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STURM-LIOUVILLE OPERATORS WITH MATRIX DISTRIBUTIONAL COEFFICIENTS

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Abstract. The paper deals with the singular Sturm-Liouville expressions

$$l(y) = -(py')' + qy$$

with the matrix-valued coefficients p, q such that

$$q = Q', \quad p^{-1},\ p^{-1}Q,\ Qp^{-1},\ Qp^{-1}Q \in L_1,$$

where the derivative of the function Q is understood in the sense of distributions. Due to a suitable regularization, the corresponding operators are correctly defined as quasi-differentials. Their resolvent convergence is investigated and all self-adjoint, maximal dissipative, and maximal accumulative extensions are described in terms of homogeneous boundary conditions of the canonical form.

1. Introduction

Many problems of mathematical physics lead to a study of Schrödinger-type operators with strongly singular (in particular distributional) potentials, see the monographs [1, 2] and the more recent papers [5, 6, 18, 19] and references therein. It should be noted that the case of very general singular Sturm-Liouville operators defined in terms of appropriate quasi-derivatives has been considered in [3] (see also the book [7] and earlier discussions of quasi-derivatives in [23, 26]). Higher-order quasi-differential operators with matrix-valued valued singular coefficients were studied in [8, 9, 21, 25].

The paper [22] started a new approach to a study of one-dimensional Schrödinger operators with distributional potential coefficients in connection with such areas as extension theory, resolvent convergence, spectral theory and inverse spectral theory. An important development was achieved in [11] (see also [12, 14]), where it was considered the case of Sturm-Liouville operators generated by the differential expression

$$l(y) = -(py')'(t) + q(t)y(t), \quad t \in \mathcal{J} \tag{1}$$

with singular distributional coefficients on a finite interval $\mathcal{J} := (a, b)$. Namely it was assumed that

$$q = Q', \quad 1/p,\ Q/p,\ Q^2/p \in L_1(\mathcal{J}, \mathbb{C}), \tag{2}$$

where the derivative of Q is understood in the sense of distributions. A more general class of second order quasi-differential operators was recently studied in [19]. In [12, 13] two-term singular differential operators

$$l(y) = i^m y^{(m)}(t) + q(t)y(t), \quad t \in \mathcal{J}, \quad m \geq 2, \tag{3}$$

with distributional coefficient q were investigated. The case of matrix operators of the form (3) was considered in [17]. Let us also mention [20] where the deficiency indices

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of matrix Sturm-Liouville operators with distributional coefficients on a half-line were studied.

The purpose of the present paper is to extend the results of [11] to the matrix Sturm-Liouville differential expressions. In Section 2 we give a regularization of the formal differential expression (1) under a matrix analogue of assumptions (2). The question of norm resolvent convergence of such singular matrix Sturm-Liouville operators is studied in Section 3. In Section 4 we consider the case of the symmetric minimal operator and describe all its self-adjoint, maximal dissipative, and maximal accumulative extensions. In addition, we study in details the case of separated boundary conditions.

2. Regularization of singular expression

For a positive integer s, denote by $M_s \equiv \mathbb{C}^{s\times s}$ the vector space of $s\times s$ matrices with complex coefficients. Let $\mathcal{J} := (a,b)$ be a finite interval. Consider Lebesgue measurable matrix functions p, Q on $\mathcal{J}$ into M_s such that p is invertible almost everywhere. In what follows we shall always assume that

$$p^{-1},\ p^{-1}Q,\ Qp^{-1},\ Qp^{-1}Q \in L_1(\mathcal{J}, M_s). \tag{4}$$

This condition should be considered as a matrix (noncommutative) analogue of the assumption (2). In particular (4) is valid under the (more restrictive) condition

$$\int_{\mathcal{J}} \| p^{-1}(t) \| \, (1 + \| Q(t) \|^2)\, dt < \infty,$$

which was (locally) assumed in the above-mentioned paper [20]. Consider the block Shin–Zettl matrix

$$A := \begin{pmatrix} p^{-1}Q & p^{-1} \\ -Qp^{-1}Q & -Qp^{-1} \end{pmatrix} \in L_1(\mathcal{J}, M_{2s}) \tag{5}$$

and the corresponding quasi-derivatives

$$D^{[0]}y = y, \quad D^{[1]}y = py' - Qy, \quad D^{[2]}y = (D^{[1]}y)' + Qp^{-1}D^{[1]}y + Qp^{-1}Qy.$$

