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TANNAKA-KREIN RECONSTRUCTION FOR COACTIONS OF FINITE QUANTUM GROUPOIDS

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Dedicated to the memory of Professor Myroslav Lvovich Gorbachuk

Abstract. We study coactions of finite quantum groupoids on unital C^*-algebras and obtain a Tannaka-Krein reconstruction theorem for them.

1. Introduction

Let us recall what Tannaka-Krein reconstruction is. In his paper [29] T. Tannaka showed that a compact group G can be reconstructed if the set $URep(G)$ of its unitary finite dimensional representations is known. Then M.G. Krein [11] gave an abstract description of $URep(G)$. Later on, mainly due to works by A. Grothendieck, P. Deligne, and N. Saavedra Rivano, these results referred to as "Tannaka-Krein reconstruction for compact groups" or "Tannaka-Krein duality for compact groups" were formulated in the language of symmetric monoidal tensor categories and extended to affine algebraic groups.

A convenient formulation of the Tannaka-Krein duality was done by S. Doplicher and J.E. Roberts who introduced the notion of a C^*-tensor category with conjugates (the basic definitions and results concerning C^*-tensor categories can be found in the book [18]). These authors proved that if such a category is symmetric, then it is equivalent to the unitary representation category of a unique compact group. In the much wider setting of *compact quantum groups* – see [18], the S.L. Woronowicz's Tannaka-Krein reconstruction theorem [35] claims that any C^*-tensor category with conjugates and with a unitary tensor functor to the category of finite dimensional Hilbert spaces (fiber functor), is equivalent to the category of unitary finite dimensional representations of a unique compact quantum group with the canonical fiber functor sending any representation to the Hilbert space where it acts.

Consider now an action α of a compact group G on a unital C^*-algebra A by automorphisms and ask if it is possible to reconstruct not only G, but the whole dynamical system (G, A, α) from a given C^*-tensor category $\mathcal{C}$ with conjugates equipped with some additional structure. One can show that the answer is positive, and this additional structure is a *module category* over $\mathcal{C}$ [9] containing a generating element. Namely, in the context of compact quantum groups, K. De Commer and M. Yamashita [6] showed that there is a one-to-one correspondence between ergodic coactions of G and semisimple irreducible module categories over $URep(G)$ with simple generators. This abstract result enabled them, as a spectacular application, to classify all ergodic coactions of the concrete compact quantum group $SU_q(2)$ in terms of weighted graphs – see [7].

Later on, the above approach was extended by S. Neshveyev [16] to general coactions of compact quantum groups, on the one hand, and general module categories over $URep(G)$

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containing generating elements, on the other hand. The main features of this construction are explained in the survey [30].

The goal of the present paper is to obtain similar reconstruction results of Tannaka-Krein type for coactions of *weak Hopf C^*-algebras* in the sense of [3] on C^*-algebras. We use systematically the term "a finite quantum groupoid" instead of "a weak Hopf C^*-algebra" because a groupoid C^*-algebra and an algebra of functions on a usual finite groupoid carry this type of a structure. These objects are important at least for two reasons. First, any fusion (i.e., semisimple finite rigid tensor) category [9] can be realized as a representation category of a weak Hopf algebra by the result of T. Hayashi [10]. Second, as shown in [21], [22], weak Hopf C^*-algebras and their coideal C^*-subalgebras play important role in the description of the Jones's tower of II_1-subfactors with finite index and finite depth.

The above application to subfactors explains the interest in the construction of concrete examples of finite quantum groupoids and in the classification of their coideal subalgebras. Some particular constructions of finite quantum groupoids were proposed in [34] and [20] (see also the survey [23] and references therein), but the general way to construct them is the application of the T. Hayashi's reconstruction theorem [10] to concrete tensor categories. This approach was used in [14], where a series of concrete finite quantum groupoids was constructed using Tambara-Yamagami categories [28]. These categories belong to the much wider family of $\mathbb{Z}/2\mathbb{Z}$-extensions of pointed fusion categories classified in [31].

The problem of the description of coideal C^*-subalgebras of a given finite quantum groupoid is even harder, and until now only two concrete families of such subalgebras constructed in [14] "by hand" are known. But such coideal C^*-subalgebras are equipped with coactions of a given finite quantum groupoid via its coproduct, so their description can be viewed as an application of the general reconstruction result for coactions – see Theorem 1.1 below. Concrete results of this type will be given in a subsequent work.

Let us describe the structure of the paper. In Section 2 we recall basic definitions and results on finite quantum groupoids following [3] and [23]. We also translate the representation theory of these objects treated in [4] and [19] into the language of unitary corepresentations and C^*-tensor categories suitable for the construction of the categorical duality. Finally, we translate into this language the reconstruction theorem proved in [10] and [27].

In Section 3 we develop the theory parallel to the one of compact quantum group coactions [2]. Doing this, we simplify significantly, in our particular case, some constructions related to coactions of general measured quantum groupoids – see [8], [34]. Let $\mathfrak{a}$ be a coaction of a finite quantum groupoid $\mathfrak{G}$ on a unital C^*-algebra A (called a $\mathfrak{G}$-C^*-algebra). We get the canonical implementation of $\mathfrak{a}$ and study the properties of the spectral subspaces (isotypical components) of A. Note that the subalgebra of fixed points of A with respect to $\mathfrak{a}$ can be strictly smaller than the spectral subspace corresponding to the trivial corepresentation of $\mathfrak{G}$ (in the compact quantum group case they are equal). This creates specific problems that we solve in Sections 4,5 and 6 devoted to the proof of our main result which is parallel to [6], Theorem 6.4 and [16], Theorem 3.3:

Theorem 1.1. *Let $\mathfrak{G}$ be a regular coconnected finite quantum groupoid. Then the following two categories are equivalent:*

(i) The category of unital $\mathfrak{G}$-C^*-algebras with unital $\mathfrak{G}$-equivariant $*$-homomorphisms as morphisms.

(ii) The category of pairs $(\mathcal{M}, M)$, where $\mathcal{M}$ is a left module C^*-category over C^*-tensor category $\mathbf{UCorep}(\mathfrak{G})$ *of unitary corepresentations of $\mathfrak{G}$ and M is a generator in $\mathcal{M}$, with equivalence classes of unitary module functors respecting the prescribed generators as morphisms.*

This proof divides into three parts. First, given a unital $\mathfrak{G}$-C^*-algebra A, we show in Section 4 that the category $\mathcal{D}_A$ of finitely generated equivariant C^*-correspondences whose morphisms are equivariant maps, is a strict left module category over **UCorep**($\mathfrak{G}$). The algebra A itself is a generator in $\mathcal{D}_A$. The idea of such a construction in the compact quantum group case was proposed in [6].

Vice versa, it is shown in [16] that any pair $(\mathcal{M}, M)$ as above generates so-called weak tensor functor. Using this functor, we construct in Section 5 an algebra whose C^*-completion is a unital $\mathfrak{G}$-C^*-algebra. Finally, we show in Section 6 that the two above mentioned constructions are mutually inverse which gives the equivalence of the categories in question.

It was shown in [15] in the compact quantum group case that **UCorep**($\mathfrak{G}$)-module categories parameterized by unitary tensor (not weak tensor !) functors correspond to Yetter-Drinfeld $\mathfrak{G}$-C^*-algebras. In a subsequent work we expect to get a similar result for finite quantum groupoids and to apply it to the description of coideal C^*-subalgebras of quotient type.

Our standard references are: [13] for general categories, [9] for tensor categories, [18] for C^*- and C^*-tensor categories, [12] for Hilbert C^*-modules, and [23] for finite quantum groupoids.

2. Finite quantum groupoids, their representations, comodules and corepresentations

1. Finite quantum groupoids. A *weak Hopf C^*-algebra* $\mathfrak{G} = (B, \Delta, S, \varepsilon)$ is a finite dimensional C^*-algebra B with the comultiplication $\Delta : B \to B \otimes B$, counit $\varepsilon : B \to C$, and antipode $S : B \to B$ such that (B, Δ, ε) is a coalgebra and the following axioms hold for all $b, c, d \in B$:

(1) Δ is a (not necessarily unital) $*$-homomorphism :

$$\Delta(bc) = \Delta(b)\Delta(c), \quad \Delta(b^*) = \Delta(b)^*,$$

(2) The unit and counit satisfy the identities (we use the Sweedler leg notation $\Delta(c) = c_1 \otimes c_2$, $(\Delta \otimes id_B)\Delta(c) = c_1 \otimes c_2 \otimes c_3$ etc.):

$$\begin{aligned} \varepsilon(bc_1)\varepsilon(c_2 d) &= \varepsilon(bcd), \\ (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) &= (\Delta \otimes id_B)\Delta(1), \end{aligned}$$

(3) S is an anti-algebra and anti-coalgebra map such that

$$\begin{aligned} m(id_B \otimes S)\Delta(b) &= (\varepsilon \otimes id_B)(\Delta(1)(b \otimes 1)), \\ m(S \otimes id_B)\Delta(b) &= (id_B \otimes \varepsilon)((1 \otimes b)\Delta(1)), \end{aligned}$$

where m denotes the multiplication.

The right hand sides of two last formulas are called *target* and *source counital maps* ε_t and ε_s, respectively. Their images are unital C^*-subalgebras of B called *target* and *source counital subalgebras* B_t and B_s, respectively.

The dual vector space $\hat{B}$ has a natural structure of a weak Hopf C^*-algebra $\hat{\mathfrak{G}} = (\hat{B}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ given by dualizing the structure operations of B

$$\begin{aligned} < \varphi\psi, b > &= < \varphi \otimes \psi, \Delta(b) >, \\ < \hat{\Delta}(\varphi), b \otimes c > &= < \varphi, bc >, \\ < \hat{S}(\varphi), b > &= < \varphi, S(b) >, \\ < \phi^*, b > &= \overline{< \varphi, S(b)^* >}, \end{aligned}$$

for all $b, c \in B$ and $\varphi, \psi \in \hat{B}$. The unit of $\hat{B}$ is ε and the counit is 1.

The counital subalgebras commute elementwise, we have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ is the center of B), coconnected if $B_t \cap B_s = \mathbb{C}$, and *biconnected* if both conditions are satisfied.

The antipode S is unique, invertible, and satisfies $(S \circ *)^2 = id_B$. We will only consider *regular* quantum groupoids, i.e., such that $S^2|_{B_t} = id$. In this case, there exists a canonical positive element H in the center of B_t such that S^2 is an inner automorphism implemented by $G = HS(H)^{-1}$, i.e., $S^2(b) = GbG^{-1}$ for all $b \in B$. The element G is called the canonical group-like element of B, it satisfies the relation $\Delta(G) = (G \otimes G)\Delta(1) = \Delta(1)(G \otimes G)$.

There exists a unique positive functional h on B, called a *normalized Haar measure* such that

$$(id_B \otimes h)\Delta = (\varepsilon_t \otimes h)\Delta, \quad h \circ S = h, \quad h \circ \varepsilon_t = \varepsilon, \quad (id_B \otimes h)\Delta(1_B) = 1_B.$$

We will denote by H_h the GNS Hilbert space generated by B and h and by $\Lambda_h : B \to H_h$ the corresponding GNS map.

2. Unitary representations. By definition, the objects of the category $URep(\mathfrak{G})$ of unitary representations of $\mathfrak{G}$ are left B-modules of finite rank such that the underlying vector space is a Hilbert space H with a scalar product $< \cdot, \cdot >$ such that

$$< b \cdot v, w > = < v, b^* \cdot w >, \quad \text{for all} \quad v, w \in H, \quad b \in B,$$

and morphisms are B-linear maps. It is a semisimple linear category whose simple objects are irreducible B-modules. It is also a tensor category: for objects $H_1, H_2 \in URep(\mathfrak{G})$, define their tensor product as the Hilbert subspace $\Delta(1_B) \cdot (H_1 \otimes H_2)$ of the usual tensor product together with the action of B given by Δ. Here we use the fact that $\Delta(1_B)$ is an orthogonal projection.

The tensor product of morphisms is the restriction of the usual tensor product of B-module morphisms. Let us note that any $H \in URep(\mathfrak{G})$ is automatically a B_t-bimodule via $z \cdot v \cdot t := zS(t) \cdot v$, $\forall z, t \in B_t, v \in E$, and that the above tensor product is in fact $\otimes_{B_t}$, moreover the B_t-bimodule structure for $H_1 \otimes_{B_t} H_2$ is given by $z \cdot \xi \cdot t = (z \otimes S(t)) \cdot \xi$, $\forall z, t \in B_t, \xi \in H_1 \otimes_{B_t} H_2$.

One deduces that the above tensor product is associative

$$(H_1 \otimes_{B_t} H_2) \otimes_{B_t} H_3 = H_1 \otimes_{B_t} (H_2 \otimes_{B_t} H_3),$$

so the associativity isomorphisms are trivial. The unit object of $URep(\mathfrak{G})$ is B_t with the action of B given by $b \cdot z := \varepsilon_t(bz)$, $\forall b \in B, z \in B_t$ and the scalar product $< z, t > = h(t^* z)$. The left and right unit morphisms are

$$l_E(z \otimes_{B_t} v) = z \cdot v \quad \text{and} \quad r_E(v \otimes_{B_t} z) = S(z) \cdot v, \quad \forall z \in B_t, \quad v \in E. \tag{1}$$

For any morphism $f : H_1 \to H_2$, define $f^* : H_2 \to H_1$ as the adjoint linear map: $< f(v), w > = < v, f^*(w) >$, $\forall v \in H_1, w \in H_2$, it is easy to check that f^* is B-linear. It is clear that $f^{**} = f$, that $(f \otimes_{B_t} g)^* = f^* \otimes_{B_t} g^*$, and that $End(H)$ is a C^*-algebra, for any object H. So $URep(\mathfrak{G})$ is a strict finite C^*-multitensor category (i.e., has all the properties of a C^*-tensor category except for one: $\mathbf{1}$ is not necessarily simple).

In order to make $URep(\mathfrak{G})$ a rigid C^*-tensor category in the sense of [18], Definition 2.1.1, we have to define the conjugate for any $H \in URep(\mathfrak{G})$. Take the dual vector space $\hat{H}$ which is naturally identified ($v \mapsto \overline{v}$) with the conjugate Hilbert space $\overline{H} :< \overline{v}, \overline{w} > = < w, v >$, $\forall v, w \in H$. The action of B on $\overline{H}$ is defined by $b \cdot \overline{v} = \overline{G^{1/2} S(b)^* G^{-1/2} \cdot v}$, where G is the canonical group-like element of $\mathfrak{G}$. Then the rigidity morphisms defined by

$$R_H(1_B) = \Sigma_i (G^{1/2} \cdot \overline{e}_i \otimes_{B_t} \cdot e_i), \quad \overline{R}_H(1_B) = \Sigma_i (e_i \otimes_{B_t} G^{-1/2} \cdot \overline{e}_i), \tag{2}$$

where $\{e_i\}_i$ is any orthogonal basis in H, satisfy all the needed properties – see [5], 3.6. Also, it is known that the B-module B_t is irreducible if and only if $B_s \cap Z(B) = \mathbb{C}1_B$, i.e., if $\mathfrak{G}$ is connected. So that, we have

Proposition 2.1. *$URep(\mathfrak{G})$ is a strict rigid finite C^*-multitensor category. It is C^*-tensor if and only if $\mathfrak{G}$ is connected.*

3. Unitary comodules.

Definition 2.2. A right unitary $\mathfrak{G}$-comodule is a pair $(H, \mathfrak{a})$, where H is a Hilbert space with scalar product $< \cdot, \cdot >$, $\mathfrak{a} : H \to H \otimes B$ is a bounded linear map between Hilbert spaces H and $H \otimes H_h = H \otimes \Lambda_h(B)$, and such that

(i) $(\mathfrak{a} \otimes id_B)\mathfrak{a} = (id_H \otimes \Delta)\mathfrak{a}$;

(ii) $(id_H \otimes \varepsilon)\mathfrak{a} = id_H$;

(iii) $< v^1, w > v^2 =< v, w^1 > S(w^2)^*, \quad \forall v, w \in H$.

A morphism of unitary $\mathfrak{G}$-comodules H_1 and H_2 is a linear map $T : H_1 \to H_2$ such that $\mathfrak{a}_{H_2} \circ T = (T \otimes id_B)\mathfrak{a}_{H_1}$ (i.e., a B-colinear map).

Right unitary $\mathfrak{G}$-comodules with **finite dimensional** underlying Hilbert spaces and their morphisms form a category which we denote by $UComod(\mathfrak{G})$.

We say that two unitary $\mathfrak{G}$-comodules are equivalent (resp., unitarily equivalent) if the space of morphisms between them contains an invertible (resp., unitary) operator.

In what follows, we will use the leg notation $\mathfrak{a}(v) = v^1 \otimes v^2$, for all $v \in H$.

Example 2.3. Let us equip a right coideal $I \subset B$ with the scalar product $< v, w >:= h(w^* v)$. Then the strong invariance of h gives

$$\begin{aligned} < v^1, w > v^2 &= (h \otimes id_B)((w^* \otimes 1_B)\Delta(v)) \\ &= (h \otimes S^{-1})(\Delta(w^*)(v \otimes 1_B)) =< v, w^1 > S(w^2)^*. \end{aligned}$$

Remark 2.4. *By (ii) any coaction $\mathfrak{a}$ is injective.*

If $(H, \mathfrak{a})$ is a right unitary $\mathfrak{G}$-comodule, then H is naturally a unitary left $\hat{\mathfrak{G}}$-module via

$$\hat{b} \cdot v := v^1 < \hat{b}, v^2 >, \quad \forall \hat{b} \in \hat{B}, \quad v \in H. \tag{3}$$

The unitarity follows from the calculation

$$\begin{aligned} &< \hat{b} \cdot v, w > =< v^1 < \hat{b}, v^2 >, w >=< \hat{b}, < v^1, w > v^2 >= \\ &\qquad =< \hat{b}, < v, w^1 > S(w^2)^* >=< v, w^1 \overline{< \hat{b}, S(w^2)^* >} >=< v, (\hat{b})^* \cdot w >, \end{aligned}$$

for all $v, w \in H$ and $\hat{b} \in \hat{B}$. In particular H is a $\hat{B}_t$-bimodule.

