ON FOURIER ALGEBRA OF A HYPERGROUP CONSTRUCTED FROM A CONDITIONAL EXPECTATION ON A LOCALLY COMPACT GROUP

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ABSTRACT. We prove that the Fourier space of a hypergroup constructed from a conditional expectation on a locally compact group has a Banach algebra structure.

1. INTRODUCTION

It is well known [4] that, for a locally compact group G, the Fourier-Stieltjes space $\mathscr{B}(G)$ and the Fourier space $\mathscr{A}(G)$, make commutative Banach algebras.

For a hypercomplex system [2] or a DJS-hypergroup [1], one can also define the Fourier-Stieltjes and the Fourier spaces. It is well known that, being Banach spaces, they do not, in general, make Banach algebras. Conditions for this to be the case were considered in [11, 12] for commutative hypergroups and special type hypergroups called ultraspherical. In [9], a general sufficient condition for the Fourier and the Fourier-Stieltjes spaces of a locally compact hypergroup to be Banach algebras was obtained. It was also proved there that there exists the Fourier Banach algebra on the double coset hypergroup constructed from a locally compact group.

In this paper we prove that the Fourier space of a DJS-hypergroup that is constructed from a locally compact group and an expectation, see [6, 8], also has a Banach algebra structure. This result is a generalization of the one obtained in [12], where the author imposes additional conditions on the hypergroup.

The paper is organized as follows. In Section 2, we recall a traditional definition of a DJS-hypergroup Q in terms of the convolution algebra structure on bounded measures on Q. Then, in Propositions 1 and 2, we give an equivalent form for a definition of a DJS-hypergroup in a dual setting in terms of a comultiplication on the C^*-algebra of continuous functions on Q vanishing at infinity and its multiplier algebra of continuous bounded functions on Q. This turns out to be more convenient for purposes of this paper.

Section 3 gives a main result of the paper. In Theorem 1, we describe construction of a DJS-hypergroup from a conditional expectation on a locally compact group. It follows from proofs that the construction of a conditional expectation is similar to the construction of orbital morphisms [5], except that the corresponding orbital mapping need not be open. Using results from [9] we prove in Theorem 2 the main result that the Fourier space of the hypergroup constructed from a conditional expectation on a locally compact hypergroup is a Banach algebra.

2. PRELIMINARY

2.1. **Hypergroups.** Let Q be a Hausdorff locally compact topological space.

The linear space of complex-valued continuous functions (resp., bounded continuous functions, continuous functions approaching zero at infinity) is denoted by $\mathscr{C}(Q)$ (resp.,

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$\mathscr{C}_b(Q)$, $\mathscr{C}_0(Q)$). The space $\mathscr{C}_b(Q)$ is endowed with the norm $\|f\|_\infty = \sup_{t \in Q} |f(t)|$. Support of a function $f \in \mathscr{C}(Q)$ is denoted by $\mathrm{supp}\,(f)$. For an open subset U of Q, $\mathscr{K}(U)$ denotes the subspace of functions $f \in \mathscr{C}(Q)$ such that $\mathrm{supp}\,(f) \subset U$, and $\mathscr{K}_+(U)$ (resp., $\mathscr{K}_+^*(U)$) denotes the subset of functions $f \in \mathscr{K}(U)$ such that $f \geq 0$ (resp., $f > 0$). For $p \in U$, we denote by $\mathscr{K}_+^p(U)$ the subset of $\mathscr{K}_+(U)$ such that $f(p) > 0$. By 1_Q, we denote the constant function, $1_Q(s) = 1$ for all $s \in Q$.

A measure is understood as a complex Radon measure [3] on Q. The linear space of complex Radon measures, over the field $\mathbb{C}$ of complex numbers, is denoted by $\mathscr{M}(Q)$. The subspace of $\mathscr{M}(Q)$ of bounded (resp., compactly supported) measures is denoted by $\mathscr{M}_b(Q)$ (resp., $\mathscr{M}_c(Q)$) and, for $\mu \in \mathscr{M}_b(Q)$, its norm is $\|\mu\| = \sup_{f \in \mathscr{K}(Q),\ \|f\|_\infty \leq 1} |\mu(f)|$. The subset of $\mathscr{M}(Q)$ of nonnegative (resp., probability) measures is denoted by $\mathscr{M}_+(Q)$ (resp., $\mathscr{M}_p(Q)$). For a measure $\mu \in \mathscr{M}_+(Q)$, its support is denoted by $\mathrm{supp}\,(\mu)$. If $\mu \in \mathscr{M}_+(Q) \cap \mathscr{M}_b(Q)$, then $\|\mu\| = \mu(1_Q)$. The Dirac measure at a point $s \in Q$ is denoted by ε_s. The integral of $f \in \mathscr{K}(F)$ with respect to a measure $\mu \in \mathscr{M}$ is denoted by $\mu(f) = \langle\, f, \mu\, \rangle = \int_F \langle\, f, \varepsilon_t\, \rangle\, d\mu(t) = \int_F f(t)\, d\mu(t)$.

The set of all nonempty compact subsets of Q is denoted by $\mathcal{K}$. We consider $\mathcal{K}$ endowed with the Michael topology [10]. Recall that this topology is generated by the subbasis of all $\mathcal{U}_V(W) = \{F \in \mathcal{K} : F \subset W \text{ and } F \cap V \neq \emptyset\}$ for open subsets V and W of Q.

A (locally compact) hypergroup is a locally compact Hausdorff topological space Q such that $\mathscr{M}_b(Q)$ is endowed with a multiplication, called convolution and denoted by $*$, satisfying the following conditions [1]:

(H_1) ($\mathscr{M}_b(Q), *$) is an associative algebra over $\mathbb{C}$.

(H_2) For all $s, t \in Q$, $\varepsilon_s * \varepsilon_t \in \mathscr{M}_p(Q)$ and $\mathrm{supp}\,(\varepsilon_s * \varepsilon_t)$ is compact.

(H_3) The mapping $(s, t) \mapsto \varepsilon_s * \varepsilon_t$ of $Q \times Q$ into $\mathscr{M}_p(Q)$ is continuous with respect to the weak topology $\sigma(\mathscr{M}_p(Q), \mathscr{C}_0(Q))$ on $\mathscr{M}_p(Q)$.

(H_4) The mapping $(s, t) \mapsto \mathrm{supp}\,(\varepsilon_s * \varepsilon_t)$ of $Q \times Q$ into $\mathcal{K}$ is continuous with respect to the Michael topology on $\mathcal{K}$.

(H_5) There exists a (necessarily unique) element $e \in Q$ such that $\varepsilon_e * \varepsilon_s = \varepsilon_s * \varepsilon_e = \varepsilon_s$ for all $s \in Q$.

(H_6) There exists a (necessarily unique) homeomorphism $s \mapsto \check{s}$ of Q into Q such that $\check{\check{s}} = s$ and $(\varepsilon_s * \varepsilon_t)^\vee = \varepsilon_{\check{t}} * \varepsilon_{\check{s}}$, where $\check{\mu}$ denotes the image of the measure μ with respect to the homeomorphism $s \mapsto \check{s}$, i.e., $\langle\, f, \check{\mu}\, \rangle = \langle\, \check{f}, \mu\, \rangle$, where $\check{f}(s) = f(\check{s})$.

(H_7) For $s, t \in Q$, $e \in \mathrm{supp}\,(\varepsilon_s * \varepsilon_t)$ if and only if $s = \check{t}$.

For a measure $\mu \in \mathscr{M}(Q)$ and $h \in \mathscr{C}(Q)$, the measure $h\mu$ is defined by $\langle\, f, h\mu\, \rangle = \langle\, fh, \mu\, \rangle$ for $f \in \mathscr{K}(Q)$; it is clear that $h\mu \in \mathscr{M}_c(Q)$ for $h \in \mathscr{K}(Q)$.

Everywhere in the sequel, we assume that the hypergroup possesses a left invariant measure, denoted by m, which means that

$$\varepsilon_s * m = m$$

for all $s \in Q$. The Banach space $L_p(Q, m)$, $1 \leq p \leq \infty$, is denoted by $L_p(Q)$.

