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ON BEHAVIOR AT INFINITY OF SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract. For a differential equation of the form $y''(t) - By(t) = 0,\ t \in (0, \infty)$, where B is a weakly positive linear operator in a Banach space $\mathfrak{B}$, the conditions on the operator B, under which this equation is uniformly or uniformly exponentially stable are given. As distinguished from earlier works dealing only with continuous at 0 solutions, in this paper no conditions on behavior of a solution near 0 are imposed.

1. On extensions of differentiable semigroups of linear operators in a Banach space

Let $\mathfrak{F}$ be a locally convex Hausdorff space. Recall (see [15]) that a one-parameter family $\{U(t)\}_{t\geq 0}$ of continuous linear operators from $\mathfrak{F}$ into $\mathfrak{F}$ forms a semigroup in $\mathfrak{F}$ if:

(i) $U(0) = I$ (I is the identity operator in $\mathfrak{F}$) ;

(ii) $\forall t, s > 0 : U(t + s) = U(t)U(s)$.

In the sequel we only consider semigroups that are strongly continuous at the point 0. A C_0-semigroup is called equicontinuous if for any continuous semi-norm $p(x)$ on $\mathfrak{F}$, there exists another continuous semi-norm $q(x)$ such that $p(U(t)x) \leq q(x)$ ($\forall t \geq 0,\ \forall x \in \mathfrak{F}$).The linear operator A defined as

$$Ax = \lim_{t\to 0} \frac{U(t)x - x}{t}, \quad \mathcal{D}(A) = \left\{ x \in \mathfrak{F} \,\middle|\, \exists \lim_{t\to 0} \frac{U(t)x - x}{t} \right\},$$

($\mathcal{D}(\cdot)$ denotes the domain of the operator) is called the generating operator or, simply, the generator of $\{U(t)\}_{t\geq 0}$. The fact that A is the generator of a semigroup $\{U(t)\}_{t\geq 0}$ is written as $U(t) = e^{tA}$.

As a rule we shall deal with C_0-semigroups in a Banach space $\mathfrak{B}$. For any such a semigroup $\{U(t)\}_{t\geq 0}$, the value

$$\omega_0 = \lim_{t\to\infty} \frac{\ln \|U(t)\|}{t}$$

is finite ($\|\cdot\|$ is the norm in $\mathfrak{B}$); it is called the type of $\{U(t)\}_{t\geq 0}$. The resolvent set of the operator A contains the half-plane $\mathrm{Re}\lambda > \omega_0$.

A C_0-semigroup $\{U(t) = e^{tA}\}_{t\geq 0}$ in $\mathfrak{B}$ is called (strongly) differentiable if for any $x \in \mathfrak{B}$, the $\mathfrak{B}$-valued function $U(t)x$ is strongly differentiable on $(0, \infty)$. As is known (see [12]), for such a semigroup

$$\forall x \in \mathfrak{B}, \forall t > 0 : U(t)x \in C^\infty(A) = \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n),$$

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the vector-valued function $U(t)x$ is infinitely differentiable on $(0,\infty)$, and

$$\forall x \in \mathfrak{B}, \forall t>0, \forall n \in \mathbb{N} : \frac{d^n U(t)x}{dt^n} = A^n U(t)x.$$

Let now $\theta \in \left(0, \frac{\pi}{2}\right]$. A C_0-semigroup $\{U(t)\}_{t\geq 0}$ on $\mathfrak{B}$ is called analytic with angle θ if the operator-valued function $U(\cdot)$ is defined in the sector $S_\theta = \{z : |\arg z| < \theta\}$ and possesses the following properties:

1) $\forall z_1, z_2 \in S_\theta : U(z_1+z_2) = U(z_1)U(z_2)$;
2) $\forall x \in \mathfrak{B} : U(z)x$ is analytic in S_θ;
3) $\forall x \in \mathfrak{B} : \|U(z)x - x\| \to 0$ as $z \to 0$ in any closed subsector of S_θ.

If in addition the family $U(z)$ is bounded on every sector S_ψ with $\psi < \theta$, then $U(t)$ is called a bounded analytic semigroup with angle θ.

For a number $\beta \geq 0$ we put

$$\mathfrak{A}_{\{\beta\}}(A) = \left\{ g \in C^\infty(A) \middle| \exists \alpha > 0,\ \exists c = c(g) > 0,\ \forall n \in \mathbb{N}_0 : \|A^n g\| \leq c\alpha^n n^{n\beta} \right\}$$

and

$$\mathfrak{A}_{(\beta)}(A) = \left\{ g \in C^\infty(A) \middle| \forall \alpha > 0,\ \exists c = c(g,\alpha) > 0,\ \forall n \in \mathbb{N}_0 : \|A^n g\| \leq c\alpha^n n^{n\beta} \right\}.$$

According to [5], elements of the spaces

$$\mathfrak{A}(A) = \mathfrak{A}_{\{1\}}(A), \quad \mathfrak{A}_c(A) = \mathfrak{A}_{(1)}(A), \quad \text{and} \quad \mathfrak{A}_e(A) = \mathfrak{A}_{\{0\}}(A)$$

are called analytic, entire, and entire of exponential type vectors of the operator A, respectively. If the C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ is analytic, then (see [5]) $\overline{\mathfrak{A}_c(A)} = \mathfrak{B}$, $\forall x \in \mathfrak{B}, \forall t > 0 : e^{tA}x \in \mathfrak{A}(A)$, and

$$\mathfrak{A}(A) = \bigcup_{t>0} e^{tA}\mathfrak{B}, \quad \mathfrak{A}_c(A) = \bigcap_{t\geq 0} e^{tA}\mathfrak{B}.$$

In what follows we may assume, without loss of generality, that A is the generator of a contraction C_0-semigroup in $\mathfrak{B}$ and $\ker e^{tA} = \{0\}$ as $t > 0$.

Let $\mathfrak{B}_{-t}(A), t > 0$, be the completion of $\mathfrak{B}$ in the norm

$$\|x\|_{\mathfrak{B}_{-t}(A)} = \|e^{tA}x\|.$$

Since the norms $\|\cdot\|_{\mathfrak{B}_{-t}(A)},\ t \in (0,\infty)$, are coordinated and comparable on $\mathfrak{B}$, we have for $t < t'$ the dense and continuous embedding $\mathfrak{B}_{-t}(A) \subseteq \mathfrak{B}_{-t'}(A)$. Set

$$\mathfrak{B}_-(A) = \operatorname*{proj\,lim}_{t\to 0} \mathfrak{B}_{-t}(A).$$

It should be noted that to obtain $\mathfrak{B}_-(A)$, it suffices to be restricted to the spaces $\mathfrak{B}_{-\frac{1}{n}}(A), n \in \mathbb{N}$. So, $\mathfrak{B}_-(A)$ is a complete countably normed space.

The operator e^{tA} admits a continuous extension $\widetilde{U}(t)$ to the space $\mathfrak{B}_{-t}(A)$. By virtue of continuity of the embedding $\mathfrak{B}_{-t}(A) \subseteq \mathfrak{B}_{-t'}(A)$ as $t < t'$, we have $\widetilde{U}(t') \restriction_{\mathfrak{B}_{-t}(A)} = \widetilde{U}(t)$.