Due to the canonical identifications $B_t \cong \hat{B}_s$ and $B_s \cong \hat{B}_t$ given by the maps $z \mapsto \hat{z} = \varepsilon(\cdot z)$ and $t \mapsto \hat{t} = \varepsilon(t \cdot)$, H is also a B_s-bimodule via $z \cdot v \cdot t = v^1 \varepsilon(z v^2 t)$, for all $z, t \in B_s$, $v \in V$. The maps $\alpha, \beta : B_s \to B(H)$ defined by $\alpha(z)v := z \cdot v$ and $\beta(z)v := v \cdot z$, for all $z \in B_s, v \in H$ are a $*$-algebra homomorphism and antihomomorphism, respectively, with commuting images. Indeed, for instance, for all $v, w \in H, z \in B_s$, one has

$$\begin{aligned} < \alpha(z)v, w >:= &< v^1 \varepsilon(z v^2), w >= \varepsilon(< v^1, w > z v^2) \\ &= \varepsilon(< v, w^1 > z S(w^2)^*) =< v, w^1 > \overline{\varepsilon(S(w^2) z^*)} \\ &= < v, w^1 \varepsilon(S(z^*) w^2) >=< v, \alpha(z^*) w^1 \varepsilon(w^2) >=< v, \alpha(z^*) w > . \end{aligned}$$

So that, $\alpha(z)^* = \alpha(z^*)$, and similarly for the map β. We have the following useful relations:

$$\mathfrak{a}(\alpha(x)\beta(y)v) = v^1 \otimes x v^2 y, \quad \forall v \in H, \quad x, y \in B_s. \tag{4}$$

and

$$\alpha(x)\beta(y)v^1 \otimes v^2 = v^1 \otimes S(x)v^2S(y), \quad \forall v \in H, \quad x, y \in B_s. \tag{5}$$

The correspondence (3) is bijective as one has the inverse formula: if $(b_i)_i$ is a basis for B and $(\hat{b}_i)$ is its dual basis in $\hat{B}$, then set

$$\mathfrak{a}(v) = \sum_i (\hat{b}_i \cdot v) \otimes b_i, \quad \forall v \in H. \tag{6}$$

Moreover, formulas (3) and (6) imply also a bijection of morphisms. Thus, we have two functors, $\mathcal{F}_1 : UComod(\mathfrak{G}) \to URep(\hat{\mathfrak{G}})$ and $\mathcal{G}_1 : URep(\hat{\mathfrak{G}}) \to UComod(\mathfrak{G})$, which are mutually inverse. So, these categories are isomorphic as linear categories, and we can transport various additional structures from $URep(\hat{\mathfrak{G}})$ to $UComod(\mathfrak{G})$.

For instance, let us define tensor product of two unitary $\mathfrak{G}$-comodules, $(H_1, \mathfrak{a}_{H_1})$ and $(H_2, \mathfrak{a}_{H_2})$. As a vector space, it is

$$H_1 \otimes_{\hat{B}_t} H_2 := \hat{\Delta}(\hat{1})(H_1 \otimes H_2) = \hat{1}_1 \cdot H_1 \otimes \hat{1}_2 \cdot H_2,$$

and is generated by the elements $x \otimes_{\hat{B}_t} y := \hat{\Delta}(\hat{1}) \cdot (x \otimes y)$, where $x \in H_1, y \in H_2$, so it can be identified with $H_1 \otimes_{B_s} H_2$ (see [26], 2.2 or [24], Chapter 4).

Lemma 2.5. *If $(H_1, \mathfrak{a})$, $(H_2, \mathfrak{b}) \in UComod(\mathfrak{G})$, then the projection $P : H_1 \otimes H_2 \to H_1 \otimes_{\hat{B}_t} H_2$ defined by $P(v) = \hat{\Delta}(\hat{1}) \cdot v$, for all $v \in H_1 \otimes H_2$, satisfies*

$$P(x \otimes y) = x^1 \otimes y^1 \varepsilon(x^2 y^2), \quad \text{for all} \quad x \in H_1, \quad y \in H_2.$$

The proof is the direct calculation using the axiom (2) of a weak Hopf algebra

$$\hat{1}_1 \cdot x \otimes \hat{1}_2 \cdot y = (x^1 \otimes y^1)\varepsilon(x^2 1_1)\varepsilon(1_2 y^2) = (x^1 \otimes y^1)\varepsilon(x^2 y^2).$$

Corollary 2.6. *The linear map $\mathfrak{a} \otimes_{B_s} \mathfrak{b}$ given by*

$$v \otimes_{B_s} w \mapsto v^1 \otimes_{B_s} w^1 \otimes v^2 w^2, \quad \forall v \in H_1, \quad w \in H_2$$

is a coaction of $\mathfrak{G}$ on $H_1 \otimes_{B_s} H_2$ (i.e., satisfies Definition 2.2, (i), (ii)).

Proof. $\forall v \in H_1, w \in H_2$, one has

$$\begin{aligned}
&((\mathfrak{a} \otimes_{B_s} \mathfrak{b}) \otimes i_B)(\mathfrak{a} \otimes_{B_s} \mathfrak{b})(v \otimes_{B_s} w) \\
&= ((\mathfrak{a} \otimes_{B_s} \mathfrak{b}) \otimes i_B)(v^1 \otimes_{B_s} w^1 \otimes v^2 w^2) \\
&= (\hat{\Delta}(\hat{1}) \otimes 1_{B\otimes B}) \cdot (\hat{\Delta}(\hat{1}).(\mathfrak{a}(v^1)^1 \otimes \mathfrak{b}(w^1)^1) \otimes \mathfrak{a}(v^1)^2 \mathfrak{b}(w^2)^2 \otimes v^2 w^2) \\
&= (\hat{\Delta}(\hat{1}) \otimes 1_{B\otimes B}) \cdot (\mathfrak{a}(v^1)^1 \otimes \mathfrak{b}(w^1)^1 \otimes \mathfrak{a}(v^1)^2 \mathfrak{b}(w^2)^2 \otimes v^2 w^2) \\
&= (\hat{\Delta}(\hat{1}) \otimes 1_{B\otimes B}) \cdot ((\mathfrak{a} \otimes i_B)\mathfrak{a}(v))_{134} (\mathfrak{b} \otimes i_B)\mathfrak{b}(w))_{234}) \\
&= (\hat{\Delta}(\hat{1}) \otimes 1_{B\otimes B}) \cdot ((i_E \otimes \Delta)\mathfrak{a}(v))_{13} (i_F \otimes \Delta)\mathfrak{b}(v))_{23}) \\
&= (\hat{\Delta}(\hat{1}) \otimes 1_{B\otimes B})(i_{E\otimes F} \otimes \Delta)((\hat{\Delta}(\hat{1}) \otimes 1_B).(\mathfrak{a}(v))_{13} \mathfrak{b}(v))_{23}) \\
&= (id_{H_1 \otimes_{B_s} H_2} \otimes \Delta)(\mathfrak{a} \otimes_{B_s} \mathfrak{b})(v \otimes_{B_s} w).
\end{aligned}$$

Moreover, using Lemma 2.5, we have

$$(id_{H_1 \otimes_{B_s} H_2} \otimes \varepsilon)(\mathfrak{a} \otimes_{B_s} \mathfrak{b})(v \otimes_{B_s} w) = v^1 \otimes w^1 \varepsilon(v^2 w^2) = P(v \otimes w) = v \otimes_{B_s} w$$

□

The direct calculation shows that the tensor product coaction is unitary. Thus, $UComod(\mathfrak{G})$ is a multitensor category whose associativity morphisms are trivial, the unit object is $(B_s, \Delta|_{B_s})$. It is simple if and only if $\mathfrak{G}$ is coconnected. The left and right unit isomorphisms are

$$l_H : B_s \otimes_{B_s} H \to H, \quad z \otimes_{B_s} v \mapsto z \cdot v, \quad r_H : H \otimes_{B_s} B_s \to H, \quad v \otimes_{B_s} z \mapsto v \cdot z. \tag{7}$$

One can check that these isomorphisms are unitary and their inverses are

$$l_H^{-1}(v) = 1_1 \otimes_{B_s} v^1 \varepsilon(1_2 v^2) \quad \text{and} \quad r_H^{-1}(v) = v^1 \otimes_{B_s} \varepsilon_s(v^2). \tag{8}$$

Let us define the conjugate object for $(H, \mathfrak{a}) \in UComod(\mathfrak{G})$. The corresponding Hilbert space is $\overline{H}$. In what follows, we use the Sweedler arrows $\hat{b} \rightharpoonup b := b_1 < \hat{b}, b_2 >$, $b \leftharpoonup \hat{b} := b_2 < \hat{b}, b_1 >$, $\forall b \in B$, $\hat{b} \in \hat{B}$.

Lemma 2.7. *The conjugate object for* $(H, \mathfrak{a})$ *in* $UComod(\mathfrak{G})$ *is* $(\overline{H}, \tilde{\mathfrak{a}})$*, where*

$$\tilde{\mathfrak{a}}(\overline{v}) = \overline{v^1} \otimes [\hat{G}^{-1/2} \rightharpoonup (v^2)^* \leftharpoonup \hat{G}^{1/2}],$$

and $\hat{G}$ *is the canonical group-like element of the dual quantum groupoid* $\hat{\mathfrak{G}}$.

Proof. The unitarity of $\mathcal{G}_1(\overline{H}, \tilde{\mathfrak{a}})$ means that $< \hat{b}\cdot\overline{v}, \overline{w} >_{\overline{H}} = < \overline{v}, \hat{b}^*\cdot\overline{w} >_{\overline{H}}$, for all $v, w \in H$. The left hand side equals to $< \overline{v}^1, \overline{w} >_{\overline{H}} < \hat{b}, \overline{v}^2 >$. And the right hand side equals to

$$\begin{aligned}
&< \overline{v}, \overline{\hat{G}^{1/2}\hat{S}(\hat{b}^*)^*\hat{G}^{-1/2} \cdot w} >_{\overline{H}} = < \hat{G}^{1/2}\hat{S}(\hat{b}^*)^*\hat{G}^{-1/2} \cdot w, v >_H \\
&= < w, \hat{G}^{-1/2}\hat{S}(\hat{b}^*)\hat{G}^{1/2} \cdot v >_H = < w, v^1 >_H \overline{< \hat{G}^{-1/2}\hat{S}(\hat{b}^*)\hat{G}^{1/2}, v^2 >} \\
&= < \overline{v}^1, \overline{w} >_{\overline{H}} < \hat{G}^{-1/2}\hat{b}\hat{G}^{1/2}, (v^2)^* > \\
&= < \overline{v}^1, \overline{w} >_{\overline{H}} < \hat{b}, [\hat{G}^{-1/2} \rightharpoonup (v^2)^* \leftharpoonup \hat{G}^{1/2}] > .
\end{aligned}$$

Comparing the above expressions, we have the result. □

The rigidity morphisms are given by (2) with B_t replaced by B_s. For any morphism f, f^* is the conjugate linear map on the corresponding Hilbert spaces, the colinearity of f implies that f^* is colinear. So that, we have

Proposition 2.8. $UComod(\mathfrak{G})$ *is a strict rigid finite* C^**-multitensor category. It is* C^**-tensor if and only if* $\mathfrak{G}$ *is coconnected.*

4. Unitary corepresentations.

Definition 2.9. A **right unitary corepresentation** of $\mathfrak{G}$ on a Hilbert space H is a partial isometry $V \in B(H) \otimes B$ such that

(i) $V_{12}V_{13} = (id_{B(H)} \otimes \Delta)(V)$.

(ii) $(id_{B(H)} \otimes \varepsilon)(V) = id_{B(H)}$.

If U and V are two right corepresentations on Hilbert spaces H_U and H_V, respectively, a morphism between them is a bounded linear map $T \in B(H_U, H_V)$ such that $(T \otimes 1_B)U = V(T \otimes 1_B)$. The vector space of such morphisms is denoted by $Mor(U, V)$. We will denote by $UCorep(\mathfrak{G})$ the category whose objects are right unitary corepresentations (H, V) on **finite dimensional** vector spaces with morphisms as above.

One says that U and V are equivalent (resp., unitarily equivalent) if $Mor(U, V)$ contains an invertible (resp., unitary) operator.

Proposition 2.10. *If* $(H, \mathfrak{a})$ *is a unitary* $\mathfrak{G}$*-comodule, let us define an operator* V *on* $H \otimes H_h$ *as follows:*

$$V(x \otimes \Lambda_h y) := x^1 \otimes \Lambda_h(x^2 y), \quad \textit{for all} \quad x \in H, \quad y \in B.$$

Then V *is a unitary corepresentation of* $\mathfrak{G}$ *on* H*, and one has*

$$V^*(x \otimes \Lambda_h y) := x^1 \otimes \Lambda_h(S(x^2)y), \quad \textit{for all} \quad x \in H, \quad y \in B.$$

Proof. Let I_h be an implementation of Δ (for example, $I_h \in B(H_h \otimes H_h) : \Lambda_{h\otimes h}(y' \otimes y) \mapsto \Lambda_{h\otimes h}(\Delta(y)(y' \otimes 1_B))$, see [32], 3.2) for details), then one has for all $x \in H, y, c \in B$:

$$\begin{aligned} V_{12}V_{13}(x \otimes \Lambda_h y \otimes \Lambda_h c) &= (V \otimes 1_B)(x^1 \otimes \Lambda_h y \otimes \Lambda_h(x^2 c)) \\ &= x^1 \otimes \Lambda_h(x^2 y) \otimes \Lambda_h(x^3 c) \\ &= x^1 \otimes \Lambda_{h\otimes h}(\Delta(x^2)(y \otimes c)) \\ &= x^1 \otimes I_h(x^2 \otimes 1_B) I_h^*(\Lambda_{h\otimes h}(y \otimes c)) \\ &= (1_{B(H)} \otimes I_h)(x^1 \otimes \{(x^2 \otimes 1_B) I_h^* \Lambda_{h\otimes h}(y \otimes c)\}) \\ &= (1_{B(H)} \otimes I_h)(V \otimes 1_B)(x \otimes I_h^*(\Lambda_{h\otimes h} y \otimes c)) \\ &= (1_{B(H)} \otimes I_h)(V \otimes 1_B)(1_{B(H)} \otimes I_h)^*(x \otimes \Lambda_{h\otimes h}(y \otimes c)) \\ &= (1_{B(H)} \otimes \Delta)(V)(x \otimes \Lambda_h y \otimes \Lambda_h c). \end{aligned}$$

Next, we have, for any decomposition $V = \sum_{i\in I} v_i \otimes b_i$ $(v_i \in B(H), b_i \in B)$

$$\begin{aligned} (id_{B(H)} \otimes \varepsilon)(V)(\xi) &= (id_{B(H)} \otimes \varepsilon)(\sum_{i\in I} v_i(\xi) \otimes b_i) \\ &= \sum_{i\in I} \varepsilon(b_i) v_i(\xi) = (id_{B(H)} \otimes \varepsilon)\mathfrak{a}(\xi) = \xi, \quad \forall \xi \in H. \end{aligned}$$

In order to show that V is a partial isometry, consider the separability element $e_s = (id_B \otimes S)\Delta(1_B)$ of the algebra B_s and the idempotents $e_{\beta,id} = (\beta \otimes id_B)(e_s) \in \beta(B_s) \otimes B_s$ and $e_{\alpha,S} = (\alpha \otimes S)(e_s) \in \alpha(B_s) \otimes B_s$. As α and β are $*$-maps, these idempotents are orthogonal projections on $H \otimes H_h$. It is straightforward to check, using (4) and (5), that:

- for all $x, y \in B_s$, one has

$$V(\alpha(x)\beta(y) \otimes 1_B) = (1_{B(H)} \otimes x)V(1_{B(H)} \otimes y), \tag{9}$$

$$(\alpha(x)\beta(y) \otimes 1_B)V = (1_{B(H)} \otimes S(x))V(1_{B(H)} \otimes S(y)). \tag{10}$$

- $Ve_{\beta,id} = V$, $e_{\alpha,S}V = V$.

Moreover, V is invertible in $B(e_{\beta,id}(H \otimes H_h), e_{\alpha,S}(H \otimes H_h))$. Indeed, consider an operator W acting on $H \otimes H_h$ defined by

$$W(v \otimes \Lambda_h(b) := v^1 \otimes \Lambda_h(S(v^2)b), \quad \forall v \in H, \quad b \in B.$$

Then we have

$$\begin{aligned} WV(v \otimes \Lambda_h(b) :=& W(v^1 \otimes \Lambda_h(v^2 b)) \\ =& v^1 \otimes \Lambda_h(S(v^2)v^3 b) = v^1 \otimes \Lambda_h(\varepsilon_s(v^2)b). \end{aligned}$$

On the other hand,

$$\begin{aligned} e_{\beta,id}(v \otimes \Lambda_h(b)) &= (v \cdot 1_1) \otimes \Lambda_h(S(1_2)b) = v^1 \otimes \Lambda_h(S(1_2) \\ &\times \varepsilon(v^2 1_1)b) = v^1 \otimes \Lambda_h(1_1)\varepsilon(v^2 S(1_2))b) = v^1 \otimes \Lambda_h(\varepsilon_s(v^2)b). \end{aligned}$$

And similarly $VW = e_{\alpha,S}$, so that W is the inverse of V. Finally, we compute, for all $v, w \in H, b, c \in B$:

$$\begin{aligned} < V(v \otimes \Lambda_h(b)), V(w \otimes \Lambda_h(c)) > &=< v^1 \otimes \Lambda_h(v^2 b), w^1 \otimes \Lambda_h(w^2 c) > \\ &=< v^1, w^1 > h(c^*(w^2)^* v^2 b) \\ &=< v, w^1 > h(c^*(w^3)^*[S(w^2)]^* b) \\ &=< v, w^1 > h(c^*[\varepsilon_s(w^2)]^* b). \end{aligned}$$

On the other hand,

$$< v \otimes \Lambda_h(b), e_{\beta,id}(w \otimes \Lambda_h(c)) >=< v \otimes \Lambda_h(b), (w \cdot 1_1) \otimes \Lambda_h(S(1_2)c) >$$
$$=< v, w \cdot 1_1 > h(c^*[S(1_2)]^* b) =< v, w^1 > \overline{\varepsilon(w^2 1_1)} h(c^2[S(1_2)]^* b).$$

These expressions are equal because $\varepsilon_s(x) := 1_1\varepsilon(x1_2) = S(1_2)\varepsilon(xS(1_1)) = S(1_2)\varepsilon(x1_1))$, for all $x \in B$. We used above the equality $\varepsilon(xS(z)) = \varepsilon(xz)$, for all $z \in B_t$ which can be obtained by applying $\varepsilon \otimes \varepsilon$ to both sides of the equality $\Delta(1_B)(S(z) \otimes 1_B) = \Delta(1_B)(1_B \otimes z)$. As $e_{\beta,id}$ is an orthogonal projection, this means that V is bounded and $V^*V = e_{\beta,id}$.

Similar reasoning shows that V^* equals to the above mentioned W. □

We also have a converse statement.