For $\mu \in \mathscr{M}_b(Q)$, denote by $\mu^\star$ the bounded measure defined by $\mu^\star(f) = \overline{\check{\mu}(\bar{f})}$ for $f \in \mathscr{K}(Q)$. It follows from the axiom (H_6) of a hypergroup that $\star$ is an involution on the algebra ($\mathscr{M}_b(Q), *$). It is well known [1] that ($\mathscr{M}_b(Q), *, \star$) is an involutive Banach algebra.

We denote $L_1(m) = \{fm : f \in L_1(Q)\}$. It is well known that ($L_1(m), *, \star$) is a closed two-sided ideal of ($\mathscr{M}_b(Q), *, \star$).

Identifying each $f \in L_1(Q)$ with the measure $fm \in L_1(m)$ yields an involutive Banach algebra structure on $L_1(Q)$, denoted by $(L_1(Q), *, {}^\star)$, where

$$(f * g)(s) = \int_Q f(t) \langle g, \check{\varepsilon}_t * \varepsilon_t \rangle \, dm(t),$$
$$f^\star(s) = \kappa^{-1}(s)\overline{f}(\check{s}),$$

where $\kappa : Q \to \mathbf{R}$ is the modular function, $m * \varepsilon_s = \Delta(s)m$, see [1]. Also denote by

$$f^\dagger(s) = \overline{f(\check{s})}.$$

It is wellknown that a function $f * f^\dagger$ is a positive definite on hypergroup Q, see [1].

Let $(Q, *, e, \check{\ })$ be a locally compact hypergroup. Define a linear map $\tilde{\Delta}$, homomorphisms ϵ and $\check{\ }$ as follows:

(1)
$$\tilde{\Delta} : \mathscr{C}_b(Q) \to \mathscr{C}_b(Q \times Q), \qquad (\tilde{\Delta}f)(p, q) = \langle f, \varepsilon_p * \varepsilon_q \rangle,$$
$$\epsilon : \mathscr{C}_b(Q) \to \mathbb{C}, \qquad \epsilon(f) = f(e),$$
$$\check{\ } : \mathscr{C}_b(Q) \to \mathscr{C}_b(Q), \qquad \check{f}(p) = f(\check{p}),$$

where $\mathscr{C}_b(Q)$ is considered as a commutative C^*-algebra.

The following lemma is clear.

Lemma 1. *Let Q be a locally compact space, $\mu \in \mathscr{M}_b(Q) \cap \mathscr{M}_+(Q)$, and $F \subset Q$ is a closed set. Then $F \cap \mathrm{supp}\,(\mu) = \emptyset$ if and only if there is $f \in \mathscr{C}_+(Q)$, $\|f\|_\infty = 1$, such that $f(r) = 0$ for all $r \in F$ and $\langle f, \mu \rangle = \|\mu\|$.*

Proposition 1. *Let $(Q, *, e, \check{\ })$ be a locally compact hypergroup. Then the maps defined by (1) have the following properties:*

$(\widetilde{H}_1)$ *the map $\tilde{\Delta}$ is linear and coassociative, that is,*

$$(\tilde{\Delta} \times \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} \times \tilde{\Delta}) \circ \tilde{\Delta};$$

$(\widetilde{H}_2)$ (a) *the map $\tilde{\Delta}$ is positive, that is, $\tilde{\Delta}(\mathscr{C}_{b,+}) \subset \mathscr{C}_{b,+}(Q \times Q)$, where $\mathscr{C}_{b,+}(Q)$ (resp., $\mathscr{C}_{b,+}(Q \times Q)$) is the cone of bounded nonnegative continuous functions on Q (resp., $Q \times Q$),*

    (b) *$\tilde{\Delta} 1_Q = 1_{Q \times Q}$,*

    (c) *for any $s, t \in Q$ there is $f \in \mathscr{K}_+(Q)$ such that $\|f\|_\infty \leq 1$ and $(\tilde{\Delta}f)(s, t) = 1$;*

$(\widetilde{H}_4)$ *if $F \subset Q$ is closed, $(s_0, t_0) \in Q \times Q$ and $f_0 \in \mathscr{K}_+(Q)$ is such that $\|f_0\|_\infty = 1$, $f_0(r) = 0$ for all $r \in F$ and $(\tilde{\Delta}f_0)(s_0, t_0) = 1$, then there exist an open neighborhood $U \subset Q \times Q$ of (s_0, t_0) and $f \in \mathscr{K}_+(Q)$ such that $\|f\|_\infty = 1$, $f(r) = 0$ for all $r \in F$ and $(\tilde{\Delta}f)(s, t) = 1$ for all $(s, t) \in U$;*

$(\widetilde{H}_5)$ *the homomorphism ϵ is a right and left counit, that is,*

(2)
$$(\epsilon \times \mathrm{id}) \circ \tilde{\Delta} = (\mathrm{id} \times \epsilon) \circ \tilde{\Delta} = \mathrm{id};$$

$(\widetilde{H}_6)$ *the homomorphism $\check{\ }$ satisfies $\mathscr{K}(Q)\check{\ } \subset \mathscr{K}(Q)$ and the following identities:*

(3)
$$\check{\ } \circ \check{\ } = \mathrm{id},$$
$$\tilde{\Delta} \circ \check{\ } = \Sigma \circ (\check{\ } \times \check{\ }) \circ \tilde{\Delta},$$

where $\Sigma : \mathscr{C}_b(Q \times Q) \to \mathscr{C}_b(Q \times Q)$ is defined by $(\Sigma f)(p, q) = f(q, p)$;

$(\widetilde{H}_7)$ *if $\Gamma^{\check{\ }} \subset Q \times Q$ is a graph of the homeomorphism $\check{\ }$, that is,*

$$\Gamma^{\check{\ }} = \{(p, \check{p}) \in Q \times Q : p \in Q\},$$

then

(4)
$$\Gamma^{\check{\ }} = \bigcap_{f \in \mathscr{K}_+^e(G)} \mathrm{supp}\,(\tilde{\Delta}(f)),$$

where $\mathscr{K}_+^e(G) = \{f \in \mathscr{K}_+(G) : f(e) > 0\}$.

Proof. First of all note that $\tilde{\Delta} f \in \mathscr{C}(Q \times Q)$ if $f \in \mathscr{C}(Q)$, as directly follows from (H_3). It also immediately follows from (H_2) that $\tilde{\Delta}$ is positive and $\tilde{\Delta} 1_Q = 1_{Q \times Q}$. To see that $\tilde{\Delta} f$ is bounded for bounded $f \in \mathscr{C}(Q)$, we have that $-\|f\|_\infty 1_Q \leq f \leq \|f\|_\infty 1_Q$ and, hence,

$$-\|f\|_\infty 1_{Q \times Q} \leq \tilde{\Delta} f \leq \|f\|_\infty 1_{Q \times Q},$$

which means that $\tilde{\Delta} f$ is bounded. It is also clear that $\check{f}$ is continuous if f is such, and that $\|\check{f}\|_\infty = \|f\|_\infty$. This means that the maps in (1) are well defined.

