K$.

The class of supernormal hypergroups of K contains that of *normal* subhypergroups H given by

$$\{x\} * H = H * \{x\}$$

for each $x \in K$.

If H is a compact normal or a supernormal subhypergroup of K, the right *coset space*

$$K/H := \{H * \{x\}\colon x \in K\}$$

carries a hypergroup structure given by

$$\varepsilon_{H*\{x\}} * \varepsilon_{H*\{y\}} := \int_K \varepsilon_{H*\{t\}}\, (\varepsilon_x * \varepsilon_y)\, \mathrm{d}t$$

for $x, y \in H$. The *center* of a hypergroup K is the subhypergroup

$$Z(K) := \{t \in K\colon \varepsilon_t * \varepsilon_x = \varepsilon_x * \varepsilon_t \text{ for all } x \in K\}\,,$$

for which $Z(K)^- = Z(K)$ holds.

A rather deep result in the analysis of hypergroups is the existence of a unique (right invariant) Haar measure on an arbitrary hypergroup K. This nonvanishing Haar measure $\omega_K \in \mathcal{M}_+(K)$ is defined by

$$\int_K f\, \mathrm{d}(\omega_K * \varepsilon_a) = \int_K f\, \mathrm{d}\omega_K$$

for all $f \in \mathrm{C}^c(K)$.

The proof of the existence of ω_K due to Yu. A. Chapovsky (2012) is still unpublished but unquestioned.

Given a compact subhypergroup H of K the *double coset hypergroup*

$$K//H := \{H * \{x\} * H : x \in K\}$$

carries the convolution

$$\varepsilon_{H*\{x\}*H} * \varepsilon_{H*\{y\}*H} := \int_H \varepsilon_{H*\{z\}*H} \, (\varepsilon_x * \omega_H * \varepsilon_y)(\mathrm{d}t)$$

and the involution

$$(H * \{x\} * H)^- := H * \{x^-\} * H$$

defined for all $x, y \in K$.

In a recent paper [2] M. Amini and C.-H. Chu introduced a new class of hypergroups in analogy to the group case.

A hypergroup is called *nilpotent (of class r)* if there exists a finite descending series

$$K = K_0 \supset K_1 \supset \cdots \supset K_{r-1} = \{e\}$$

of supernormal subhypergroups $K_1, \ldots, K_{r-1}$ such that $K_{i-1}/K_i \subset Z(K/K_i)$ for $i = 1, \ldots, r$ and there is no such series of length $< r$. Note that $K_{i-1} \subset Z(K)$ and K_{i-1}/K_i is a group for $i < r$.

In the case of a locally compact group G and a compact subgroup H of G the double coset hypergroup $G//H$ is commutative if and only if (G, H) is a Gelfand pair. If (G, H) is not necessarily a Gelfand pair, then $G//H$ is a nilpotent hypergroup provided G is a nilpotent group.

Now we are prepared for the discussion of Liouville properties for a given hypergroup K and a measure $\mu \in \mathcal{M}^1(K)$.

Given a measure $\mu \in \mathcal{M}^b(K)$ and (Borel) function $f \in \mathrm{M}(K, \mathfrak{B}(K))$ one introduces the convolution $f * \mu$ by

$$f * \mu(x) := \int_G f(x * y^-) \, \mu(\mathrm{d}y)$$

for all $x \in K$.

A bounded continuous function f on K is said to be *right uniformly continuous* if the mapping

$$y \mapsto f * \varepsilon_{y^-}$$

on G is continuous.

The space of all (bounded) right uniformly continuous functions on K will be abbreviated by $\mathrm{C}^{ru}(K)$.

8.1. Definition. Let K be a hypergroup and $\mu \in \mathcal{M}^1(K)$. A function $f \in \mathrm{M}(K, \mathfrak{B}(K))$ is called μ-harmonic if

$$f = f * \mu\,.$$

We agree on the symbols $\mathrm{H}^c_\mu(K)$, $\mathrm{H}^{bc}_\mu(K)$ and $\mathrm{H}^{bu}_\mu(K)$ for the spaces of continuous, bounded continuous and bounded right uniformly continuous functions on K respectively. Similarly we abbreviate the *Liouville property* that μ-harmonic functions on K are constant, by $(\mathrm{LP}^{\mathrm{c}})$, $(\mathrm{LP}^{\mathrm{bc}})$ and $(\mathrm{LP}^{\mathrm{bu}})$ respectively.