On the space $\mathfrak{B}_-(A)$ we define the operators $U(t), t \geq 0$, in the following way:

(1) $$\forall x \in \mathfrak{B}_-(A) : U(t)x = \widetilde{U}(t)x \ \text{ if } \ t > 0; \quad U(0)x = x.$$

The following assertion was proved in [5].

Proposition 1. *The family $\{U(t)\}_{t\geq 0}$ forms an equicontinuous C_0-semigroup on the space $\mathfrak{B}_-(A)$, possessing the following properties:*

1) $U(t)\mathfrak{B}_-(A) \subseteq \mathfrak{B}$ as $t > 0$;
2) $\forall x \in \mathfrak{B} : U(t)x = e^{tA}x$;
3) $\forall x \in \mathfrak{B}_-(A),\ \forall t, s > 0 : U(t+s)x = e^{tA}U(s)x = e^{sA}U(t)x$.

Denote by $\widehat{A}$ the generator of the semigroup $\{U(t)\}_{t\geq 0}$.

Proposition 2. *If the semigroup $\{e^{tA}\}_{t\geq 0}$ is analytic on $(0,\infty)$, then $\mathfrak{B} \subset \mathfrak{B}_-(A)$ strongly, and the operator $\widehat{A}$ is continuous in $\mathfrak{B}_-(A)$. Moreover, for arbitrary fixed $x \in \mathfrak{B}_-(A)$ and $t > 0$, $U(t)x \in \mathfrak{A}(A)$ and the vector-valued function $U(t)x$ is analytic in $\mathfrak{A}(A)$ on $(0,\infty)$.*

2. On solutions of an abstract elliptic equation on $(0,\infty)$ in a Banach space

1. Denote by $E(\mathfrak{B}), R_B(\cdot)$ and $\rho(B)$ the set of all operators closed in $\mathfrak{B}$, the resolvent, and the resolvent set of the operator B, respectively, and consider the second-order equation

$$y''(t) - By(t) = 0, \quad t \in (0,\infty), \tag{2}$$

where B is a weakly positive operator in $\mathfrak{B}$, that is, $B \in E(\mathfrak{B})$, $\rho(B) \supset (-\infty, 0)$, and there exists a constant $M > 0$ such that

$$\forall \lambda > 0 : \|R_B(-\lambda)\| \leq \frac{M}{\lambda}.$$

If, in addition, $0 \in \rho(B)$, then the operator B is called positive.

As was shown in [10], for a weakly positive operator B, the powers $B^\alpha, 0 < \alpha < 1$, are defined, and $A = -B^{1/2}$ is a generating operator of a bounded analytic C_0-semigroup in $\mathfrak{B}$.

By a solution of equation (2) on $(0,\infty)$ we mean a twice continuously differentiable function $y(t) : (0,\infty) \mapsto \mathcal{D}(B)$ satisfying (2) on $(0,\infty)$.

Theorem 1. *Let B be a weakly positive operator in $\mathfrak{B}$. A function $y(t) : (0,\infty) \mapsto \mathcal{D}(B)$ is a solution of equation (2) on $(0,\infty)$ if and only if it admits a representation in the form*

$$y(t) = \exp(t\hat{A})f + \frac{\sinh(tA)}{A}g, \quad f \in \mathfrak{B}_-(A), \quad g \in \mathfrak{A}_c(A), \tag{3}$$

where $A = -B^{1/2}$,

$$\frac{\sinh(zA)}{A} = \int_0^z \coth(tA)\,dt = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}A^{2k},$$

$$\cosh(zA) = \frac{1}{2}[\exp(zA) + \exp(-zA)] = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}A^{2k}.$$

So, every solution of equation (2) on $(0,\infty)$ is an analytic on $(0,\infty)$ vector-valued function in the space $\mathfrak{A}(A)$.

Proof. Let $y(t)$ be a solution of (2) on $(0,\infty)$. In view of the relation $A^2 = B$, equation (2) may be rewritten as

$$\left(\frac{d}{dt} + A\right)\left(\frac{d}{dt} - A\right)y(t) = 0.$$

Put $z(t) = \left(\frac{d}{dt} - A\right)y(t)$. Then $z(t)$ is a solution of the equation

$$\frac{dz(t)}{dt} = -Az(t), \quad t \in (0,\infty),$$

with the operator $A = -(-A)$, generating a bounded analytic C_0-semigroup. As was shown in [4],

$$z(t) = \exp(-tA)g, \quad g \in \mathfrak{A}_c(A);$$

so the vector-valued function $y(t)$ satisfies the equation

$$\left(\frac{d}{dt} - A\right) y(t) = \exp(-tA)g$$

on $(0,\infty)$. Taking into account the equality

$$\int_0^z \exp((z-2s)A)\,ds = \frac{\sinh(zA)}{A},$$

we obtain

$$\begin{aligned} y(t) &= U(t)f + \int_0^t e^{(t-s)A} \exp(-sA)g\,ds \\ &= U(t)f + \int_0^t \exp((t-2s)A)g\,ds = e^{t\hat{A}}f + \frac{\sinh(tA)}{A}g, \end{aligned}$$

where $f \in \mathfrak{B}_-(A),\ g \in \mathfrak{A}_c(A)$.

It is easily verified directly that a vector-valued function of the form (3) is a solution of (2) on $(0,\infty)$. □

Corollary 1. *Each solution of equation* (2) *on* $(0,\infty)$ *has a boundary value in the space* $\mathfrak{B}_-(A)$ *as* $t \to 0$, *and it is an analytic on* $(0,\infty)$ *vector-valued function in the space* $\mathfrak{B}$. *In order that a solution admit an extension to an entire in* $\mathfrak{B}$ *vector-valued function, it is necessary and sufficient that* $y(0) \in \mathfrak{A}_c(A)$.

2. The homogeneous Dirichlet problem for equation (2) consists in finding its solution $y(t)$ satisfying the condition

$$y(t) \to 0 \quad \text{in the space} \quad \mathfrak{B}_-(A) \quad \text{as} \quad t \to 0. \tag{4}$$

Theorem 1 shows that in the case, where B is weakly positive, this problem has a lot of solutions. The solutions are of the form

$$y(t) = \frac{\sinh(tA)}{A}g, \quad g \in \mathfrak{A}_c(A). \tag{5}$$

It is reasonable to ask what conditions on behavior of a solution at infinity guarantee its uniqueness.

Consider at first the case where $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space and B is a self-adjoint operator in it. Denote by $E(\lambda)$ its spectral function.

Theorem 2. *Let B be a self-adjoint operator in $\mathfrak{H}$. Assume also that for a solution $y(t)$ of the homogeneous Dirichlet problem for equation* (2), *the following estimate is fulfilled:*

$$\forall \varepsilon > 0\ \exists c_\varepsilon > 0 : \|y(t)\| \le c_\varepsilon e^{\varepsilon t}. \tag{6}$$

Then $y(t) = tg,\ g \in \ker B$. *In particular, if* $\ker B = \{0\}$, *then* $y(t) \equiv 0$.