For $q = Q'$ the Sturm-Liouville expression (1) is defined by

$$l[y] := -D^{[2]}y. \tag{6}$$

The quasi-differential expression (6) gives rise to the *maximal* quasi-differential operator in the Hilbert space $L_2(\mathcal{J}, \mathbb{C}^s) =: L_2$

$$L_{\max} : y \to l[y], \quad \mathrm{Dom}(L_{\max}) := \left\{ y \in L_2 \;\middle|\; y,\, D^{[1]}y \in AC([a,b], \mathbb{C}^s), D^{[2]}y \in L_2 \right\}.$$

The *minimal* quasi-differential operator is defined as a restriction of the operator $L_{\max}$ onto the set

$$\mathrm{Dom}(L_{\min}) := \left\{ y \in \mathrm{Dom}(L_{\max}) \;\middle|\; D^{[k]}y(a) = D^{[k]}y(b) = 0, k = 0,1 \right\}.$$

Note that under the assumption

$$p^{-1}, q \in L_1(\mathcal{J}, M_s),$$

the operators $L_{\max}, L_{\min}$ introduced above coincide with the standard maximal and minimal matrix Sturm-Liouville operators. The regularization of the formally adjoint differential expression

$$l^{+}y := -(p^{*}y')'(t) + q^{*}(t)y(t)$$

can be defined in an analogous way (here $A^{*} = \overline{A^{T}}$ is the conjugate transposed matrix to A). Let $D^{\{k\}}$ $(k = 0,1,2)$ be the Shin–Zettl quasi-derivatives associated with l^{+}. Denote by $L^{+}_{\max}$ and $L^{+}_{\min}$ the maximal and the minimal operators generated by this expression on the space L_2. The following results are proved in [8] (see also [21]) in the case of general quasi-differential matrix operators.

Lemma 1. (Green's formula). *For any* $y \in \operatorname{Dom}(L_{\max})$, $z \in \operatorname{Dom}(L^{+}_{\max})$ *there holds*

$$\int_a^b \left(D^{[2]}y \cdot \overline{z} - y \cdot \overline{D^{\{2\}}z}\right) dt = (D^{[1]}y \cdot \overline{z} - y \cdot \overline{D^{\{1\}}z})\,\Big|_{t=a}^{t=b}.$$

Lemma 2. *For any* $(\alpha_0, \alpha_1), (\beta_0, \beta_1) \in \mathbb{C}^{2s}$ *there exists a function* $y \in \operatorname{Dom}(L_{\max})$ *such that*

$$D^{[k]}y(a) = \alpha_k, \quad D^{[k]}y(b) = \beta_k, \quad k = 0, 1.$$

Theorem 1. *The operators* $L_{\min}$, $L^{+}_{\min}$, $L_{\max}$, $L^{+}_{\max}$ *are closed and densely defined on* $L_2([a,b], \mathbb{C}^s)$, *and satisfy*

$$L^{*}_{\min} = L^{+}_{\max}, \quad L^{*}_{\max} = L^{+}_{\min}.$$

In the case of Hermitian matrices p *and* Q *the operator* $L_{\min} = L^{+}_{\min}$ *is symmetric with the deficiency indices* $(2s, 2s)$, *and*

$$L^{*}_{\min} = L_{\max}, \quad L^{*}_{\max} = L_{\min}.$$

3. Convergence of resolvents

Let $l_\varepsilon[y] = -D_\varepsilon^{[2]}y$, $\varepsilon \in [0, \varepsilon_0]$, be quasi-differential expressions with the coefficients $p_\varepsilon, Q_\varepsilon$ satisfying (4). These expressions generate the minimal operators $L^{\varepsilon}_{\min}$, $L^{\varepsilon}_{\max}$ in L_2. Consider the quasi-differential operators

$$L_\varepsilon y = l_\varepsilon[y], \quad \operatorname{Dom}(L_\varepsilon) = \{ y \in \operatorname{Dom}(L^{\varepsilon}_{\max}) |\, \alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0 \}.$$

Here $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2s \times 2s}$ are complex matrices and

$$\mathcal{Y}_\varepsilon(a) := \left\{ y(a), D_\varepsilon^{[1]}y(a) \right\}, \quad \mathcal{Y}_\varepsilon(b) := \left\{ y(b), D_\varepsilon^{[1]}y(b) \right\}.$$

