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ASYMPTOTIC PROPERTIES OF THE p-ADIC FRACTIONAL INTEGRATION OPERATOR

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To the blessed memory of M. L. Gorbachuk

Abstract. We study asymptotic properties of the p-adic version of a fractional integration operator introduced in the paper by A. N. Kochubei, Radial solutions of non-Archimedean pseudo-differential equations, *Pacif. J. Math.* **269** (2014), 355–369.

1. Introduction

1.1. In analysis of complex-valued functions on the field $\mathbb{Q}_p$ of p-adic numbers (or, more generally, on a non-Archimedean local field), the basic operator is Vladimirov's fractional differentiation operator D^α, $\alpha > 0$, defined via the Fourier transform or, for wider classes of functions, as a hypersingular integral operator [1, 6]. Properties of this p-adic pseudo-differential operator were studied by Vladimirov (see [6]) and found to be more complicated than those of its classical counterparts. For example, as an operator on $L^2(\mathbb{Q}_p)$, it has a point spectrum of infinite multiplicity. However, it was shown in [2] to behave much simpler on radial functions $x \to f(|x|_p)$.

In particular, in [2] the first author introduced a right inverse I^α to the operator D^α on radial functions, which can be seen as a p-adic analog of the Riemann-Liouville fractional integral of real analysis (including the case $\alpha = 1$ of the usual antiderivative). Just as the Riemann-Liouville fractional integral is a source of many problems of analysis, that must be true for the operator I^α.

In this paper we study asymptotic properties of the function $I^\alpha f$ for a given asymptotic expansion of f; for the asymptotic properties of Riemann-Liouville fractional integral see [3, 4, 7].

1.2. Let us recall the main definitions and notation used below.

Let p be a prime number. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \quad \text{if} \quad x = p^\nu \frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p. It is well known that $\mathbb{Q}_p$ is a locally compact topological field with the topology determined by the metric $|x - y|_p$, and that there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$. We will denote by dx the Haar measure on the additive group of $\mathbb{Q}_p$ normalized by the condition $\int_{|x|_p \le 1} dx = 1$.

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The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \quad \text{if and only if} \quad x = 0,$$
$$|xy|_p = |x|_p \cdot |y|_p,$$
$$|x+y|_p \le \max(|x|_p, |y|_p).$$

The latter property called the ultrametric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ and unusual geometric properties. Note also the following consequence of the ultrametric inequality:

$$|x+y|_p = \max(|x|_p, |y|_p) \quad \text{if} \quad |x|_p \ne |y|_p.$$

We will often use the integration formulas (see [1, 5, 6])

$$\int\limits_{|x|_p \le p^n} |x|_p^{\alpha-1} dx = \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha n}, \quad \text{here and below} \quad n \in \mathbb{Z}, \quad \alpha > 0,$$

in particular,

$$\int\limits_{|x|_p \le p^n} dx = p^n,$$
$$\int\limits_{|x|_p = p^n} dx = (1 - \frac{1}{p}) p^n,$$
$$\int\limits_{|x|_p = 1} |1-x|_p^{\alpha-1} = \frac{p-2+p^{-\alpha}}{p(1-p^{-\alpha})}.$$

See [1, 6] for further details of analysis of complex-valued functions on $\mathbb{Q}_p$.

From now on, we consider the case $\alpha > 1$. The integral operator I^α introduced in [2] has the form

$$(I^\alpha f(x)) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{|y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) f(y)\, dy, \tag{1}$$

where f is a locally integrable function on $\mathbb{Q}_p$. See [2] for its connection to the Vladimirov operator D^α and applications to non-Archimedean counterparts of ordinary differential equations. Note that our results can be generalized easily to the case of general non-Archimedean local fields.

2. Asymptotics at the origin

Let $0 < M_0 < M_1 < M_2 < \cdots$, $M_n \to \infty$. Then the sequence $f_n(x) = |x|_p^{M_n}$ is an asymptotic scale for $x \to 0$ (see, for example, §16 of [4] for the main notions regarding asymptotic expansions).