Proposition 2.11. *Any unitary corepresentation V of $\mathfrak{G}$ on a Hilbert space H generates a unitary comodule $(H, \mathfrak{a})$, where $\mathfrak{a}(v) = V(v \otimes \Lambda_h(1_B))\ \forall v \in H$.*

Proof. The first two conditions of Definition 2.2 follow from the first two conditions of Definition 2.9. The relation between V and the coaction $\mathfrak{a} : v \mapsto v^1 \otimes v^2$ is given by $V(v \otimes \Lambda_h(b)) = v^1 \otimes \Lambda_h(v^2 b)$. We have seen already that the operator W acting on $H \otimes H_h$ and defined by

$$W(v \otimes \Lambda_h(b)) = v^1 \otimes \Lambda_h(S(v^2)b), \quad \forall v \in H, \quad b \in B,$$

satisfies the relations $VW = e_{a,S}$ and $WV = e_{b,id}$. As V is a partial isometry with initial and final Hilbert subspaces $e_{a,S}(H \otimes H_h)$ and $e_{b,id}(H \otimes H_h)$, respectively, we have $W = V^*$. Then for all $v, w \in H$ and $c \in B$, the equality

$$< V(v \otimes \Lambda_h(1_B)), w \otimes \Lambda_h(c) >=< v \otimes \Lambda_h(1_B), V^*(w \otimes \Lambda_h(c)) >,$$

can be rewritten as

$$< v^1, w > h(c^* v^2) =< v, w^1 > h(c^*[S(w^2)]^*),$$

which implies the unitarity of the $\mathfrak{G}$-comodule in question. □

Let $(H_1, \mathfrak{a})$ and $(H_2, \mathfrak{b})$ be two unitary $\mathfrak{G}$-comodules, and let T be in $\mathcal{B}(H_1, H_2)$ intertwining $\mathfrak{a}$ and $\mathfrak{b}$, then one has, for all $x \in H_1, b \in B$

$$\begin{aligned} V_{H_2}(T \otimes 1)(x \otimes \Lambda_h(b)) &= (Tx)^1 \otimes \Lambda_h((Tx)^2 b) \\ &= (1_{H_2} \otimes \pi'(b))((Tx)^1 \otimes \Lambda_h((Tx)^2) \\ &= (1_{H_2} \otimes \pi'(b))(id_F \otimes \Lambda_h)(\mathfrak{b}(Tx)) \\ &= (1_{H_2} \otimes \pi'(b))(id_F \otimes \Lambda_h)((T \otimes 1)\mathfrak{a}(x)) \\ &= (1_{H_2} \otimes \pi'(b))(T \otimes 1)(id_{H_1} \otimes \Lambda_h)(\mathfrak{a}(x)) \\ &= (T \otimes 1)(1_{H_1} \otimes \pi'(b))(id_{H_1} \otimes \Lambda_h)(\mathfrak{a}(x)) \\ &= (T \otimes 1)V_{H_1}(x \otimes \Lambda_h(b)). \end{aligned}$$

Hence, $T \in Mor(V_{H_1}, V_{H_2})$.

Corollary 2.12. *The correspondence $\mathcal{F}_2$ defined by $\mathcal{F}_2(H, \mathfrak{a}) = (V, H)$ and $\mathcal{F}_2(T) = T$ for all objects $(H, \mathfrak{a})$ and morphisms T of $UComod(\mathfrak{G})$, is a functor from $UComod(\mathfrak{G})$ to $UCorep(\mathfrak{G})$ viewed as semisimple linear categories. The correspondence $\mathcal{G}_2$ between unitary corepresentations of $\mathfrak{G}$ and $\mathfrak{G}$-comodules given by Proposition 2.11 clearly extends to morphisms and defines a functor inverse to $\mathcal{F}_2$, so $UComod(\mathfrak{G})$ and $UCorep(\mathfrak{G})$ are isomorphic as linear categories. Then we can equip $UCorep(\mathfrak{G})$ with tensor product and duality by transporting these structures from $Comod(\mathfrak{G})$.*

If $(U, H_U),\ (V, H_V) \in UCorep(\mathfrak{G})$, let us define their tensor product.

Lemma 2.13. *One has* $(P \otimes id_B)U_{13}V_{23} = U_{13}V_{23}(P \otimes id_B) = U_{13}V_{23}$, *where* $U_{13}V_{23} \in B(H_U \otimes H_V) \otimes B$ *and* P *was defined in Lemma 2.5.*

Proof. There exist finite families $\{b_k\}$ and $\{b'_k\}$ in B_s such that $\Sigma_k b'_k b_k = \Sigma_k b_k b'_k = 1_B$, and for all $x \in H_U$ and all $y \in H_V$ one has

$$P(x \otimes y) = \hat{\Delta}(\hat{1}) \cdot (x \otimes y) = \Sigma_k \beta(b_k)x \otimes \alpha'(b'_k)y,$$

where α' is the $*$-representation of B_s corresponding to (V, H_2). Using four times (10), one has

$$\begin{aligned}
(P \otimes id_B)U_{13}V_{23} &= \Sigma_k(\beta(b_k) \otimes \alpha'(b'_k) \otimes id_B)U_{13}V_{23} \\
&= \Sigma_k(\beta(b_k) \otimes id_{H_V} \otimes id_B)U_{13}(id_{H_U} \otimes \alpha'(b'_k) \otimes id_B)V_{23} \\
&= \Sigma_k(\beta(b_k) \otimes id_{H_V} \otimes id_B)U_{13}(id_{H_U} \otimes id_{H_V} \otimes S(b'_k)V_{23} \\
&= \Sigma_k(\beta(b_k)\beta(b'_k) \otimes id_{H_V} \otimes id_B)U_{13}V_{23} \\
&= \Sigma_k(\beta(b'_k b_k) \otimes id_{H_V} \otimes id_B)U_{13}V_{23} = U_{13}V_{23} \\
&= \Sigma_k U_{13}(id_{H_U} \otimes id_{H_V} \otimes b_k b'_k)V_{23} \\
&= \Sigma_k U_{13}(\beta(b_k) \otimes id_{H_V} \otimes 1_B)V_{23}(id_{H_U} \otimes \alpha'(b'_k) \otimes id_B) \\
&= \Sigma_k U_{13}V_{23}(\beta(b_k) \otimes \alpha'(b'_k) \otimes id_B) = U_{13}V_{23}(P \otimes id_B).
\end{aligned}$$

□

Lemma 2.13 justifies the following:

Definition 2.14. If $(U, H_U),\ (V, H_V) \in UCorep(\mathfrak{G})$, their tensor product is the bounded linear map:

$$U \odot V = U_{13}V_{23} = (P \otimes id_B)U_{13}V_{23}(P \otimes id_B)$$

viewed as an element of $B(H_U \otimes_{B_s} H_V) \otimes B$.

Proposition 2.15. $U \odot V \in UCorep(\mathfrak{G})$, *it acts on* $H_U \otimes_{B_s} H_V$ *and:*

$$\mathcal{G}_2(U, H_1) \otimes_{B_s} \mathcal{G}_2(V, H_2) = \mathcal{G}_2(U \odot V, H_1 \otimes_{B_s} H_2).$$

Proof. If $(U, H_U),\ (V, H_V) \in UCorep(\mathfrak{G})$, let $U = \sum_i u_i \otimes b_i,\ V = \sum_j u_j \otimes b_j$ be decompositions of U and V. Then $U \odot V = \sum_{i,j} u_i \otimes v_j \otimes b_i b_j$, and let us define $\theta_{U \odot V} \in B(H_U \otimes_{B_s} H_V, H_U \otimes_{B_s} H_V \otimes B)$ by

$$\theta_{U \odot V}(x \otimes_{B_s} y) = \sum_{i,j} u_i \otimes v_j(P(x \otimes y)) \otimes b_i b_j.$$

Then, using Lemma 2.13, one has

$$\begin{aligned}
\theta_{U \odot V}(x \otimes_{B_s} y) &= \sum_{i,j} u_i \otimes v_j(x \otimes y) \otimes b_i b_j \\
&= \sum_{i,j} P(u_i(x) \otimes v_j(y)) \otimes b_i b_j = (a_U \otimes \mathfrak{a}_V)(x \otimes_{B_s} y),
\end{aligned}$$

and the result follows. □

The unit object $\mathbf{1}$ of $UCorep(\mathfrak{G})$ with respect to $\odot$ acts on B_s and is defined by $z \otimes b \mapsto 1_1 \otimes 1_2 zb$, for all $z \in B_s,\ b \in B$. It is simple if and only if $\mathfrak{G}$ is coconnected. The conjugate object for $(V, H) \in UCorep(\mathfrak{G})$ is the unitary corepresentation acting on $\overline{H}$ via $\overline{V}(\overline{x} \otimes \Lambda_h(y)) = \overline{x}^1 \otimes \Lambda_h((\overline{x}^2)^* y)$, where $\tilde{\mathfrak{a}}(\overline{x})$ is described in Lemma 2.7, and the rigidity morphisms are the same as in $UCorep(\mathfrak{G})$. For any morphism f, again f^* is the conjugate bounded linear map on the corresponding Hilbert spaces. So that, we have

Proposition 2.16. *$UCorep(\mathfrak{G})$ is a strict rigid finite C^*-multitensor category. It is C^*-tensor if and only if $\mathfrak{G}$ is coconnected.*

The simple objects of this category are exactly irreducible corepresentations of $\mathfrak{G}$. Let us denote by Ω the set of equivalence classes of irreducibles and choose a representative U^x in any class $x \in \Omega$. The regular corepresentation of $\mathfrak{G}$ is decomposed as follows:

$$W = \oplus_{x\in\Omega} dim(x) U^x, \tag{11}$$

where $dim(x)$ is the dimension of the Hilbert space on which U^x acts.

Definition 2.17. Let $(U, H_U) \in UCorep(\mathfrak{G})$ and $\{m_{i,j}\}_{i,j=1}^n$ be the matrix units of $B(H_U)$ with respect to some orthonormal basis $\{e_i\}_{i=1}^n$ in H_U. Then

$$U = \Sigma_{i,j=1}^n m_{i,j} \otimes U_{i,j},$$

where $U_{i,j}$ $(i,j = 1,\dots,n)$ are called the matrix coefficients of U with respect to $\{e_i\}$. Put $B_U := Span(U_{i,j})_{i,j=1}^n$; in particular, we denote B_{U^x} by B_x.

Remark 2.18. *Let us summarize some properties of matrix coefficients of U^x $(x \in \Omega)$ which can be proved in a standard way.*

(i) $B_{\oplus_{k=1}^p U_k} = span\{B_{U_1},\dots,B_{U_p}\}$ for any finite direct sum of unitary corepresentations. In particular, (11) implies that $B = \oplus_{x\in\Omega} B_x$.

(ii) Decomposition $U \odot V = \oplus_z d_z U^z$ with multiplicities d_z implies that $B_U B_V \subset \oplus_z B_z$, where z parameterizes the irreducibles of the above decomposition.

(iii) The definition of a unitary corepresentation written in terms of $U^x_{i,j}$:

$$\Delta(U^x_{i,j}) = \Sigma_{k=1}^{dim(x)} U^x_{i,k} \otimes U^x_{k,j}, \quad \varepsilon(U^x_{i,j}) = \delta_{i,j}, \quad U^x_{i,j} = S(U^x_{j,i})^*,$$

for all $i,j = 1,\dots,dim(x)$, gives: $B_x \otimes B_x = \Delta(1_B)(B_x \otimes B_x)$, $\Delta(B_x) \subset \Delta(1_B)(B_x \otimes B_x)$ and $B_U = S(B_U)^*$. We also have $B_{\overline{U}} = (B_U)^*$.

Example 2.19. In the case of the trivial corepresentation of $\mathfrak{G}$ associated with $(\Delta_{|B_s}, B_s)$, we will use the notation B_ε instead of B_U. Let $\{b_i\}_{i=1}^{dimB_s}$ be an orthonormal basis in B_s with respect to the scalar product $< z,t >= \varepsilon(t^*z)$ $\forall z,t \in B_s$. Then one can write $\Delta(1_B) = \Sigma_{i=1}^{dimB_s} b_i^* \otimes S(b_i)$ (see [23], 2.3.3), which implies: $\Delta(b_j^*) = \Sigma_{i=1}^{dimB_s}(b_i^* \otimes S(b_i)b_j^*)$, so $U^\varepsilon_{i,j} = S(b_i)b_j^*$, for all $i,j = 1,\dots,dimB_s$. This means that B_ε is the unital C^*-algebra B_tB_s.

5. Fiber functor and reconstruction theorem. Let Q and R be two unital C^*-algebras. By definition, a (Q,R)-correspondence is a right Hilbert R-module $\mathcal{E}$ (see [12]) with a unital $*$-homomorphism $\varphi : Q \to \mathcal{L}(\mathcal{E})$, where $\mathcal{L}(\mathcal{E})$ is the C^*-algebra of all bounded R-linear adjointable operators on $\mathcal{E}$. If $Q = R$, we call it an R-correspondence. R-correspondences form a C^*-multitensor category $Corr(R)$ with interior tensor product $\otimes_R$ and adjointable R-bilinear maps as morphisms.

There exists another definition of a (Q,R)-correspondence, due to Alain Connes, this is a triple (H,α,β) where H is a Hilbert space equipped with unital $*$-homomorphism $\alpha : Q \to B(H)$ and $*$-anti-homomorphism $\beta : R \to B(H)$ whose images commute in $B(H)$. Then H is a (Q,R)-bimodule via $q \cdot v \cdot r := \alpha(q)\beta(r)v$, for all $q \in Q, r \in R, v \in H$.

In this paper, we are especially interested in the particular case, when $Q = R$ is a finite dimensional C^*-algebra equipped with a faithful tracial state ϕ. Below we treat this particular case in detail.

Lemma 2.20. *Both definitions of an R-correspondence are equivalent.*

Proof. (i) If $(H,\alpha,\beta)\in Corr(R)$, define, for any $\eta\in H$, an operator $\Pi(\eta):H_\phi\to H$ by $\Pi(\eta)\Lambda_\phi(r):=\beta(r)\eta$, for all $r\in R$, where H_ϕ is the GNS Hilbert space generated by (R,ϕ). Then define an R-valued scalar product:

$$<\xi,\eta>_R:=\Pi(\xi)^*\Pi(\eta),\quad \text{for all}\quad \xi,\eta\in H.$$

It is clear that $<\xi,\eta>_R$ is in fact in $\pi_\phi(R)$. Finally, $\Pi(\beta(b)\eta)=\Pi(\eta)\pi_\varepsilon(b)$, so $<\xi,\beta(b)\eta>_R=<\xi,\eta>_R\ \pi_\phi(b)$, for all $\xi,\eta\in H, b\in R$. Moreover, together with the unital $*$-representation α we have on H the structure of an R-correspondence in the sense of the first definition.

(ii) Vice versa, if H is an R-correspondence in this last sense, then one can define a usual scalar product $<\xi,\eta>=\phi(<\eta,\xi>_R)$, for all $\eta,\xi\in H$, and there are clearly a unital $*$-homomorphism $\alpha:R\to B(H)$ and a unital $*$-anti-homomorphism $\beta:R\to B(H)$ whose images commute in $B(H)$. Thus, (H,α,β) is an R-correspondence in the sense of A. Connes. □

A morphism between (H,α,β) and (K,α',β') is a map $T\in B(H,K)$ intertwining α and α' and also β and β', then $Corr(R)$ is a semisimple linear category. If $(H,\alpha,\beta),(K,\alpha',\beta')\in Corr(R)$, we define their tensor product

$$(H,\alpha,\beta)\otimes_R(K,\alpha',\beta')=((\beta\otimes\alpha')(e)(H\otimes K),\alpha\otimes 1_K,1_H\otimes\beta'),$$

where e is the symmetric separability idempotent for R, so $e_{\beta,\alpha'}=(\beta\otimes\alpha')(e)$ is an orthogonal projection. For the sake of simplicity we shall denote $H_1\otimes_R H_2:=e_{\beta,\alpha'}(H\otimes K)$, and $v\otimes_R w=e_{\beta,\alpha'}(v\otimes w)$, for all $v\in H, w\in K$. The unit object is R with the GNS scalar product defined by ϕ. The unit isomorphisms are as follows:

$$l_H(z\otimes_R v):=z\cdot v\quad\text{and}\quad r_H(v\otimes_R z):=v\cdot z,\quad \forall z\in R,\quad v\in H.$$

They are isometric, for example

$$||l_H(z\otimes_R v)||^2:=||z\cdot v||_H^2=\phi(1_R)||z\cdot v)||^2=||1_R\otimes(z\cdot v)||^2=||z\otimes_R v||^2.$$

The conjugate of a morphism $T:H_1\to H_2$ is just the adjoint operator $T^*:H_2\to H_1$, so $Corr(R)$ is a C^*-multitensor category. We denote by $Corr_f(R)$ its full subcategory with finite dimensional underlying Hilbert spaces. The unit object is simple if and only if R is a full matrix algebra.

For all objects of the three above categories: $URep(\mathfrak{G})$, $UComod(\mathfrak{G})$, and $UCorep(\mathfrak{G})$, the underlying Hilbert spaces are B_s-correspondences, so each of these categories has a forgetful C^*-tensor functor with values in $Corr_f(B_s)$.

In order to reformulate in suitable terms the reconstruction theorem of Tannaka-Krein type for finite quantum groupoids proved initially in [10], [27], recall the construction of the canonical **Hayashi functor** $\mathcal{H}$.

Let $\mathcal{C}$ be a rigid finite C^*-tensor category and $\Omega=Irr(\mathcal{C})$ be an exhaustive set of representatives of equivalence classes of its simple objects. Let R be the C^*-algebra $R=\mathbb{C}^\Omega=\bigoplus_{x\in\Omega}\mathbb{C}p_x$, where $p_x=p_x^*$ are mutually orthogonal idempotents: $p_xp_y=\delta_{x,y}p_x$, for all $x,y\in\Omega$. Then $\mathcal{H}$ is a functor from $\mathcal{C}$ to $Corr_f(R)$ defined by

$$\mathcal{H}(x)=H_x=\bigoplus_{y,z\in\Omega}\mathcal{C}(z,y\otimes x),\quad\text{for every}\quad x\in\Omega,$$

where $\mathcal{C}(x,y)$ is the vector space of morphisms $x\to y$. The R-bimodule structure on H_x is given by

$$p_y\cdot H_x\cdot p_z=\mathcal{C}(z,y\otimes x),\quad\text{for all}\quad x,y,z\in\Omega.$$

If $y\in\Omega$ and $f\in\mathcal{C}(x,y)$, then $\mathcal{H}(f):H_x\to H_y$ is defined by

$$\mathcal{H}(f)(g)=(id_z\otimes f)\circ g,\quad\text{for any}\quad z,t\in\Omega\quad\text{and}\quad g\in p_z\cdot H_x\cdot p_t.$$

The inverse natural isomorphisms $J^{-1}_{x,y} : H_x \otimes H_y \to H_x \underset{R}{\otimes} H_y$ are

$$J^{-1}_{x,y}(v \otimes w) = a_{z,x,y} \circ (v \otimes id_y) \circ w \in p_z \cdot H(x \otimes y) \cdot p_t,$$

for all $v \in p_z \cdot H_x \cdot p_t, w \in p_t \cdot H_y \cdot p_s, z, s, t \in \Omega$. Here $a_{z,x,y}$ are the associativity isomorphisms of $\mathcal{C}$.

We define the scalar product on H_x as follows. If $x, y, z \in \Omega$ and $f, g \in \mathcal{C}(z, y \otimes x)$, then $g^* \in \mathcal{C}(y \otimes x, z)$ and $g^* \circ f \in End(z) = \mathbb{C}$, so one can put $< f, g >_x = g^* \circ f$. The subspaces $\mathcal{C}(z, y \otimes x)$ are declared to be orthogonal, so $H_x \in Corr_f(R)$. Dually, $\overline{H}_x \in Corr_f(R)$ via $z_1 \cdot \overline{v} \cdot z_2 = \overline{z_2^* \cdot v \cdot z_1^*}$, for all $z_1, z_2 \in R, v \in H_x$. Now one can check that $\mathcal{H} : \mathcal{C} \to Corr_f(R)$ is a unitary tensor functor in the sense of [18] 2.1.3.