Proof. It follows from (6) that

$$e^{-2\varepsilon t}\|y(t)\|^2 = \int_0^\infty \frac{\sinh^2 \lambda t}{\lambda^2} e^{-2\varepsilon t}\, d(E(\lambda)g,g) \le c_\varepsilon^2,$$

whence

$$\int_0^\infty e^{2(\lambda-\varepsilon)t}\, d(E(\lambda)g,g) \le \text{ const}.$$

By the Fatou theorem $(E(\lambda)g,g) = \text{const}$ as $\lambda \ge \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(E(\lambda)g,g) = \text{const}$ on $(0,\infty)$. Thus, the function $(E(\lambda)g,g)$ may have a jump only in the point 0, that is, $Ag = 0$. Then (5) implies the equality $y(t) = tg$. □

The next theorems characterize in detail a vector g in representation (5) of a solution of the homogeneous Dirichlet problem for equation (2) depending on behavior of $y(t)$ as $t \to \infty$.

Theorem 3. *Let B be a self-adjoint operator in $\mathfrak{H}$. A vector g in* (5) *belongs to the space $\mathfrak{A}_e(A)$, that is, has an exponential type if and only if*

$$\exists c > 0, \exists \alpha > 0 : \|y(t)\| \le c e^{\alpha t} \tag{7}$$

for sufficiently large $t > 0$.

Proof. Suppose the solution (5) of problem (4) to be fulfilled (7). Then

$$e^{-2\alpha t}\|y(t)\|^2 = \int_0^\infty \left(\frac{\sinh \lambda t}{\lambda}\right)^2 e^{-2\alpha t}\, d(E(\lambda)g, g) = \int_0^{\alpha^2} + \int_{\alpha^2}^\infty e^{-2\alpha t}\, d(E(\lambda)g, g)$$
$$= \frac{1}{4}\int_0^{\alpha^2} + \frac{1}{4}\int_{\alpha^2}^\infty \frac{e^{2(\lambda-\alpha)t} + e^{-2(\lambda+\alpha)t} - 2}{\lambda^2}\, d(E(\lambda)g, g) \le c^2.$$

Passing to the limit as $t \to \infty$, we conclude on the basis of the Fatou theorem that the measure $d(E(\lambda)g, g)$ is concentrated only on the interval $[0, \alpha^2]$, that is, $g = E(\Delta)h,\ h \in \mathfrak{H},\ \Delta \subseteq [0, \alpha^2]$. Thus, g is an entire vector of exponential type for the operator A.

On the other hand, if $g = E(\Delta)h,\ h \in \mathfrak{H},\ \Delta \subset [0, \alpha)$ is a finite interval, then for sufficiently large $t > 0$,

$$\|y(t)\|^2 = \int_0^\infty \left(\frac{\sinh \lambda t}{\lambda}\right)^2 d(E(\lambda)g, g) = \int_0^\alpha \left(\frac{\sinh \lambda t}{\lambda}\right)^2 d(E(\lambda)h, h) \le c e^{2\lambda t}.$$

This completes the proof of the theorem. □

Theorem 3 shows that a solution of the homogeneous Dirichlet problem for equation (2) may have an exponential growth at infinity if and only if the vector g in representation (5) is entire of exponential type for the operator A. It is not difficult to prove that for an arbitrary monotonically increasing positive function $\gamma(t)$, there exists a solution $y(t)$ of the Dirichlet problem (2,4) on $(0, \infty)$ satisfying the inequality $\|y(t)\| > \gamma(t)$ when $t \le 1$. The solutions of finite order growth at infinity which type is higher than exponential one are described in the following way.

Let $\gamma(t) > 0$ be a continuous on $[0, \infty)$ function such that

$$\forall \lambda > 0 : G^2(\lambda) = \int_0^\infty \left(\frac{\sinh \lambda t}{\lambda}\right)^2 \gamma(t)\, dt < \infty. \tag{8}$$

This means that $\gamma(t)$ decreases at infinity faster than any exponential. Denote by Y_γ the set of all solutions $y(t)$ of problem (2,4) for which

$$\|y\|^2_{Y_\gamma} = \int_0^\infty \|y(t)\|^2 \gamma(t)\, dt < \infty.$$

The set Y_γ forms a Hilbert space with respect to the scalar product

$$(y, z)_{Y_\gamma} = \int_0^\infty (y(t), z(t))\gamma(t)\, dt.$$

Evidently, Y_γ contains all the solutions of (2,4), increasing exponentially at ∞.

Theorem 4. *Let B be a self-adjoint operator in $\mathfrak{H}$. The vector g in representation* (5) *of a solution $y(t)$ of the homogeneous Dirichlet problem for equation* (2) *belongs to the Hilbert space*

$$\mathfrak{H}_G(A) = \mathcal{D}\left(G(A)\right), \quad (f, x)_{\mathfrak{H}_G(A)} = \left(G(A)f, G(A)x\right),$$

constructed by the function $G(\lambda)$ from (8), *if and only if $y(t) \in Y_\gamma$. Moreover, formula* (5) *establishes an isometric isomorphism of the spaces Y_γ and $\mathfrak{H}_G(A)$*

Proof. The proof follows from the equalities

$$\begin{aligned}\|y\|^2_{Y_\gamma} &= \int_0^\infty \|y(t)\|^2\gamma(t)\,dt = \int_0^\infty \gamma(t)\left(\int_0^\infty \left(\frac{\sinh\lambda t}{\lambda}\right)^2 d(E(\lambda)g,g)\right)dt\\ &= \int_0^\infty \left(\int_0^\infty \left(\frac{\sinh\lambda t}{\lambda}\right)^2 \gamma(t)\,dt\right) d(E(\lambda)g,g)\\ &= \int_0^\infty G^2(\lambda)\,d(E(\lambda)g,g) = \|g\|^2_{\mathfrak{H}_G(A)}.\end{aligned}$$

In particular, if

$$\gamma(t) = \gamma_\mu(t) = e^{-2\mu t^p}, \quad p > 1, \quad \mu > 0,$$

then for

$$G_\mu(\lambda) = \int_0^\infty \left(\frac{\sinh\lambda t}{\lambda}\right)^2 \gamma_\mu(t)\,dt$$

the estimates

$$c_{\varepsilon,1}\exp\left(k_1\lambda^{p/2(p-1)}\left(\mu^{1/(1-p)}-\varepsilon\right)\right) \le G_\mu(\lambda) \le c_{\varepsilon,2}\exp\left(k_2\lambda^{p/2(p-1)}\left(\mu^{1/(1-p)}+\varepsilon\right)\right)$$

($c_{\varepsilon,i} > 0,\ k_i > 0$ are constants, $\varepsilon > 0$ is arbitrarily small) by means of which it is possible to find the conditions on the growth of $y(t)$ as $t \to \infty$, under which the vector g belongs to one of the classes $\mathfrak{A}_{\{\beta\}}(A)$ or $\mathfrak{A}_{(\beta)}(A)$ with $\beta < 1$, are fulfilled. □

Corollary 2. *In order that a solution $y(t)$ of problem $(2,4)$ admit a representation of the form* (5) *with $g \in \mathfrak{A}_{\{\beta\}}(A)$ $\left(\mathfrak{A}_{(\beta)}(A)\right)$ with $\beta < 1$, it is necessary and sufficient that there exist constants $c > 0$ and $\mu > 0$ ($\forall \mu > 0$ there exist $c > 0$) such that*

$$\|y(t)\| \le c\exp\left(\mu t^{1/(1-\beta)}\right)$$

(β and p are associated with each other by the relation $p = \frac{1}{1-\beta}$).