To prove $(\widetilde{H}_1)$, let $f \in \mathscr{C}_b(Q)$ and consider

$$\big((\tilde{\Delta} \times \mathrm{id}) \circ \tilde{\Delta} f\big)(s_1, s_2, s_3) = \langle \tilde{\Delta} f, (\varepsilon_{s_1} * \varepsilon_{s_2}) \otimes \varepsilon_{s_3} \rangle = \langle f, \varepsilon_{s_1} * \varepsilon_{s_2} * \varepsilon_{s_3} \rangle.$$

Similarly,

$$\big((\mathrm{id} \times \tilde{\Delta}) \circ \tilde{\Delta} f\big)(s_1, s_2, s_3) = \langle \tilde{\Delta} f, \varepsilon_{s_1} \otimes (\varepsilon_{s_2} * \varepsilon_{s_3}) \rangle = \langle f, \varepsilon_{s_1} * \varepsilon_{s_2} * \varepsilon_{s_3} \rangle.$$

It has already been mentioned, that all statements in $(\widetilde{H}_2)$ except for (c) hold true. To prove (c) we use compactness of $F = \mathrm{supp}\,(\varepsilon_s * \varepsilon_t)$ in (H_2) a function $f \in \mathscr{K}_+(Q)$ such that $f(p) = 1$ for all $p \in F$. Then

$$(\tilde{\Delta} f)(s, t) = \langle f, \varepsilon_s * \varepsilon_t \rangle = \int_Q f(p) \, d(\varepsilon_s * \varepsilon_t)(p) = \int_F f(p) \, d(\varepsilon_s * \varepsilon_t)(p)$$

$$= \int_F d(\varepsilon_s * \varepsilon_t)(p) = \langle 1_Q, \varepsilon_s * \varepsilon_t \rangle$$

$$= \langle \tilde{\Delta} 1_Q, \varepsilon_s \otimes \varepsilon_t \rangle = 1_{Q \times Q}(s, t) = 1.$$

Let us prove $(\widetilde{H}_4)$. Denote $E_0 = \mathrm{supp}\,(\varepsilon_{s_0} * \varepsilon_{t_0})$. If F is closed and f_0 is such that $\|f_0\|_\infty = 1$, $f_0(r) = 0$ for all $r \in F$ and $(\tilde{\Delta} f_0)(s_0, t_0) = \langle f_0, \varepsilon_{s_0} * \varepsilon_{t_0} \rangle = 1$, then it follows from Lemma 1 that $F \cap E_0 = \emptyset$, hence E_0 is a subset of the open set $Q \setminus F$. Let $V \subset Q$ be an open neighborhood of E_0 such that $\overline{V} \subset Q \setminus F$. By (H_4) there is an open neighborhood $U \subset Q \times Q$ of (s_0, t_0) such that $\mathrm{supp}\,(\varepsilon_s * \varepsilon_t) \subset V$ for all $(s, t) \in U$. Let $f \in \mathscr{C}_+(Q)$, $0 \leq f(r) \leq 1$, be such that $f(r) = 0$ for all $r \in F$ and $f(r) = 1$ for all $r \in \overline{V}$. Then, for any $(s, t) \in U$, we have

$$(\tilde{\Delta} f)(s, t) = \langle f, \varepsilon_s * \varepsilon_t \rangle = \langle 1_Q, \varepsilon_s * \varepsilon_t \rangle = 1.$$

It is easy to see that $(\widetilde{H}_5)$ holds. Indeed. let $f \in \mathscr{C}_b(Q)$ and $s \in Q$. Then

$$\big((\epsilon \times \mathrm{id}) \circ \tilde{\Delta} f\big)(s) = \langle f, \varepsilon_e * \varepsilon_s \rangle = \langle f, \varepsilon_s \rangle = f(s).$$

The second part of (2) is proved similarly.

Property $(\widetilde{H}_6)$ is a direct consequence of (H_6).

Let us prove $(\widetilde{H}_7)$. Let $f \in \mathscr{K}_+^e(G)$. Then

$$(\tilde{\Delta} f)(s, \check{s}) = \langle f, \varepsilon_s * \varepsilon_{\check{s}} \rangle > 0,$$

since $e \in \mathrm{supp}\,(\varepsilon_s * \varepsilon_{\check{s}})$ by (H_7). Thus $(s, \check{s}) \in \mathrm{supp}\,(\tilde{\Delta} f)$ and

$$\Gamma^{\check{}} \subset \bigcap_{f \in \mathscr{K}_+^e(G)} S(\tilde{\Delta} f).$$

Conversely, let $s_0 \neq t_0$, hence $(s_0, \check{t}_0) \notin \Gamma^{\check{}}$. By (H_7), $e \notin \mathrm{supp}\,(\varepsilon_{s_0} * \varepsilon_{\check{t}_0})$. Since $\mathrm{supp}\,(\varepsilon_s * \varepsilon_{\check{t}})$ is compact there are open neighborhoods U of e and V of $\mathrm{supp}\,(\varepsilon_s * \varepsilon_{\check{t}})$ such that $U \cap V = \emptyset$. Let $f \in \mathscr{K}_+(U)$ be such that $f(e) = 1$. Hence $f \in \mathscr{K}_+^e(G)$ and, as it follows from (H_4), there is an open neighborhood $W \subset Q \times Q$ of $(s_0, \check{t}_0)$ such that

$\operatorname{supp}(\varepsilon_s * \varepsilon_{\check{t}}) \subset V$ for all $(s, \check{t}) \in W$, hence $\operatorname{supp}(f) \cap \operatorname{supp}(\varepsilon_s * \varepsilon_{\check{t}}) = \emptyset$ for such $(s, \check{t})$. Thus

$$(\tilde{\Delta} f)(s, \check{t}) = \langle f, \varepsilon_s * \varepsilon_{\check{t}} \rangle = 0,$$

and $(s_0, \check{t}_0) \notin \operatorname{supp}(\tilde{\Delta} f)$. This means that

$$(s, \check{t}) \notin \bigcap_{f \in \mathscr{K}_+^e(G)} \operatorname{supp}(\tilde{\Delta} f), \quad (s, \check{t}) \in W,$$

which finishes the proof. $\qquad \square$

Remark 1. It immediately follows from $(\widetilde{H}_2)$ (a) and $(\widetilde{H}_2)$ (b) that $(\widetilde{H}_2)$ (c) is equivalent to the condition that there is $f \in \mathscr{K}_+(Q)$ such that $\tilde{\Delta} f = \|\tilde{\Delta}\|$.

Proposition 2. *Let Q be a Hausdorff locally compact space, and let a linear map $\tilde{\Delta} \colon \mathscr{C}_b(Q) \to \mathscr{C}_b(Q \times Q)$, $*$-homomorphisms $\check{} \colon \mathscr{C}_b(Q) \to \mathscr{C}_b(Q)$ and $\epsilon \colon \mathscr{C}_b(Q) \to \mathbb{C}$ be given and satisfy the properties $(\widetilde{H}_1) - (\widetilde{H}_7)$ in Proposition 1. For $\mu_1, \mu_2, \mu \in \mathscr{M}_b(Q)$, define $\mu_1 * \mu_2 \in \mathscr{M}_b(Q)$ and $\check{\mu} \in \mathscr{M}_b(Q)$ by*

(5)
$$\langle f, \mu_1 * \mu_2 \rangle = \int_{Q \times Q} (\tilde{\Delta} f)(s_1, s_2) \, d\mu_1(s_1) d\mu_2(s_2),$$
$$\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle,$$

where $f \in \mathscr{K}(Q)$, and let $e = \operatorname{supp}(\epsilon)$. Then $(Q, *, e, \check{})$ is a locally compact hypergroup.

Proof. First consider (H_1) and show that $(\mathscr{M}_b(Q), *)$ is indeed an associative algebra, if the convolution of measures is given by the first formula in (5).

If $\mu_1, \mu_2 \in \mathscr{M}_b(Q)$, then the measures $|\mu_1|$ and $|\mu_2|$ are also bounded. Moreover, by (5) and using $(\widetilde{H}_2)$, we have

$$\left| \langle f, \mu_1 * \mu_2 \rangle \right| = \left| \int_{Q \times Q} (\tilde{\Delta} f)(s_1, s_2) \, d\mu_1(s_1) d\mu_2(s_2) \right|$$
$$\leq \int_{Q \times Q} (\tilde{\Delta} |f|)(s_1, s_2) \, d|\mu_1|(s_1) d|\mu_2|(s_2)$$
$$\leq \int_{Q \times Q} \tilde{\Delta}(\|f\|_\infty 1_Q)(s_1, s_2) \, d|\mu_1|(s_1) d|m_2|(s_2)$$
$$= \|f\|_\infty \int_{Q \times Q} 1_Q(s_1) 1_Q(s_2) \, d|\mu_1|(s_1) d|\mu_2|(s_2)$$
$$= \|f\|_\infty \|\mu_1\| \|\mu_2\|,$$

which shows that $\|\mu_1 * \mu_2\| \leq \|\mu_1\| \|\mu_2\|$, hence $\mu_1 * \mu_2 \in \mathscr{M}_b(Q)$.

Associativity of the algebra $(\mathscr{M}_b(Q), *)$ is immediately implied by $(\widetilde{H}_1)$.