Theorem 2.21. *Let $\mathcal{C}$ be a rigid finite C^*-tensor category and $\Omega = Irr(\mathcal{C})$. Let R be the C^*-algebra $\mathbb{C}^\Omega$ and $\mathcal{H} : \mathcal{C} \to Corr_f(R)$ be the Hayashi functor. Then the vector space*

$$B = \bigoplus_{x \in \Omega} \overline{H}_x \otimes H_x \tag{12}$$

has a regular biconnected finite quantum groupoid structure $\mathfrak{G}$ such that $\mathcal{C} \cong UCorep(\mathfrak{G})$ as C^*-tensor categories.

Proof. A rigid finite C^*-tensor category $\mathcal{C}$ is semisimple and spherical, so [26], Theorems 1.1 and 1.2 claims that B has a structure of a selfdual regular biconnected semisimple weak Hopf algebra. The algebra of the dual quantum groupoid $\hat{\mathfrak{G}}$ is (see [27], [14])

$$\hat{B} = \bigoplus_{x \in \Omega} B(H_x), \tag{13}$$

the duality is given, for all $x, y \in \Omega, A \in B(H_y), v, w \in H_x$ by

$$< A, \overline{w} \otimes v >= \delta_{x,y} < Av, w >_x .$$

$\hat{B}$ is clearly a C^*-algebra with the obvious matrix product and involution, its coproduct is given (see [14] Theorem 1.3.4) by

$$\hat{\Delta}(\hat{b}) = \sum_{i \in I} (s(r_i) \otimes t(p_i)) J \hat{b} J^{-1}, \quad \text{for any} \quad \hat{b} \in \hat{B},$$

where $\sum_{i \in I} (r_i \otimes p_i)$ is the symmetric separability element of R hence $\sum_{i \in I} (s(r_i) \otimes t(p_i)) = \hat{\Delta}(\hat{1})$ is an orthogonal projection in $\hat{B} \otimes \hat{B}$; moreover $J = \bigoplus_{x,y \in \Omega} \mathcal{H}_{x,y}$ is a unitary as a direct sum of unitaries. Then one can easily deduce that $\hat{\Delta}(\hat{b}^*) = \hat{\Delta}(\hat{b})^*$, so both $\hat{\mathfrak{G}}$ and $\mathfrak{G}$ are finite quantum groupoids.

The explicit structure of $\mathfrak{G}$ is given in [26], Theorems 1.1 and 1.2. If $v, w \in H_x, g, h \in H_y$ and $\{e_j^x\}$ is an orthogonal basis in H_x $(\forall x, y \in \Omega)$, then

$$\Delta(\overline{w} \otimes v) = \bigoplus_j (\overline{w} \otimes e_j^x)_x \otimes (\overline{e_j^x} \otimes v)_x, \tag{14}$$

$$\varepsilon(\overline{w} \otimes v) =< v, w >_x, \tag{15}$$

$$(\overline{w} \otimes v)_x \cdot (\overline{g} \otimes h)_y = (\overline{J^{-1}_{x,y}(w \otimes g)} \otimes J^{-1}_{x,y}(v \otimes h))_{x \otimes y} \in \overline{H_{x \otimes y}} \otimes H_{x \otimes y}, \tag{16}$$

$$1_B = \bigoplus_{x \in \Omega} (\rho_x \otimes \rho_x^{-1})_{\mathbf{1}}, \tag{17}$$

where ρ_x is the unit constraint attached to x, so $\rho_x^{-1} \in p_x \cdot H_{\mathbf{1}} \cdot p_x$ and $\rho_x = \overline{\rho_x^{-1}}$. In order to define the antipode, consider the natural isomorphisms $\Phi_x : H_x \to \overline{H}_{x^*}$ and $\Psi_x : \overline{H}_x \to H_{x^*}$ given by

$$\Phi_x = \rho_y(id_y \otimes \overline{ev_x}) \circ a_{y,x,x^*} \circ (v \otimes id_{x^*}), \Psi_x = (\overline{v} \otimes id_{x^*}) \circ a^{-1}_{y,x,x^*} \circ (id_y \otimes coev_x) \circ \rho_y^{-1}.$$

Here ev_x and $coev_x$ $(x \in \Omega)$ are the rigidity morphisms. Then we define

$$S(\overline{w} \otimes v) = [\Phi_x(v) \otimes \Psi_x(\overline{w})]_{x^*}. \tag{18}$$

Any H_x is a right B-comodule via

$$\mathfrak{a}_x(v) = \sum_j e_j^x \otimes \overline{e_j^x} \otimes v, \quad \text{where} \quad v \in H_x,$$

one checks that it is unitary which gives the equivalence $\mathcal{C} \cong UCorep(\mathfrak{G})$. □

3. Coactions of finite quantum groupoids on unital C*-algebras

1. Canonical implementation of a coaction.

Definition 3.1. A right coaction of a finite quantum groupoid $\mathfrak{G}$ on a unital $*$-algebra A, is a $*$-homomorphism $\mathfrak{a} : A \to A \otimes B$ such that
1) $(\mathfrak{a} \otimes i)\mathfrak{a} = (id_A \otimes \Delta)\mathfrak{a}$.
2) $(id_A \otimes \varepsilon)\mathfrak{a} = id_A$.
3) $\mathfrak{a}(1_A) \in A \otimes B_t$.
One also says that $(A, \mathfrak{a})$ is a $\mathfrak{G}$-$*$-algebra.

Remark 3.2. *If A is a C^*-algebra, then $\mathfrak{a}$ is automatically continuous, even an isometry by 2.4 and* [25] *1.5.7.*

Proposition 3.3. *Any right coaction of $\mathfrak{G}$ on a unital $*$-algebra A is simplifiable: the set $\mathfrak{a}(A)(1_A \otimes B) = \{\mathfrak{a}(a)(1_A \otimes b) \mid a \in A, b \in B\}$ generates $\mathfrak{a}(1_A)(A \otimes B)$ as a vector space.*

Proof. Using Sweedler notations (which makes sense here as B is finite dimensional), one has

$$\begin{aligned}
\mathfrak{a}(1_A)(a \otimes 1_B) &= (id_A \otimes \varepsilon \otimes id_B)(\mathfrak{a} \otimes id_B)[\mathfrak{a}(1_A)(a \otimes 1_B)] \\
&= (id_A \otimes \varepsilon \otimes id_B)[(id_A \otimes \Delta)\mathfrak{a}(1_A))\mathfrak{a}(a) \otimes 1d_B)] \\
&= (id_A \otimes \varepsilon \otimes id_B)[(1_A{}^1 \otimes \Delta(1_A{}^2)(a^1 \otimes a^2 \otimes 1_B)] \\
&= (id_A \otimes \varepsilon \otimes id_B)[(1_A{}^1 \otimes \Delta(1_B)(1_A{}^2 \otimes 1_B)(a^1 \otimes a^2 \otimes 1_B)] \\
&= (id_A \otimes \varepsilon \otimes id_B)[(1_A \otimes \Delta(1_B))(1_A{}^1 a^1 \otimes 1_A{}^2 a^2 \otimes 1_B)] \\
&= (id_A \otimes \varepsilon \otimes id_B)[(1_A \otimes \Delta(1_B))(a^1 \otimes a^2 \otimes 1_B)] \\
&= (id_A \otimes \varepsilon_t)\mathfrak{a}(a).
\end{aligned}$$

Definition 2.1.1 (3) of [23] gives that

$$\begin{aligned}
\mathfrak{a}(1_A)(a \otimes 1_B) &= (id_A \otimes m)(id_A \otimes id_B \otimes S)(id_A \otimes \Delta)\mathfrak{a}(a) \\
&= (id_A \otimes m)(id_A \otimes id_B \otimes S)(\mathfrak{a} \otimes id_B)\mathfrak{a}(a) \\
&= (id_A \otimes m)(\mathfrak{a}(a^1) \otimes S(a^2)).
\end{aligned}$$

Finally, the trivial equality: $(id_A \otimes m)(x \otimes y \otimes z) = (x \otimes y)(1_A \otimes z)$ implies

$$\mathfrak{a}(1_A)(a \otimes 1_B) = \mathfrak{a}(a^1)(1 \otimes S(a^2)).$$

So $\mathfrak{a}(1_A)(a \otimes 1_B)$ belongs to the vector space generated by $\mathfrak{a}(A)(1_A \otimes B)$. □

Let us introduce the unital $*$-homomorphism $\alpha : B_s \to A : \alpha(x) := x \cdot 1_A$. Equalities (4) and (5) show that, for all $x \in B_s$ and $a \in A$

$$\mathfrak{a}(\alpha(x)a) = (1_A \otimes x)\mathfrak{a}(a), \tag{19}$$

$$(\alpha(x) \otimes 1_B)\mathfrak{a}(a) = (1_A \otimes S(x))\mathfrak{a}(a). \tag{20}$$

It is helpful to note that

$$\mathfrak{a}(1_A) = (\alpha \otimes id_B)\Delta(1_B). \tag{21}$$

Indeed

$$\begin{aligned}\alpha(1_1) \otimes 1_2 :=& 1_1 \cdot 1_A \otimes 1_2 = (id_A \otimes \varepsilon)[(1_A \otimes 1_1)\mathfrak{a}(1_A)] \otimes 1_2 \\ =& 1_A^1 \otimes (\varepsilon \otimes id_B)\Delta(1_A^2) = \mathfrak{a}(1_A).\end{aligned}$$

Lemma 3.4. *(cf.* [34] *3.1.5, 3.1.6). If* $(A, \mathfrak{a})$ *is a* $\mathfrak{G}$-$*$*-algebra* A*, then*

(i) The set $A^{\mathfrak{a}} = \{a \in A | \mathfrak{a}(a) = \mathfrak{a}(1_A)(a \otimes 1_B)\}$ *is a unital* $*$*-subalgebra of* A *(it is a unital* C^**-subalgebra of* A *when* A *is a* C^**-algebra) commuting pointwise with* $\alpha(B_s)$*.*

(ii) The map $T^{\mathfrak{a}} := (id_A \otimes h)\mathfrak{a}$ *(where* h *is the normalized Haar measure of* $\mathfrak{G}$*) is a conditional expectation from* A *to* $A^{\mathfrak{a}}$*; it is faithful when* A *is a* C^**-algebra.*

Proof. (i) For all $a \in A^{\mathfrak{a}}$ and $x \in B_s$, one has

$$\mathfrak{a}(a\alpha(x)) = \mathfrak{a}(1_A)(a \otimes 1_B)(1_A \otimes x)\mathfrak{a}(1_A) = \mathfrak{a}(\alpha(x)a),$$

so $A^{\mathfrak{a}}$ commutes pointwise with $\alpha(B_s)$, then it is stable with respect to the multiplication and the $*$-operation in A; moreover if A is a C^*-algebra, it is clearly norm closed in A, so this is a unital C^*-subalgebra of A.

(ii) Since $h_{|B_t} = \varepsilon_{|B_t}$ (see [23], 7.3.2), one has $T^{\mathfrak{a}}(1_A) := (id_A \otimes h)\mathfrak{a}(1_A) = 1_A$, from where, for all $a \in A^{\mathfrak{a}}$:

$$T^{\mathfrak{a}}(a) = (id_A \otimes h)(\mathfrak{a}(1_A)(a \otimes 1_B)) = (id_A \otimes h)(\mathfrak{a}(1_A))a = a.$$

Now, if $E_t = (id_B \otimes h)\Delta$ is the target Haar conditional expectation of $\mathfrak{G}$, one has, for all $a \in A$

$$\begin{aligned}\mathfrak{a}(T^{\mathfrak{a}}(a)) &= \mathfrak{a}((id_A \otimes h)\mathfrak{a}(a)) = (id_A \otimes id_B \otimes h)((\mathfrak{a} \otimes id_B)\mathfrak{a}(a)) \\ &= (id_A \otimes id_B \otimes h)(id_A \otimes \Delta)\mathfrak{a}(a) = (id_A \otimes E_t)\mathfrak{a}(a) \\ &= (id_A \otimes E_t)(\mathfrak{a}(1_A)\mathfrak{a}(a)) \\ &= (id_A \otimes E_t)(\mathfrak{a}(1_A)(a^1 \otimes a^2)) = \mathfrak{a}(1_A)(a^1 \otimes E_t(a^2)) \\ &= \mathfrak{a}(1_A)(1_A \otimes E_t(a^2))(a^1 \otimes 1_B) = \mathfrak{a}(1_A)(\beta(S(E_t(a^2))) \otimes 1_B)(a^1 \otimes 1_B) \\ &= \mathfrak{a}(1_A)(\beta(S(E_t(a^2)))a^1 \otimes 1_B).\end{aligned}$$

Using the fact proved above that $(id_A \otimes h)(\mathfrak{a}(1_A)) = 1_A$, this implies that

$$\begin{aligned}(id_A \otimes h)\mathfrak{a}(T^{\mathfrak{a}}(a)) &= (id_A \otimes h)\mathfrak{a}(1_A)(\beta(S(E_t(a^2)))a^1 \otimes 1_B) \\ &= \beta(S(E_t(a_2)))a^1.\end{aligned}$$

But since $h \circ E_t = h$, one has also

$$\begin{aligned}(id_A \otimes h)\mathfrak{a}(T^{\mathfrak{a}}(a)) &= (id_A \otimes h)\mathfrak{a}(1_A)(a_1 \otimes E_t(a_2)) \\ &= (id_A \otimes h)(\mathfrak{a}(1_A)(a_1 \otimes a_2)) \\ &= (id_A \otimes h)(\mathfrak{a}(1_A)\mathfrak{a}(a)) = T^{\mathfrak{a}}(a).\end{aligned}$$

One deduces that $T^{\mathfrak{a}}(a) = \beta(S(E_t(a_2)))a^1$ and

$$\mathfrak{a}(T^{\mathfrak{a}}(a)) = \mathfrak{a}(1_A)(\beta(S(E_t(a_2)))a_1 \otimes 1_B) = \mathfrak{a}(1_A)(T^{\mathfrak{a}}(a) \otimes 1_B).$$

This implies that $T^{\mathfrak{a}}(A) = A^{\mathfrak{a}}$, moreover, $T^{\mathfrak{a}} \circ T^{\mathfrak{a}} = T^{\mathfrak{a}}$. Finally, for all $c, d \in A^{\mathfrak{a}}$ and $a \in A$, one has

$$\begin{aligned} T^{\mathfrak{a}}(cad) &= (id_A \otimes h)\mathfrak{a}(cad) = (id_A \otimes h)(\mathfrak{a}(c)\mathfrak{a}(a)\mathfrak{a}(d)) \\ &= (id_A \otimes h)((1_B \otimes c)\mathfrak{a}(a)(1_B \otimes d)) = cT^{\mathfrak{a}}(a)d. \end{aligned}$$

When A is a C^*-algebra, $T^{\mathfrak{a}}$ is faithful because $\mathfrak{a}$ and h are faithful. □

Definition 3.5. Let $(A, \mathfrak{a})$ be a unital $\mathfrak{G}$-$*$-algebra, then unital $*$-subalgebra

$$A^{\mathfrak{a}} = \{a \in A/\mathfrak{a}(a) = \mathfrak{a}(1_A)(a \otimes 1_B)\}$$

is called the subalgebra of invariants (or fixed points) of $(A, \mathfrak{a})$.

Proposition 3.6. *Let $(A, \mathfrak{a})$ be a unital $\mathfrak{G}$-C^*-algebra and ϕ be an element in A^*, then the following assertions are equivalent:*
i) for any $a \in A$ one has: $(\phi \otimes i_B)\mathfrak{a}(a) \in B_s$;
ii) $\phi \circ T^{\mathfrak{a}} = \phi$;
iii) there exists a linear form ω on $A^{\mathfrak{a}}$ such that $\phi = \omega \circ T^{\mathfrak{a}}$;
iv) for any $x, y \in A$, one has

$$(\phi \otimes id_B)(\mathfrak{a}(x)(y \otimes 1_B)) = (\phi \otimes S)((x \otimes 1_B)\mathfrak{a}(y)).$$

Proof. Clearly, ii) and iii) are equivalent. If ii) is true and if $E_s = (h \otimes i_B)\Delta$ is the source Haar conditional expectation of $\mathfrak{G}$, then i) is true because, for all $\omega' \in B^*$ and $a \in A$, one has

$$\begin{aligned} \omega'((\phi \circ i_B)\mathfrak{a}(a)) &= (\phi \circ \omega')\mathfrak{a}(a) = (\phi \circ \omega')(T^{\mathfrak{a}} \otimes i_B)\mathfrak{a}(a) \\ &= (\phi \circ \omega')((id_A \otimes h)\mathfrak{a} \otimes id_B)\mathfrak{a}(a) \\ &= (\phi \circ \omega')(id_A \otimes h \otimes id_B)(\mathfrak{a} \otimes id_B)\mathfrak{a}(a) \\ &= (\phi \circ \omega')(id_A \otimes h \otimes id_B)(id_A \otimes \Delta)\mathfrak{a}(a) \\ &= (\phi \circ \omega')(id_A \otimes (h \otimes id_B)\Delta)\mathfrak{a}(a) \\ &= (\phi \circ \omega')(id_A \otimes E_s)\mathfrak{a}(a) \\ &= \omega'(E_s((\phi \circ id_B)\mathfrak{a}(a))). \end{aligned}$$

If i) is true, one has

$$\begin{aligned} \phi(a) &= \phi((id_A \otimes \varepsilon)\mathfrak{a}(a)) = \varepsilon((\phi \otimes id_B)\mathfrak{a}(a)) = \varepsilon(E_s(\phi \otimes id_B)\mathfrak{a}(a)) \\ &= (\phi \otimes \varepsilon)(id_A \otimes E_s)\mathfrak{a}(a) = (\phi \otimes \varepsilon)(id_A \otimes (h \otimes id_B)\Delta)\mathfrak{a}(a) \\ &= (\phi \otimes \varepsilon)(id_A \otimes h \otimes id_B)(id_A \otimes \Delta)\mathfrak{a}(a) \\ &= (\phi \otimes \varepsilon)(id_A \otimes h \otimes id_B)(\mathfrak{a} \otimes id_B)\mathfrak{a}(a) \\ &= (\phi \otimes \varepsilon)(T^{\mathfrak{a}} \otimes id_B)\mathfrak{a}(a) = (\phi \circ T^{\mathfrak{a}})(id_A \otimes \varepsilon)\mathfrak{a}(a) = (\phi \circ T^{\mathfrak{a}})(a), \end{aligned}$$

which is ii), so the three first assertions are equivalent.