3. Consider now the homogeneous Neumann problem

$$\left\{\begin{array}{rcl} y''(t) &=& By(t),\ t \in (0,\infty),\\ \lim_{t\to+0} y'(t) &=& 0\end{array}\right. \tag{9}$$

(the limit is taken in the space $\mathfrak{B}_-(A)$).

There is a one-to-one correspondence between solutions of problems (9) and (2,4), namely, if $z(t)$ is a solution of the homogeneous Dirichlet problem, then $y(t) = z'(t)$ is a solution of the homogeneous Neumann problem, and conversely, if $y(t)$ is a solution of the homogeneous Neumann problem, then $z(t) = \int_0^t y(\xi)\,d\xi$ is a solution of the homogeneous Dirichlet problem. It follows from this that the vector-valued function $y(t)$ is a solution of problem (9) if and only if it is representable in the form

$$y(t) = \cosh(tA)x, \quad x \in \mathfrak{A}_c(A).$$

Using this representation we can obtain analogs of Theorems 2 - 4 for solutions of the homogeneous Neumann problem.

4. We now pass to the non-homogeneous Dirichlet problem for equation (2). It consists in finding a twice continuously differentiable on $(0,\infty)$ function $y(t) : (0,\infty) \mapsto \mathcal{D}(B)$ such that

$$\left\{\begin{array}{rcl} y''(t) &=& By(t),\ t \in (0,\infty),\\ \lim_{t\to+0} y'(t) &=& g \in \mathfrak{B}_-(A)\end{array}\right. \tag{10}$$

(the limit is understood in the $\mathfrak{B}_-(A)$-topology). Assuming that $y(t)$ is a solution of (10), we observe that the vector-valued function $y(t) - U(t)g$ is a solution of problem (2,4) and, therefore,

$$y(t) - U(t)g = \frac{\sinh(tA)}{A} f, \quad f \in \mathfrak{A}_c(A).$$

Taking into account the boundedness of $U(t)g$ and Theorems 1-4 we arrive at the following conclusion.

Theorem 5. *A vector-valued function $y(t)$ is a solution of the non-homogeneous Dirichlet problem* (10) *if and only if it can be represented in the form*

$$y(t) = U(t)g + \frac{\sinh(tA)}{A} f, \quad g \in \mathfrak{B}_-(A), \quad f \in \mathfrak{A}_c(A).$$

If $\ker A = \{0\}$*, then, under condition* (6)*, this problem is solvable uniquely.*

As has been shown in [13], if the operator B is weakly positive, then there exists a unique twice strongly differentiable on $[0, \infty)$ solution of the problem

$$\begin{cases} y''(t) &= By(t), \ t \in [0, \infty), \\ y(0) &= y_0 \in \mathcal{D}(A), \quad \sup_{t \ge 0} \|y(t)\| < \infty. \end{cases}$$

This result was made more precise in [8] in such a way: a function $y(t) \in C^2([0, \infty), \mathfrak{B})$ for which $y''(t) = By(t)$ (B is weakly positive) and $\|y(t)\| = o(t)$ $(t \to \infty)$, $y(0) = 0$, is identically equal to 0. This refinement admits the following generalization.

Theorem 6. *Let B be a weakly positive operator in $\mathfrak{B}$. Then a vector-valued function $y(t) \in C^2([0, \infty), \mathfrak{B})$, satisfying the conditions $y''(t) = By(t)$, $y(0) = 0$, $\|y(t)\| = o(t^n)$ $(t \to \infty)$ with some $n \in \mathbb{N}$, has the form $y(t) = tf$, $f \in \ker A$. In particular, if* $\ker A = \{0\}$*, then $y(t) \equiv 0$.*

Proof. Fix an arbitrary $a > 0$. Then (see [7]) there exist commuting with each other and A operators $F(t, a)$, which fulfill a role of $\frac{\sinh(a-t)A}{\sinh(aA)}$, such that the operator-valued function $F(t, a)$ is analytic on $(0, a]$ and strongly continuous on $[0, a]$ in t, $F(0, a) = I$ and $F(a, a) = 0$; the solution $y(t)$ of equation (2) on the interval $[0, a]$, satisfying the condition $y(0) = 0$, is represented in the form

$$y(t) = F(a - t, a)y(a), \quad t \in [0, a]. \tag{11}$$

Moreover,

$$\left\| F^{(n)}(t, a) \right\| \le \frac{b^{n+1} n^n (1 + c)^{n+2}}{t^n}, \quad n = 0, 1, \ldots,$$

where $b > 0$ does not depend on a and n. In consequence of the relation $\|y(a)\| = o(a^n)$, we have

$$\left\| y^{(n)}(t) \right\| \le \frac{c\|y(a)\| a^n}{(a - t)^n a^n} \to 0 \quad \text{as} \quad a \to \infty, \quad t \in [0, a),$$

whence $y^{(n)}(t) \equiv 0$.

So,

$$y(t) = f_0 + tf_1 + \cdots + t^{n-1} f_{n-1}, \quad f_i \in \mathcal{D}(B) \ (i = 0, \ldots, n - 1).$$

The latter follows from the possibility to present $y(t)$ as $y(t) = g_0 + (t - t_0)g_1 + \cdots + (t - t_0)^{n-1} g_{n-1}$ $(t_0 > 0)$, the closure of B and the commutation of B with $F(t, a)$ which cause the inclusions $g_i = \frac{y^{(i)}(t_0)}{i!} \in \mathcal{D}(B)$ and $f_k \in \mathcal{D}(B)$ as a linear combination of g_i.