Consider (H_2). Since $\tilde{\Delta}$ is positive by $(\widetilde{H}_2)$, the measure $\varepsilon_s * \varepsilon_t$ is nonnegative for $s, t \in Q$. Moreover,

$$\langle 1_Q, \varepsilon_s * \varepsilon_t \rangle = \langle 1_Q \otimes 1_Q, \varepsilon_s \otimes \varepsilon_t \rangle = 1,$$

hence $\varepsilon_s * \varepsilon_t \in \mathscr{M}_p(Q)$.

Let us show that $\operatorname{supp}(\varepsilon_s * \varepsilon_t)$ is compact for any $s, t \in Q$. Indeed, using $(\widetilde{H}_2)$ (c) there is $f \in \mathscr{K}_+(Q)$ such that $\|f\|_\infty \leq 1$ and $(\tilde{\Delta} f)(s, t) = 1$. This means that $\langle f, \varepsilon_s * \varepsilon_t \rangle = 1$. Denote $F = \operatorname{supp}(f)$ and let $g \in \mathscr{K}_+(Q)$ be arbitrary such that $\|g\|_\infty \leq 1$ and $\operatorname{supp}(g) \subset Q \setminus F$. Then $f + g \leq 1_Q$, that is, $g \leq 1_Q - f$, and

$$0 \leq \langle g, \varepsilon_s * \varepsilon_t \rangle \leq \langle 1_Q - f, \varepsilon_s * \varepsilon_t \rangle = \langle 1_Q, \varepsilon_s * \varepsilon_t \rangle - \langle f, \varepsilon_s * \varepsilon_t \rangle = 0.$$

This shows that $\langle g, \varepsilon_s * \varepsilon_t \rangle = 0$ for any such g, which means that $\operatorname{supp}(\varepsilon_s * \varepsilon_t) \subset F$. Since F is compact and support of a measure is closed, $\operatorname{supp}(\varepsilon_s * \varepsilon_t)$ is compact.

Property (H_3) is immediate, since $\tilde{\Delta} f \in \mathscr{C}_b(Q \times Q)$ by definition for any $f \in \mathscr{C}_0(Q)$.

Consider (H_4). Let $(s_0, t_0) \in Q \times Q$, and denote $F_0 = \operatorname{supp}(\varepsilon_{s_0} * \varepsilon_{t_0})$, which is a compact set by (H_2). Let $V, W \subset Q$ be open such that $\mathcal{U}_V(W)$ is an open neighborhood of F_0 in the Michael topology, that is, $F_0 \subset W$ and $F_0 \cap V \neq \emptyset$. Choose $p \in F_0 \cap V$ and let $f \in \mathscr{K}_+^p(V)$. Then, since $p \in F_0$,

$$(\tilde{\Delta} f)(s_0, t_0) = \langle f, \varepsilon_{s_0} * \varepsilon_{t_0} \rangle > 0,$$

and, by continuity of $\tilde{\Delta} f$, we can find a neighborhood $U_1 \subset Q \times Q$ of (s_0, t_0) such that $(\tilde{\Delta} f)(s, t) > 0$ for all $(s, t) \in U_1$. This would imply that $\langle f, \varepsilon_s * \varepsilon_t \rangle > 0$, hence $\operatorname{supp}(f) \cap \operatorname{supp}(\varepsilon_s * \varepsilon_t) \neq \emptyset$, in particular, $V \cap \operatorname{supp}(\varepsilon_s * \varepsilon_t) \neq \emptyset$ for all $(s, t) \in U_1$.

To proceed, let $W \subset Q$ be open and $\operatorname{supp}(\varepsilon_{s_0} * \varepsilon_{t_0}) \subset W$. Let $F = Q \setminus \operatorname{supp}(\varepsilon_{s_0} * \varepsilon_{t_0})$. Then there is a function $f_0 \in \mathscr{C}_+(Q)$, $0 \leq f(r) \leq 1$, $r \in Q$, such that $f(r) = 0$ for $r \in F$ and $f(r) = 1$ for $r \in \operatorname{supp}(\varepsilon_{s_0} * \varepsilon_{t_0})$. We have $\langle f_0, \varepsilon_{s_0} * \varepsilon_{t_0} \rangle = \langle 1_Q, \varepsilon_{s_0} * \varepsilon_{t_0} \rangle = 1 = \|\varepsilon_{s_0} * \varepsilon_{t_0}\|$. Hence, there is a neighborhood $U \subset Q \times Q$ of the point (s_0, t_0) and a function $f \in \mathscr{C}_+$, $\|f\|_\infty = 1$, such that $f(r) = 0$ for all $r \in F$ and $\langle f, \varepsilon_s * \varepsilon_t \rangle = 1$ for all $(s, t) \in U$. By Lemma 1, $F \cap \operatorname{supp}(\varepsilon_s * \varepsilon_t) = \emptyset$, hence $\operatorname{supp}(\varepsilon_s * \varepsilon_t) \subset W$ for all $(s, t) \in U$.

(H_5) (resp., (H_6)) is immediately implied by $(\widetilde{H}_5)$ (resp., $(\widetilde{H}_6)$.

Let us prove (H_7). Let $s \neq t$ and show that $e \notin \operatorname{supp}(\varepsilon_s * \varepsilon_{\check{t}})$. Indeed, $(s, \check{t}) \notin \Gamma^{\smile}$, hence there exists $f \in \mathscr{K}_+^e(G)$ such that $(s, \check{t}) \notin \operatorname{supp}(\tilde{\Delta} f)$. This means that

$$(\tilde{\Delta} f)(s, \check{t}) = \langle f, \varepsilon_s * \varepsilon_{\check{t}} \rangle = 0.$$

This means that $f \restriction_{\operatorname{supp}(\varepsilon_s * \varepsilon_{\check{t}})} = 0$ but $f(e) > 0$. Hence $e \notin \operatorname{supp}(\varepsilon_s * \varepsilon_{\check{t}})$.

Conversely, let us show that $e \in \operatorname{supp}(\varepsilon_s * \varepsilon_{\check{s}})$ for any $s \in Q$. Indeed, if U is an arbitrary open neighborhood of e, choose a function $f \in \mathscr{K}_+^e(U)$. Since $(s, \check{s}) \in \Gamma^{\smile}$, it follows from (4) that $(s, \check{s}) \in \operatorname{supp}(f)$, which means that

$$(\tilde{\Delta} f)(s, \check{s}) = \langle f, \varepsilon_s * \varepsilon_{\check{s}} \rangle > 0,$$

and $e \in \operatorname{supp}(\varepsilon_s * \varepsilon_{\check{s}})$. $\qquad\square$

2.2. Fourier and Fourier-Stieltjes algebras.

Definition 1. Let Q be a locally compact hypergroup and H a Hilbert space. A representation π of the hypergroup Q on the Hilbert space H is an involution homomorphism $\pi : \mathscr{M}_b(Q) \to B(H)$ of the involution algebra $(\mathscr{M}_b(Q), *, ^*)$ into the C^*-algebra $B(H)$ of all linear bounded operators on H. The set of all representations of Q will be denoted by Σ.

Let us remark that definition 1 can be rewritten in terms of the comultiplication in (1) as follows [7]: a weakly continuous map $\pi : Q \to B(H)$ is ca;;ed a representation of the hypergroup Q if the following conditions are satisfied:

  (i) $\pi(e) = I$, where I is the identity operator on H;
  (ii) $\pi(p)^* = \pi(p^*)$ for all $p \in Q$;
  (iii) for any $\xi, \eta \in H$, we have $\tilde{\Delta}(\pi(\cdot)\xi, \eta)_H(p, q) = (\pi(p)\pi(q)\xi, \eta)_H$.

Definition 2. Let $\tilde{m}$ be a left Haar measure on the hypergroup Q. A left (resp., right) regular representation is a mapping $\lambda : Q \ni p \mapsto L_p \in B(L_2(Q))$ (resp., $\mu : Q \ni p \mapsto R_p \in B(L^2(Q))$) defined, for $f \in \mathscr{K}(Q)$, by $(L_p f)(q) = (\tilde{\Delta} f)(p^*, q)$ (resp., $(R_p f)(q) = (\tilde{\Delta} f)(p, q) \kappa^{\frac{1}{2}}(p))$ and then extended to $L^2(Q)$ by continuity in virtue of the estimates $\|L_p f\|_2 \leq \|f\|_2$ and $\|R_p f\|_2 \leq \|f\|_2$ [7].

