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INITIAL-BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL PARABOLIC EQUATIONS IN HÖRMANDER SPACES

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To the memory of Professor M. L. Gorbachuk

Abstract. We investigate a general nonhomogeneous initial-boundary value problem for a two-dimensional parabolic equation in some anisotropic Hörmander inner product spaces. We prove that the operators corresponding to this problem are isomorphisms between appropriate Hörmander spaces.

1. Introduction

The modern theory of general parabolic initial-boundary problems has been developed for the classical scales of Hölder–Zygmund and Sobolev function spaces [1, 5, 6, 8, 11, 15, 17, 26, 46]. The central result of this theory states that these problems are well posed in the sense of Hadamard in appropriate pairs of the function spaces belonging to these scales.

In 1963 Hörmander [9] proposed a broad generalization of Sobolev spaces in the framework of Hilbert spaces. He introduced the spaces

$$\mathcal{B}_{2,\mu} := \bigl\{ w \in \mathcal{S}'(\mathbb{R}^k) : \mu(\xi)\widehat{w}(\xi) \in L_2(\mathbb{R}^k,\, d\xi) \bigr\},$$

for which a general Borel measurable weight function $\mu : \mathbb{R}^k \to (0,\infty)$ serves as an index of regularity of a distribution w. These spaces and their versions within the category of normed spaces have found various applications to analysis and partial differential equations [7, 12, 18, 33, 34, 36, 37, 44, 45].

Recently Mikhailets and Murach [27–29, 31, 35] have built a theory of solvability of general elliptic systems and elliptic boundary-value problems on Hilbert scales of spaces $H^{s;\varphi} := \mathcal{B}_{2,\mu}$ for which the index of regularity is of the form

$$\mu(\xi) := (1 + |\xi|^2)^{s/2}\varphi((1 + |\xi|^2)^{1/2}).$$

Here, s is a real number, and φ is a function varying slowly at infinity in the sense of Karamata [13]. This theory is based on the method of interpolation with a function parameter between Hilbert spaces, specifically between Sobolev spaces.

Generally, the method of interpolation between normed spaces proved to be very useful in the theory of elliptic [2, 16, 43] and parabolic [17, 26] partial differential equations. Using the method of interpolation with a function parameter between Hilbert spaces, Los, Mikhailets, and Murach proved theorems on solvability of parabolic problems in $2b$-anisotropic Hörmander spaces $H^{s,s/(2b);\varphi}$, where $2b$ is a parabolic weight and where the parameters s and φ are the same as those in the above mentioned elliptic theory. They considered general parabolic problems with homogeneous initial conditions (Cauchy data) [23, 24] and parabolic problems of second order with nonhomogeneous initial conditions [19, 21, 22, 25].

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In this paper we consider a general nonhomogeneous initial-boundary value problem for a two-dimensional parabolic equation. The purpose of the paper is to prove that the operator corresponding to this problem sets isomorphisms between appropriate above-mentioned $2b$-anisotropic Hörmander spaces.

2. Statement of the problem

Let $\Omega := (0,l)\times(0,\tau)$, where positive numbers l and τ are chosen arbitrarily. Consider the following linear parabolic initial–boundary value problem in an open rectangle Ω:

$$\begin{aligned} &A(x,t,D_x,\partial_t)u(x,t)\\ &\equiv \sum_{\alpha+2b\beta\le 2m} a^{\alpha,\beta}(x,t)\,D_x^{\alpha}\partial_t^{\beta}u(x,t)=f(x,t)\quad\text{in}\quad\Omega, \end{aligned}\tag{2.1}$$

$$\begin{aligned} &B_{j,0}(t,D_x,\partial_t)u(x,t)\big|_{x=0}\\ &\equiv \sum_{\alpha+2b\beta\le m_j} b_{j,0}^{\alpha,\beta}(t)\,D_x^{\alpha}\partial_t^{\beta}u(x,t)\big|_{x=0}=g_{j,0}(t)\quad\text{and} \end{aligned}\tag{2.2}$$

$$\begin{aligned} &B_{j,1}(t,D_x,\partial_t)u(x,t)\big|_{x=l}\\ &\equiv \sum_{\alpha+2b\beta\le m_j} b_{j,1}^{\alpha,\beta}(t)\,D_x^{\alpha}\partial_t^{\beta}u(x,t)\big|_{x=l}=g_{j,1}(t)\\ &\text{for}\quad 0<t<\tau\quad\text{and}\quad j=1,\dots,m, \end{aligned}\tag{2.3}$$

$$\partial_t^{k}u(x,t)\big|_{t=0}=h_k(x)\quad\text{for}\quad 0<x<l\quad\text{and}\quad k=0,\dots,\varkappa-1.\tag{2.4}$$