Theorem 1. *Suppose that a function f admits an asymptotic series expansion*

$$f \sim \sum_{n=0}^{\infty} a_n |x|_p^{M_n}, \quad |x|_p \to 0, \quad a_n \in \mathbb{C}.$$

Then

$$(I^\alpha f(x)) \sim \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \sum_{n=0}^{\infty} a_n b_n |x|_p^{M_n+\alpha}, \quad |x|_p \to 0, \tag{2}$$

where

$$b_n = \frac{p^{-\alpha+1}-1}{(1-p^{-\alpha})p} + (1-p^{-1}) \sum_{k=1}^{\infty} (1-p^{-k(\alpha-1)}) p^{-k(M_n+1)}.$$

Proof. We have

$$f=\sum_{n=0}^{N} a_n|x|_p^{M_n}+R_N(x), \quad R_N(x)=o(|x|_p^{M_N}), \quad |x|_p\to 0.$$

Then $I^\alpha f=I^\alpha_{(1)}+I^\alpha_{(2)}$,

$$I^\alpha_{(1)}=\frac{1-p^{-\alpha}}{1-p^{\alpha-1}}\int\limits_{|y|_p\le|x|_p}(|x-y|_p^{\alpha-1}-|y|_p^{\alpha-1})\Big(\sum_{n=0}^{N}a_n|y|_p^{M_n}\Big)\,dy,$$

$$I^\alpha_{(2)}=\frac{1-p^{-\alpha}}{1-p^{\alpha-1}}\int\limits_{|y|_p\le|x|_p}(|x-y|_p^{\alpha-1}-|y|_p^{\alpha-1})R_N(y)\,dy.$$

After the change of variables $y=sx$ we get

$$\begin{aligned}I^\alpha_{(1)}&=\frac{1-p^{-\alpha}}{1-p^{\alpha-1}}|x|_p^\alpha\int\limits_{|s|_p\le 1}(|1-s|_p^{\alpha-1}-|s|_p^{\alpha-1})\Big(\sum_{n=0}^{N}a_n|x|_p^{M_n}|s|_p^{M_n}\Big)ds\\&=\frac{1-p^{-\alpha}}{1-p^{\alpha-1}}|x|_p^\alpha(A+B),\end{aligned}$$

where

$$\begin{aligned}A=&\int\limits_{|s|_p<1}\left(1-|s|_p^{\alpha-1}\right)\left(\sum_{n=0}^{N}a_n|x|_p^{M_n}|s|_p^{M_n}\right)ds\\&=\sum_{n=0}^{N}a_n|x|_p^{M_n}\sum_{k=1}^{\infty}\left(1-p^{-k(\alpha-1)}\right)p^{-kM_n}\int\limits_{|s|_p=p^{-k}}ds\\&=(1-p^{-1})\sum_{n=0}^{N}a_n|x|_p^{M_n}\sum_{k=1}^{\infty}\left(1-p^{-k(\alpha-1)}\right)p^{-k(M_n+1)},\end{aligned}$$

$$B=\sum_{n=0}^{N}a_n|x|_p^{M_n}\int\limits_{|s|_p=1}\left(|1-s|_p^{\alpha-1}-1\right)ds=\frac{p^{-\alpha+1}-1}{(1-p^{-\alpha})p}\sum_{n=0}^{N}a_n|x|_p^{M_n}.$$

On the other hand, since $|R_N(x)|\le C|x|_p^{M_{N+1}}$, we find that for some constant $C_1>0$,

$$\left|I^\alpha_{(2)}\right|\le C_1|x|_p^{\alpha+M_{N+1}}\int\limits_{|s|_p\le 1}(|1-s|_p^{\alpha-1}-|s|_p^{\alpha-1})|s|_p^{M_{N+1}}ds=O(|x|_p^{\alpha+M_{N+1}}).$$

The above calculations result in the asymptotic relation (2). □

3. Asymptotics at infinity

For positive functions φ,ψ, we write $\varphi(x)\asymp\psi(x)$, $|x|_p\to\infty$, if $c\psi(x)\le\varphi(x)\le d\psi(x)$, for large values of $|x|_p$, $x\in\mathbb{Q}_p$, for some positive constants c,d.