Let us recall that each representation of a hypergroup can be uniquely continued to a nondegenerate representation of the Banach $*$-algebra $L^1(Q)$ and, conversely, each nondegenerate representation of the $*$-algebra $L^1(Q)$ gives rise to a representation of the

hypergroup Q [7]. By $C^*(Q)$, we denote the full C^*-algebra of the hypergroup Q, that is the closure of $L^1(Q)$ with respect to the C^*-norm $\|f\| = \sup_{\pi \in \Sigma} \pi(f)$.

The reduced C^*-algebra of the hypergroup Q is denoted by $C_r^*(Q)$ that is the C^*-closure of the $*$-algebra generated by the family $\{L_p : p \in Q\}$ of operators on $L^2(Q)$ with respect to the norm of $B(L^2(Q))$.

Definition 3. The Banach space dual to the full C^*-algebra $C^*(Q)$ will be called the Fourier-Stieltjes space and denoted by $\mathscr{B}(Q)$. The Banach space dual to the reduced C^*-algebra of the hypergroup Q, $C_r^*(Q)$, will be denoted by $\mathscr{B}_\lambda(Q)$.

Definition 4. Let (H_1, π_1) and (H_2, π_2) be two representations of a hypergroup Q. The representation π_1 is said to be weakly contained in the representation π_2 if the kernel of the representation π_1 contains the kernel of the representation π_2.

It is well known, see [11] that for any $\alpha \in \mathscr{B}(Q)$ (respectively, $\alpha \in \mathscr{B}_\lambda(Q)$) there is a representation (H, π) of the hypergroup Q (respectively, a representation (H, π) weakly contained in the left regular representation) and two vectors $\xi, \eta \in H$ such that the function $a \in C_b(Q)$ given by

$$a(p) = (\pi(p)\xi, \eta)_H, \quad p \in Q,$$

defines a linear functional α by

$$\alpha(f) = \int_Q a(p) f(p) dm(p), \quad f \in L_1(Q),$$

such that its norm is given by

$$\|\alpha\| = \sup_{f \in L_1(Q), \|f\|'_\Sigma = 1} \left| \int_Q a(p) f(p) dm(p) \right| = \|\xi\|_H \|\eta\|_H,$$

where $\Sigma' = \Sigma$ (respectively, $\Sigma' = \lambda$).

Definition 5. The Fourier space of a hypergroup Q will be called the closure of the space generated by the elements $f * f^\dagger$, $f \in \mathscr{K}(Q)$ with respect to the norm of the space $\mathscr{B}_\lambda(Q)$, and denoted by $\mathscr{A}(Q)$.

2.3. **Tensor product and conditional expectation.** Let A and B be C^*-algebras. The tensor product of A and B, denoted by $A \otimes B$, is the completion of the algebraic tensor product $A \odot B$ with respect to the min-C^*-norm on $A \odot B$,

$$\|\sum_{i=1}^n a_i \otimes b_i\|_{\min} = \sup_{\pi_A \in \Sigma_A, \pi_B \in \Sigma_B} \left\| \sum_{i=1}^n \pi_A(a_i) \otimes \pi_B(b_i) \right\|,$$

where Σ_A (resp., Σ_B) is the set of all representations of the C^*-algebra A (resp., B) [14].

For a C^*-algebra A, the C^*-algebra of multipliers of A is denoted by $M(A)$, see [14] for details.

Let A be a C^*-algebra and $B \subset A$ a C^*-subalgebra of A. A bounded linear map $P \colon A \to B$ is called a *conditional expectation* if it satisfies the following properties [16]:

  (i) P is a projection onto and has norm 1, that is, $P^2 = P$ and $\|P\| = 1$;
  (ii) P is positive, that is $P(a^*a) \geq 0$ for any $a \in A$;
  (iii) $P(b_1 a b_2) = b_1 P(a) b_2$ for any $a \in A$ and $b_1, b_2 \in B$;
  (iv) $P(a^*)P(a) \leq P(a^*a)$ for all $a \in A$.

It follows from (ii) and the polarization identity that

  (v) $P(a^*) = P(a)^*$, $a \in A$.

It also follows from [15] that (i) implies (ii), (iii), and (iv).

3. Main results

Let G be a locally compact group with a Haar measure m, A denote the C^*-algebra $\mathscr{C}_b(G)$, A_0 its C^*-subalgebra $\mathscr{C}_0(G)$, and let $\mathscr{K}(G)$ be the ideal of A consisting of functions with compact supports. Let $P\colon A \to A$ be a conditional expectation. The restriction of P to $\mathscr{C}_0(G)$ is still denoted by P.

Proposition 3. *Let B be a C^*-subalgebra of $\mathscr{C}_0(G)$, $P\colon \mathscr{C}_0(G) \to B$ be a conditional expectation. Let Q denote the spectrum of the commutative C^*-algebra B, and identify B with $\mathscr{C}_0(Q)$ via the Gelfand transform. Let $\iota\colon \mathscr{C}_0(Q) \cong B \to A$ denote the embedding map. Then we have the following.*

  (i) *There is a unique continuous surjection $\varphi\colon G \to Q$ such that $\iota(\tilde{f})(p) = \tilde{f}(\varphi(p))$ for any $\tilde{f} \in \mathscr{C}_0(Q)$ and any $p \in G$. If $p, q \in G$ are such that $\varphi(p) = \varphi(q)$, then $\iota(\tilde{f})(p) = \iota(\tilde{f})(q)$.*

  (ii) *Denote by $P^*\colon \mathscr{M}_b(Q) \to \mathscr{M}_b(G)$ the linear map adjoint to P,*

$$\langle\, f, P^*(\tilde{\mu})\,\rangle = \langle\, P(f), \tilde{\mu}\,\rangle, \qquad \tilde{\mu} \in \mathscr{M}_b(Q), \quad f \in \mathscr{C}_b(G),$$

and let $\varphi_*\colon \mathscr{M}_b(G) \to \mathscr{M}_b(Q)$ be the linear map induced by φ,

$$\langle\, \tilde{f}, \varphi_*(\mu)\,\rangle = \langle\, \tilde{f} \circ \varphi, \mu\,\rangle, \qquad \tilde{f} \in \mathscr{C}_b(Q), \quad \mu \in \mathscr{M}_b(G).$$

Then

$$\varphi_* \circ P^* = \mathrm{id}_{\mathscr{M}_b(Q)}.$$

  (iii) *Let $s \in Q$, and denote*

$$O_s = \{p \in G : \varphi(p) = s\}.$$

If $P(\mathscr{K}_+(G)) \subset \mathscr{K}_+(G)$ and, for $p \in G$ and $f \in \mathscr{K}_+(G)$, $P(f)(\varphi(p)) > 0$ whenever $f(p) > 0$, then O_s is compact.

  (iv) *For any $s \in Q$,*

$$\mathrm{supp}\,(P^*(\varepsilon_s)) \subset O_s.$$

If $P(f)(\varphi(p)) > 0$ whenever $f(p) > 0$ for $p \in G$ and $f \in \mathscr{K}_+(G)$, then

$$\mathrm{supp}\,(P^*(\varepsilon_s)) = O_s.$$

Proof. (i) Regarding G as $\mathrm{Spec}\,(\mathscr{C}_0(G))$, the spectrum of the commutative C^*-algebra $\mathscr{C}_0(G)$, we define

$$\varphi(p)(\tilde{f}) = p\!\restriction_B (\tilde{f})$$

for $\tilde{f} \in B$ and $p \in \mathrm{Spec}\,(\mathscr{C}_0(G))$. Hence, by the definition, for $\tilde{f} \in \mathscr{C}_0(Q)$ and $p \in G$, we have

$$\iota(\tilde{f})(p) = \tilde{f}(\varphi(p)) = p\!\restriction_B (\tilde{f}).$$

If $s \in Q = \mathrm{Spec}\,(B)$, then it follows from [13] that s can be extended to an element $p \in \mathrm{Spec}\,(\mathscr{C}_0(G))$ such that $p\!\restriction_B = s$, hence φ is onto.