Here b, m, and all m_j are arbitrarily fixed integers such that $m\ge b\ge 1$, $\varkappa := m/b\in\mathbb{Z}$, and $m_j\ge 0$. All coefficients of the partial differential expressions $A := A(x,t,D_x,\partial_t)$ and $B_{j,k} := B_{j,k}(t,D_x,\partial_t)$, with $j\in\{1,\dots,m\}$ and $k\in\{0,\,1\}$, are supposed to be complex-valued and infinitely smooth functions; namely, $a^{\alpha,\beta}\in C^{\infty}(\overline{\Omega})$ and $b_{j,k}^{\alpha,\beta}\in C^{\infty}[0,\tau]$, where $\overline{\Omega} := [0,l]\times[0,\tau]$ as usual. We use the notation $D_x := i\,\partial/\partial x$ and $\partial_t := \partial/\partial t$ for partial derivatives and take summation over the integer-valued indexes $\alpha,\beta\ge 0$ satisfying the conditions indicated.

Recall [1, § 9, Subsec. 1] that the initial–boundary value problem (2.1)–(2.4) is said to be parabolic in Ω if the following three conditions are fulfilled:

(i) Given any $x\in[0,l]$, $t\in[0,\tau]$, $\xi\in\mathbb{R}$, and $p\in\mathbb{C}$ with $\operatorname{Re}p\ge 0$, we have

$$\begin{aligned} &A^{(0)}(x,t,\xi,p)\\ &\equiv \sum_{\alpha+2b\beta=2m} a^{\alpha,\beta}(x,t)\,\xi^{\alpha}p^{\beta}\neq 0\quad\text{whenever}\quad |\xi|+|p|\neq 0. \end{aligned}$$

(ii) Let $x\in\{0,l\}$, $t\in[0,\tau]$, and $p\in\mathbb{C}\setminus\{0\}$ with $\operatorname{Re}p\ge 0$ be arbitrary. Then the polynomial $A^{(0)}(x,t,\xi,p)$ in $\xi\in\mathbb{C}$ has m roots $\xi_j^{+}(x,t,p)$, $j=1,\dots,m$, with positive imaginary part and m roots with negative imaginary part provided that each root is taken the number of times equal to its multiplicity.

(iii) Assume that x, t, and p are the same as ones considered in (ii). Let $k := 0$ if $x=0$, and let $k := 1$ if $x=l$. Then the polynomials

$$B_{j,k}^{(0)}(t,\xi,p)\equiv \sum_{\alpha+2b\beta=m_j} b_{j,k}^{\alpha,\beta}(t)\,\xi^{\alpha}p^{\beta},\quad j=1,\dots,m,$$

in ξ are linearly independent modulo

$$\prod_{j=1}^{m}\bigl(\xi-\xi_j^{+}(x,t,p)\bigr).$$

We investigate parabolic problem (2.1)–(2.4) in appropriate Hörmander inner product spaces considered in the next section.

Note, in the paper, all functions (and distributions) are supposed to be complex-valued.

3. Hörmander spaces

Here, we will define the Hörmander inner product spaces being used in the paper. The regularity properties of the distributions belonging to these spaces are characterized by two number parameters and a function parameter. The latter runs over a certain function class $\mathcal{M}$, which is defined as follows.

The class $\mathcal{M}$ consists of all functions $\varphi : [1,\infty) \to (0,\infty)$ such that

a) φ is Borel measurable on $[1,\infty)$;
b) both the functions φ and $1/\varphi$ are bounded on each compact interval $[1,d]$, with $1 < d < \infty$;
c) φ is a slowly varying function at infinity in the sense of J. Karamata; i.e.,

$$\lim_{r\to\infty} \frac{\varphi(\lambda r)}{\varphi(r)} = 1 \quad \text{for every} \quad \lambda > 0. \tag{3.1}$$

Remark 3.1. The theory of slowly varying functions is set forth in the monographs [4,40]. We give an important and standard example of functions satisfying (3.1) if we put

$$\varphi(r) := (\log r)^{\theta_1} (\log\log r)^{\theta_2} \dots (\underbrace{\log\dots\log}_{k \text{ times}} r)^{\theta_k} \quad \text{for} \quad r \gg 1, \tag{3.2}$$

where the parameters $k \in \mathbb{N}$ and $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$ are chosen arbitrarily. The functions (3.2) form the logarithmic multiscale, which has a number of applications in the theory of function spaces. Some other examples of slowly varying functions can be found in [4, Sec. 1.3.3] and [33, Sec. 1.2.1].