Clearly, $L^{\varepsilon}_{\min} \subset L_\varepsilon \subset L^{\varepsilon}_{\max}$, $\varepsilon \in [0, \varepsilon_0]$. Denote by $\rho(L)$ the resolvent set of the operator L. Recall that L_ε is said to converge to L_0 in the norm resolvent sense, $L_\varepsilon \overset{R}{\Rightarrow} L_0$, if there is a number $\mu \in \rho(L_0)$, such that $\mu \in \rho(L_\varepsilon)$ for all sufficiently small ε, and

$$\|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \to 0, \quad \varepsilon \to 0+. \tag{7}$$

It should be noted that if $L_\varepsilon \overset{R}{\Rightarrow} L_0$, then the condition (7) is fulfilled for all $\mu \in \rho(L_0)$ (see [15]).

Theorem 2. *Suppose* $\rho(L_0)$ *is not empty and, for* $\varepsilon \to 0+$, *the following conditions hold:*

(1) $\|p_\varepsilon^{-1} - p_0^{-1}\|_1 \to 0$,
(2) $\|p_\varepsilon^{-1}Q_\varepsilon - p_0^{-1}Q_0\|_1 \to 0$,
(3) $\|Q_\varepsilon p_\varepsilon^{-1} - Q_0 p_0^{-1}\|_1 \to 0$,
(4) $\|Q_\varepsilon p_\varepsilon^{-1} Q_\varepsilon - Q_0 p_0^{-1} Q_0\|_1 \to 0$,
(5) $\alpha(\varepsilon) \to \alpha(0)$, $\quad \beta(\varepsilon) \to \beta(0)$,

where $\|\cdot\|_1$ *is the norm in the space* $L_1(\mathcal{J}, M_s)$. *Then* $L_\varepsilon \overset{R}{\Rightarrow} L_0$.

Essentially, the proof of Theorem 2 repeats the arguments of [11] where the scalar case $s = 1$ was considered. Nevertheless the result seems to be new even in the case of one-dimensional Schrödinger operators with distributional matrix-valued potentials (p_ε is the identity matrix in $\mathbb{C}^s$). Recall the following definition [16].

Definition 1. Denote by $\mathcal{M}^m(\mathcal{J}) =: \mathcal{M}^m$, $m \in \mathbb{N}$, the class of matrix-valued functions

$$R(\cdot;\varepsilon) : [0,\varepsilon_0] \to L_1(\mathcal{J}, \mathbb{C}^{m\times m})$$

parametrized by ε such that the solution of the Cauchy problem

$$Z'(t;\varepsilon) = R(t;\varepsilon)Z(t;\varepsilon), \quad Z(a;\varepsilon) = I,$$

satisfies the limit condition

$$\lim_{\varepsilon\to 0+} \|Z(\cdot;\varepsilon) - I\|_\infty = 0,$$

where $\|\cdot\|_\infty$ is the sup-norm.

We need the following result [16].

Theorem 3. *Suppose that the vector boundary-value problem*

$$y'(t;\varepsilon) = A(t;\varepsilon)y(t;\varepsilon) + f(t;\varepsilon), \quad t \in \mathcal{J}, \quad \varepsilon \in [0,\varepsilon_0], \tag{8}$$

$$U_\varepsilon y(\cdot;\varepsilon) = 0, \tag{9}$$

where the matrix-valued functions $A(\cdot,\varepsilon) \in L_1(\mathcal{J}, \mathbb{C}^{m\times m})$, the vector-valued functions $f(\cdot,\varepsilon) \in L_1(\mathcal{J}, \mathbb{C}^m)$, and the linear continuous operators

$$U_\varepsilon : C(\overline{\mathcal{J}}; \mathbb{C}^m) \to \mathbb{C}^m, \quad m \in \mathbb{N},$$

satisfy the following conditions.

1) *The homogeneous limit boundary-value problem* (8), (9) *with $\varepsilon = 0$ and $f(\cdot;0) \equiv 0$ has only a trivial solution;*
2) $A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^m$;
3) $\|U_\varepsilon - U_0\| \to 0, \quad \varepsilon \to 0+$.