The last statement in the item is a direct implication of the definition of φ.

(ii) If $\tilde{f} \in \mathscr{C}_0(Q)$, then $(P \circ \iota)(\tilde{f}) = \tilde{f}$, since P is a projection. Hence, for $\tilde{f} \in \mathscr{C}_0(Q)$ and $\tilde{\mu} \in \mathscr{M}_b(Q)$, we have

$$\langle\, \tilde{f}, (\varphi_* \circ P^*)(\tilde{\mu})\,\rangle = \langle\, \tilde{f} \circ \varphi, P^*(\tilde{\mu})\,\rangle = \langle\, \iota(\tilde{f}), P^*(\tilde{\mu})\,\rangle = \langle\, (P \circ \iota)(\tilde{f}), \tilde{\mu}\,\rangle = \langle\, \tilde{f}, \tilde{\mu}\,\rangle.$$

(iii) Indeed, for $p \in O_s$, let $f \in \mathscr{K}_+(G)$ such that $f(p) = 1$. Then, by the assumption, $P(f)(s) = P(f)(\varphi(p)) > 0$. Hence, for any $q \in O_s$, $P(f)(q) = P(f)(s) > 0$, and $O_s \subset \mathrm{supp}\,(P(f))$. Since $O_s = \varphi^{-1}(s)$ is closed and $\mathrm{supp}\,P(f)$ is compact, O_s is compact.

(iv) We first prove that $\mathrm{supp}\,(P^*(\varepsilon_s)) \subset O_s$. Let $p \in \mathrm{supp}\,(P^*(\varepsilon_s))$ and assume that $p \notin O_s$, hence $\varphi(p) \neq s$. Choose an open neighborhood V of $\varphi(p)$ such $s \notin V$, and let

$U = \varphi^{-1}(V)$. Then $p \in U$ and, for any $f \in \mathscr{K}_+(U)$, $\mathrm{supp}\,(P(f)) \subset V$, hence $P(f)(s) = 0$. Thus

$$\langle\, f, P^*(\varepsilon_s)\,\rangle = \langle\, P(f), \varepsilon_s\,\rangle = 0,$$

and $p \notin \mathrm{supp}\,(P^*(\varepsilon_s))$.

It remains to prove, with the assumption made, that $O_s \subset \mathrm{supp}\,(P^*(\varepsilon_s))$. Let $p \in O_s$ and U be an open neighborhood of p. Choose a function $f \in \mathscr{K}_+(U)$ such that $f(p) = 1$. By the assumption, $P(f)(s) > 0$. Hence

$$\langle\, f, P^*(\varepsilon_s)\,\rangle = \langle\, P(f), \varepsilon_s\,\rangle = P(f)(s) > 0,$$

and $p \in \mathrm{supp}\, P^*(\varepsilon_s)$. $\hfill\square$

Remark 2. In the sequel, we will identify elements of $\mathscr{C}(Q)$ with those in B, according to Proposition 3 (i).

Theorem 1. *Let G be a locally compact group, $P\colon \mathscr{C}_b(G) \to \mathscr{C}_b(G)$ be a conditional expectation. Assume that P satisfies the assumptions in Proposition 3 (iii), and let the following hold:*

$$(6)\qquad \big((P \times \mathrm{id}) \circ \Delta \circ P\big)(f) = \big((\mathrm{id} \times P) \circ \Delta \circ P\big)(f) = \big((P \times P) \circ \Delta\big)(f),$$
$$P(\check{f}) = \big(P(f)\big)^{\check{}},$$

for all $f \in \mathscr{C}_b(G)$, where $\Delta\colon \mathscr{C}_b(G) \to \mathscr{C}_b(G \times G)$ is the group comultiplication on $\mathscr{C}_b(G)$ defined by $(\Delta f)(p_1, p_2) = f(p_1 p_2)$, and $\check{f}(p) = f(p^{-1})$, $p_1, p_2, p \in G$.

Let $B = P(\mathscr{C}_0(G))$, denote by Q the spectrum of the commutative C^*-algebra B, and identify B with $\mathscr{C}_0(Q)$. Define $\tilde{\Delta}\colon \mathscr{C}_b(Q) \to \mathscr{C}_b(Q \times Q)$ by

$$\tilde{\Delta}(\tilde{f}) = \big((P \times P) \circ \Delta\big)(\tilde{f}), \quad \tilde{f} \in \mathscr{C}_b(Q).$$

For $s \in Q$, define $\check{s} \in Q$ by

$$\check{s}(\tilde{f}) = \check{\tilde{f}}(s), \quad \tilde{f} \in B.$$

Let $\tilde{e} \in Q$ be defined by

$$\tilde{e}(\tilde{f}) = \tilde{f}(e), \quad \tilde{f} \in B,$$

where e is the identity of the group G.

Then $(Q, \tilde{e}, \check{})$ is a locally compact hypergroup with comultiplication $\tilde{\Delta}$.

If $m \circ P = m$, where m is a left Haar measure on the group G, then $\tilde{m} \in \mathscr{M}(Q)$ defined by $\tilde{m} = \varphi_*(m) = m{\restriction}_B$ is a left Haar measure on Q.

Proof. To prove the theorem, using Proposition 2, we will check the conditions $(\tilde{H}_1)$—$(\tilde{H}_7)$ in Proposition 1.

$(\tilde{H}_1)$ We have

$$(\tilde{\Delta} \times \mathrm{id}) \circ \tilde{\Delta} = (P \times P \times \mathrm{id}) \circ (\Delta \times \mathrm{id}) \circ (P \times P) \circ \Delta$$
$$= (P \times P \times P) \circ (\Delta \times \mathrm{id}) \circ \Delta.$$

On the other hand,

$$(\mathrm{id} \times \tilde{\Delta}) \circ \tilde{\Delta} = (\mathrm{id} \times P \times P) \circ (\mathrm{id} \times \Delta) \circ (P \times P) \circ \Delta$$
$$= (P \times P \times P) \circ (\mathrm{id} \times \Delta) \circ \Delta.$$

Now, the result follows from coassociativity of Δ.

$(\tilde{H}_2)$ (b) Since $P(1_G) = 1_Q$ and $\Delta 1_G = 1_{G \times G}$, the claim is immediate.

$(\tilde{H}_2)$ (a) It is clear that $\tilde{\Delta}$ is positive, being a composition of the positive maps Δ and $P \times P$. It is also immediate from positivity and $(\tilde{H}_2)$ (b) that $\tilde{\Delta}(\tilde{f}) \in \mathscr{C}_{b,+}(Q \times Q)$ for $\tilde{f} \in \mathscr{C}_{b,+}(Q)$.

$(\tilde{H}_2)$ (c) For any $\tilde{f} \in \mathscr{K}_+(Q)$ and $s, t \in Q$, we have

$$(\tilde{\Delta} f)(s,t) = \langle \tilde{\Delta}\,\tilde{f}, \varepsilon_s \otimes \varepsilon_t \rangle = \langle ((P \times P) \circ \Delta)(\tilde{f}), \varepsilon_s \otimes \varepsilon_t \rangle$$
$$= \langle \Delta\,\tilde{f}, P^*(\varepsilon_s) \otimes P^*(\varepsilon_t) \rangle = \langle \tilde{f}, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle,$$

where $*_G$ is the convolution of measures on G defined by

$$\langle f, \mu *_G \nu \rangle = \int_{G^2} f(pq)\,d\mu(p)d\nu(q), \qquad f \in \mathscr{C}_b(G), \quad \mu, \nu \in \mathscr{M}_c(G).$$

It follows from Proposition 3 that $P^*(\varepsilon_s)$ and $P^*(\varepsilon_t)$ are compact, hence the set $F = \mathrm{supp}\,(P^*(\varepsilon_s) *_G P^*(\varepsilon_t))$ is also compact. It also follows from the first identity in (6) that, if $\tilde{f} = P(f)$, $f \in \mathscr{C}_b(G)$, then

$$\langle \tilde{f}, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle = \langle f, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle.$$