Let $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, and $\gamma := 1/(2b)$. By definition, the linear space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ consists of all tempered distributions $w \in \mathcal{S}'(\mathbb{R}^2)$ such that their Fourier transform $\widetilde{w}$ (in two variables) is locally Lebesgue integrable over $\mathbb{R}^2$ and satisfies the condition

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} r_\gamma^{2s}(\xi,\eta)\, \varphi^2(r_\gamma(\xi,\eta))\, |\widetilde{w}(\xi,\eta)|^2\, d\xi d\eta < \infty. \tag{3.3}$$

Here and below we use the notation

$$r_\gamma(\xi,\eta) := \bigl(1 + |\xi|^2 + |\eta|^{2\gamma}\bigr)^{1/2} \quad \text{for each} \quad \xi,\eta \in \mathbb{R}.$$

The space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ is endowed with the inner product

$$(w_1,w_2)_{H^{s,s\gamma;\varphi}(\mathbb{R}^2)} := \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} r_\gamma^{2s}(\xi,\eta)\, \varphi^2(r_\gamma(\xi,\eta))\, \widetilde{w_1}(\xi,\eta)\, \overline{\widetilde{w_2}(\xi,\eta)}\, d\xi d\eta,$$

where $w_1, w_2 \in H^{s,s\gamma;\varphi}(\mathbb{R}^2)$. It induces the norm

$$\|w\|_{H^{s,s\gamma;\varphi}(\mathbb{R}^2)} := (w,w)^{1/2}_{H^{s,s\gamma;\varphi}(\mathbb{R}^2)},$$

which is equal to the square root of the left-hand side of inequality (3.3).

Note that $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ is the inner product Hörmander space $\mathcal{B}_{2,\mu}(\mathbb{R}^2)$ which corresponds to the function parameter

$$\mu(\xi,\eta) := r_\gamma^{s}(\xi,\eta)\, \varphi(r_\gamma(\xi,\eta)) \quad \text{for} \quad \xi,\eta \in \mathbb{R}.$$

We refer the reader to the monographs by L. Hörmander [9, Sec. 2.2], [10, Sec. 10.1], and to the paper by L. R. Volevich and B. P. Paneah [45], where such spaces are investigated systematically. It follows from properties of Hörmander spaces that the space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ is Hilbert and separable, is embedded continuously in $\mathcal{S}'(\mathbb{R}^2)$, and the set $C_0^\infty(\mathbb{R}^2)$ is dense in $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$.

If $\varphi(r)\equiv 1$, then $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ becomes the anisotropic Sobolev space of order $(s,s\gamma)$; we denote this space by $H^{s,s\gamma}(\mathbb{R}^2)$.

Every space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$, with $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$, is closely connected to anisotropic Sobolev spaces. Specifically, we have the continuous and dense embeddings

$$H^{s_1,s_1\gamma}(\mathbb{R}^2)\hookrightarrow H^{s,s\gamma;\varphi}(\mathbb{R}^2)\hookrightarrow H^{s_0,s_0\gamma}(\mathbb{R}^2)\quad\text{whenever}\quad s_0<s<s_1. \tag{3.4}$$

They follow from the next property of $\varphi\in\mathcal{M}$: for each $\varepsilon>0$ there exist a number $c=c(\varepsilon)\geq 1$ such that $c^{-1}r^{-\varepsilon}\leq\varphi(r)\leq c\,r^{\varepsilon}$ for all $r\geq 1$ (see [40, Sec. 1.5, Subsec. 1]).

Consider the class of Hilbert function spaces

$$\bigl\{H^{s,s\gamma;\varphi}(\mathbb{R}^2):\,s\in\mathbb{R},\,\varphi\in\mathcal{M}\,\bigr\}. \tag{3.5}$$

Owing to the embeddings (3.4) we may assert that in (3.5) the function parameter φ defines a supplementary (subpower) smoothness with respect to the basic (power) anisotropic $(s,s\gamma)$-smoothness. Specifically, if $\varphi(r)\to\infty$ [$\varphi(r)\to 0$] as $r\to\infty$, then φ defines a positive [negative] supplementary smoothness. In other words, φ refines the power smoothness $(s,s\gamma)$.