Theorem 2. *Suppose that $a\le f(x)\le b$ $(a,b>0)$ for $|x|_p<1$, $|f(x)|\le C|x|_p^{-M}$, $M>1$, $C>0$, for $|x|_p\ge 1$. Then*

$$(I^\alpha f)(x)\asymp|x|_p^{\alpha-1}, \quad |x|_p\to\infty. \tag{3}$$

Proof. Let us rewrite (1) with $|x|_p \geq 1$ in the form $I^\alpha f = J^\alpha_{(1)} f + J^\alpha_{(2)} f$, where

$$\left(J^\alpha_{(1)} f\right)(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{|y|_p<1} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) f(y)\,dy,$$

$$\left(J^\alpha_{(2)} f\right)(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{1\leq|y|_p\leq|x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) f(y)\,dy.$$

Then

$$\left(J^\alpha_{(1)} f\right)(x) \asymp \int\limits_{|y|_p<1} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right)\,dy \asymp |x|_p^{\alpha-1}.$$

Next, if $|x|_p = p^N$, $N \geq 0$, then

$$\left|\left(J^\alpha_{(2)} f\right)(x)\right| \leq C \int\limits_{1\leq|y|_p\leq|x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right)|y|_p^{-M} dy$$

$$= C\left\{\sum_{j=0}^{N-1} \int\limits_{|y|_p=p^j} \left(|x|_p^{\alpha-1} - |y|_p^{\alpha-1}\right)|y|_p^{-M} dy\right.$$

$$\left. + \int\limits_{|y|_p=p^N} \left(|x-y|_p^{\alpha-1} - p^{N(\alpha-1)}\right) p^{-MN} dy\right\}$$

$$= C\left\{(1-\frac{1}{p})\sum_{j=0}^{N-1} p^j \left(p^{N(\alpha-1)} - p^{j(\alpha-1)}\right) p^{-Mj}\right.$$

$$\left. + p^{-MN} \int\limits_{|y|_p=p^N} |x-y|_p^{\alpha-1} dy - (1-\frac{1}{p}) p^{\alpha N - MN}\right\}.$$

Calculating the integral as above and finding the sums of geometric progressions we see that $\left|\left(J^\alpha_{(2)} f\right)(x)\right| \leq \text{const}\cdot|x|_p^{\alpha-1}$, which proves (3). □

4. Logarithmic asymptotics

If a function f decays slower than it did under the assumptions of Theorem 2, then a richer asymptotic behavior is possible. Let us consider the case where $f(t) \geq 0$,

$$f(x) \sim |x|_p^{-\beta} \sum_{n=0}^{\infty} a_n (\log|x|_p)^{\gamma-n}, \quad |x|_p \to \infty, \tag{4}$$

where $0 \leq \beta < 1$, $\gamma \geq 0$, $a_n \in \mathbb{R}$.

First we need some auxiliary results.

Lemma 1. *Let* $0 \leq f(x) = o\left(|x|_p^{-\lambda}\right)$, $|x|_p \to \infty$, *where* $0 < \lambda < 1$. *Then*

$$G_1(r) \overset{def}{=} \int\limits_{|y|_p\leq r} f(y)\,dy = o(r^{1-\lambda}), \quad r \to \infty. \tag{5}$$

Proof. Let $n_0 = [\log_p r]$. Then $p^{n_0} \leq r \leq p^{n_0+1}$. It is known (see Section 1) that

$$\int\limits_{|y|_p\leq p^\nu} |y|_p^{-\lambda} dy = \frac{1-p^{-1}}{1-p^{\lambda-1}} p^{(1-\lambda)\nu}, \quad \nu \in \mathbb{Z}, \tag{6}$$

so that

$$G_2(r) \overset{\text{def}}{=} \int\limits_{|y|_p \le r} |y|_p^{-\lambda}\, dy = O(r^{1-\lambda}), \quad r \to \infty. \tag{7}$$