A measure $\mu \in \mathcal{M}^b(K)$ is called *nondegenerate* if the subset

$$S(\mu) := \left(\bigcup_{n \geq 1} (\mathrm{supp}(\mu))^n \right)^c = \left(\bigcup_{n \geq 1} \mathrm{supp}(\mu^n) \right)^c$$

coincides with K, and *adapted* if the subhypergroup

$$K(\mu) := \left(\bigcup_{n \geq 1} (\mathrm{supp}(\mu) \cup \mathrm{supp}(\mu)^-)^n \right)^c$$

generated by $\mathrm{supp}(\mu)$ equals K.

As in the group case one can show that if a hypergroup K admits an adapted $\mu \in \mathcal{M}^b(K)$, then K is σ-compact, hence metrizable.

8.2. Theorem. Let K be a commutative hypergroup and μ an adapted measure in $\mathcal{M}^1(K)$.

Then ($\mathrm{LP}^{\mathrm{bu}}$) holds.

The proof of the theorem runs in analogy to the Choquet-Deny theorem in [10] and its extension in [12] by C.-H. Chu and T. Hilberdink, where the authors make use of the periodicity of a function $f \in \mathrm{H}_\mu^{bu}(K)$ in the form

$$f(x) = f(z * x)$$

for each $x \in K$ and $z \in Z(K) \cap S(\mu)^-$. Based on the commutative case one can proceed to the more general case.

8.3. Theorem. Let K be a nilpotent hypergroup and μ a nondegenerate measure in $\mathcal{M}^1(K)$.

Then ($\mathrm{LP}^{\mathrm{bu}}$), hence ($\mathrm{LP}^{\mathrm{bc}}$) holds.

The proof of the last statement relies on the method of regularization. In fact, given $f \in \mathrm{H}_\mu^{bc}(K)$ the function $h^\sim * f$ for $h \in \mathrm{L}^1(K, \omega_K)$ defined by

$$h^\sim * f(x) := \int_K h^\sim(x * y^-) f(y)\, \omega_K(\mathrm{d}y)$$

for all $x \in K$ belongs to $\mathrm{H}_\mu^{bu}(K)$.

8.4. Theorem. Let K be a compact hypergroup and $\mu \in \mathcal{M}^1(K)$ adapted.

Then (LP^{c}) holds.

This statement is an easy consequence of the Peter-Weyl Theorem for compact hypergroups.

Compact hypergroups are contained in the larger class of central hypergroups introduced by W. Hauenschild, E. Kaniuth and A. Kumar in [30].

Let $G(K)$ be the *maximal subgroup*

$$\{x \in K : \varepsilon_x * \varepsilon_{x^-} = \varepsilon_e\}$$

of a hypergroup K. Then K is called *central* if

$$K/(G(K) \cap Z(K)) \text{ is compact}.$$

Central hypergroups are *unimodular*, where the notion of a modular function is understood just as in the group case. If Z is a closed subgroup of $G(K) \cap Z(K)$ such that K/Z is compact, then $K = CZ$ for some compact subset of K. Note that there exists a non-compact, non-commutative central hypergroup. The relationship between central hypergroups and central groups becomes apparent in the example of the double coset $K//H$, where H is a compact subgroup of K. In this case $K//H$ is a central hypergroup.

The following result is due to S. Glasner in [27]. See also M. Amini in [1].

8.5. Theorem. Let K be a central hypergroup and μ an adapted measure in $\mathcal{M}^1(K)$. Then (LP$^{\mathrm{bu}}$) holds.

The proof makes use of an intermediate fact:
Let $P(K)$ denote the set of all continuous positive definite functions on K. For an adapted $\mu \in \mathcal{M}^1(K)$ any μ-harmonic function in the linear hull of $\mathrm{L}^1(K, \omega_K) \cap P(K)$ is constant.

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Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

E-mail address: herbert.heyer@uni-tuebingen.de

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