Since $y(0) = 0$, we have $f_0 = 0$, that is,

$$y(t) = tf_1 + t^2 f_2 + \cdots + t^{n-1} f_{n-1}.$$

Substituting this expression for $y(t)$ into equation (2), we obtain the identity

$$2f_2 + 3\cdot 2tf_3 + \cdots + (n-1)(n-2)t^{n-3}f_{n-1} = tBf_1 + \cdots + t^{n-1}Bf_{n-1},$$

whence $f_{2i} = 0$ and, moreover, $f_{n-2} = f_{n-1} = 0$. Suppose for definiteness that n is even. Then

$$Bf_{n-3} = 0, \quad Bf_{n-5} = (n-3)(n-4)f_{n-3}, \ \dots \ Bf_1 = 6f_3.$$

Thus, the vector f_{n-3} is an eigenvector of the operator B with zero eigenvalue, the vectors $f_{n-5}, \dots, f_3, f_1$ are its root vectors, and

$$(B+\lambda I)^{-1}f_{n-(2k+1)} = \frac{f_{n-(2k+1)}}{\lambda} - \frac{(n-2k+1)(n-2k)f_{n-(2k-1)}}{\lambda^2}, \quad k = 2, 3, \dots, \frac{n}{2}.$$

In view of weak positivity of B, this equality is possible only for $f_{n-(2k-1)} = 0$. From this it follows that $y(t) = tf$. □

5. Note that Theorems 2 and 3 can be generalized to the case of a weakly positive B, but in this case the corresponding estimates imply the representation $y(t) = tx$, where $x \in \ker B$.

Theorem 7. *Let B be a positive operator in $\mathfrak{B}$, and let $y(t)$ be a solution of the homogeneous Dirichlet problem for equation* (2)*. Then the following equivalence relations are fulfilled for $\alpha \geq 1$:*

$$y'(0) \in \mathfrak{A}_{\{\beta\}}(A) \iff \exists a > 0, \exists c > 0 : \|y(t)\| \leq ce^{at^\alpha}, \quad t \in [0,\infty),$$

and

$$y'(0) \in \mathfrak{A}_{(\beta)}(A) \iff \forall a > 0, \exists c = c(a) > 0 : \|y(t)\| \leq ce^{at^\alpha}, \quad t \in [0,\infty),$$

where α is related to β by the equality $\beta = \frac{\alpha-1}{\alpha}$.

Proof. Suppose that

$$\forall a > 0, \exists c = c(a) > 0 : \|y(t)\| \leq ce^{at^\alpha}, \quad t \in [0,\infty).$$

Then for an arbitrary $a > 0$, there exists a constant $\widetilde{c}_a > 0$ such that the vector-valued function

$$z(t) = \exp(-tA)x_0, \quad x_0 = A^{-1}y'(0) \in \mathfrak{A}_c(A),$$

satisfies the inequality

$$\|z(t)\| \leq \widetilde{c}_a e^{at^\alpha}, \quad t > 0. \tag{12}$$

Taking into account that

$$z(t) = e^{(t_0-t)A}z(t_0), \quad t \in [0,t_0] \quad (t_0 \text{ is arbitrary fixed}),$$

and

$$\|A^n x_0\| \leq c^n n^n t_0^{-n}\|z(t_0)\|$$

with a certain constant $c > 0$ (see [14]) and setting $t_0 = \left(\frac{n}{a}\right)^{1/\alpha}$, we obtain

$$\|A^n x_0\| \leq \widetilde{c}_a c^n \left(a^{1/\alpha}e\right)^n n^{n\frac{\alpha-1}{\alpha}}.$$

Since a is arbitrary, $ca^{1/\alpha}e$ can be chosen arbitrarily, too. Therefore, $x_0 \in \mathfrak{A}_{(\beta)}(A)$ with $\beta = \frac{\alpha-1}{\alpha}$ and so, $y'(0) = Ax_0 \in \mathfrak{A}_{(\beta)}(A)$.

Conversely, let $y'(0) \in \mathfrak{A}_{(\beta)}(A)$ $(0 \leq \beta < 1)$. Then

$$\forall a > 0 \ \exists c_a > 0 : \|A^n y'(0)\| \leq c_a \left(\frac{a^\alpha}{e\alpha}\right)^n n^{n\beta}, \quad \alpha = \frac{1}{1-\beta}\ , \tag{13}$$

whence

$$\|\exp(-tA)y'(0)\| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \left\|A^k y'(0)\right\| \leq c_a \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{a^\alpha}{e\alpha}\right)^k k^{k\beta} \quad \text{as} \quad t > 0. \tag{14}$$

Consider the entire function

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(\frac{a^\alpha}{e\alpha}\right)^n n^{n\beta}$$

of order

$$\rho = \rho(\varphi) = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln(n!^{1/n}/\frac{a^\alpha}{e\alpha} n^\beta)} = \frac{1}{1-\beta} = \alpha.$$

Its type $\sigma = \sigma(\varphi)$ is determined from the equality

$$(e\sigma\alpha)^{1/\alpha} = \overline{\lim_{n\to\infty}} \left(n^{\frac{1}{\alpha}} \sqrt[n]{\frac{\left(\frac{a^\alpha}{e\alpha}\right)^n n^{n\beta}}{n!}} \right),$$

so, $\sigma = a$.

It follows from inequality (14) that

$$\forall \varepsilon > 0 \ \exists c_\varepsilon > 0 : \|\exp(-tA)y'(0)\| \leq c_\varepsilon e^{(a+\varepsilon)t^\alpha}.$$

Because of

$$y(t) = \frac{\exp(tA) - \exp(-tA)}{2} A^{-1} y'(0)$$

and $\left\|\exp(tA)A^{-1}y'(0)\right\| \leq c$ as $t > 0$, we have

$$\|y(t)\| \leq \widetilde{c_\varepsilon} e^{(a+\varepsilon)t^\alpha}.$$

The first assertion of this theorem is proved similarly. □

3. On behavior at infinity of stable solutions of elliptic differential equations in a Banach space

By a stable solution of equation (2) on $(0,\infty)$ we mean its solution bounded in a neighborhood of a point at infinity.

We say that equation (2) is:

1) *uniformly stable* if

$$\lim_{t\to\infty} y(t) = 0 \tag{15}$$

for any stable solution $y(t)$ of this equation;

2) *uniformly exponentially stable* if

$$\exists \omega > 0 : \lim_{t\to\infty} e^{\omega t} y(t) = 0 \tag{16}$$

for all stable solutions of (2).

If $\dim \mathfrak{B} < \infty$, both the definitions are equivalent. But this is, in general, not the case if $\dim \mathfrak{B} = \infty$.

Since no condition on behavior near 0 of a stable solution is imposed, it is possible for such a solution to have a singularity when approaching to 0, that is, $\lim_{t\to 0} y(t) = \infty$; moreover, the order of growth of $y(t)$ as $t \to 0$ may be arbitrary.

In the case where the non-homogeneous Dirichlet problem for equation (2) is well-posed (the corresponding homogeneous problem is uniquely solvable), by Theorem 6, it suffices in the definitions 1), 2) to require for equalities (15),(16) to be fulfilled at least for all analytic stable solutions. More exactly, the following theorem takes place.

Theorem 8. *Let B be a weakly positive operator in $\mathfrak{B}$ and $A = -B^{1/2}$. In order that equation* (2) *be uniformly* (*uniformly exponentially*) *stable, it is necessary and sufficient that equality* (15) (*equality* (16)) *hold true for all analytic on* $[0,\infty)$ *stable solutions of* (2).