Using this scale, let us introduce some function spaces related to the parabolic problem under consideration. As before, $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. We define the normed linear space

$$\begin{gathered}H^{s,s\gamma;\varphi}(\Omega):=\bigl\{w\restriction\Omega:\,w\in H^{s,s\gamma;\varphi}(\mathbb{R}^2)\bigr\},\\ \|u\|_{H^{s,s\gamma;\varphi}(\Omega)}:=\inf\bigl\{\|w\|_{H^{s,s\gamma;\varphi}(\mathbb{R}^2)}:\,w\in H^{s,s\gamma;\varphi}(\mathbb{R}^2),\;\;w=u\;\text{ in }\;\Omega\bigr\},\end{gathered} \tag{3.6}$$

with $u\in H^{s,s\gamma;\varphi}(\Omega)$. In other words, $H^{s,s\gamma;\varphi}(\Omega)$ is the factor space of the space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ by its subspace

$$H^{s,s\gamma;\varphi}_{Q}(\mathbb{R}^2):=\bigl\{w\in H^{s,s\gamma;\varphi}(\mathbb{R}^2):\,\operatorname{supp}w\subseteq Q:=\mathbb{R}^2\setminus\Omega\bigr\}. \tag{3.7}$$

Thus, $H^{s,s\gamma;\varphi}(\Omega)$ is a separable Hilbert space. The norm (3.6) is induced by the inner product

$$(u_1,u_2)_{H^{s,s\gamma;\varphi}(\Omega)}:=(w_1-\Upsilon w_1,w_2-\Upsilon w_2)_{H^{s,s\gamma;\varphi}(\mathbb{R}^2)},$$

where $w_j\in H^{s,s\gamma;\varphi}(\mathbb{R}^2)$, $w_j=u_j$ in Ω for each $j\in\{1,\,2\}$, and Υ is the orthogonal projector of the space $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ onto its subspace (3.7).

It follows directly from the definition of $H^{s,s\gamma;\varphi}(\Omega)$ and properties of $H^{s,s\gamma;\varphi}(\mathbb{R}^2)$ that the space $H^{s,s\gamma;\varphi}(\Omega)$ is continuously embedded in the linear topological space $\mathcal{D}'(\Omega)$ of all distributions on Ω and that the set

$$C^\infty(\overline{\Omega}):=\bigl\{w\restriction\overline{\Omega}:\,w\in C_0^\infty(\mathbb{R}^2)\bigr\}$$

is dense in $H^{s,s\gamma;\varphi}(\Omega)$.

It remains to introduce the function spaces in which the right-hand sides of the boundary-value and initial-value conditions (2.2), (2.3) and (2.4) are considered. Let $s\in\mathbb{R}$ and $\varphi\in\mathcal{M}$. By definition, the linear space $H^{s;\varphi}(\mathbb{R})$ consists of all tempered distributions $h\in\mathcal{S}'(\mathbb{R})$ such that their Fourier transform $\widehat{h}$ is locally Lebesgue integrable over $\mathbb{R}$ and satisfies the condition

$$\int\limits_{-\infty}^{\infty}\langle\xi\rangle^{2s}\,\varphi^2(\langle\xi\rangle)\,|\widehat{h}(\xi)|^2\,d\xi<\infty.$$

Here, as usual, $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ is the smooth modulus of $\xi \in \mathbb{R}$. The space $H^{s;\varphi}(\mathbb{R})$ is endowed with the inner product

$$(h_1, h_2)_{H^{s;\varphi}(\mathbb{R})} := \int\limits_{-\infty}^{\infty} \langle\xi\rangle^{2s}\, \varphi^2(\langle\xi\rangle)\, \widehat{h_1}(\xi)\, \overline{\widehat{h_2}(\xi)}\, d\xi,$$

where $h_1, h_2 \in H^{s;\varphi}(\mathbb{R})$. It induces the norm

$$\|h\|_{H^{s;\varphi}(\mathbb{R})} := (h,h)^{1/2}_{H^{s;\varphi}(\mathbb{R})}.$$

Notice that $H^{s;\varphi}(\mathbb{R})$ is the inner product Hörmander space $\mathcal{B}_{2,\mu}(\mathbb{R})$ corresponding to the function parameter $\mu(\xi) := \langle\xi\rangle^s\varphi(\langle\xi\rangle)$ of $\xi \in \mathbb{R}$ (see the references [9,10,45] mentioned above). Therefore $H^{s;\varphi}(\mathbb{R})$ is a separable Hilbert space embedded continuously in $\mathcal{S}'(\mathbb{R})$, and the set $C_0^\infty(\mathbb{R})$ is dense in $H^{s;\varphi}(\mathbb{R})$.