Then, for a small enough ε, there exist Green matrices $G(t,s;\varepsilon)$ for problems (8), (9) and

$$\|G(\cdot,\cdot;\varepsilon) - G(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to 0+, \tag{10}$$

where $\|\cdot\|_\infty$ is the norm in the space $L_\infty(\mathcal{J}\times\mathcal{J}, \mathbb{C}^{m\times m})$.

It follows from [24] that conditions (1)–(4) of Theorem 2 imply

$$A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^{2s},$$

where the block Shin–Zettl matrix $A(\cdot;\varepsilon)$ is given by the formula

$$A(\cdot;\varepsilon) := \begin{pmatrix} p_\varepsilon^{-1}Q_\varepsilon & p_\varepsilon^{-1} \\ -Q_\varepsilon p_\varepsilon^{-1}Q_\varepsilon & -Q_\varepsilon p_\varepsilon^{-1} \end{pmatrix}. \tag{11}$$

In particular $A(\cdot;0) = A$ (see (5)). The following two lemmas reduce Theorem 2 to Theorem 3.

Lemma 3. *The function $y(t)$ is a solution of the boundary-value problem*

$$l_\varepsilon[y](t) = f(t;\varepsilon) \in L_2, \quad \varepsilon \in [0,\varepsilon_0], \tag{12}$$

$$\alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0, \tag{13}$$

if and only if the vector-valued function $w(t) = (y(t), D_\varepsilon^{[1]}y(t))$ is a solution of the boundary-value problem

$$w'(t) = A(t;\varepsilon)w(t) + \varphi(t;\varepsilon), \tag{14}$$

$$\alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0, \tag{15}$$

where the matrix-valued function $A(\cdot;\varepsilon)$ is given by (11) and $\varphi(\cdot;\varepsilon) := (0, -f(\cdot;\varepsilon))$.

Proof. Consider the system of equations

$$\begin{cases} (D_\varepsilon^{[0]}y(t))' = p_\varepsilon^{-1}(t)Q_\varepsilon(t)D_\varepsilon^{[0]}y(t) + p_\varepsilon^{-1}(t)D_\varepsilon^{[1]}y(t), \\ (D_\varepsilon^{[1]}y(t))' = -Q_\varepsilon(t)p_\varepsilon^{-1}(t)Q_\varepsilon(t)D_\varepsilon^{[0]}y(t) - Q_\varepsilon(t)p_\varepsilon^{-1}(t)D_\varepsilon^{[1]}y(t) - f(t;\varepsilon). \end{cases}$$

Let $y(\cdot)$ be a solution of (12), then the definition of a quasi-derivative implies that $y(\cdot)$ is a solution of this system. On the other hand, denoting $w(t) = (D_\varepsilon^{[0]}y(t), D_\varepsilon^{[1]}y(t))$ and $\varphi(t;\varepsilon) = (0, -f(t;\varepsilon))$, we rewrite this system in the form of equation (14). Taking into account that $\mathcal{Y}_\varepsilon(a) = w(a)$, $\mathcal{Y}_\varepsilon(b) = w(b)$, one can see that the boundary conditions (13) are equivalent to the boundary conditions (15). $\square$

Lemma 4. *Let a Green matrix*

$$G(t,s,\varepsilon) = (g_{ij}(t,s,\varepsilon))_{i,j=1}^2 \in L_\infty(\mathcal{J}\times\mathcal{J}, \mathbb{C}^{2s\times 2s})$$

exist for the problem (14), (15) for small enough ε. Then there exists a Green function $\Gamma(t,s;\varepsilon)$ for the semi-homogeneous boundary-value problem (12), (13) and

$$\Gamma(t,s;\varepsilon) = -g_{12}(t,s;\varepsilon) \quad a.e.$$

Proof. According to the definition of a Green matrix, a unique solution of the problem (14), (15) can be written in the form

$$w_\varepsilon(t) = \int_a^b G(t,s;\varepsilon)\varphi(s;\varepsilon)ds, \quad t\in\mathcal{J}.$$

Due to Lemma 3, the latter equality can be rewritten in the form

$$\begin{cases} D_\varepsilon^{[0]}y_\varepsilon(t) = \displaystyle\int_a^b g_{12}(t,s;\varepsilon)(-f(s;\varepsilon))\,ds, \\ D_\varepsilon^{[1]}y_\varepsilon(t) = \displaystyle\int_a^b g_{22}(t,s;\varepsilon)(-f(s;\varepsilon))\,ds, \end{cases}$$

where $y_\varepsilon(\cdot)$ is a unique solution of (12), (13). This implies the statement of Lemma 4. $\square$