Further, if iv) is true, then we have, applying it to $x \in A$ and $y = 1_B$

$$(\phi \otimes id_B)\mathfrak{a}(x) = (\phi \otimes S)((x \otimes 1_B)\mathfrak{a}(1_A)),$$

which implies i). Suppose now that i) is true (and so ii) and iii) as well). First, for all $a \in A, z \in B_t$, the equality (20) gives

$$a^1 S(z) \otimes a^2 = a^1 \otimes a^2 z.$$

Next, the equality $y^1 \otimes \varepsilon_t(y^2) = (1_A^1 y) \otimes 1_A^2$ (which can be proven directly), the equality $\varepsilon_t(b) = b_1 S(b_2), \forall b \in B$ and assertion i) give

$$\begin{aligned}(\phi \otimes id_B)(\mathfrak{a}(x)(y \otimes 1_B)) &= \phi(x^1 y)x^2 = \phi(x^1 1_A^1 y)x^2 1_A^2 = \phi(x^1 y^1)x^2 \varepsilon_t(y^2) \\ &= \phi(x^1 y^1)x^2 y^2 S(y^3) = \phi((xy^1)^1)(xy^1)^2 S(y^3) \\ &= \phi((xy^1)^1)\varepsilon_s((xy^1)^2)S(y^3).\end{aligned}$$

Now, using the definition of ε_s and the equality $\varepsilon_t(bz) = \varepsilon_t(bS(z))$ which is true for all $b \in B, z \in B_t$, we have

$$\begin{aligned}(\phi \otimes id_B)(\mathfrak{a}(x)(y \otimes 1_B)) &= \phi((xy^1)^1)\varepsilon((xy^1)^2(1_B)_2)(1_B)_1 S(y^3) \\ &= \phi((xy^1)^1)\varepsilon((xy^1)^2 S((1_B)_2))(1_B)_1 S(y^3) \\ &= (\phi \otimes \varepsilon)(\mathfrak{a}(xy^1)(1_A \otimes S((1_B)_2)))(1_B)_1 S(y^3) \\ &= \phi((i \otimes \varepsilon)(\mathfrak{a}(xy^1)(1_A \otimes S((1_B)_2))))(1_B)_1 S(y^3) \\ &= \phi((xy^1) \cdot S((1_B)_2))(1_B)_1 S(y^3),\end{aligned}$$

which equals, due to the relation $(ac) \cdot t = a(c \cdot t), \forall a, c \in A$, to

$$\begin{aligned}&\phi(x(y^1 \cdot S((1_B)_2)))(1_B)_1 S(y^3) = \phi(x(i \otimes \varepsilon)(\alpha(y^1)(1 \otimes S((1_B)_2))))(1_B)_1 S(y^3) \\ &\quad = \phi(xy^1 \varepsilon(y^2 S((1_B)_2)))(1_B)_1 S(y^3) \\ &\quad = (\phi \otimes \varepsilon \otimes S)(xy^1 \otimes y^2 S((1_B)_2 \otimes y^3 S((1_B)_1) \\ &\quad = (\phi \otimes \varepsilon \otimes S)((x \otimes 1 \otimes 1)(\alpha \otimes i)\alpha(y)(1 \otimes \varsigma(S \otimes S)(\Delta(1_B)))) \\ &\quad = (\phi \otimes \varepsilon \otimes S)((x \otimes 1 \otimes 1)(i \otimes \Delta)\alpha(y)(i \otimes \Delta)(1 \otimes 1_B)) \\ &\quad = (\phi \otimes S)((x \otimes 1)(i \otimes (i \otimes \varepsilon)\Delta)\alpha(y)) \\ &\quad = (\phi \otimes S)((x \otimes 1)\alpha(y)).\end{aligned}$$

□

Corollary 3.7. *Let $\mathfrak{a}(1_A) = 1_A^1 \otimes 1_A^2$ be a decomposition of $\mathfrak{a}(1_A)$ in Sweedler leg notations, and let ϕ be a positive faithful form on A satisfying the conditions of Proposition 3.6, then 1_A^1 is in the centralizer of ϕ.*

Proof. Due to i), one has for all $x \in A$: $(\phi \otimes i)\mathfrak{a}(x) \in B_t$, so $(\phi \otimes S)\mathfrak{a}(x) = (\phi \otimes S^{-1})\mathfrak{a}(x)$, hence by iv) applied twice:

$$\begin{aligned}(\phi \otimes i)(\mathfrak{a}(1_A)(x \otimes 1_B)) &= (\phi \otimes S)\mathfrak{a}(x) = (\phi \otimes S^{-1})\mathfrak{a}(x) \\ &= (\phi \otimes i)((x \otimes 1_B)\mathfrak{a}(1_A)),\end{aligned}$$

which gives the result. □

Definition 3.8. A linear form on A satisfying the conditions of Proposition 3.6 is called an invariant form with respect to $\mathfrak{a}$.

Example 3.9. The Haar measure h is an invariant faithful form on B with respect to the coaction Δ of $\mathcal{G}$ on B.

Definition 3.10. If $A^{\mathfrak{a}} = \mathbb{C}1_A$, we say that the coaction $\mathfrak{a}$ is ergodic.

Example 3.11. Let $I \subset B$ be a unital right coideal C^*-subalgebra with the coaction $\mathfrak{a} = \Delta|_I$. Then $I^{\mathfrak{a}} = I \cap B_t$, so this coaction is ergodic if and only if $I \cap B_t = \mathbb{C}1_B$, i.e., if and only if I is connected.

Remark 3.12. *Lemma 3.6 iii) shows that the set of $\mathfrak{a}$-invariant faithful states on A is not empty. Moreover, if $\mathfrak{a}$ is ergodic, then the linear form h_A on A defined by $T^{\mathfrak{a}}(x) = h_A(x)1_A (\forall x \in A)$ is the unique $\mathfrak{a}$-invariant faithful state.*

Definition 3.13. Let H be a Hilbert space and $\mathfrak{a}$ be a coaction of $\mathfrak{G}$ on a unital C^*-subalgebra A of $B(H)$, then an implementation of $\mathfrak{a}$ is a unitary corepresentation V of $\mathfrak{G}$ on H such that, for all $a \in A$, one has

$$\mathfrak{a}(a) = V(a \otimes 1_B)V^*.$$

Let us construct a canonical implementation for any coaction.

Proposition 3.14. *Let $\mathfrak{a}$ be a coaction of $\mathfrak{G}$ on A and ϕ a faithful $\mathfrak{a}$-invariant state on A, then the operator V defined on $H_\phi \otimes H_h$ by*

$$V(a \otimes b) := \mathfrak{a}(a)(1_A \otimes b), \quad \text{for all} \quad a \in A, \quad b \in B,$$

is a unitary corepresentation of $\mathfrak{G}$ implementing $\mathfrak{a}$.

Proof. For the proof that V is a corepresentation of $\mathfrak{G}$, see the proof of Proposition 2.10. Then Proposition 3.6 and Corollary 3.7 imply

$$\begin{aligned} < V(a \otimes b), V(a \otimes b) > &= (\phi \otimes h)((1_A \otimes b^*)\mathfrak{a}(a^*a)(1_A \otimes b)) \\ &= h[b^*(\phi \otimes id_b)(\mathfrak{a}(a^*a)(1_A \otimes 1_B))b)] \\ &= h[b^*(\phi \otimes S)[(a^*a \otimes 1_B)\mathfrak{a}(1_A)]b] \\ &= (\phi \otimes h)[(a^*a1_A^1 \otimes b^*S(1_A^2)b] \\ &=< J_\phi \sigma_{i/2}^\phi(1_A^1)^* J_\phi a \otimes S(1_A^2)b, a \otimes b > \\ &=< (J_\phi(1_A^1)^* J_\phi \otimes S(1_A^2))(a \otimes b), a \otimes b >, \end{aligned}$$

for all $a \in A, b \in B$, from where

$$V^*V = (j_\phi \otimes S)\mathfrak{a}(1_A).$$

Here $j_\phi(x) := J_\phi x^* J_\phi$ is the Tomita involution associated with ϕ. Then V is a partial isometry, by Proposition 3.3 its image is $\mathfrak{a}(1_A)(H_\phi \otimes H_h)$, so $VV^* = \mathfrak{a}(1_A)$. Put $\beta := j_\phi \circ \alpha$, then by Tomita's theory β is a faithful anti-representation of B_s whose image commutes in $B(H_\phi)$ with $Im\ \alpha$.

Now, for any $x, a \in A$ and $b \in B$, one has: $\mathfrak{a}(x)V(a \otimes b) = \mathfrak{a}(x)\mathfrak{a}(a)(1_A \otimes b) = \mathfrak{a}(xa)(1_A \otimes b) = V(ax \otimes b) = V(x \otimes 1)(a \otimes b)$. Hence, $\mathfrak{a}(x)V = V(x \otimes 1)$, and one deduces that:

$$\mathfrak{a}(x) = \mathfrak{a}(x)\mathfrak{a}(1) = \mathfrak{a}(x)VV^* = V(x \otimes 1)V^*.$$

□

Example 3.15. If I is a right coideal *-subalgebra of B and $\Delta|_I$ is a coaction of $\mathfrak{G}$ on it, the above formula gives the unitary corepresentation of $\mathfrak{G}$ which is a canonical implementation of Δ. In particular, if $I = B$ (resp., $I = B_s$), we have the regular (resp., the trivial) unitary corepresentation of $\mathfrak{G}$.

2. Spectral subspaces of A. For any $(U, H_U) \in UCorep(\mathfrak{G})$, H_U is a $\mathfrak{G}$-comodule via $\delta_U : v \mapsto U(v \otimes 1_B)$. In terms of the matrix coefficients $U_{i,j}$ $(i, j = 1, ..., n)$ with respect to some orthonormal basis $\{e_i\}_{i=1}^n$ in H_U, this means that $\delta_U(e_j) = \Sigma_{i=1}^n e_i \otimes U_{i,j}$.

Definition 3.16. Let A be a unital $\mathfrak{G}$-C^*-algebra A. We call the spectral subspace of A corresponding to (U, H_U) the linear span A_U of the images of all $\mathfrak{G}$-comodule maps $H_U \to A$.

For instance, if U is the trivial corepresentation which is associated with $(\Delta_{|B_s}, B_s)$, so $H_U = B_s$, we will use the notation A_ε instead of A_U, and we have $\alpha(B_s) \subset A_\varepsilon$. Indeed, $\alpha : B_s \to A$ is a $\mathfrak{G}$-comodule map: $\mathfrak{a}(\alpha(x)) = (1_A \otimes x)\mathfrak{a}(1_A) = (1_A \otimes x)(\alpha \otimes id_B)\Delta(1_B)$ –see (21).

Proposition 3.17. (cf. [2], Proposition 13). *One can characterize the spectral subspaces as follows:*

$$A_U := \{a \in A | \mathfrak{a}(a) \in \mathfrak{a}(1_A)(A \otimes B_U)\}.$$

Proof. (i) Let $R : H_U \to A$ be a $\mathfrak{G}$-comodule map. Then

$$\mathfrak{a}(a) = \mathfrak{a}(R(v)) = \mathfrak{a}(1_A)(R \otimes id)\delta_U(v) \in \mathfrak{a}(1_A)(R \otimes id)(H_U \otimes B_U),$$

where $a = R(v),\ v \in H_U$, and

$$\mathfrak{a}(1_A)(R \otimes id)(H_U \otimes B_U) \subset \mathfrak{a}(1_A)(A \otimes B_U).$$

(ii) Vice versa, let $a \in A$ be such that $\mathfrak{a}(a) \in \mathfrak{a}(1_A)(A \otimes B_U) \subset A \otimes B_U$, so $\mathfrak{a}(a) = \Sigma_{i,j}(a_{i,j} \otimes U_{i,j})$. Then, on the one hand,

$$(\mathfrak{a} \otimes id_B)\mathfrak{a}(a) = \Sigma_{i,j}(\mathfrak{a}(a_{i,j}) \otimes U_{i,j}),$$

and, on the other hand, using Remark 2.18 (iii),

$$(\mathfrak{a} \otimes id_B)\mathfrak{a}(a) = \Sigma_{i,j}(a_{i,j} \otimes \Delta(U_{i,j})) = \Sigma_{i,j,k}(a_{i,j} \otimes U_{i,k} \otimes U_{k,j}),$$

from where $\mathfrak{a}(a_{k,j}) = \Sigma_i(a_{i,j} \otimes U_{i,k})$, for all $k, j = 1, \dots, dim(H_U)$. But $\mathfrak{a}(1_A)^2 = \mathfrak{a}(1_A)$, so in fact $\mathfrak{a}(a_{k,j}) = \mathfrak{a}(1_A)(\Sigma_i a_{i,j} \otimes U_{i,k})$. We have $a = \Sigma_j a_{j,j}$ because the images of both sides of this equality under $\mathfrak{a}$ coincide and $\mathfrak{a}$ is injective. So it suffices to show that any $a_{j,j}$ is the image of some vector from H_U under some $\mathfrak{G}$-comodule map to A. But the map defined by $e_k \mapsto a_{k,j}$, for all $j, k = 1, \dots, dim(H_U)$ (where $\{e_k\}_{k=1}^{dim(H_U)}$ is the above orthonormal basis in H_U), is clearly a $\mathfrak{G}$-comodule map and $a_{j,j}$ is the image of the vector e_j. □

Corollary 3.18. *(i) All A_U are closed.*

(ii) $A = \oplus_{x \in \Omega} A_{U^x}$.

(iii) $A_{U^x} A_{U^y} \subset \oplus_z A_{U^z}$, where z runs over the set of all irreducible direct summands of $U^x \odot U^y$.

(iv) $\mathfrak{a}(A_U) \subset \mathfrak{a}(1_A)(A_U \otimes B_U)$ and $A_{\overline{U}} = (A_U)^*$.

(v) A_ε is a unital C^*-algebra.

Proof. (i) $\mathfrak{a}$ is continuous and $dim(B_U) < \infty$, so all A_U are closed.

(ii) Follows from Remark 2.18 (i).

(iii) Follows from Remark 2.18 (ii).

(iv) Remark 2.18 (iii) implies:

$$\mathfrak{a}(a^1) \otimes a^2 = a^1 \otimes \Delta(a^2) \in A \otimes B_U \otimes B_U,$$

so $\mathfrak{a}(a^1) \in A \otimes B_U$. As $\mathfrak{a}(1_A)$ is an idempotent, we have $\mathfrak{a}(a^1) \in \mathfrak{a}(1_A)(A \otimes B_U)$ which means that $a^1 \in A_U$. Then the second statement follows.

(v) Follows from Example 2.19. □

Example 3.19. Let (ε, B_s) be the trivial corepresentation of $\mathfrak{G}$, so $B_\varepsilon = B_s B_t$ is a unital C^*-algebra (see Example 2.19). The definition of A_ε shows that it is a unital C^*-subalgebra of A. It contains a unital C^*-subalgebra $\alpha(B_s)A^{\mathfrak{a}}$ invariant with respect to $\mathfrak{a}$. Indeed, if $z \in B_s, a \in A^{\mathfrak{a}}$, we have, using (21)

$$\mathfrak{a}(\alpha(z)a) = (1_A \otimes z)\alpha(1_A)(a \otimes 1_B) \in \alpha(1_A)(\alpha(B_s)A^{\mathfrak{a}} \otimes B_s B_t).$$

We will show that for coconnected finite quantum groupoids $A_\varepsilon = \alpha(B_s)A^{\mathfrak{a}}$.

4. From coactions to module categories over $UCorep(\mathfrak{G})$

1. Equivariant C^*-correspondences. The next definition is parallel to the definitions given in [1] and [6].

Definition 4.1. Given a $\mathfrak{G} - C^*$-algebra $(A, \mathfrak{a})$, we call a right Hilbert A-module $\mathcal{E}$ A-equivariant if it is equipped with a map $\mathfrak{a}_\mathcal{E} : \mathcal{E} \mapsto \mathcal{E} \otimes B$ such that

1) $(\mathfrak{a}_\mathcal{E} \otimes id_B)\mathfrak{a}_\mathcal{E} = (id_\mathcal{E} \otimes \Delta)\mathfrak{a}_\mathcal{E}$; $(id_\mathcal{E} \otimes \varepsilon)\mathfrak{a}_\mathcal{E} = id_H$;

2) $\mathfrak{a}_\mathcal{E}(\xi \cdot a) = \mathfrak{a}_\mathcal{E}(\xi) \cdot \mathfrak{a}(a)$, for all $a \in A, \xi \in \mathcal{E}$;

3) $< \mathfrak{a}_\mathcal{E}(\xi), \mathfrak{a}_\mathcal{E}(\eta) >_{A \otimes B} = \mathfrak{a}(< \xi, \eta >_A)$, for all $\xi, \eta \in \mathcal{E}$, where the exterior product $\mathcal{E} \otimes B$ [12], Chapter 4, is considered as a right Hilbert $A \otimes B$-module.

Let $\mathcal{D}_A$ be the category of **finitely generated** A-equivariant Hilbert A-modules and morphisms: equivariant A-linear maps. These maps are automatically adjointable – see [12], Chapter 1, so $\mathcal{D}_A$ is a C^*-category.

Remark 4.2. *Condition 1) implies that $\mathcal{E}$ is canonically a B_s-bimodule, given by $x.\xi.y = \xi^1 \varepsilon(x\xi^2 y)$ $\forall x, y \in B_s, \forall \xi \in \mathcal{E}$. So $\mathcal{E} \otimes B$ is a $B_s \otimes B$-bimodule, where B is a B-bimodule via right and left multiplication. Then one proves using (21) and (5) that $\mathfrak{a}_\mathcal{E}(\xi) \cdot \mathfrak{a}(1_A) = \mathfrak{a}_\mathcal{E}(\xi)$, for all $\xi \in \mathcal{E}$, and that the vector space $(\mathcal{E} \otimes B) \cdot \mathfrak{a}(1_A)$ is generated by $\mathfrak{a}_\mathcal{E}(\mathcal{E})(1_A \otimes B)$ – see the proof of Proposition 3.3.*

Lemma 4.3. *Any $\mathcal{E} \in \mathcal{D}_A$ satisfies the following conditions:*

(i) $(z \cdot \zeta) \cdot a = z \cdot (\zeta \cdot a)$, for all $z \in B_s, a \in A$.

(ii) $< z \cdot \zeta, \eta >_A = < \zeta, z^* \cdot \eta >_A$, for all $z \in B_s, \zeta, \eta \in \mathcal{E}$.

Proof. (i) We have

$$z \cdot (\zeta \cdot a) = (\zeta \cdot a)^1 \varepsilon(z(\zeta \cdot a)^2) = (id_\mathcal{E} \otimes \varepsilon)[((z \cdot \zeta)^1 \otimes (z \cdot \zeta)^2) \cdot \mathfrak{a}(a)] = (z \cdot \zeta) \cdot a.$$

(ii) The needed equality is equivalent to

$$\mathfrak{a}_\mathcal{E}(z \cdot \zeta, \eta >_A) = \mathfrak{a}_\mathcal{E}(< \zeta, z^* \cdot \eta >_A),$$

which is the same as

$$< \mathfrak{a}_\mathcal{E}(z \cdot \zeta), \mathfrak{a}_\mathcal{E}(\eta) >_{A \otimes B} = < \mathfrak{a}_\mathcal{E}(\zeta), \mathfrak{a}_\mathcal{E}(z^* \cdot \eta) >_{A \otimes B}$$

or

$$< \zeta^1, \eta^1 >_A < z\zeta^2, \eta^2 >_B = < \zeta^1, \eta^1 >_A < \zeta^2, z^* \eta^2 >_B .$$

As we see, the A-valued scalar products coincide and both B-valued scalar products are equal to $(\zeta^2)^* z^* \eta^2$ which finishes the proof. □

This lemma shows that any $\mathcal{E} \in \mathcal{D}_A$ is automatically a (B_s, A)-correspondence (see the definition in Section 2); we call such an object an equivariant (B_s, A)-correspondence and denote it by ${}_{B_s}\mathcal{E}_A$.