Proof. As has been noted above, under the conditions of the theorem on the operator B, the set of all continuous at 0 stable solutions $y(t)$ of equation (2) is described by formula (3) where $f = y_0 = y(0)$ goes through the whole space $\mathfrak{B}$. If the semigroup $\{e^{tA}\}_{t\geq 0}$ is analytic and y_0 passes through the set of all analytic vectors of the operator A, that is, $y_0 \in \mathfrak{A}(A)$, then formula (3) gives all analytic on $[0,\infty)$ stable solutions of (2).

Let $y(t)$ be an arbitrary stable solution of (2) on $(0,\infty)$. Then, because of Proposition 1 and Theorem 1,

$$\exists y_0 \in \mathfrak{A}(A) : y(t) = U(t)y_0 = e^{(t-t_0)A}U(t_0)y_0, \quad t > t_0. \tag{17}$$

Since $U(t_0)y_0 \in \mathfrak{A}(A)$ and for an arbitrary fixed $t_0 > 0$, $t - t_0 \to \infty$ as $t \to \infty$, formula (17) implies that if relation (15) is fulfilled for all continuous at 0 stable solutions of equation (2), then $y(t) \to 0$ for any stable solution of this equation on $(0,\infty)$. The equality

$$e^{\omega t}\|y(t)\| = e^{\omega t_0}e^{\omega(t-t_0)}\|e^{(t-t_0)A}U(t_0)y_0\| \tag{18}$$

shows that if formula (16) is fulfilled for any continuous at 0 stable solution of (2), then it is valid for an arbitrary stable solution on $(0,\infty)$.

Suppose now that the semigroup $\{e^{tA}\}_{t\geq 0}$ is analytic. By Proposition 2,

$$\forall t > 0, \forall y_0 \in \mathfrak{B}_-(A) : e^{tA}y_0 \in \mathfrak{A}(A).$$

It follows from (17) and (18) that if relations (15) and (16) are fulfilled for all analytic on $[0,\infty)$ stable solutions of (2), then they are valid for any stable one on $(0,\infty)$. □

In accordance with [13], a C_0-semigroup $\{U(t)\}_{t\geq 0}$ on $\mathfrak{B}$ is:

(i) uniformly stable if

$$\forall x \in \mathfrak{B} : \lim_{t\to\infty} \|U(t)x\| = 0;$$

(ii) uniformly exponentially stable if

$$\exists M > 0, \exists \omega > 0, \forall t \geq 0 : \|U(t)\| \leq Me^{-\omega t}$$

(M and ω are constants).

As all continuous at 0 stable solutions $y(t)$ of equation (2) are described by formula (3) where y_0 runs through the whole $\mathfrak{B}$, Theorem 1 may be reformulated in terms of stability of a C_0-semigroup. Namely, the following assertion holds.

Corollary 3. *Let B be a weakly positive operator in $\mathfrak{B}$. Then for equation* (2) *to be uniformly* (*uniformly exponentially*) *stable, it is sufficient that the semigroup $\{e^{tA}\}_{t\geq 0}$ be uniformly* (*uniformly exponentially*) *stable. If the semigroup $\{e^{tA}\}_{t\geq 0}$ is analytic, it is sufficient in the relations*

$$\forall y_0 = y(0) \in \mathfrak{B} : e^{tA}y_0 \to 0 \quad \text{as} \quad t \to 0 \quad (e^{\omega t}e^{tA}y_0 \to 0 \quad \text{as} \quad t \to 0) \tag{19}$$

confine ourselves to $y_0 \in \mathfrak{A}(A)$.

Note also that a number of works of various mathematicians were devoted to searching uniform and uniform exponential stability criterions for C_0-semigroups (see, for instance, [13, 1, 14]). In what follows some new ones are given.

Denote by $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_c(\cdot), \sigma_r(\cdot)$ and $\rho(\cdot)$ the spectrum, the point, continuous, residual spectra and the resolvent set of an operator, respectively.

Theorem 9. *Let B be a weakly positive operator in $\mathfrak{B}$ and $A = -B^{1/2}$. In order that a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ be uniformly stable, it is necessary that $0 \in \sigma_c(A) \cup \rho(A)$. If the semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly exponentially stable, then $0 \in \rho(A)$. In order that $\{e^{tA}\}_{t\geq 0}$ be uniformly but not uniformly exponentially stable, it is necessary that $0 \in \sigma_c(A)$. In the case where $\{e^{tA}\}_{t\geq 0}$ is bounded analytic, all conditions mentioned above are sufficient, too.*

Proof. Let the semigroup $\{e^{tA}\}_{t\geq 0}$ be uniformly stable. Assume that $0 \in \sigma_p(A)$. Then there exists $x \in \mathcal{D}(A),\ x \neq 0$, such that $Ax = 0$. It follows from this that $\lim_{t\to\infty} e^{tA}x = x \neq 0$ contrary to the uniform stability of $\{e^{tA}\}_{t\geq 0}$.

Suppose now $0 \in \sigma_r(A)$. Then $\overline{\mathcal{R}(A)} \neq \mathfrak{B}$ ($\mathcal{R}(\cdot)$ is the range of an operator). This implies that

$$\exists f \in \mathfrak{B}^* \ (f \neq 0),\ \forall x \in \mathcal{D}(A) : f(Ax) = 0. \tag{20}$$

Consider the function $\varphi_x(t) = f(e^{tA}x)$. Since $e^{tA}\mathcal{D}(A) \subset \mathcal{D}(A)$ as $t > 0$, the function $\varphi_x(t)$ is continuously differentiable on $[0,\infty)$ and $\varphi_x'(t) = f(Ae^{tA}x) \equiv 0$. So $\varphi_x(t) = c_x = \mathit{const}$ on $[0,\infty)$. Because of $\varphi_x(0) = f(x) = \lim_{t\to 0} f(e^{tA}x) = 0$, we have $f(x) = 0$ for any $x \in \mathcal{D}(A)$. Taking into account that $\overline{\mathcal{D}(A)} = \mathfrak{B}$ and the continuity of f, we may conclude that $f = 0$ which contradicts to (20). Thus, in the case of uniform stability of $\{e^{tA}\}_{t\geq 0},\ 0 \in \sigma_c(A) \cup \rho(A)$.

Next, suppose $\{e^{At}\}_{t\geq 0}$ to be uniformly exponentially stable. Then $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\omega\} \subset \rho(A)$. As $\omega > 0$, we have $0 \in \rho(A)$. It follows from this that if $\{e^{tA}\}_{t\geq 0}$ is uniformly but not uniformly exponentially stable, then $0 \in \sigma_c(A)$.