By our assumption, for any $n \in \mathbb{N}$, there exists such $r_0 = r_0(n)$ that $f(x) < \frac{1}{n}|x|_p^{-\lambda}$ for $|x|_p > r_0$. Then we can write

$$\frac{G_1(r)}{G_2(r)} = \frac{G_1(r_0(n)) + (G_1(r) - G_1(r_0(n)))}{G_2(r_0(n)) + (G_2(r) - G_2(r_0(n)))} \le \frac{G_1(r_0(n)) + \frac{1}{n}G_3(n,r)}{G_2(r_0(n)) + G_3(n,r)},$$

where

$$G_3(n,r) = \int\limits_{r_0 \le |y|_p \le r} |y|_p^{-\lambda}\, dy.$$

It follows from (6) that $G_3(n,r) \to \infty$, so that

$$0 \le \limsup_{r\to\infty} \frac{G_1(r)}{G_2(r)} \le \frac{1}{n},$$

where n is arbitrary. Therefore

$$\lim_{r\to\infty} \frac{G_1(r)}{G_2(r)} = 0,$$

which gives, together with (7), the required asymptotic relation (5). $\square$

Lemma 2. *Let $0 \le \beta < 1$, $k \in \mathbb{N}$. For any $\varepsilon > 0$, such that $\beta + \varepsilon < 1$,*

$$K_r \overset{def}{=} \int\limits_{|t|_p \le r^{-1}} (|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1})|t|_p^{-\beta}|\log|t|_p|^k dt = O\big(r^{\beta+\varepsilon-1}\big), \quad r \to \infty. \tag{8}$$

Proof. Assuming that $r > 2$, we have $|t|_p < \frac{1}{2}$, so that $|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1} = 1 - |t|_p^{\alpha-1} \le 1$, and we find that

$$K_r \le \int\limits_{|t|_p \le r^{-1}} |t|_p^{-\beta}|\log|t|_p|^k dt \le \int\limits_{|t|_p \le r^{-1}} |t|_p^{-\beta-\varepsilon} dt,$$

if r is large enough, and the relation (8) follows from the integration formula (6). $\square$

Now we are ready to consider the asymptotics of $I^\alpha f$ for a function f satisfying (4). Below we use the notation

$$\binom{\gamma}{n} = \frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!}$$

for any real positive number γ and $n \in \mathbb{N}$.

Theorem 3. *If a function $f \ge 0$ satisfies the asymptotic relation (4), then*

$$(I^\alpha f)(x) \sim \frac{1-p^{-\alpha}}{1-p^{\alpha-1}}|x|_p^{\alpha-\beta} \sum_{n=0}^{\infty} B_n (\log|x|_p)^{\gamma-n}, \quad |x|_p \to \infty, \tag{9}$$

where

$$B_n = \sum_{k=0}^{n} a_{n-k}\binom{\gamma+k-n}{k}\Omega(k,\alpha,\beta),$$

$$\Omega(k,\alpha,\beta) = \int\limits_{|t|_p \le 1} (|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1})|t|_p^{-\beta}(\log|t|_p)^k dt.$$

Proof. Let us write $(I^\alpha f)(x)$ for $|x|_p \ge 1$ as the sum of two integrals I_1 and I_2, with the integration over $\{y:\ |y|_p < |x|_p^{1/2}\}$ and $\{y:\ |x|_p^{1/2} \le |y|_p \le |x|_p\}$ respectively.

Denote $\mathcal{K}(x,y) = |x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}$. Considering I_1, for $|y|_p \le |x|_p$, we have

$$|\mathcal{K}(x,y)| \le |x|_p^{\alpha-1}. \tag{10}$$

Indeed, if $|x|_p > 1$, then $|y|_p < |x|_p$, $\mathcal{K}(x,y) = |x|_p^{\alpha-1} - |y|_p^{\alpha-1}$, and we get (10). If $|x|_p = 1$, $|y|_p < 1$, then $0 < \mathcal{K}(x,y) = 1 - |y|_p^{\alpha-1} < |x|_p^{\alpha-1}$.