Example 4.4. A $\mathfrak{G} - C^*$-algebra $(A, \mathfrak{a})$ itself with the A-valued scalar product $< a, b >_A = a^* b$ $(\forall a, b \in A)$, is an equivariant (B_s, A)-correspondence.

Theorem 4.5. *If (V, H_V) is a unitary corepresentation of $\mathfrak{G}$, then H_V is an equivariant B_s-correspondence (B_s is equipped with the coaction $\Delta|_{B_s}$ of $\mathfrak{G}$).*

Proof. Proposition 2.11 shows that $(H_V, \mathfrak{a}_V)$ is a unitary $\mathfrak{G}$-comodule (where $\mathfrak{a}_V(\eta) = V(\eta \otimes \Lambda_h(1_B))$, $\forall \eta \in H_V$) so H_V is a B_s-correspondence in the sense of A. Connes. Then the Hilbert B_s-module structure on H_V is described in the proof of Lemma 2.20.

Applying the relations (4) and (5), one has

$$\mathfrak{a}_V(\eta) \cdot \Delta(1_B) = \mathfrak{a}_V(\eta) \cdot (1_1 \otimes 1_2) = \mathfrak{a}_V(\eta) \cdot (1_B \otimes S(1_1)1_2) = \mathfrak{a}_V(\eta),$$

which implies, for all $\eta \in H_V, t \in B_s$

$$\mathfrak{a}_V(\eta \cdot t) = \mathfrak{a}_V(\eta) \cdot (\Delta(1_B)(1 \otimes t)) = \mathfrak{a}_V(\eta) \cdot \Delta(t).$$

Now, consider V as an element of $B(H_V \otimes H_h)$, where H_h is the GNS Hilbert space constructed by (B, h), the canonical multiplicative isometry I_h of $\mathfrak{G}$ (see [34], Proposition 2.2.4) and its normalized fixed vector e (see [32], [33] 2.3 and 2.4)). Applying [33], Lemma 2.1.1, one has, for all $b' \in B'$ (the commutant of B in $B(H_h)$), $\xi, \eta \in \mathfrak{H}$, and $x, x' \in B_s$

$$
\begin{aligned}
&< \Delta(<\xi,\eta>_\alpha)(\Lambda_\varepsilon x \otimes e), \Lambda_\varepsilon x' \otimes b'e > \\
&\quad = < \Delta(1_B)(1_B \otimes <\xi,\eta>_{B_s})(\Lambda_\varepsilon x \otimes e), \Lambda_\varepsilon x' \otimes b'e > \\
&\quad = (h \otimes \omega_e)((x'^* \otimes b'^*)\Delta(1_B)(1_B \otimes <\xi,\eta>_{B_s})(x \otimes 1_B)) \\
&\quad = (h \otimes \omega_e)(\Delta(1_B)(1_B \otimes <\xi,\eta>_\alpha)(xx'^* \otimes b'^*)) \\
&\quad = \omega_e((h \otimes id_B)(\Delta(1_B)(xx'^* \otimes 1_B) <\xi,\eta>_{B_s} b'^*)) \\
&\quad = \omega_e(S(xx'^*) <\xi,\eta>_{B_s} b'^*).
\end{aligned}
$$

On the other hand, taking two decompositions: $V(\xi \otimes e) = \sum_{j\in J} (\xi_j \otimes b_j e)$ and $V(\eta \otimes e) = \sum_{i\in I} (\eta_i \otimes b_i e)$, one computes

$$
\begin{aligned}
&<< \mathfrak{a}_V(\xi), \mathfrak{a}_V(\eta) >_{B_s \otimes B} (\Lambda_\varepsilon x \otimes e), \Lambda_\varepsilon x' \otimes b'e > \\
&= \sum_{i\in I, j\in J} << \xi_j \otimes b_j, \eta_i \otimes b_i > (\Lambda_\varepsilon x \otimes e), \Lambda_\varepsilon x' \otimes b'e > \\
&= \sum_{i\in I, j\in J} < (R(\xi_j)^* R(\eta_i) \otimes b_j^* b_i)(\Lambda_\varepsilon x \otimes e), \Lambda_\varepsilon x' \otimes b'e > \\
&= \sum_{i\in I, j\in J} < (R(\xi_j)^* R(\eta_i) \Lambda_\varepsilon x, \Lambda_\varepsilon x' >< b_j^* b_i e, b'e > \\
&= \sum_{i\in I, j\in J} < R(\eta_i) \Lambda_\varepsilon x, R(\xi_j) \Lambda_\varepsilon x' >< b_i e, b_j b'e > \\
&= \sum_{i\in I, j\in J} < \beta(x)\eta_i), \beta(x')\xi_j >< b_i e, b' b_j e > \\
&= < \sum_{i\in I} (\beta(x) \otimes 1_B)(\eta_i \otimes b_i e), \sum_{j\in J} (\beta(x') \otimes b')(\xi_j \otimes b_j e) > \\
&= < (\beta(x) \otimes 1_B)(V(\eta \otimes e), (\beta(x') \otimes b')V(\xi \otimes e) > \\
&= < V(\eta \otimes S(x)e), V(\xi \otimes S(x')b'e) > \\
&= < e_{\beta,i}(\eta \otimes S(x)e), e_{\beta,i}(\xi \otimes S(x')b'e) > \\
&= << \xi,\eta >_{B_s} S(x)e, S(x')b'e > = << \xi,\eta >_{B_s} S(xx'^*)e, b'e > \\
&= \omega_e(S(xx'^*) <\xi,\eta>_{B_s} b'^*) = \omega_e(S(xx'^*) <\xi,\eta>_{B_s} b'^*).
\end{aligned}
$$

Thus, $< \mathfrak{a}_V(\xi), \mathfrak{a}_V(\eta) >_{B_s \otimes B} = \Delta(<\xi,\eta>_{B_s})$. □

Proposition 4.6. *Given an equivariant B_s-correspondence ${}_{B_s}\mathcal{E}_{B_s}$, define on $\mathcal{E}$ the scalar product inherited from its B_s-scalar product: $<\xi,\eta> = \varepsilon(<\eta,\xi>_{B_s})$, for all $\xi,\eta \in \mathcal{E}$. Then $V \in B(\mathcal{E} \otimes H_h)$ defined by*

$$V(\eta \otimes \Lambda_h(b)) = (id_{\mathcal{E}} \otimes \Lambda_h)(\mathfrak{a}_{\mathcal{E}}(\eta) \cdot (1 \otimes b)), \quad \textit{for all} \quad \eta \in \mathcal{E}, \quad b \in B,$$

is a unitary corepresentation of $\mathfrak{G}$.

Proof. As $\mathcal{E}$ satisfies the condition 1) of Definition 4.1, it has a B_s-bimodule structure defined by the maps $\alpha, \beta : B_s \to \mathcal{L}(\mathcal{E})$. In particular, $\beta(n)\xi = \xi \cdot n$, for all $n \in B_s$ and

$\xi \in \mathcal{E}$. Definition 4.1 2) shows that the right B_s-module structure given by β is the same as the initial B_s-bimodule structure on $\mathcal{E}$. With the new scalar product on $\mathcal{E}$, one has

$$\begin{aligned} < \beta(n)\xi, \xi > &= \varepsilon(< \xi, \beta(n)\xi >_{B_s}) = \varepsilon(< \xi, \xi \cdot n >_{B_s}) \\ &= \varepsilon(< \xi, \xi >_{B_s} n) = \varepsilon(n < \xi, \xi >_{B_s})) \\ &= \varepsilon(< \xi \cdot n^*, \xi >_{B_s})) \\ &= \varepsilon(< \beta(n^*)\xi, \xi >_{B_s})) =< \xi, \beta(n^*)\xi > . \end{aligned}$$

Hence, β is a unital $*$-anti-representation of B_s on $\mathcal{E}$, and $e_{\beta,i}$ is an orthogonal projection. Moreover, as $\mathcal{E}$ satisfies the condition 1) of Definition 4.1, then V defined above satisfies the conditions (i) and (ii) of Definition 2.9 - see the proof of Proposition 2.10. On the other hand

$$\begin{aligned} < V^*V(\eta \otimes e), (\eta \otimes e) > &=< V(\eta \otimes e), V(\eta \otimes e) > \\ &=< \sum_{i \in I} \eta_i \otimes b_i e, \sum_{i \in I} \eta_i \otimes b_i e > \\ &= (\varepsilon \otimes h)(< \mathfrak{a}_{\mathcal{E}}(\eta), \mathfrak{a}_{\mathcal{E}}(\eta) >_{B_s \otimes B}) \\ &= (\varepsilon \otimes h)(\mathfrak{a}_{triv}(< \eta, \eta >_{B_s})) = h(< \eta, \eta >_{B_s}) \\ &=<< \eta, \eta >_{B_s} e, e >=< e_{\beta,i}(\eta \otimes e), \eta \otimes e > . \end{aligned}$$

As e is separating for B, this implies that V is a partial isometry whose initial support is $e_{\beta,i}$. □

Theorem 4.5 and Proposition 4.6 allow to define two functors : $\mathcal{F}_3 : UCorep(\mathfrak{G}) \to \mathcal{D}_{B_s}$ and $\mathcal{G}_3 : \mathcal{D}_{B_s} \to UCorep(\mathfrak{G})$ on the level of objects, and the morphisms in both cases are just B-comodule maps. These functors are inverse to one another. Indeed, since the B-comodule structure is the same in both cases, the only thing to explain is the relation between the usual scalar product in H_V and the corresponding B_s-valued scalar product, but this explanation was done in the proof of Lemma 2.20. Thus, we have

Theorem 4.7. *The categories $UCorep(\mathfrak{G})$ and $\mathcal{D}_{B_s}$ are isomorphic.*

In particular, the unit object $\mathbf{1} \in \mathcal{D}_{B_s}$ is $(B_s, \Delta|_{B_s})$ with the B_s-valued scalar product $< b, c >= b^*c$, for all $b, c \in B_s$, and the tensor product is the interior tensor product of B_s-correspondences.

2. Module categories over $UCorep(\mathfrak{G})$ associated with equivariant C^*-correspondences.

Definition 4.8. [6]. Let $\mathcal{C}$ be a C^*-multitensor category with unit object $\mathbf{1}$. A C^*-category $\mathcal{M}$ is called a left $\mathcal{C}$-module C^*-category if there is a bilinear $*$-functor $\boxtimes$: $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ with natural unitary transformations $(X \otimes Y) \boxtimes M \to X \boxtimes (Y \boxtimes M)$ and $\mathbf{1} \boxtimes M \to M$ $(X, Y \in \mathcal{C}, M \in \mathcal{M})$ making $\mathcal{M}$ a left module category over $\mathcal{C}$ – see [9], Chapter 7. If $\mathcal{C}$ is strict, we say that $\mathcal{M}$ is strict (resp., indecomposable) if these natural transformations are identities (resp., if, for all non-zero $M, N \in \mathcal{M}$, there is $X \in \mathcal{C}$ such that $\mathcal{M}(X \boxtimes M, N) \neq 0$).

We say that an object $M \in \mathcal{M}$ generates $\mathcal{M}$ if any object of $\mathcal{M}$ is isomorphic to a subobject of $X \boxtimes M$ for some $X \in \mathcal{C}$. $\mathcal{M}$ is said to be semisimple if the underlying C^*-category is semisimple.

We will always consider C^*-categories closed with respect to subobjects, i.e., such that for any object M and any projection $p \in End(M)$, there are an object N and isometry $v \in \mathcal{M}(N, M)$ satisfying $p = vv^*$ (if necessary, one can complete given C^*-category with respect to subobjects).

One naturally defines a morphism $F : \mathcal{M}_1 \to \mathcal{M}_2$ between two $\mathcal{C}$-module C^*-categories as a morphism of the underlying C^*-categories equipped with a unitary natural equivalence $F(X \boxtimes M) \to X \boxtimes F(M),\ \forall\ X \in \mathcal{C},\ M \in \mathcal{M}$ satisfying some coherence conditions (see [6], 2.17).

Lemma 4.9. *$\mathcal{D}_A$ is a strict left module category over $UCorep(\mathfrak{G})$ defined by interior tensor product of C^* correspondences over B_s.*

Proof. Given $H_V \in \mathcal{D}_{B_s}$ and ${}_{B_s}\mathcal{E}_A \in \mathcal{D}_A$, equip the vector space $H_V \otimes_{B_s} \mathcal{E}$ with A-valued scalar product – see [12], Proposition 4.5

$$(22) \qquad < v \otimes_{B_s} \zeta, w \otimes_{B_s} \eta >_A = < \zeta, < v, w >_{B_s} \cdot \eta >_A, \quad \forall v, w \in H_V, \quad \zeta, \eta \in \mathcal{E},$$

which gives it the $B_s - A$-correspondence structure, and also with the algebraic structure of tensor product of the corresponding B-comodules. One can check that we obtain a new object $H_V \otimes_{B_s} \mathcal{E} \in \mathcal{D}_A$, and that this construction is natural both in V and $\mathcal{E}$. Thus, we have defined a functor $\boxtimes : UCorep(\mathfrak{G}) \times \mathcal{D}_A \to \mathcal{D}_A$ having the needed properties. Indeed, the first of them is true because $H_{U \odot V} = H_U \otimes_{B_s} H_V$ and because of the associativity of $\otimes_{B_s}$, and the second one can be proved by direct computation.

Finally, $\boxtimes$ sends adjoint morphisms to adjoint, so it is a $*$-functor. □

Let us show that A viewed as an object of $\mathcal{D}_A$ (see Example 4.4) is a generator for $\mathcal{D}_A$. More precisely, if $V \in UCorep(\mathfrak{G})$, then $H_V \otimes_{B_s} A \in \mathcal{D}_A$ and the corresponding right coaction of B on $H_V \otimes_{B_s} A$ defines a left action of $\hat{B}$ on it: $\hat{b} \cdot v := v^1 < \hat{b}, v^2 >$, for all $v \in H_V \otimes_{B_s} A, \hat{b} \in \hat{B}$. If $p \in \mathcal{L}(H_V \otimes_{B_s} A)$ is a $\hat{B}$-invariant orthogonal projection, then one can check that $H_{V,p} = p(H_V \otimes_{B_s} A)$ is a subobject of $H_V \otimes_{B_s} A$ in $\mathcal{D}_A$.

Lemma 4.10. (cf. [17], Lemma 3.2). *For any $\mathcal{E} \in \mathcal{D}_A$, there is $V \in UCorep(\mathfrak{G})$ and a $\hat{B}$-invariant projection $p \in \mathcal{L}(H_V \otimes_{B_s} A)$ such that $\mathcal{E}$ is isomorphic to $H_{V,p}$.*

Proof. For any fixed $\zeta \in \hat{B} \cdot \mathcal{E} = \mathcal{E}$, the finite dimensional vector space $\hat{B} \cdot \zeta$ is a $\hat{B}$-module, so there is a finite dimensional $\hat{B}$-submodule $\mathcal{E}_0$ of $\mathcal{E}$ such that $\mathcal{E}_0 \cdot A = \mathcal{E}$. In particular, there are unital $*$-representations of $B_s \cong \hat{B}_t$ and $B_t \cong \hat{B}_s$ on $\mathcal{E}_0$, so it is a B_s-bimodule. Constructing on this space a B_s-valued scalar product like in the proof of Lemma 2.20, we turn $\mathcal{E}_0$ into an equivariant B_s correspondence, and Proposition 4.6 allows to construct $V \in UCorep(\mathfrak{G})$ such that the left $\hat{B}$-modules H_V and $\mathcal{E}_0$ are isomorphic. Fix an isomorphism $T_0 : H_V \to \mathcal{E}_0$ and define $T : H_V \otimes_{B_s} A \to \mathcal{E}$ by $T(v \otimes_{B_s} a) = (T_0 v) \cdot a$. This is a surjective morphism of A-modules. Since $H_V \otimes_{B_s} A$ is a finitely generated Hilbert A-module, it makes sense to consider the polar decomposition $T^* = u|T^*|$. Then $|T^*|$ is an invertible endomorphism of the A-module $\mathcal{E}$, and $u : \mathcal{E} \to H_V \otimes_{B_s} A$ is an A-module mapping such that $u^* u = \iota$. Property (iii) in Definition 4.1 and non-degeneracy ensure that T^*, $|T^*|$, $u = T^*|T^*|^{-1}$, and u^* are morphisms of A-equivariant Hilbert modules. In particular, $u : \mathcal{E} \to H_{V,p}$ is an isomorphism such that $p = uu^*$. □

Remark 4.11. *$End_{\mathcal{D}_A}(A) = A^{\mathfrak{a}}$. In particular, a coaction $\mathfrak{a}$ is ergodic if and only if the generator A of the module category $\mathcal{D}_A$ is simple.*

Indeed, if $T \in End_{\mathcal{D}_A}(A)$, then $\mathfrak{a}(T(1_A)) = (T \otimes id_B)\mathfrak{a}(1_A) \in A \otimes B_t$. So $T(1_A) \in A^{\mathfrak{a}}$ because $(id_A \otimes h)\mathfrak{a}(T(1_A)) = (id_A \otimes \varepsilon)\mathfrak{a}(T(1_A)) = T(1_A)$.

Vice versa, arbitrary $a \in A^{\mathfrak{a}}$ generates an equivariant endomorphism of A via $T : 1_A \mapsto a$.

We can summarize the above considerations as follows:

Theorem 4.12. *Given a regular coconnected finite quantum groupoid $\mathfrak{G}$, consider two categories:*

(i) The category $\mathfrak{G}-Alg$ of unital $\mathfrak{G}-C^*$-algebras together with unital $\mathfrak{G}$-equivariant $*$-homomorphisms as morphisms.

(ii) The category $UCorep(\mathfrak{G})-Mod$ of pairs $(\mathcal{M}, M)$, where $\mathcal{M}$ is a left $UCorep(G)$-module C^*-category and M is its generator, with equivalence classes of unitary $Rep(G)$-module functors respecting the generators as morphisms.

Let us associate with any $\mathfrak{G}-C^*$-algebra $(A, \mathfrak{a})$ the C^*-category $\mathcal{D}_A$ of finitely generated A-equivariant (B_s, A)-correspondences with its generator A, and with any morphism $f : A_0 \to A_1$ in $\mathfrak{G}-Alg$ the morphism $\mathcal{E} \mapsto \mathcal{E} \otimes_{A_0} A_1$ from $\mathcal{D}_{A_0}$ to $\mathcal{D}_{A_1}$. This defines a functor $\mathcal{T} : \mathfrak{G}-Alg \to UCorep(\mathfrak{G})-Mod$.

The only thing to check is that $\mathcal{T}$ is well defined on the level of morphisms. This is straightforward because A_1 is a left A_0-module via morphism f. This construction was discussed in [6], Chapter 7 as "extension of scalars".

5. From module categories over $UCorep(\mathfrak{G})$ to coactions

In Sections 5 and 6 we use the approach proposed in [16] with certain modifications reflecting the difference between CQG and finite quantum groupoids and the fact that we are considering left module categories and right coactions instead of right module categories and left coactions as in [16].