If $\varphi(r) \equiv 1$, then $H^{s;\varphi}(\mathbb{R})$ becomes the Sobolev space $H^s(\mathbb{R})$ of order s. Analogously to (3.4), we have the continuous and dense embedding

$$H^{s_1}(\mathbb{R}) \hookrightarrow H^{s;\varphi}(\mathbb{R}) \hookrightarrow H^{s_0}(\mathbb{R}) \quad \text{whenever} \quad s_0 < s < s_1, \quad \varphi \in \mathcal{M}. \tag{3.8}$$

The class of Hilbert function spaces

$$\big\{H^{s;\varphi}(\mathbb{R}) :\, s \in \mathbb{R},\, \varphi \in \mathcal{M}\,\big\} \tag{3.9}$$

is called the refined Sobolev scale over $\mathbb{R}$ (see [32, Sec. 3.2] and [33, Sec.1.3.3]).

Using this scale, introduce one-dimensional analogs of the spaces considered above. Let real $d > 0$. We define the normed linear space

$$H^{s;\varphi}(0,d) := \big\{h \restriction (0,d) :\, h \in H^{s;\varphi}(\mathbb{R})\big\},$$

$$\|v\|_{H^{s;\varphi}(0,d)} := \inf\big\{\|h\|_{H^{s;\varphi}(\mathbb{R})} :\, h \in H^{s;\varphi}(\mathbb{R}),\ \ h = v \ \text{in} \ (0,d)\big\},$$

with $v \in H^{s;\varphi}(0,d)$. This space is separable Hilbert as it is the factor space of $H^{s;\varphi}(\mathbb{R})$ by

$$\big\{h \in H^{s;\varphi}(\mathbb{R}) :\, \operatorname{supp} h \subseteq (-\infty, 0] \cup [d, \infty)\big\}. \tag{3.10}$$

It follows directly from the definition of $H^{s;\varphi}(0,d)$ and properties of $H^{s;\varphi}(\mathbb{R})$ that the set

$$C^\infty[0,d] := \big\{h \restriction [0,d] :\, h \in C_0^\infty(\mathbb{R})\big\}$$

is dense in $H^{s;\varphi}(0,d)$.

Note that the classes of isotropic inner product spaces

$$\big\{H^{s;\varphi}(\mathbb{R}) : s \in \mathbb{R},\ \varphi \in \mathcal{M}\big\} \quad \text{and} \quad \big\{H^{s;\varphi}(0,d) : s \in \mathbb{R},\ \varphi \in \mathcal{M}\big\}$$

were selected, investigated, and systematically applied to elliptic differential operators and elliptic boundary-value problems by Mikhailets and Murach [32,33].

If $\varphi \equiv 1$, then the considered spaces $H^{s,s\gamma;\varphi}(\Omega)$ and $H^{s;\varphi}(0,d)$ become the Sobolev spaces $H^{s,s\gamma}(\Omega)$ and $H^s(0,d)$ respectively. It follows directly from (3.4) that

$$H^{s_1,s_1\gamma}(\Omega) \hookrightarrow H^{s,s\gamma;\varphi}(\Omega) \hookrightarrow H^{s_0,s_0\gamma}(\Omega) \quad \text{whenever} \quad s_0 < s < s_1. \tag{3.11}$$

Analogously,

$$H^{s_1}(0,d) \hookrightarrow H^{s;\varphi}(0,d) \hookrightarrow H^{s_0}(0,d) \quad \text{whenever} \quad s_0 < s < s_1; \tag{3.12}$$

see [33, Theorems 2.3(iii) and 3.3(iii)]. These embeddings are continuous and dense. Of course, if $s = 0$, then $H^{s,s\gamma}(\Omega)$ and $H^s(0,d)$ are the Hilbert spaces $L_2(\Omega)$ and $L_2(0,d)$ of all square integrable functions given on the corresponding measurable sets.

In the Sobolev case of $\varphi \equiv 1$, we will omit the index φ in designations of function spaces that will be introduced on the base of the Hörmander spaces $H^{s,s\gamma;\varphi}(\Omega)$ and $H^{s;\varphi}(0,d)$.