Proof of Theorem 2. Consider the matrices

$$Q_{\varepsilon(t),\mu} = Q_\varepsilon(t) + \mu t I,\ p_{\varepsilon(t),\mu} = p_\varepsilon(t)$$

corresponding to the operators $L_\varepsilon + \mu I$. Clearly assumption (4) and conditions (1)–(4) of Theorem 2 do not depend on μ and we can assume without loss of generality that $0 \in \rho(L_0)$. It follows that the homogeneous boundary-value problem

$$l_0[y](t) = 0, \quad \alpha(0)\mathcal{Y}_0(a) + \beta(0)\mathcal{Y}_0(b) = 0$$

has only a trivial solution. Due to Lemma 3 the homogeneous boundary-value problem

$$w'(t) = A(t;0)w(t), \quad \alpha(0)w(a) + \beta(0)w(b) = 0$$

also has only a trivial solution. By conditions (1)–(4) of Theorem 2 we have that $A(\cdot;\varepsilon) - A(\cdot;0) \in \mathcal{M}^{2s}$, where $A(\cdot;\varepsilon)$ is given by formula (11). Thus the statement of Theorem 2 implies that the problem (14), (15) satisfies conditions of Theorem 3. It follows that Green matrices $G(t,s;\varepsilon)$ of the problems (14), (15) exist. Taking into account Lemma 4 and (10) we have that

$$\begin{aligned} \|L_\varepsilon^{-1} - L_0^{-1}\| &\le \|L_\varepsilon^{-1} - L_0^{-1}\|_{HS} = \|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_2 \\ &\le (b-a)\|\Gamma(\cdot,\cdot;\varepsilon) - \Gamma(\cdot,\cdot;0)\|_\infty \to 0, \quad \varepsilon \to 0+. \end{aligned}$$

Here $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. $\square$

Remark 1. It follows from the proof that $(L_\varepsilon - \mu)^{-1} \to (L_0 - \mu)^{-1}$ in a Hilbert-Schmidt norm for all $\mu \in \rho(L_0)$.

4. Extensions of symmetric minimal operator

In what follows we additionally suppose that the matrix functions p, Q and, consequently, the distribution $q = Q'$ are Hermitian. By Theorem 1 the minimal operator $L_{\min}$ is symmetric and one may consider a problem of describing (in terms of homogeneous boundary conditions) all self-adjoint, maximal dissipative, and maximal accumulative extensions of the operator $L_{\min}$. Let us recall following definition.

Definition 2. Let L be a closed densely defined symmetric operator on a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficient indices. A triplet (H, Γ_1, Γ_2), where H is an auxiliary Hilbert space and Γ_1, Γ_2 are linear mappings of $\mathrm{Dom}(L^*)$ onto H, is called a *boundary triplet* of the symmetric operator L, if

(1) for any $f, g \in \mathrm{Dom}\,(L^*)$,

$$(L^* f, g)_{\mathcal{H}} - (f, L^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_2 g)_H - (\Gamma_2 f, \Gamma_1 g)_H\,,$$

(2) for any $f_1, f_2 \in H$ there is a vector $f \in \mathrm{Dom}\,(L^*)$ such that $\Gamma_1 f = f_1$, $\Gamma_2 f = f_2$.

The definition of a boundary triplet implies that $f \in \mathrm{Dom}\,(L)$ if and only if $\Gamma_1 f = \Gamma_2 f = 0$. A boundary triplet exists for any symmetric operator with equal non-zero deficient indices (see [10] and references therein). The following result is crucial for the rest of the paper.

Lemma 5. *A triplet* $(\mathbb{C}^{2s}, \Gamma_1, \Gamma_2)$*, where* Γ_1, Γ_2 *are the linear mappings*

$$\Gamma_1 y := \left(D^{[1]}y(a), -D^{[1]}y(b)\right), \quad \Gamma_2 y := (y(a), y(b))\,,$$

from $\mathrm{Dom}(L_{\max})$ *onto* $\mathbb{C}^{2s}$ *is a boundary triplet for the operator* $L_{\min}$.