Definition 5.1. Let R be a C^*-algebra and let $(\mathcal{C}, \otimes, \mathbf{1})$ be a strict C^*-tensor category, a weak tensor functor from $\mathcal{C}$ to $Corr(R)$ is a linear functor $F : \mathcal{C} \to Corr(R)$ together with natural R-bilinear isometries $J = J_{U,V} : F(U) \otimes_R F(V) \to F(U \otimes V)$ satisfying the following conditions:

(i) $F(\mathbf{1}) = R$;

(ii) $F(T)^* = F(T^*)$ for any morphism T in $\mathcal{C}$;

(iii) $J : R \otimes_R F(U) \to F(\mathbf{1} \otimes U) = F(U)$ maps $r \otimes X$ into Xr, and $J : F(U) \otimes_R R \to F(U \otimes \mathbf{1}) = F(U)$ maps $X \otimes r$ into rX, for all $X \in F(U)$;

(iv) $J(id \otimes J) = J(J \otimes id)$;

(v) for all $U, V \in \mathcal{C}$ and every vector $Y \in F(U)$, the right R-linear map $S_Y = S_{Y,U} : F(U) \to F(U \otimes V)$ mapping $X \in F(U)$ into $J(X \otimes Y)$ is adjointable, and $J(id \otimes S_Y^*) = S_Y^* \circ J$.

Remark 5.2. *(i) Any unitary tensor functor $F : \mathcal{C} \to Corr(R)$ is a weak tensor functor – if the conditions (i)–(iv) are satisfied and the maps J are surjective, then the condition (v) is also satisfied.*

(ii) If we consider F as a functor into the category of vector spaces, then S_Y is a natural transformation from F to $F(\cdot \otimes V)$, and we have

$$S_Y^* F(T \otimes id) = F(T) \circ S_Y^*, \quad \textit{for all morphisms in } \mathcal{C}. \tag{23}$$

We will also need the following modification of [16], Proposition 3.1:

Proposition 5.3. *Let $\mathcal{M}$ be a strict left module C^*-category over a strict C^*-tensor category $\mathcal{C}$, M be an object in $\mathcal{M}$, and denote by R the unital C^*-algebra $End(M)$. Then the map $F(U) = \mathcal{M}(M, U \boxtimes M)$ $\forall U \in \mathcal{C}$ defines a weak tensor functor $F : \mathcal{C} \to Corr(R)$, where $X = F(U)$ is a right R-module via the composition of morphisms, a left R-module via $rX = (id \otimes r)X$, the R-valued inner product is given by $< X, Y >= X^*Y$, the action of F on morphisms is defined by $F(T)X = (T \otimes id)X$, and $J_{X,Y}(X \otimes Y) = (id \otimes Y)X$, for all $X \in F(U), Y \in F(V), X, Y \in \mathcal{C}$.*

Let us note that $S_Y(X) = (id \otimes Y)X$ and $S_Y^*(Z) = (id \otimes Y^*)Z$, where $Z \in F(U \otimes V)$.

Now we will describe step by step the reconstruction procedure. Let $\mathcal{M}$ be a strict left $UCorep(\mathfrak{G})$-module C^*-category with generator M.

Let Ω be an exhaustive set of representatives of the equivalence classes of irreducible objects in $UCorep(\mathfrak{G})$. Consider the following vector space:

$$A = \bigoplus_{x\in\Omega} A_{U^x} := \bigoplus_{x\in\Omega} (F(U^x) \otimes \overline{H_x}), \tag{24}$$

and also a much larger vector space:

$$\tilde{A} = \bigoplus_{U\in\|UCorep(G)\|} A_U := \bigoplus_{U\in\|UCorep(G)\|} (F(U) \otimes \overline{H_U}), \tag{25}$$

where $F(U) = \bigoplus_i F(U_i)$ corresponds to the decomposition $U = \bigoplus U_i$ into irreducibles, and $\|UCorep(G)\|$ is an exhaustive set of representatives of the equivalence classes of objects in $UCorep(G)$ (these classes constitute a countable set). $\tilde{A}$ is a unital associative algebra with the product

$$(X \otimes \overline{\xi})(Y \otimes \overline{\eta}) = (id \otimes Y)X \otimes (\overline{\xi} \otimes_{B_s} \overline{\eta}), \quad \forall (X \otimes \overline{\xi}) \in A_U, \quad (Y \otimes \overline{\eta}) \in A_V,$$

and the unit

$$1_{\tilde{A}} = id_M \otimes \overline{1_B}.$$

Note that $(id \otimes Y)X = J_{X,Y}(X \otimes Y) \in F(U \odot V)$. Then, for any $U \in UCorep(G)$, choose isometries $w_i : H_i \to H_U$ defining the decomposition of U into irreducibles, and define the projection $p : \tilde{A} \to A$ by

$$p(X \otimes \xi) = \Sigma_i (F(w_i^*)X \otimes \overline{w_i^*\xi}), \quad \forall (X \otimes \overline{\xi}) \in A_U, \tag{26}$$

which does not depend on the choice of w_i. Indeed, for any other choice of isometries v_j there exists a unitary matrix u_{ij} such that $w_i = \Sigma_{i,j} u_{ij} v_j$. Note also that if $w : H_U \to H_V$ is an isometry between $U, V \in Corep(\mathfrak{G})$, then

$$p(F(w)X \otimes \overline{w\xi}) = p(X \otimes \overline{\xi}), \quad \forall (X \otimes \overline{\xi}) \in A_U. \tag{27}$$

Lemma 5.4. *A is a unital associative algebra with the product $x \cdot y := p(xy)$, for all $x, y \in A$.*

Proof. It suffices to check that $p(p(a)p(b)) = p(a)p(b)$, for all $a, b \in \tilde{A}$. Let $a = (X \otimes \overline{\xi}) \in A_U, b = (Y \otimes \overline{\eta}) \in A_V$, where $U, V \in UCorep(G)$. Choose isometries u_i and v_j corresponding to the decompositions $U = \bigoplus U_i$ and $V = \bigoplus U_j$ into irreducibles, and let $w_{i,j,k}$ be isometries corresponding to the decomposition of $U_i \odot V_j$ into irreducibles. Then

$$\begin{aligned} p(a)p(b) :=& \Sigma_i (F(u_i^*)X \otimes \overline{u_i^*\xi}))\Sigma_j (F(v_j^*)Y \otimes \overline{v_i^*\eta}) \\ =& \Sigma_{i,j}((id \otimes F(v_j^*)Y)F(u_i^*)X \otimes \overline{u_i^*\xi \otimes v_j^*\eta}) \\ =& \Sigma_{i,j,k}(F(w_{i,j,k}^*)(id \otimes F(v_j^*)Y)F(u_i^*)X \otimes \overline{w_{i,j,k}^*(u_i^*\xi \otimes v_j^*\eta)}). \end{aligned} \tag{28}$$

On the other hand, if we apply p to (28), we get the same result. □

In particular, the vector subspace $A_\varepsilon = R \otimes \overline{H}_\varepsilon$ (where $R = End(M)$ and $H_\varepsilon = B_s$) is a unital C^*-subalgebra of A and any $F(U)$ is an R-correspondence (see Proposition 5.3).

Lemma 5.5. *If X is in $F(U)$, then $X^\bullet = S_X^* F(R_U)(1_B)$ is the unique element from $F(\overline{U})$ satisfying*

$$< X^\bullet, Y >= F(R_U^*)J(Y \otimes X), \quad \textit{for all} \quad Y \in F(\overline{U}),$$

where R_U and $\overline{R}_U$ come from (2). We also have

$$< X, Y >= F(\overline{R}_U^*)J(Y \otimes X^\bullet), \quad \forall Y \in F(U).$$

Proof. We compute

$$< X^\bullet, Y >=< S_X^* F(R_U)(1_B), Y > =< F(R_U)(1_B), S_X(Y) > \\ = F(R_U^*)J(Y \otimes X).$$

The uniqueness follows from the faithfulness of the inner product. As for the last statement, we compute:

$$\begin{aligned} F(\overline{R}_U^*)J(Y \otimes X^\bullet) &= F(\overline{R}_U^*)J(Y \otimes S_X^* F(R_U)(1_B)) \\ &= F(\overline{R}_U^*)S_X^* J(Y \otimes F(R_U)(1_B)) \\ &= S_X^* F(\overline{R}_U^* \otimes id)F(id \otimes R_U)Y, \end{aligned}$$

where we have used (23). The latest expression equals to $S_X^* Y$, where $S_X^* : R \to F(U)$ is given by $r \to J(X \otimes r) = r \cdot X$, so $S_X^* Y =< X, Y >$. □

Similarly, for any $\xi \in H_U$ define $\xi^\bullet \in H_{\overline{U}}$ by

$$\xi^\bullet = (\overline{\xi} \otimes id_U)\overline{R}_U(1_B) = \overline{\hat{G}^{1/2} \cdot \xi} \text{ (see (2)),} \quad \text{so} \quad < \eta, \xi^\bullet >= \overline{R}_U^*(\xi \otimes \eta) \quad \forall \eta \in H_{\overline{U}},$$

and consider the map $\bullet : \tilde{A} \to \tilde{A}$

$$(X \otimes \overline{\xi})^\bullet := X^\bullet \otimes \overline{\xi^\bullet}.$$

Lemma 5.6. *A is a unital $*$-algebra with the above product and the involution $x^* := p(x^\bullet)$, for all $x \in A$.*

Proof. First, we prove that $p(p(a)^\bullet) = p(a^\bullet)$, for all $a \in \tilde{A}$. Take $a = (X \otimes \overline{\xi}) \in A_U$ and choose isometries u_i corresponding to the decompositions of $U = \bigoplus U_i$ and into irreducibles. Then for the standard duality morphisms we have $R_U = \Sigma_i(\overline{w}_i \otimes w_i)R_i$ and $\overline{R}_U = \Sigma_i(w_i \otimes \overline{w}_i)\overline{R}_i$, where $R_i := R_{U_i}, \overline{R}_i := \overline{R}_{U_i}$. Then

$$\begin{aligned} F(R_U^*)(Y \otimes X) &= \Sigma_i F(R_i^*)J(F(\overline{w}_i^*)Y \otimes F(w_i^*)X) \\ &= \Sigma_i < (F(w_i^*)X)^\bullet, F(\overline{w}_i^*)Y >, \end{aligned}$$

so $X^\bullet = \Sigma_i F(\overline{w}_i)(F(w_i^*)X)^\bullet$. Similarly, $\xi^\bullet = \Sigma_i \overline{w}_i(w_i^*\xi)^\bullet$, therefore, applying p to $a^\bullet$ and using (27), we have

$$\begin{aligned} p(a^\bullet) &= \Sigma_i p(F(\overline{w}_i)(F(w_i^*)X)^\bullet \otimes \overline{\overline{w}_i(w_i^*\xi)^\bullet}) \\ &= \Sigma_i p((F(w_i^*)X)^\bullet \otimes \overline{(w_i^*\xi)^\bullet}). \end{aligned}$$

On the other hand, the last expression equals to $p(p(a)^\bullet)$.

Next, in order to prove that $(p(a)p(b))^* = p(b)^*p(a)^*$, it suffices to prove that $p((a \cdot b)^\bullet) = p(b^\bullet \cdot a^\bullet)$, for all $a, b \in \tilde{A}$. Take $a = \overline{(X \otimes \xi)} \in A_U$ and $b = \overline{(Y \otimes \eta)} \in A_V$. The unitary $\sigma : H_{\overline{V}} \otimes H_{\overline{U}} \to H_{\overline{U \odot V}}$ mapping $\overline{\theta} \otimes \overline{\zeta}$ into $\overline{\zeta \otimes \theta}$ defines an equivalence between $\overline{V} \odot \overline{U}$ and $\overline{U \odot V}$, and we have

$$R_{U \odot V} = (\sigma \otimes id \otimes id)(id \otimes R_U \otimes id)R_V \quad \text{and} \quad \overline{R}_{U \odot V} = (id \otimes id \otimes \sigma)(id \otimes \overline{R}_V \otimes id)\overline{R}_U.$$

Then we compute using Lemma 5.5, relations $S_{J(X \otimes Y)} = S_Y S_X$ and (23)

$$\begin{aligned} J(X \otimes Y)^\bullet &= S_{J(X \otimes Y)}^* F(R_{U \odot V})(1_B) \\ &= S_X^* S_Y^* F(\sigma \otimes id \otimes id)F(id \otimes R_U \otimes id)F(R_V)(1_B) \\ &= F(\sigma)S_X^* F(id \otimes R_U)S_Y^* F(R_V)(1_B) = F(\sigma)S_X^* F(id \otimes R_U)(Y^\bullet) \\ &= F(\sigma)S_X^* J(Y^\bullet \otimes F(R_U)(1_B)) = F(\sigma)J(Y^\bullet \otimes S_X^* F(R_U)(1_B)) \\ &= F(\sigma)J(Y^\bullet \otimes X^\bullet). \end{aligned}$$

Similarly, $(\xi \otimes \eta)^\bullet = \sigma(\eta^\bullet \otimes \xi^\bullet)$, from where

$$(a \cdot b)^\bullet = (F(\sigma) \otimes \overline{\sigma})(J(Y^\bullet \otimes X^\bullet) \otimes (\eta^\bullet \otimes \xi^\bullet)) = (F(\sigma) \otimes \overline{\sigma})(b^\bullet \cdot a^\bullet).$$

Applying now p, we get $p((a \cdot b)^\bullet) = p(b^\bullet \cdot a^\bullet)$.

In order to show that $** = id$ on A, we will show that $p(a^{\bullet\bullet}) = p(a)$, for all $a \in \tilde{A}$. Take $a = (X \otimes \xi) \in A_U$ and consider the unitary $u : H_U \to H_{\overline{\overline{U}}} : \xi \mapsto \overline{\overline{\xi}}$. Then $\overline{R}_U = (u \otimes id)\overline{R}_U$, hence, applying twice Lemma 5.5, we have

$$< X^{\bullet\bullet}, Y > = F(R^*_{\overline{U}})J(Y \otimes X^\bullet) = F(\overline{R}^*_U)F(u^* \otimes id)J(Y \otimes X^\bullet)$$
$$= F(\overline{R}^*_U)J(F(u^*)Y \otimes X^\bullet) =< X, F(u^*)Y >, \quad \text{for any} \quad Y \in F(\overline{\overline{U}}).$$

So $X^{\bullet\bullet} = F(u)X$. We also have $\xi^{\bullet\bullet} = \overline{\overline{\xi}} = u\xi$, from where $a^{\bullet\bullet} = (F(u) \otimes \overline{u})a$, and applying p to both sides of this equality we get $p(a^{\bullet\bullet}) = p(a)$. □

Now define a linear map $\mathfrak{a} : A \to A \otimes B$ by $\mathfrak{a}(X \otimes \overline{\xi}) = X \otimes (- \otimes id_B)U^x(\xi \otimes 1_B)$ or, in other words, by

$$\mathfrak{a}(X \otimes \overline{\xi_i}) = X \otimes \Sigma_j(\overline{\xi_j} \otimes U^x_{j,i}), \tag{29}$$

where $X \in F(U^x)$, $\{\xi_j\}$ is any orthonormal basis in H_x and $U^x_{i,j}$ are matrix coefficients of U_x with respect to this basis (see Definition 2.17).

Lemma 5.7. *(i) The map $\mathfrak{a}$ is a right coaction of $\mathfrak{G}$ on A.*

(ii) A admits a unique C^*-completion $\overline{A}$ such that $\mathfrak{a}$ extends to a continuous coaction of $\mathfrak{G}$ on it.

Proof. (i) Clearly, $(A, \mathfrak{a})$ is a right B-comodule. In order to show that $\mathfrak{a}$ is an algebra homomorphism, remark that $\tilde{A}$ is a right B-comodule via extension $\tilde{\mathfrak{a}}$ of $\mathfrak{a}$ which is defined as in (29), but with arbitrary $U \in UCorep(\mathfrak{G})$. It follows from (29) that $p : \tilde{A} \to A$ is a comodule map, and from the formula $U \odot V = U_{13}V_{23}$ that $\tilde{\mathfrak{a}}$ is a homomorphism, hence $\mathfrak{a}$ is also a homomorphism.

In order to check that $\mathfrak{a}$ is $*$-preserving, it suffices to show that $\tilde{\mathfrak{a}}(a)^{\bullet\otimes *} = \tilde{\mathfrak{a}}(a^\bullet)$, for all $a = (X \otimes \overline{\xi}) \in A_U,\ U \in UCorep(\mathfrak{G})$. This is equivalent to

$$U(\xi \otimes 1_B)^{\bullet\otimes *} = \overline{U}(\overline{\hat{G}^{1/2} \cdot \xi} \otimes 1_B), \quad \forall \xi \in H_U,$$

which follows from Lemma 2.7 and a few relations that are easy to check: $(\hat{b} \cdot v^1) \otimes v^2 = v^1 \otimes (v^2 \leftharpoonup \hat{b})$, $(\hat{b} \cdot v)^1 \otimes (\hat{b} \cdot v)^2 = v^1 \otimes (\hat{b} \rightharpoonup v^2)$, $(\hat{b} \rightharpoonup b)^* = \hat{S}(\hat{b})^* \rightharpoonup b^*$, and $(b \leftharpoonup \hat{b})^* = b^* \leftharpoonup \hat{S}(\hat{b})^*$, for all $v \in H_U,\ b \in B$, and $\hat{b} \in \hat{B}$.

Finally, $\mathfrak{a}(1_A) = id_{\mathbf{1}} \otimes (- \otimes id_B)U^\varepsilon(1_B \otimes 1_B) = id_{\mathbf{1}} \otimes (- \otimes id_B)\Delta(1_B)$, so $\mathfrak{a}(1_A) \in id_{\mathbf{1}} \otimes \overline{B_s} \otimes B_t$.

(ii) By Lemma 3.4, the set $A^\mathfrak{a}$ of all fixed points is a unital $*$-subalgebra of A commuting with $\alpha(B_s)$. Moreover, the conditional expectation $T^\mathfrak{a} := (id_A \otimes h)\mathfrak{a}$ (where h is the normalized Haar measure of $\mathfrak{G}$) from A onto $A^\mathfrak{a}$ gives rise to a $A^\mathfrak{a}$-valued (pre)inner product for A defined by

$$< a, b >_T = T^\alpha(a^* b), \quad \text{for all} \quad a, b \in A.$$

Note that if $a = (X \otimes \overline{\xi}) \in A_U$, then $T^\mathfrak{a}(p(a)) = \Sigma_i(F(w_i^*)X \times \overline{w_i^*\xi})$, where w_i are isometries corresponding to the decomposition of U into irreducibles such that $\Sigma_i(w_i w_i^*)$ is the projection onto the component of ε. This implies the mutual orthogonality of the spaces $A_{U^x}\ \forall x \in \Omega$, but $1_A \in A_\varepsilon$ hence $T^\mathfrak{a}(A_{U^x}) = 0$ for all $x \neq \varepsilon$. The component $A_\varepsilon = End(M) \otimes B_s$ is a unital C^*-algebra and using (29), by restriction $\mathfrak{a}$ is a coaction of $\mathcal{G}$ on the C^*-algebra A_ε and $T^\mathfrak{a}(A_\varepsilon) \subset A_\varepsilon$, which implies that $A^\mathfrak{a} = T^\mathfrak{a}(A) = T^\mathfrak{a}(A_\varepsilon) \subset A_\varepsilon$ and by Lemma 3.4, $A^\mathfrak{a}$ is a unital C^*-subalgebra of A_ε. Therefore A is a right pre-Hilbert $A^\mathfrak{a}$-module.