4. Main result

Here, we formulate an isomorphism theorem for the parabolic problem (2.1)–(2.4) in Hörmander spaces introduced above.

First note, in order that a regular enough solution u of the problem (2.1)–(2.4) exist, the right-hand sides it's problem should satisfy certain compatibility conditions (see, e.g., [1, Section 11] or [15, Chapter 4, Section 5]). These conditions consist in that the partial derivatives $\partial_t^k u(x,t)\big|_{t=0}$, which could be found from the parabolic equation (2.1) and initial conditions (2.4), should satisfy the boundary conditions (2.3), (2.4) and some relations that are obtained by means of the differentiation of the boundary conditions with respect to t. To write these compatibility conditions we use Sobolev inner product spaces.

Let

$$\sigma_0 := \max\{2m, m_1+1, \ldots, m_m+1\}.$$

Note, if $m_j \leq 2m-1$ for each $j \in \{1,\ldots,m\}$, then $\sigma_0 = 2m$.

We associate the linear mapping

$$\begin{aligned} u \mapsto \Lambda u := \big(&Au, B_{1,0}u, B_{1,1}u, \ldots, B_{m,0}u, B_{m,1}u, \\ &u\restriction_{[0,l]}, \ldots, (\partial_t^{\varkappa-1}u)\restriction_{[0,l]}\big), \quad u \in C^\infty(\overline{\Omega}) \end{aligned} \tag{4.1}$$

with the problem (2.1)–(2.4).

Let real $s \geq \sigma_0$; the mapping (4.1) extends uniquely (by continuity) to a bounded linear operator

$$\begin{aligned} \Lambda :\ H^{s,s/(2b)}(\Omega)\ \to \mathcal{H}^{s-2m,(s-2m)/(2b)} &:= H^{s-2m,(s-2m)/(2b)}(\Omega) \\ &\oplus \bigoplus_{j=1}^{m} \big(H^{(s-m_j-1/2)/(2b)}(0,\tau)\big)^2 \oplus \bigoplus_{k=0}^{\varkappa-1} H^{s-2bk-b}(0,l). \end{aligned} \tag{4.2}$$

This follows directly from [41, Chapter I, Lemma 4, and Chapter II, Theorems 3 and 7]. Choosing any function $u(x,t)$ from the space $H^{s,s/(2b)}(\Omega)$, we define the right-hand sides

$$\begin{gathered} f \in H^{s-2m,(s-2m)/(2b)}(\Omega), \quad g_{j,\lambda} \in H^{(s-m_j-1/2)/(2b)}(0,\tau), \quad h_k \in H^{s-2bk-b}(0,l) \\ \text{for all} \quad \lambda \in \{0,1\}, \quad j \in \{1,\ldots,m\} \quad \text{and} \quad k \in \{0,\ldots,\varkappa-1\} \end{gathered} \tag{4.3}$$

of the problem by the formula

$$(f, g_{1,0}, g_{1,1}, \ldots, g_{m,0}, g_{m,1}, h_0, \ldots, h_{\varkappa-1}) := \Lambda u$$

with the help of this bounded operator.

Compatibility conditions for functions f, $g_{j,\lambda}$ and h_k naturally arise in such a way. According to [41, Chapter II, Theorem 7], the traces $\partial_t^k u(\cdot,0) \in H^{s-2bk-b}(0,l)$ are well defined by closure for all $k \in \mathbb{Z}$ such that $0 \leq k < s/(2b) - 1/2$ (and only for these k). These traces should be expressed from the equation (2.1) and the initial data (2.4) by functions f and h_k as follows.

The parabolicity condition (i) in the case of $\xi = 0$ and $p = 1$ means that the coefficient $a^{0,\varkappa}(x,t) \neq 0$ for all $x \in [0,l]$ and $t \in [0,\tau]$. Therefore we can resolve the parabolic equation (2.1) with respect to $\partial_t^{\varkappa} u(x,t)$; namely, we can write

$$\partial_t^{\varkappa} u(x,t) = \sum_{\substack{\alpha+2b\beta\leq 2m,\\ \beta\leq\varkappa-1}} a_0^{\alpha,\beta}(x,t)\, D_x^{\alpha}\partial_t^{\beta} u(x,t) + (a^{0,\varkappa}(x,t))^{-1} f(x,t) \tag{4.4}$$