It follows from (10) that

$$0 \le I_1 \le C|x|_p^{\alpha-1} \int\limits_{|y|_p < |x|_p^{1/2}} f(y)\,dy,$$

and by (4) and Lemma 1, for any small $\varepsilon > 0$,

$$I_1 = o\left(|x|_p^{\alpha-\beta+\frac{\beta+\varepsilon-1}{2}}\right), \quad |x|_p \to \infty. \tag{11}$$

Considering I_2 we write

$$f(t) = |t|_p^{-\beta} \sum_{n=0}^{N} a_n (\log|t|_p)^{\gamma-n} + R_N(t), \quad R_N(t) = O(|t|_p^{-\beta}(\log|t|_p)^{\gamma-N-1}), \quad |t|_p \to \infty.$$

Denote

$$\begin{aligned} L(\alpha,\beta,\gamma,x) &= \int\limits_{|x|_p^{1/2} \le |y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) |y|_p^{-\beta} (\log|y|_p)^\gamma \, dy \\ &= |x|_p^{\alpha-\beta} (\log|x|_p)^\gamma \int\limits_{|x|_p^{-1/2} \le |t|_p \le 1} \left(|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1}\right) |t|_p^{-\beta} \left(1 + \frac{\log|t|_p}{\log|x|_p}\right)^\gamma dt, \end{aligned}$$

where on the domain of integration,

$$\left|\frac{\log|t|_p}{\log|x|_p}\right| \le \frac{1}{2},$$

and we may write, for a non-integer γ, the convergent binomial series

$$\left(1 + \frac{\log|t|_p}{\log|x|_p}\right)^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} \left(\frac{\log|t|_p}{\log|x|_p}\right)^k.$$

Note that we can use the Taylor formula with the integral form of the remainder

$$(1+s)^\gamma = \sum_{k=0}^{N} \binom{\gamma}{k} s^k + \frac{\gamma(\gamma-1)\cdots(\gamma-N)}{N!} \int\limits_0^s (1+\sigma)^{\gamma-N-1} (s-\sigma)^N \, d\sigma,$$

where

$$\begin{aligned} \int\limits_0^s (1+\sigma)^{\gamma-N-1}(s-\sigma)^N \, d\sigma &= s^{N+1} \int\limits_0^1 (1+s\tau)^{\gamma-N-1} (1-\tau)^N \, d\tau \\ &= s^{N+1} \int\limits_0^1 (1+s(1-\tau))^{\gamma-N-1} \tau^N \, d\tau. \end{aligned}$$

If $-\frac{1}{2} < s < \frac{1}{2}$, $0 < \tau < 1$, then $\frac{1}{2} \le 1 + s(1-\tau) \le \frac{3}{2}$. Therefore

$$\left(1 + \frac{\log|t|_p}{\log|x|_p}\right)^\gamma = \sum_{k=0}^{N} \binom{\gamma}{k} \left(\frac{\log|t|_p}{\log|x|_p}\right)^k + S_N(t,x),$$

$$S_N(t,x) = O\left(\left(\frac{\log|t|_p}{\log|x|_p}\right)^{N+1}\right), \quad |x|_p \to \infty,$$

and this asymptotics is uniform with respect to $t, |t|_p \in [|x|_p^{-1/2}, 1]$.

Substituting and using Lemma 2 we obtain the expansion

$$(12) \qquad L(\alpha,\beta,\gamma,x) = |x|_p^{\alpha-\beta} \sum_{k=0}^{N} \binom{\gamma}{k} \Omega(k,\alpha,\beta)(\log|x|_p)^{\gamma-k} + o(|x|_p^{\alpha-\beta}(\log|x|_p)^{\gamma-N}), \quad |x|_p \to \infty.$$

We have

$$I_2 = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \sum_{n=0}^{N} a_n L(\alpha,\beta,\gamma-n,x) + \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{|x|_p^{1/2} \le |y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) R_N(y)\,dy,$$

where

$$\int\limits_{|x|_p^{1/2} \le |y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}\right) R_N(y)\,dy \le CL(\alpha,\beta,\gamma-N-1,x) = O\left(|x|_p^{\alpha-\beta}(\log|x|_p)^{\gamma-N-1}\right), \quad |x|_p \to \infty.$$

The last estimate is a consequence of (12).