Hence, let $f \in \mathscr{K}_+(G)$ be such that $f(p) = 1$ for all $p \in \mathrm{supp}\,(P^*(\varepsilon_s) *_G P^*(\varepsilon_t))$. Then, by the assumption, $\tilde{f} = P(f) \in \mathscr{K}_+(Q)$, and

$$\langle \tilde{f}, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle = \langle f, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle = \langle 1_G, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle$$
$$= \langle 1_{Q \times Q}, \varepsilon_s \otimes \varepsilon_t \rangle = 1.$$

$(\tilde{H}_4)$ Let $F \subset Q$ be closed, $(s_0, t_0) \in Q \times Q$, and $\tilde{f}_0 \in \mathscr{K}_+(Q)$ be such that $\|\tilde{f}_0\| \leq 1$, $f_0(r) = 0$ for $r \in F$ and $(\tilde{\Delta}\,\tilde{f}_0)(s_0, t_0) = 1$. Since $\tilde{\Delta}\,\tilde{f}_0 \in \mathscr{C}_+(Q \times Q)$ there is a neighborhood $U \subset Q \times Q$ of the point s_0, t_0 such that $(\tilde{\Delta}\,\tilde{f}_0)(s,t) > \frac{1}{2}$ for all $(s,t) \in U$. This means that

$$\langle \tilde{f}_0, P^*(\varepsilon_s) *_G P^*(\varepsilon_t) \rangle > \frac{1}{2}, \quad (s,t) \in U.$$

This means that the set E,

$$E = \bigcup_{(s,t) \in U} \mathrm{supp}\,(P^*(\varepsilon_s) *_G P^*(\varepsilon_t)) \subset \mathrm{supp}\,\tilde{f}_0,$$

is compact. Choose $\tilde{f} \in \mathscr{K}_+(Q)$ to be such that $\tilde{f}(r) = 1$ if $r \in E$, and $\tilde{f}(r) = 0$ if $r \in F$. This function satisfies the condition in $(\tilde{H}_4)$.

$(\tilde{H}_5)$ For $\tilde{f} \in \mathscr{C}_0(Q)$, we have

$$(\epsilon \times \mathrm{id}) \circ \tilde{\Delta}(\tilde{f}) = (\epsilon \times \mathrm{id}) \circ (P \times P) \circ \Delta(P(\tilde{f}))$$
$$= (\epsilon \times P) \circ \Delta(P(\tilde{f})) = P(P(\tilde{f})) = \tilde{f}.$$

$(\tilde{H}_6)$ First of all note that $\check{\tilde{f}} \in \mathscr{K}(Q)$ for $\tilde{f} \in \mathscr{K}(Q)$, since $P(\check{\tilde{f}}) = P(\tilde{f})\check{\ } = \check{\tilde{f}}$ by (6).

It is clear that the first identity in (3) holds. To prove the second identity in (3), consider

$$(\tilde{\Delta}\check{\tilde{f}})(s,t) = ((P \times P) \circ \Delta\check{\tilde{f}})(s,t) = ((P \times P) \circ \Sigma \circ (\check{\ } \times \check{\ }) \circ \Delta)(\tilde{f})(s,t)$$
$$= ((P \times P) \circ \Delta)(\tilde{f})(\check{t}, \check{s}) = (\tilde{\Delta}\,\tilde{f})(\check{t}, \check{s})$$

for $s, t \in Q$, $\tilde{f} \in B$.

$(\tilde{H}_7)$ We will prove that (H_7) holds true. To this end, it is sufficient to prove that $e \in O_s \cdot O_{\check{t}}$, $s, t \in Q$, if and only if $s = t$, since the set $O_s \cdot O_{\check{t}}$ is compact because such are O_s and $O_{\check{t}}$. Since $O_{\check{t}} = O_t^{-1}$, $p^{-1} \in O_{\check{s}}$ for $p \in O_s$, hence $e \in O_s \cdot O_{\check{s}}$. On the other hand, if $e \in O_s \cdot O_{\check{t}}$, then there is $p \in O_s \cap O_t$, that is, $s = t$.

Let us prove that $\tilde{m} = \varphi_*(m)$ is a left invariant measure on Q, that is, for any $\tilde{f}, \tilde{g} \in \mathscr{K}(Q)$ the following relation holds:

$$(7) \qquad (\mathrm{id} \times \tilde{m})\big(\tilde{\Delta}(\tilde{f}) \cdot (1 \otimes \tilde{g})\big) = (\mathrm{id} \times \tilde{m})\big((1 \otimes \tilde{f}) \cdot (\check{\ } \times \mathrm{id}) \circ \tilde{\Delta}(\tilde{g})\big).$$

Consider the left-hand side of (7), use the definitions of $\tilde{m}$ and $\tilde{\Delta}$, and let $\tilde{f} = Pf$, $\tilde{g} = Pg$, $f, g \in \mathscr{K}(G)$,

$$(\mathrm{id} \times m)\big(((P \times P) \circ \Delta(Pf)) \cdot (1 \otimes Pg)\big)$$
$$= (\mathrm{id} \times m)\big((P \times \mathrm{id}) \circ \Delta(Pf) \cdot (1 \otimes Pg)\big)$$
$$= P\big((\mathrm{id} \times m)(\Delta(Pf) \cdot (1 \otimes Pg))\big).$$

Now using a relation similar to (7) that holds for m on G and that $P \circ \,\check{} = \check{}\, \circ P$ we get

$$P\big((\mathrm{id} \times m)(\Delta(Pf) \cdot (1 \otimes Pg))\big)$$
$$= P\big((\mathrm{id} \times m)((1 \otimes Pf) \cdot (\check{} \times \mathrm{id}) \circ \Delta(Pg))\big)$$
$$= (\mathrm{id} \times m)\big((1 \otimes Pf) \cdot (\check{} \circ P \times \mathrm{id}) \circ \Delta(Pg)\big)$$
$$= (\mathrm{id} \times m)\big((1 \otimes Pf) \cdot (\check{} \circ P \times P) \circ \Delta(Pg)\big)$$
$$= (\mathrm{id} \times m \circ P)\big((1 \otimes Pf) \cdot (\check{} \times P) \circ \Delta(Pg)\big)$$
$$= (\mathrm{id} \times \tilde{m})\big((1 \otimes Pf) \cdot (\check{} \times \mathrm{id}) \circ \tilde{\Delta}(Pg)\big).$$

$\square$

Lemma 2. *Let $\langle Pf, m \rangle = \langle f, m \rangle$ for all $f \in \mathscr{K}(G)$. Then the conditional expectation $P: \mathscr{K}(G) \to B$ can be extended by continuity to an idempotent on $L_1(G, m)$ and an orthogonal projection on $L_2(G, m)$, still denoted by P.*

Proof. Let $f \in \mathscr{K}(G)$. Since $-|f| \leq f \leq |f|$ and P preserves the cone of nonnegative functions, we have $|P(f)| \leq P(|f|)$. Hence,

$$\|P(f)\|_1 = \int_G |P(f)|(p)\, dm(p) \leq \int_G P(|f|)(p)\, dm(p)$$
$$= \int_G |f|(p)\, dm(p) = \|f\|_1,$$

which shows that P is continuous with respect to the L_1-norm.

Now, since $|P(f)|^2 \leq P(|f|^2)$, we have

$$\|P(f)\|_2^2 = \int_G |P(f)(p)|^2\, dm(p) \leq \int_G P(|f|^2)(p)\, dm(p)$$
$$= \int_G |f(p)|^2\, dm(p) = \|f\|_2^2.$$

Hence, P is continuous with respect to the L_2-norm on $\mathscr{K}(G)$.

It is clear that the continuous extension of P to $L_2(G, m)$ is a projection. To see that it is orthogonal, let $f_1, f_2 \in \mathscr{K}(G)$ and consider

$$(Pf_1, f_2)_{L_2(G,m)} = \int_G (Pf_1)(p)\overline{f_2(p)}\, dm(p) = \int_G P\big((Pf_1)(p)\overline{f_2(p)}\big)\, dm(p)$$
$$= \int_G (Pf_1)(p)\overline{(Pf_2)(p)}\, dm(p) = \int_G P(f_1(p)\overline{(Pf_2)(p)})\, dm(p)$$
$$= \int_G f_1(p)\overline{(Pf_2)(p)}\, dm(p) = (f_1, Pf_2)_{L_2(G,m)}.$$

$\square$

Remark 3. In what follows, we will identify the space $L_1(Q, \tilde{m})$ (resp., $L_2(Q, \tilde{m})$) with a closed subspace of $L_1(G, m)$ (resp., $L_2(G, m)$).