The map T^α is completely positive, the C^*-algebra $A^\mathfrak{a}$ is unital, and the number of the components A_{U^x} is finite, so the multiplication on the left gives a faithful $*$-representation $A \to \mathcal{L}(A)$. One can extend $\mathfrak{a}$ to the C^*-completion $\overline{A}$ of A using the reasoning from the proof of [6], Proposition 4.4. The map V on $A \otimes B$ defined by $X(a \otimes b) = \mathfrak{a}(a)(1_A \otimes b)$, extends (due to the invariance of h) to a partial isometry on the right Hilbert $A^{\mathfrak{a}}$-module $A \otimes H_h$. The direct calculation shows that the formula $\overline{\mathfrak{a}} : a \mapsto V(a \otimes 1_B)V^*$ gives the needed extension of the coaction. □

6. Equivalence of categories

Definition 6.1. Let $(A, \mathfrak{a})$ be a unital $\mathfrak{G}$-C^*-algebra and A_ε be its spectral C^*-subalgebra corresponding to the trivial corepresentation ε. The **spectral functor** associated with $(A, \mathfrak{a})$ is a functor $F : UCorep(\mathfrak{G}) \to Corr(A_\varepsilon)$ defined as follows: for any $U \in UCorep(\mathfrak{G})$, put $F(U) = \{X \in H_U \otimes_{B_s} A | U_{13}X_{12} = (id_A \otimes \mathfrak{a})(X)\} = \{X = \Sigma_i(\xi_i \otimes_{B_s} a_i) | \mathfrak{a}(a_i) = \Sigma_j(a_j \otimes U_{ij}),\ \forall\, i\}$, where $\{\xi_i\}$ is an orthonormal basis in H_U. Then $F(\varepsilon) = A_\varepsilon$, all $F(U)$ are A_ε-bimodules, and A_ε-valued inner product of $X = \Sigma_i(\xi_i \otimes_{B_s} a_i)$, $Y = \Sigma_i(\xi_i \otimes_{B_s} b_i) \in F(U)$ defined by $< X, Y >:= \Sigma_i(a_i^* b_i)$, does not depend on the choice of $\{\xi_i\}$. Putting also $F(T) := T \otimes id$ for morphisms, we have a unitary functor respecting tensor products: if $X = \Sigma_i(\xi_i \otimes_{B_s} a_i) \in F(U)$, $Y = \Sigma_j(\eta_j \otimes_{B_s} b_j) \in F(V)$, $U, V \in UCorep(\mathfrak{G})$, then the maps $J_{U,V} : X \otimes Y \mapsto Y_{23}X_{13}$ are A_ε-bilinear isometries between $F(U) \otimes_{A_\varepsilon} F(V)$ and $F(U \odot V)$.

Remark 6.2. *1) The spectral functor (F, J) associated with a $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ is a weak unitary tensor functor. Indeed, properties (i) - (iv) are immediate, and (v) follows by observing that the adjoint of the map*

$$S_Y : F(U) \to F(U \odot V) : \quad X \to Y_{23}X_{13},$$

is given by $S_Y^*(Z) = Y_{23}^* Z$. Namely, if $Y = \Sigma_i(\eta_j \otimes_{B_s} a_j)$ and $Z = \Sigma_{i,j}(\xi_i \otimes_{B_s} \eta_j \otimes_{B_s} z_{i,j})$ for some orthonormal bases $\{\xi_i\} \in H_U$ and $\{\eta_j\} \in H_V$, then

$$S_Y^* Z = \Sigma_{i,j}(\xi_i \otimes_{B_s} a_j^* z_{i,j}) \in F(U). \tag{30}$$

2) The spectral subspaces A_U can be recovered from $F(U)$ using the canonical surjective maps

$$F(U) \otimes \overline{H}_U \to A_U,$$

which are isomorphisms for irreducible U.

Theorem 6.3. *Fix a regular coconnected finite quantum groupoid $\mathfrak{G}$ and a C^*-algebra C. By associating to a $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ its spectral functor, we get a bijection between isomorphism classes of triples $(A, \mathfrak{a}, \psi)$, where $\psi : C \to A$ is an embedding such that $A_\varepsilon = \psi(C)$, and natural unitary monoidal isomorphism classes of weak tensor functors $UCorep(\mathfrak{G}) \to Corr(C)$.*

Proof. Isomorphic $\mathfrak{G}$-C^*-algebras produce naturally unitarily monoidally isomorphic weak unitary tensor functors, and vice versa. It remains to show that up to some isomorphisms these constructions are mutually inverse.

Let $(A, \mathfrak{a})$ be a $\mathfrak{G}$-C^*-algebra with its spectral C^*-subalgebra A_ε corresponding to the trivial corepresentation of $\mathfrak{G}$, and let F be the associated spectral functor. As F is a weak unitary tensor functor, Lemmas 5.6 and 5.7 allow to construct a unital $\mathfrak{G}$-$*$-algebra $(A_F, \mathfrak{a}_F)$. One can check that linear maps sending $p(X \otimes \overline{\xi})$ to $(\overline{\xi} \otimes id)X \in A_U$, for any $(X \otimes \overline{\xi}) \in F(U) \otimes \overline{H}_U$ ($U \in UCorep(\mathfrak{G})$), define a unital $\mathfrak{G}$-equivariant homomorphism of algebras. In order to show that it is $*$-preserving, fix irreducibles U^x and an orthonormal basis $\{\xi_i\}$ in H_x $\forall x \in \hat{G}$. For an element $X = \Sigma_i(\xi_i \otimes_{B_s} a_i) \in F(U^x)$, we compute, using Lemma 5.5 and identity (30)

$$X^\bullet = S_X^* F(R_{U^x})(1_B) = S_X^*(\Sigma_j(\overline{\hat{G}^{-1/2}\xi_j} \otimes \xi_j \otimes 1_B)) = \Sigma_j(\overline{\hat{G}^{-1/2}\xi_j} \otimes a_j^*).$$

Then the image of the element $(X \otimes \overline{\xi})^* = p(X^\bullet \otimes \overline{\xi^\bullet}) = p(X^\bullet \otimes \overline{\overline{\hat{G}^{1/2}\xi}}) \in A_F$ equals to

$$\Sigma_j(\overline{\hat{G}^{1/2}\xi}, \overline{\hat{G}^{-1/2}\xi_j})a_j^* = (\Sigma_j(\xi_j, \xi)a_j)^*,$$

which shows that the homomorphism is $*$-preserving. Passing to the C^*-completion, we have the first part of the proof.

Conversely, let us start with a unitary weak tensor functor F, construct a unital $\mathfrak{G}$-C^*-algebra $(A_F, \mathfrak{a}_F)$, and consider the spectral functor F' associated with it. For any irreducible $U^x \in UCorep(\mathfrak{G})$, $x \in \Omega$, fix an orthonormal basis $\{\xi_i\} \in H_x$, then the space $F'(U_x)$ consists of vectors of the form $\Sigma_i(\xi_i \otimes X \otimes \overline{\xi}_i)$, where $X = F(U^x)$. The map $X \mapsto \Sigma_i(\xi_i \otimes X \otimes \overline{\xi}_i)$ from $F(U^x)$ to $F'(U^x)$ is clearly A_ε-bilinear, let us check that it is isometric. Taking $X' = \Sigma_i(\xi_i \otimes X \otimes \overline{\xi}_i)$, $Y' = \Sigma_i(\xi_i \otimes Y \otimes \overline{\xi}_i)$ in $F'(U^x)$, we compute

$$\begin{aligned} < X', Y' > &= \Sigma_i(X \otimes \overline{\xi}_i)^*(Y \otimes \overline{\xi}_i) \\ &= p(\Sigma_i(X^\bullet \otimes \overline{\overline{\hat{G}^{1/2}\xi_i}})(Y \otimes \overline{\xi}_i) = p(J(X^\bullet \otimes Y) \otimes \overline{R_{U^x}(1_B)}). \end{aligned}$$

Lemma 5.5 and the fact that the morphism $R_{U^x} : B_s \to \overline{U^x} \odot U^x$ is an isometry imply that the last expression equals to $< X, Y >$, so the isomorphisms $F(U^x) \cong F'(U^x)$ are unitary and extend uniquely to a natural unitary isomorphism of functors F and F'. Finally, one can check directly that this isomorphism is monoidal. □

Proposition 6.4. *Let $\mathfrak{G}$ be a regular coconnected finite quantum groupoid and $\mathcal{M}$ be a strict right $UCorep(\mathfrak{G})$-module C^*-category with generator M. If $(A, \mathfrak{a})$ is a unital $\mathfrak{G}$-C^*-algebra constructed by this data in Lemma 5.7, then the category $\mathcal{D}_A$ (see Definition 4.1) is unitarily equivalent, as a $UCorep(\mathfrak{G})$-module C^*-category, to $\mathcal{M}$, via an equivalence sending A to M.*

Proof. As we have seen, (F, J) is a weak tensor functor. Note that there are canonical isomorphisms of vector spaces

$$F(U) \cong \mathcal{D}_A(A, U \otimes_{B_s} A)$$

that map $\Sigma_i(\xi_i \otimes_{B_s} a_i) \in F(U)$ into the morphism $a \mapsto \Sigma_i(\xi_i \otimes_{B_s} a_i a)$. Therefore, the spectral functor is naturally unitarily monoidally isomorphic to the weak tensor functor $F' : UCorep(\mathfrak{G}) \to Corr(R)$ defined by $\mathcal{D}_A$ as in Proposition 5.3, where $R = End(A)$. If $\psi : F' \to F$ is such an isomorphism, then $\psi : A = F'(U^\varepsilon) \to F(U^\varepsilon) = A$ is the identity map since it is a bimodule map such that $\psi \circ J = J'(\psi \otimes \psi)$.

Let us now define a functor of linear categories $E : \tilde{D}_A \to \tilde{\mathcal{M}}$, where $\tilde{D}_A \subset D_A$ and $\tilde{\mathcal{M}} \subset \mathcal{M}$ are full subcategories consisting of objects $U \otimes_{B_s} A$ and $U \boxtimes M$, respectively. We put $E(U \otimes_{B_s} A) = U \boxtimes M$ on objects and $E(T) = \psi(T)$ on morphisms $T \in D_A(A, U \otimes_{B_s} A)$. More generally, if $T \in D_A(U \otimes_{B_s} A, V \otimes_{B_s} A)$, where $U, V \in UCorep(\mathfrak{G})$, then $(id_{\overline{U}} \otimes T)(R_U \otimes id_A) \in D_A(A, \overline{U} \odot V \otimes_{B_s} A)$ is Frobenius reciprocity isomorphism with inverse sending $S \in D_A(A, \overline{U} \odot V \otimes_{B_s} A)$ to $(\overline{R}_U^* \otimes id \otimes id)(id_U \otimes S)$. We can define similar isomorphisms in $\mathcal{M}$ and then define linear isomorphisms

$$E : \mathcal{D}_A(U \otimes_{B_s} A, V \otimes_{B_s} A) \to \mathcal{M}(U \otimes_{B_s} M, V \otimes_{B_s} M)$$

by $E(T) = (\overline{R}_U^* \otimes id \otimes id)[id_U \otimes \psi((id_{\overline{U}} \otimes T)(R_U \otimes id_A))]$.

Let us note that the naturality of ψ implies that if $T : U \otimes_{B_s} A \to V \otimes_{B_s} A$, $S : V \to W$, where $U, V, W \in UCorep(\mathfrak{G})$, then

$$E(id_W \otimes T) = id_W \otimes E(T) \quad \text{and} \quad E((S \otimes id)T) = (S \otimes id)E(T). \tag{31}$$

Consider now morphisms $Q : U \otimes_{B_s} A \to V \otimes_{B_s} A$ and $T : V \otimes_{B_s} A \to W \otimes_{B_s} A$, and define the morphisms $P = (id_{\overline{U}} \otimes Q)(R_U \otimes id_A) : A \to (\overline{U} \odot V) \otimes_{B_s} A$ and $S =$

$(id_{\overline{V}} \otimes T)(R_V \otimes id_A) : A \to (\overline{V} \odot W) \otimes_{B_s} A$, which give

$$\begin{aligned} TQ =&(\overline{R}_V^* \otimes id_W \otimes id_A)(id_V \otimes S)(\overline{R}_U^* \otimes id_V \otimes id_A)(id_U \otimes P) \\ =&(\overline{R}_V^* \otimes id_W \otimes id_A)(\overline{R}_U^* \otimes id_V \otimes id_{\overline{V}} \otimes id_W \otimes id_A)(id_U \otimes id_{\overline{U}} \otimes S)(id_U \otimes P) \\ =&(\overline{R}_U^* \otimes \overline{R}_V^* \otimes id_W \otimes id_A)(id_U \otimes J'(P \otimes S)), \end{aligned}$$

where $J'(P \otimes S) = (id_{\overline{U}} \otimes id_V \otimes S)P : A \to \overline{U} \otimes V \otimes \overline{V} \otimes W \otimes A$. A similar calculation gives

$$E(T)E(Q) = (\overline{R}_U^* \otimes \overline{R}_V^* \otimes id_W \otimes id_M)(id_U \otimes J(\psi(P) \otimes \psi(S))),$$

from where, using (31) and monoidality of ψ, we get $E(TQ) = E(T)E(Q)$, which means that $\psi J'(P \otimes S) = J(\psi(P) \otimes \psi(S))$. Therefore, E is a functor, and since it is surjective on objects and fully faithful, it is an equivalence of linear categories $\tilde{\mathcal{D}}_A$ and $\tilde{M}$.

Next, let us show that E is unitary, i.e., $E(T^*) = E(T)^*$ on morphisms. First, let $T : A \to U \otimes_{B_s} A$. Since ψ is unitary and $\psi|_A = id_A$, we have for any $S : A \to U \otimes_{B_s} A$:

$$\begin{aligned} E(T)^*E(S) = \psi(T)^*\psi(S) =< \psi(T), \psi(S) >=< T, S > \\ = T^*S = E(T^*S) = E(T^*)E(S). \end{aligned}$$

As S is arbitrary, this implies that $E(T^*) = E(T)^*$, and using (31), we also have $E((T \otimes id)^*) = E(T \otimes id)^*$. But any morphism in $\tilde{\mathcal{D}}_A$ is a composition of two morphisms: one of the above form $T \otimes id_V$ and another of the form $id_M \otimes S$ for some morphism S in $UCorep(\mathfrak{G})$. As a consequence of (31), we have $E(id_M \otimes S)^* = (id_M \otimes S)^* = E((id_M \otimes S)^*)$, it follows that E is unitary.

Further, if we define $J = J_{U \otimes A, V} : V \otimes_{B_s} E(U \otimes_{B_s} A) \to E((V \odot U) \otimes_{B_s} A)$ to be the identity maps, the relations (31) show that we get a natural isomorphism of bilinear functors $\cdot \otimes E(\cdot)$ and $E(\cdot \otimes \cdot)$. Therefore, (E, J) is a unitary equivalence of $UCorep(\mathfrak{G})$-C^*-module categories $\tilde{\mathcal{D}}_A$ and $\tilde{\mathcal{M}}$.

Finally, since $\mathcal{D}_A$ and $\mathcal{M}$ are completions of these categories with respect to subobjects, the equivalence between $\tilde{\mathcal{D}}_A$ and $\tilde{\mathcal{M}}$ extends uniquely, up to a natural unitary isomorphism, to a unitary equivalence between the $UCorep(\mathfrak{G})$-C^*-module categories $\mathcal{D}_A$ and $\mathcal{M}$. □

Now we are ready to prove Theorem 1.1.

Proof. Due to the previous proposition, it remains to show that two unital $\mathfrak{G}$-C^*-algebras, $(A_1, \mathfrak{a}_1)$ and $(A_2, \mathfrak{a}_2)$, are isomorphic if and only if the pairs $(\mathcal{D}_{A_1}, A_1)$ and $(\mathcal{D}_{A_2}, A_2)$ are unitarily equivalent.

First, given such equivalent pairs, we have the isomorphism of the corresponding spectral subalgebras $(A_1)_\varepsilon = End(M_1)$ and $(A_2)_\varepsilon = End(M_2)$. Identifying the above algebras via this isomorphism, we have a natural unitary monoidal isomorphism of the weak tensor functors constructed in Proposition 5.3 which implies a natural unitary monoidal isomorphism of the corresponding spectral functors. Now theorem 6.3 gives the needed isomorphism of unital $\mathfrak{G}$-C^*-algebras. Conversely, isomorphic unital $\mathfrak{G}$-C^*-algebras clearly produce unitarily equivalent classes of pairs of the form $(\mathcal{M}, M)$. □

Note that: (i) one can precise the definition of the equivalence of module functors between pairs $(\mathcal{M}, M)$ as in [6], Theorem 6.4; (ii) under the above equivalence, the unital C^*-algebra A_ε is isomorphic to $End_{\mathcal{M}}(M) \otimes B_s$.

Corollary 6.5. *Let $\mathcal{M}$ be a strict left module C^*-category over a strict rigid finite C^*-tensor category $\mathcal{C}$, M be a generator in $\mathcal{M}$, and denote by R the unital C^*-algebra $End(M)$. Then there exist a regular biconnected finite quantum groupoid $\mathfrak{G}$ (even with commutative base) and a unital $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ such that $\mathcal{C}$ is equivalent to*

$UCorep(\mathfrak{G})$ as C^*-tensor categories and $\mathcal{M}$ is equivalent to $\mathcal{D}_A$ as left $UCorep(\mathfrak{G})$-module C^*-categories via an equivalence that maps M to A.

Indeed, the existence of $\mathfrak{G}$ is guaranteed by Theorem 2.21, and the second statement – by Proposition 6.4.

Corollary 6.6. *If $\mathfrak{G}$ is regular and coconnected, then $A_\varepsilon = A^{\mathfrak{a}}\alpha(B_s)$.*

Indeed, we have seen that $A_\varepsilon = End_{\mathcal{M}}(M) \otimes B_s$ and that $A^{\mathfrak{a}} = End_{\mathcal{M}}(M)$.

Example 6.7. The C^*-algebra B with coproduct Δ viewed as $\mathfrak{G}$-C^*-algebra, corresponds to the $UCorep(\mathfrak{G})$-module C^*-category $\mathcal{C}orr_f(B_s)$ with generator $M = B_s$: for any element $U \in UCorep(\mathfrak{G})$ and $N \in Corr_f(B_s)$, one defines $U \boxtimes N := F(U) \otimes_{B_s} N$, where the functor $F : UCorep(\mathfrak{G}) \to Corr_f(B_s)$ ($F(U) = H_U$) is the forgetful functor. Indeed, if one identifies $\mathcal{M}(B_s, H_U)$ with H_U, we get an isomorphism of the algebra $\tilde{A}$ constructed from the pair $(\mathcal{M}, M)$ onto $\tilde{B} = \bigoplus_U (H_U \otimes \overline{H}_U)$ and then an isomorphism $A \cong B = \bigoplus_{x\in\hat{G}} (H_x \otimes \overline{H}_x)$ such that $p : \tilde{A} \to A$ turns into the map $\tilde{B} \to B$ sending $\xi \otimes \overline{\eta} \in H_U \otimes \overline{H}_U$ into the matrix coefficient $U_{\xi,\eta}$.

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