for some functions $a_0^{\alpha,\beta} \in C^\infty(\overline{\Omega})$. Using initial data (2.4), equality (4.4) and, in the case $k > \varkappa$ the equalities obtained from (4.4) by differentiating it $(k-\varkappa)$ times with respect to t, we obtain the recurrent formula for traces $\partial_t^k u(x,0)$:

$$
\begin{aligned}
&\partial_t^k u(x,0) = h_k(x) \quad \text{if} \quad k \in \{0,\dots,\varkappa-1\},\\
&\partial_t^k u(x,0) = \sum_{\substack{\alpha+2b\beta\le 2m,\\ \beta\le\varkappa-1}} \sum_{q=0}^{k-\varkappa} \binom{k-\varkappa}{q} \partial_t^{k-\varkappa-q} a_0^{\alpha,\beta}(x,0)\, D_x^\alpha \partial_t^{\beta+q} u(x,0)\\
&\qquad + \partial_t^{k-\varkappa}\big((a^{0,\varkappa}(x,0))^{-1} f(x,0)\big) \quad \text{if} \quad k \ge \varkappa
\end{aligned}
\tag{4.5}
$$

for each $k \in \mathbb{Z}$ such that $0 \le k < s/(2b) - 1/2$. These equalities holding for almost all $x \in (0,l)$.

Besides, according to (4.3) and Sobolev embedding theorem, for each $j \in \{1,\dots,m\}$ and $\lambda \in \{0,1\}$ the traces $\partial_t^k g_{j,\lambda}(0) \in \mathbb{C}$ are well defined for all $k \in \mathbb{Z}$ such that $0 \le k < (s - m_j - 1/2 - b)/(2b)$ (and only for these k). We can express these traces in terms of the function $u(x,t)$ and its time derivatives; namely,

$$
\begin{aligned}
\partial_t^k g_{j,\lambda}(0) &= \big(\partial_t^k B_{j,\lambda} u(x,t)\big)|_{t=0}\\
&= \sum_{\alpha+2b\beta\le m_j} \sum_{q=0}^{k} \binom{k}{q} \partial_t^{k-q} b_{j,\lambda}^{\alpha,\beta}(0) D_x^\alpha \partial_t^{\beta+q} u(x,0),
\end{aligned}
\tag{4.6}
$$

where $x = 0$ if $\lambda = 0$ and $x = l$ if $\lambda = 1$. Here, all the functions $u(x,0)$, $\partial_t u(x,0)$, $\dots$, $\partial_t^{[m_j/(2b)]+k} u(x,0)$ of $x \in (0,l)$ are expressed in terms of the functions $f(x,t)$ and $h_0(x)$, $\dots$, $h_{\varkappa-1}(x)$ by the recurrent formula (4.5). Here and below $[m_j/(2b)]$ denotes the integer part of $m_j/(2b)$.

Substituting (4.5) in the right-hand side of formula (4.6), we obtain the compatibility conditions

$$
\begin{aligned}
&\partial_t^k g_{j,0}|_{t=0} = B_{j,0,(k)}[v_0,\dots,v_{[m_j/(2b)]+k}],\\
&\partial_t^k g_{j,1}|_{t=0} = B_{j,1,(k)}[v_0,\dots,v_{[m_j/(2b)]+k}],\\
&\text{with} \quad k \in \mathbb{Z} \quad \text{and} \quad 0 \le k < \frac{s - m_j - 1/2 - b}{2b} \quad \text{and} \quad j \in \{1,\dots,m\}.
\end{aligned}
\tag{4.7}
$$

Here, for all above-mentioned j and k the functions v_0, $\dots$, $v_{[m_j/(2b)]+k}$ are defined on $(0,l)$ by the recurrent formula

$$
\begin{aligned}
&v_\mu(x) := h_\mu(x) \quad \text{if} \quad \mu \in \{0,\dots,\varkappa-1\},\\
&v_\mu(x) := \sum_{\substack{\alpha+2b\beta\le 2m,\\ \beta\le\varkappa-1}} \sum_{q=0}^{\mu-\varkappa} \binom{\mu-\varkappa}{q} \partial_t^{\mu-\varkappa-q} a_0^{\alpha,\beta}(x,0)\, D_x^\alpha v_{\beta+q}(x)\\
&\qquad + \partial_t^{\mu-\varkappa}\big((a^{0,\varkappa}(x,0))^{-1} f(x,0)\big) \quad \text{if} \quad \mu \ge \varkappa;
\end{aligned}
\tag{4.8}
$$