Proof. According to Theorem 1, $L_{\min}^* = L_{\max}$. Due to Lemma 1,

$$(L_{\max} y, z) - (y, L_{\max} z) = \left(y \cdot \overline{D^{[1]}z} - D^{[1]}y \cdot \overline{z}\right)\Big|_a^b.$$

But

$$(\Gamma_1 y, \Gamma_2 z) = D^{[1]}y(a) \cdot \overline{z(a)} - D^{[1]}y(b) \cdot \overline{z(b)},$$
$$(\Gamma_2 y, \Gamma_1 z) = y(a) \cdot \overline{D^{[1]}z(a)} - y(b) \cdot \overline{D^{[1]}z(b)}.$$

This means that condition 1) is fulfilled. Condition 2) is true due to Lemma 2. □

Let K be a linear operator on $\mathbb{C}^{2s}$. Denote by L_K the restriction of $L_{\max}$ onto the set of functions $y \in \mathrm{Dom}(L_{\max})$ satisfying the homogeneous boundary condition in the canonical form

$$(K - I)\,\Gamma_1 y + i\,(K + I)\,\Gamma_2 y = 0. \tag{16}$$

Similarly, L^K denotes the restriction of $L_{\max}$ onto the set of the functions $y \in \mathrm{Dom}(L_{\max})$ satisfying the boundary condition

$$(K - I)\,\Gamma_1 y - i\,(K + I)\,\Gamma_2 y = 0. \tag{17}$$

Clearly, L_K and L^K are the extensions of L for any K. Recall that a densely defined linear operator T on a complex Hilbert space $\mathcal{H}$ is called *dissipative* (resp. *accumulative*) if

$$\Im\,(Tx, x)_{\mathcal{H}} \geq 0 \quad (\text{resp.} \leq 0), \quad \text{for all} \quad x \in \mathrm{Dom}(T)$$

and it is called *maximal dissipative* (resp. *maximal accumulative*) if, in addition, T has no non-trivial dissipative (resp. accumulative) extensions in $\mathcal{H}$. Every symmetric operator is both dissipative and accumulative, and every self-adjoint operator is a maximal dissipative and maximal accumulative one. Lemma 5 together with results of [10, Ch. 3] leads to the following description of dissipative, accumulative, and self-adjoint extensions of $L_{\min}$.

Theorem 4. *Every L_K with K being a contracting operator in $\mathbb{C}^{2s}$, is a maximal dissipative extension of $L_{\min}$. Similarly every L^K with K being a contracting operator in $\mathbb{C}^{2s}$, is a maximal accumulative extension of the operator $L_{\min}$. Conversely, for any maximal dissipative (respectively, maximal accumulative) extension $\widetilde{L}$ of the operator $L_{\min}$ there exists a contracting operator K such that $\widetilde{L} = L_K$ (respectively, $\widetilde{L} = L^K$). The extensions L_K and L^K are self-adjoint if and only if K is a unitary operator on $\mathbb{C}^{2s}$. These correspondences between operators $\{K\}$ and the extensions $\{\widetilde{L}\}$ are all bijective.*

Remark 2. It follows from Theorem 2 and Theorem 4 that the mapping $K \to L_K$ is not only bijective but also continuous. More accurately, if contracting operators K_n converge to an operator K, then $L_{K_n} \overset{R}{\Rightarrow} L_K$. The converse is also true, because the set of contracting operators in the space $\mathbb{C}^{2s}$ is a compact set. This means that the mapping

$$K \to (L_K - \lambda)^{-1}, \quad \operatorname{Im} \lambda < 0,$$

is a homeomorphism for any fixed λ. Analogous result is true for L^K.

Now we pass to a description of separated boundary conditions. Denote by f_a the germ of a continuous function f at the point a.

Definition 3. The boundary conditions that define the operator $L \subset L_{\max}$ are called *separated* if for arbitrary functions $y \in \operatorname{Dom}(L)$ and any $g, h \in \operatorname{Dom}(L_{\max})$, such that

$$g_a = y_a, \quad g_b = 0, \quad h_a = 0, \quad h_b = y_b$$

we have $g, h \in \operatorname{Dom}(L)$.