Let now $\{e^{tA}\}_{t\geq 0}$ be bounded analytic and $0 \in \sigma_c(A) \cup \rho(A)$, hence, $\overline{\mathcal{R}(A)} = \mathfrak{B}$. Then for every $g \in \mathcal{R}(A)$, there exists $x \in \mathcal{D}(A)$ such that $g = Ax$. The boundedness and analyticity of $\{e^{tA}\}_{t\geq 0}$ imply the relation

$$\left\|e^{tA}g\right\| = \left\|e^{tA}Ax\right\| \leq \frac{c_x\|x\|}{t} \to 0 \quad \text{as} \quad t \to \infty \quad (0 < c_x = \mathit{const}).$$

Since $\overline{\mathcal{R}(A)} = \mathfrak{B}$, we make sure, on the basis of the principle of uniform boundedness (Banach-Steinhaus theorem), that $e^{tA}g \to 0$ for any $g \in \mathfrak{B}$, that is, the semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly stable. If $\{e^{tA}\}_{t\geq 0}$ is bounded analytic and $0 \in \rho(A)$, then the spectrum $\sigma(A)$ of the operator A lies in the sector $S(\varphi,\delta) = \{\lambda \in \mathbb{C} : |\arg(\lambda+\delta)| < \pi-\varphi\}$ with some $\delta > 0$ and $\varphi \in (0,\pi]$. For this reason, $S(A) = \sup_{\lambda\in\sigma(A)}\mathrm{Re}\lambda < 0$. Taking into account that, by virtue of analyticity of the semigroup $\{e^{tA}\}_{t\geq 0},\ S(A) = -\omega(A)$, we arrive at the conclusion that this semigroup is uniformly exponentially stable. This implies also that if a bounded analytic semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly but not uniformly exponentially stable, then $0 \in \sigma_c(A)$. □

Theorem 10. *Let $\{e^{tA}\}_{t\geq 0}$ be a C_0-semigroup on $\mathfrak{B}$, and $\gamma(t) > 0$ a continuous on $[0,\infty)$ function such that $\gamma(t) \to 0$ as $t \to \infty$. If*

$$\forall x \in \mathfrak{B}, \exists c = c(x) > 0 : \left\|e^{tA}x\right\| \leq c\gamma(t), \quad t \in [0,\infty), \tag{21}$$

then $\{e^{tA}\}_{t\geq 0}$ is uniformly exponentially stable. In the case where the semigroup $\{e^{tA}\}_{t\geq 0}$ is differentiable (*analytic*) *on $(0,\infty)$, it suffices for inequality* (21) *to be fulfilled at least for $x \in C^\infty(A)$ ($x \in \mathfrak{A}(A)$).*

Proof. Denote by $C_\gamma([0,\infty),\mathfrak{B})$ the Banach space of all continuous on $[0,\infty)$ vector-valued functions $y(t)$ for which

$$\|y\|_\gamma = \sup_{t\geq 0}\frac{\|y(t)\|}{\gamma(t)} < \infty.$$

The operator

$$C : \mathfrak{B} \mapsto C_\gamma([0,\infty),\mathfrak{B}), \quad Cx = e^{At}x,$$

admits a closure. Really, suppose $x_n \to 0$ in $\mathfrak{B}$ and $e^{tA}x_n \to y(t)$ in $C_\gamma([0,\infty),\mathfrak{B})$. As $e^{tA}x_n \to 0$ uniformly on each compact set from $[0,\infty)$, we have $y(t) \equiv 0$. Since the operator C is defined on the whole $\mathfrak{B}$ we make sure, in view of Closed Graph Theorem, that C is continuous. So,

$$\exists d > 0 : \left\|e^{tA}x\right\|_\gamma \le d\|x\|,$$

whence

$$\left\|e^{tA}x\right\| \le d\gamma(t).$$

Taking into account that

$$\omega_0 = \inf_{t>0} \frac{\ln\left\|e^{tA}\right\|}{t} > 0$$

(see [13]), we obtain

$$\left\|e^{tA}\right\| \le c_{\omega_0-\varepsilon}e^{-(\omega_0-\varepsilon)t}, \quad 0 < c_{\omega_0-\varepsilon} = const, \quad 0 < \varepsilon < \omega_0,$$

which means that the semigroup $\{e^{tA}\}_{t\ge0}$ (and so, equation (2)) is uniformly exponentially stable.

Assume now that the semigroup $\{e^{tA}\}_{t\ge0}$ is differentiable and inequality (21) holds true only for $x \in C^\infty(A)$. Fix $t_0 > 0$. By Proposition 2

$$\forall x \in \mathfrak{B} : g = e^{t_0A}x \in C^\infty(A).$$

So,

$$\forall t \ge t_0 : \left\|e^{tA}x\right\| = \left\|e^{(t-t_0)A}e^{t_0A}x\right\| \le c_g\gamma(t-t_0).$$

Putting

$$\gamma_1(t) = \begin{cases} \gamma(0) & \text{as} \quad 0 \le t \le t_0 \\ \gamma(t-t_0) & \text{as} \quad t > t_0 \end{cases}, \quad \widetilde{c_x} = \max\left\{\frac{1}{\gamma(0)}\max_{t\in[0,t_0]}\left\|e^{At}x\right\|, c_g\right\},$$

we obtain

$$\forall x \in \mathfrak{B}, \forall t \in [0,\infty) : \left\|e^{tA}x\right\| \le \widetilde{c_x}\gamma_1(t),$$

that is, $\{e^{tA}\}_{t\ge0}$ is uniformly exponentially stable.

In the case when $\{e^{tA}\}_{t\ge0}$ is analytic, the proof scheme is the same. □

Theorem 10 shows that if the semigroup $\{e^{tA}\}_{t\ge0}$ is uniformly but not uniformly exponentially stable, then its orbits $e^{tA}x$ may tend to 0 anyhow slowly when approaching to infinity. But it is impossible for such a semigroup to have an exponential decrease for all its orbits. Indeed, suppose that

$$\forall x \in \mathfrak{B},\ \exists c = c(x) > 0,\ \exists \omega_x > 0 : \left\|e^{tA}x\right\| \le ce^{-\omega_x t}.$$

Then

$$\forall x \in \mathfrak{B} : \left\|e^{tA}x\right\| \le c_1\frac{1}{1+t}, \quad 0 < c_1 = c\sup_{t\in[0,\infty)}\left\{(1+t)e^{-\omega_x t}\right\}.$$

Setting in Theorem 10 $\gamma(t) = \frac{1}{1+t}$, we conclude that $\{e^{tA}\}_{t\ge0}$ is uniformly exponentially stable contrary to the above assumption.

The next theorem gives one more criterion of uniform exponential stability.

Theorem 11. *Let $\{e^{tA}\}_{t\ge0}$ be a C_0-semigroup on $\mathfrak{B}$. If*

$$\forall x \in \mathfrak{B},\ \exists p_x > 0 : \int_0^\infty \left\|e^{tA}x\right\|^{p_x} dt < \infty, \tag{22}$$

then this semigroup is uniformly exponentially stable. If $\{e^{tA}\}_{t\ge0}$ is differentiable (*analytic*), *it is sufficient that inequality* (22) *be valid at least for infinitely differentiable* (*analytic*) *vectors of the operator A.*

Proof. Consider first the case where the semigroup $\{e^{tA}\}_{t\geq 0}$ is bounded: $\left\|e^{tA}\right\| \leq c = const,\ t \in (0,\infty)$. We may assume, without restriction of generality, that $c = 1$ because we can introduce in $\mathfrak{B}$ the equivalent to $\|\cdot\|$ norm

$$\|x\|_1 = \sup_{t\in[0,\infty)} \left\|e^{tA}x\right\|,$$

with respect to which $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup. Then $\left\|e^{tA}x\right\|$ does not increase for any $x \in \mathfrak{B}$ and, therefore, condition (22) implies

$$\forall x \in \mathfrak{B},\ \exists c_x > 0 : \left\|e^{tA}x\right\| \leq c_x(1+t)^{-\frac{1}{p_x}},$$

whence

$$\forall x \in \mathfrak{B},\ \exists \widetilde{c_x} > 0 : \left\|e^{tA}x\right\| \leq \widetilde{c_x}\frac{1}{\ln(2+t)},$$

where

$$\widetilde{c_x} = \sup_{t\in[0,\infty)} \frac{\ln(2+t)}{(1+t)^{\frac{1}{p_x}}} c_x.$$

By Theorem 10, the semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly exponentially stable.