For a hypergroup Q, consider the product hypergroup $Q \times Q$ [1], and let $\delta \colon \mathcal{M}_b(Q) \to \mathcal{M}_b(Q \times Q)$ denote a linear extension of the map defined by

$$(8) \qquad\qquad \delta(\varepsilon_s) = \varepsilon_s \otimes \varepsilon_s, \quad s \in Q,$$

that is, for $\mu \in \mathcal{M}_b(Q)$ and $F \in \mathcal{K}(Q \times Q)$,

$$\langle\, \delta(\mu), F \,\rangle = \int_Q F(s,s)\, d\mu(s).$$

Definition 6. Let B be a $*$-algebra and A a C^*-algebra. A linear map $\phi \colon B \to A$ will be called *positive*, if $\phi(b^*b) \geq 0$ for all $b \in B$, and *completely positive*, if the map $\mathrm{id} \otimes \phi \colon M_n(\mathbb{C}) \otimes B \to M_n(\mathbb{C}) \otimes A$ is positive for all $n \in \mathbf{N}$, where $M_n(\mathbb{C})$ is the C^*-algebra of $(n \times n)$-matrices over $\mathbb{C}$.

It follows from [14] that a map $\phi \colon B \to A$ from a $*$-algebra B into a C^*-algebra A is completely positive if and only if, for any $n \in \mathbf{N}$, $b_i \in B$ and $a_i \in A$, $i = 1, \ldots, n$, we have

$$(9) \qquad\qquad \sum_{i,j=1}^n a_j^* \phi(b_j^\star b_i) a_i \geq 0.$$

Theorem 2. *Let P be a conditional expectation on $A_0 = \mathcal{C}_0(Q)$ that satisfy all the conditions in Theorem 1. Let Q be the corresponding hypergroup. Denote by λ_Q the left regular representation of $\mathcal{M}_b(Q)$ on $L_2(Q)$,*

$$(\lambda_Q(\tilde{\mu}).\, \tilde{f})(t) = \int_Q \langle\, \tilde{f}, \check{\varepsilon}_s *_Q \varepsilon_t \,\rangle\, d\tilde{\mu}(s), \qquad \tilde{\mu} \in \mathcal{M}_b(Q), \quad \tilde{f} \in L_2(Q), \quad t \in Q,$$

and let δ be defined by (8). Then the linear map

$$(\lambda_Q \otimes \lambda_Q) \circ \delta \colon L_1(\tilde{m}) \to B(L_2(Q) \otimes L_2(Q))$$

is completely positive.

Proof. We will identify $L_2(Q)$ with the closed subspace $P(L_2(G))$ of $L_2(G)$, and the Banach $*$-algebra $L_1(Q, \tilde{m})$ with the Banach $*$-subalgebra $P(L_1(G, m))$ of $L_1(G, m)$. With such an identification, we have

$$(\tilde{f}, \tilde{g})_{L_2(Q)} = (\tilde{f}, \tilde{g})_{L_2(G)}, \quad \tilde{f}, \tilde{g} \in L_2(Q),$$

$$(10) \qquad \lambda_Q(\tilde{\mu}).\, \tilde{f} = \lambda_G(P^*(\tilde{\mu})).\, \tilde{f}, \quad \tilde{\mu} \in \mathcal{M}_b(Q), \quad \tilde{f} \in L_2(Q),$$

$$\delta(P^*(\varepsilon_s)) = P^*(\varepsilon_s) \otimes P^*(\varepsilon_s), \quad s \in Q.$$

To prove the theorem, using (9), it is sufficient to show that

$$\sum_{i,j=1}^n \big(\, A_j^* \cdot (\lambda_Q \otimes \lambda_Q) \circ \delta(\tilde{\mu}_j^\star * \tilde{\mu}_i) \cdot A_i.\, F \,\big|\, F \,\big)_{L_2(Q) \otimes L_2(Q)}$$

$$= \sum_{i,j=1}^n \big(\, (\lambda_Q \otimes \lambda_Q) \circ \delta(\tilde{\mu}_j^\star * \tilde{\mu}_i) \cdot A_i.\, F \,\big|\, A_j.\, F \,\big)_{L_2(Q) \otimes L_2(Q)} \geq 0$$

for any $A_i \in B(L_2(Q) \otimes L_2(Q))$, $\tilde{\mu}_i \in L_1(\tilde{m})$, $F \in L_2(Q) \otimes L_2(Q)$, $i = 1, \ldots, n$.

Hence letting $A_i F = F_i \in L_2(Q) \otimes L_2(Q)$, and $\mu_i = P^*(\tilde{\mu}_i) \in L_1(m_G)$, and using (10), we have

$$\sum_{i,j=1}^n \left((\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i). F_i \mid F_j \right)_{L_2(Q) \otimes L_2(Q)}$$

$$= \sum_{i,j=1}^n \int_G \left((\lambda_G(P^*(\varepsilon_{\varphi(p)})) \otimes \lambda_G(P^*(\varepsilon_{\varphi(p)}))). F_i \mid F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$d(\mu_j^\star * \mu_i)(p).$$

Let $\mu_i = f_i m$, $f_i \in P(L_1(G)) \subset L_1(G)$, $i = 1, \ldots, n$. Then $\mu_j^* = f_j^* m$ and

$$\mu_j^\star * \mu_i = (f_j^* * f_i)m.$$

Using left invariance of the measure m we thus have

$$\sum_{i,j=1}^n \left((\lambda_Q \otimes \lambda_Q) \circ \delta(\mu_j^\star * \mu_i). F_i \mid F_j \right)_{L_2(Q) \otimes L_2(Q)}$$

$$= \sum_{i,j=1}^n \int_G \left((\lambda_G(P^*(\varepsilon_{\varphi(p)})) \otimes \lambda_G(P^*(\varepsilon_{\varphi(p)}))). F_i \mid F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot (f_j^\star * f_i)(p) \, dm(p)$$

$$= \sum_{i,j=1}^n \int_{G^2} \left((\lambda_G(P^*(\varepsilon_{\varphi(p)})) \otimes \lambda_G(P^*(\varepsilon_{\varphi(p)}))). F_i \mid F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot f_j^*(q) f_i(q^{-1}p) \, dm(q) dm(p)$$

$$= \sum_{i,j=1}^n \int_{G^2} \left((\lambda_G(p) \otimes \lambda_G(p)). F_i \mid F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot f_j^*(q) f_i(q^{-1}p) \, dm(q) dm(p)$$

$$= \sum_{i,j=1}^n \int_{G^2} \left((\lambda_G(qp) \otimes \lambda_G(qp)). F_i \mid F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot f_j^*(q) f_i(p) \, dm_G(q) dm_G(p)$$

$$= \sum_{i,j=1}^n \int_{G^2} \left((\lambda_G(p) \otimes \lambda_G(p)). F_i \mid (\lambda_G(q^{-1}) \otimes \lambda_G(q^{-1}). F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot f_j^*(q) f_i(p) \, dm_G(q) dm_G(p)$$

$$= \sum_{i,j=1}^n \int_{G^2} \left((\lambda_G(p) \otimes \lambda_G(p)). F_i \mid (\lambda_G(q^{-1}) \otimes \lambda_G(q^{-1}). F_j \right)_{L_2(G) \otimes L_2(G)}$$
$$\cdot \overline{f}_j(q^{-1}) f_i(p) \, dm_G(q^{-1}) dm_G(p)$$

$$= \left\| \sum_{i=1}^n \int_G f_i(p) (\lambda_G(p) \otimes \lambda_G(p)) \, dm_G(p). F_i \right\|_{L_2(G) \otimes L_2(G)}^2 \geq 0.$$

$$\square$$

Corollary 1. *The Fourier space $\mathscr{A}(Q)$ is a Banach algebra.*

Proof. The statement of this corollary follows from the results of [9]. $\square$

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