Theorem 5. *Let K be a linear operator on $\mathbb{C}^{2s}$. Boundary conditions (16), (17) defining L_K and L^K respectively are separated if and only if K is block diagonal, i.e.,*

$$K = \begin{pmatrix} K_a & 0 \\ 0 & K_b \end{pmatrix}, \tag{18}$$

where K_a, K_b are arbitrary $s \times s$ matrices.

Proof. We consider the operators L_K, the case of L^K can be treated in a similar way. The assumption $y_c = g_c$ implies that

$$y(c) = g(c), \quad (D^{[1]}y)(c) = (D^{[1]}g)(c), \quad c \in [a, b]. \tag{19}$$

Let K have the form (18). Then (16) can be written in the form of a system,

$$\begin{cases} (K_a - I)D^{[1]}y(a) + i(K_a + I)y(a) = 0, \\ -(K_b - I)D^{[1]}y(b) + i(K_b + I)y(b) = 0. \end{cases}$$

Clearly these conditions are separated. Conversely, suppose that boundary conditions (16) are separated. The matrix $K \in \mathbb{C}^{2s \times 2s}$ can be written in the form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

We need to prove that $K_{12} = K_{21} = 0$. Let us rewrite (16) in the form of the system

$$\begin{cases} (K_{11} - I)D^{[1]}y(a) - K_{12}D^{[1]}y(b) + i(K_{11} + I)y(a) + iK_{12}y(b) = 0, \\ K_{21}D^{[1]}y(a) - (K_{22} - I)D^{[1]}y(b) + iK_{21}y(a) + i(K_{22} + I)y(b) = 0. \end{cases}$$

The fact that the boundary conditions are separated implies that a function g such that $g_a = y_a, g_b = 0$ also satisfies this system. It follows from (19) that for any $y \in \mathrm{Dom}(L_K)$

$$\begin{cases} K_{11}\left[D^{[1]}y(a) + iy(a)\right] = D^{[1]}y(a) - iy(a), \\ K_{21}\left[D^{[1]}y(a) + iy(a)\right] = 0. \end{cases}$$

This means that for any $y \in \mathrm{Dom}(L_K)$

$$D^{[1]}y(a) + iy(a) \in \mathrm{Ker}(K_{21}). \tag{20}$$

For any $z = (z_1, z_2) \in \mathbb{C}^{2s}$, consider the vectors $-i\,(K+I)\,z$ and $(K-I)\,z$. Due to Lemma 5 and the definition of the boundary triplet, there exists a function $y_z \in \mathrm{Dom}(L_{\max})$ such that

$$\begin{cases} -i\,(K+I)\,z = \Gamma_1 y_z, \\ \quad (K-I)\,z = \Gamma_2 y_z. \end{cases} \tag{21}$$

Clearly y_z satisfies (16) and $y_z \in \mathrm{Dom}(L_K)$. Rewrite (21) in the form of the system

$$\begin{cases} -i(K_{11}+I)z_1 - iK_{12}z_2 = D^{[1]}y_z(a), \\ -iK_{21}z_1 - i(K_{22}+I)z_2 = -D^{[1]}y_z(b), \\ \quad (K_{11}-I)z_1 + K_{12}z_2 = y_z(a), \\ \quad K_{21}z_1 + (K_{22}-I)z_2 = y_z(b). \end{cases}$$

The first and the third equations of the system above imply that for any $z_1 \in \mathbb{C}^s$

$$D^{[1]}y_z(a) + iy_z(a) = -2iz_1.$$

Due to (20) we have that $\mathrm{Ker}(K_{21}) = \mathbb{C}^s$ and therefore $K_{21} = 0$. Similarly one can prove that $K_{12} = 0$. □

Remark 3. It follows from Lemma 5 and Theorem 1 of [4] that there is a one-to-one correspondence between the generalized resolvents R_λ of $L_{\min}$ and the boundary-value problems

$$l[y] = \lambda y + h,\ \ (K(\lambda) - I)\,\Gamma_1 y + i\,(K(\lambda) + I)\,\Gamma_2 y = 0.$$

Here $\mathrm{Im}\,\lambda < 0$, $h \in L_2$, and $K(\lambda)$ is an operator-valued function on the space $\mathbb{C}^{2s}$, regular in the lower half-plane, such that $||K(\lambda)|| \le 1$. This correspondence is given by the identity

$$R_\lambda h = y, \quad \mathrm{Im}\,\lambda < 0.$$

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