Let now $\{e^{tA}\}_{t\geq 0}$ be not bounded on $[0,\infty)$. Since the growth of $\{e^{tA}\}_{t\geq 0}$ at infinity is not higher than exponential, we have

$$\exists \omega > 0,\ \exists c > 0 : \left\|e^{tA}\right\| \leq ce^{\omega t}.$$

Suppose that for some $x \in \mathfrak{B}$ relation (22) is fulfilled, but $\left\|e^{tA}x\right\|$ does not tend to 0 at infinity. Then there exists a sequence $t_i \to \infty$ such that $\left\|e^{t_iA}x\right\| \geq \delta$ with some $\delta > 0$. Choose this sequence so that $t_{i+1} - t_i > \omega^{-1}$. Then for $s \in \triangle_i = [t_i - \omega^{-1}, t_i]$, we obtain

$$\delta \leq \left\|e^{t_iA}x\right\| \leq \left\|e^{(t_i-s)A}\right\| \left\|e^{sA}x\right\| \leq ce^{\omega\omega^{-1}} \left\|e^{sA}x\right\| = ce \left\|e^{sA}x\right\|.$$

It follows from this that

$$\forall s \in \triangle_i : \left\|e^{sA}x\right\| \geq (ce)^{-1}\delta.$$

So,

$$\int_0^\infty \left\|e^{tA}x\right\|^{p_x} dt \geq \sum_{i\in\mathbb{N}} \int_{\triangle_i} \left\|e^{tA}x\right\|^{p_x} dt = \infty$$

contrary to (22). Thus, for an arbitrary $x \in \mathfrak{B}$, $\left\|e^{tA}x\right\|$ is nonincreasing and the investigation amounts to the considered above case of a bounded semigroup.

The latter assertion of the theorem follows from the identity

$$\int_{t_0}^\infty \left\|e^{tA}x\right\|^q dt = \int_{t_0}^\infty \left\|e^{(t-t_0A)}e^{t_0A}x\right\|^q dt = \int_0^\infty \left\|e^{\xi A}e^{t_0A}x\right\|^q d\xi$$

($t_0 > 0$ and $q > 0$ are arbitrary) and the fact that $e^{t_0A}x \in C^\infty(A)$ $(e^{t_0A}x \in \mathfrak{A}(A))$ if $\{e^{tA}\}_{t\geq 0}$ is differentiable (analytic). □

It should be noted that Theorem 11 is a generalization of the corresponding results of Datko [2], Pazy [11], M. Krein [9] where it was required the existence of one the same p in (22) for all $x \in \mathfrak{B}$. In Theorem 11 p may be different for different x. Moreover, if $\{e^{tA}\}_{t\geq 0}$ is infinitely differentiable (analytic), it is sufficient for (22) to be fulfilled at least for infinitely differentiable (analytic) vectors of the operator A.

The next assertion follows from the proof of Theorem 10.

Corollary 4. *Let $\gamma(t)$ be a continuous monotone nondecreasing function on $[0,\infty)$ such that $\gamma(t) \to \infty$ as $t \to \infty$. If for any solution $y(t)$ of the well-posed non-homogeneous Dirichlet problem* (10)

$$\exists c = c(y) : \|y(t)\| \leq c\gamma(t) \quad as \quad t \geq 1,$$

then equation (2) *is uniformly exponentially stable.*

Observe also that in the case where A is the generator of a uniformly exponentially stable C_0-semigroup, every solution $y(t)$ of equation (2) tends to 0 exponentially at infinity. Namely,

$$\forall a < -\omega_0 : \lim_{t\to\infty} y(t)e^{at} = 0.$$

As for uniformly but not uniformly exponentially stable semigroups, Theorem 10 shows that this is not the case. The question arises of finding a connection between the order of decrease to 0 of solutions $y(t)$ when approaching to ∞ and the properties of their initial data $y(0)$. Taking into account [6], we arrive, by virtue of Theorems 2 and 3, at the next assertion.

Theorem 12. *Let $A = -B^{1/2}$, where the operator B is weakly positive in $\mathfrak{B}$, and $0 \in \sigma_c(A)$. If $y(t)$ is a continuous at 0 solution of equation* (2)*, then the following equivalence relations take place:*

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n y(t) = 0 \Longleftrightarrow y(0) \in C^\infty(A^{-1});$$

$$\exists a > 0 : \lim_{t\to\infty} e^{a\sqrt{t}} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{A}(A^{-1});$$

$$\forall a > 0 : \lim_{t\to\infty} e^{a\sqrt{t}} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{A}_c(A^{-1}).$$

If $y(t)$ is exponentially decreasing at ∞, then

$$\exists a > 0 : \lim_{t\to\infty} e^{at} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{A}_e(A^{-1}),$$

where

$$\mathfrak{A}_e(A^{-1}) = \left\{x \in C^\infty(A^{-1}) \middle| \exists\alpha > 0, \exists c = c(x), \forall n \in \mathbb{N} : \|A^{-n}x\| \le c\alpha^n\right\}$$

is the space of entire vectors of exponential type for the operator A^{-1}.

If the semigroup $\{e^{tA}\}_{t\ge 0}$ is bounded analytic, then the operator A^{-1} generates an analytic semigroup, too (see [3]), and, as was shown there, $\overline{\mathfrak{A}_c(A^{-1})} = \mathfrak{B}$; moreover, the set of stable solutions of equation (2) behaving like $e^{-a\sqrt{t}}$ when $t \to \infty$, is dense in the set of all its stable solutions. As for the set of stable solutions decreasing at ∞ exponentially, it may consist only of the trivial one $y(t) \equiv 0$ even in the case where the analyticity angle of $\{e^{tA}\}_{t\ge 0}$ is equal to $\frac{\pi}{2}$. But if in the latter case

$$\int_0^1 \ln\ln M(s)\,ds < \infty, \quad M(s) = \sup_{\mathrm{Im}\lambda \ge s} \left\|A(A-\lambda I)^{-1}\right\|,$$

then $\overline{\mathfrak{A}_c(A^{-1})} = \mathfrak{B}$, and the set of stable solutions decreasing exponentially to 0 when approaching to ∞ is wide enough.

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