Now the asymptotic relations (11) and (12) imply the required relation (9). □

In our final result, we give a modification of Theorem 3 for the case where $\beta = 1$.

Theorem 4. *Suppose that f is nonnegative,*

$$f(x) \sim |x|_p^{-1} \sum_{n=0}^{\infty} a_n (\log|x|_p)^{\gamma-n}, \quad |x|_p \to \infty.$$

Then

$$(13) \qquad (I^\alpha f)(x) \sim \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \Big[|x|_p^{\alpha-1} \int\limits_{|y|_p \le |x|_p} f(y)dy + \sum_{n=0}^{\infty} \widetilde{B}_n (\log|x|_p)^{\gamma-n}\Big], \quad |x|_p \to \infty,$$

where

$$\widetilde{B}_n = \sum_{k=0}^{n} a_{n-k} \binom{\gamma+k-n}{k} \widetilde{\Omega}(k,\alpha),$$

$$\widetilde{\Omega}(k,\alpha) = \int\limits_{|t|_p \le 1} \left(|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1\right) |t|_p^{-1} (\log|t|_p)^k\,dt.$$

Proof. Let us write $I^\alpha f = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}}(J_1 + J_2 + J_3)$, where

$$J_1 = \int\limits_{|y|_p \le |x|_p^{1/2}} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}\right) f(y)\,dy,$$

$$J_2 = \int\limits_{|x|_p^{1/2} \le |y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}\right) f(y)\,dy,$$

$$J_3 = |x|_p^{\alpha-1} \int\limits_{|y|_p \le |x|_p} f(y)\,dy.$$

Choosing $\varepsilon > 0$, such that $1+\varepsilon < \alpha$, we see that $f(x) = o\left(|x|_p^{-1+\varepsilon}\right)$, $|x|_p \to \infty$. By Lemma 1,

$$\int\limits_{|y|_p \le |x|_p^{1/2}} f(y)\,dy = o\big(|x|_p^{\frac{\varepsilon}{2}}\big), \quad |x|_p \to \infty.$$

For the kernel of the above integral operator we get, considering various cases, the estimate

$$\big||x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}\big| \le 2|y|_p^{\alpha-1}.$$

It follows from Lemma 1 that

$$|J_1| \le 2 \int\limits_{|y|_p \le |x|_p^{1/2}} |y|_p^{\alpha-1} f(y)\,dy = o\big(|x|_p^{\frac{\alpha-1+\varepsilon}{2}}\big), \quad |x|_p \to \infty. \tag{14}$$

By our assumption,

$$f(t) = |t|_p^{-1} \sum_{n=0}^{N} a_n (\log|t|_p)^{\gamma-n} + R_N(t), \quad R_N(t) = O\big(|t|_p^{-1}(\log|t|_p)^{\gamma-N-1}\big), \quad |t|_p \to \infty.$$

Let us consider the expression

$$\widetilde{L}(\alpha,\gamma,x) = \int\limits_{|x|_p^{1/2} \le |y|_p \le |x|_p} \left(|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}\right) |y|_p^{-1} (\log|y|_p)^{\gamma} dy$$

$$= |x|_p^{\alpha-1} \int\limits_{|x|_p^{-1/2} \le |t|_p \le 1} (|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1)|t|_p^{-1}(\log|x|_p + \log|t|_p)^{\gamma}\,dt.$$

It follows from the first integration formula from Section 1 that

$$\int\limits_{|t|_p \le |x|_p^{-1/2}} (|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1)|t|_p^{-1}(\log|t|_p)^k\,dt = o\big(|x|_p^{\frac{1-\alpha+\varepsilon}{2}}\big), \quad |x|_p \to \infty.$$

This implies (just as in the proof of Theorem 3) the expansion

$$\widetilde{L}(\alpha,\gamma,x) \sim |x|_p^{\alpha-1} \sum_{k=0}^{\infty} \binom{\gamma}{k} (\log|x|_p)^{\gamma-k} \widetilde{\Omega}(k,\alpha), \quad |x|_p \to \infty.$$

Taking into account (14), we come to (13). □

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