and we put

$$
B_{j,\lambda,(k)}[v_0,\dots,v_{[m_j/(2b)]+k}] = \sum_{\alpha+2b\beta\le m_j} \sum_{q=0}^{k} \binom{k}{q} \partial_t^{k-q} b_{j,\lambda}^{\alpha,\beta}(0) \big(D_x^\alpha v_{\beta+q}(x)\big)\big|_{x=d},
$$

where $d = 0$ if $\lambda = 0$ and $d = l$ if $\lambda = 1$. Note, that

$$
v_\mu \in H^{s-b-2b\mu}(0,l) \quad \text{for each} \quad \mu \in \mathbb{Z} \cap [0, s/(2b) - 1/2)
\tag{4.9}
$$

due to (4.3). The right-hand sides of the equalities (4.7) is well defined because the function $D_x^\alpha v_{\beta+q}(x)$ belongs to

$$
H^{s-\alpha-b-2b(\beta+q)}(0,l) \subseteq H^{s-m_j-2bk-b}(0,l)
$$

due to (4.9) and therefore the trace $\big(D_x^{\alpha} v_{\beta+q}(x)\big)\big|_{x=d}$ is defined whenever $s-m_j-2bk-b-1/2>0$. Note that if $s\leq\min\{m_j\}+b+1/2$, then there are no compatibility conditions.

We put $E:=\{\sigma_0+r-1/2 : 1\leq r\in\mathbb{Z}\}$. Note that E is the set of all possible discontinuities of the function that assigns the number of compatibility conditions (4.7) to $s\geq\sigma_0$.

Our main result on the parabolic problem (2.1)–(2.4) consists in that the linear mapping (4.1) extends uniquely to an isomorphism between appropriate pairs of Hörmander spaces introduced in the previous section. Let us indicate these spaces. We arbitrarily choose a real number $s>\sigma_0$ and function parameter $\varphi\in\mathcal{M}$. We take $H^{s,s/(2b);\varphi}(\Omega)$ as the source space of this isomorphism; otherwise speaking, $H^{s,s/(2b);\varphi}(\Omega)$ serves as a space of solutions u to the problem. To introduce the target space of the isomorphism, consider the Hilbert space

$$\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$$
$$:=H^{s-2m,(s-2m)/(2b);\varphi}(\Omega)\oplus\bigoplus_{j=1}^{m}\big(H^{(s-m_j-1/2)/(2b);\varphi}(0,\tau)\big)^2\oplus\bigoplus_{k=0}^{\varkappa-1}H^{s-2bk-b;\varphi}(0,l).$$

In the Sobolev case of $\varphi\equiv 1$ this space coincides with the target space of the bounded operator (4.2). The target space of the isomorphism is embedded in $\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$ and is denoted by $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$. We separately define this space in the $s\notin E$ case and $s\in E$ case.

Suppose first that $s\notin E$. By definition, the linear space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ consists of all vectors

$$F=\big(f,g_{1,0},g_{1,1},\ldots,g_{m,0},g_{m,1},h_0,\ldots,h_{\varkappa-1}\big)\in\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi},$$

that satisfy the compatibility conditions (4.7). As we have noted, these conditions are well defined for every $F\in\mathcal{H}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b)}$ for sufficiently small $\varepsilon>0$. Hence, they are also well defined for every $F\in\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$ due to the continuous embedding

$$\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}\hookrightarrow\mathcal{H}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b)}. \tag{4.10}$$

The latter follows directly from (3.11) and (3.12). Thus, our definition is reasonable.

We endow the linear space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ with the inner product and norm in the Hilbert space $\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$. The space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ is complete, i.e. a Hilbert one. Indeed, if the number $\varepsilon>0$ is sufficiently small, then

$$\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}=\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}\cap\mathcal{Q}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b)}.$$

Here, the space $\mathcal{Q}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b)}$ is complete because the differential operators and traces operators used in the compatibility conditions are bounded on the corresponding pairs of Sobolev spaces. Therefore the right-hand side of this equality is complete with respect to the sum of the norms in the components of the intersection, this sum being equivalent to the norm in $\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$ due to (4.10). Thus, the space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ is complete (with respect to the latter norm).

If $s\in E$, then we define the Hilbert space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ by means of the interpolation between its analogs just introduced. Namely, we put

$$\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}:=\big[\mathcal{Q}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b);\varphi},\mathcal{Q}^{s-2m+\varepsilon,(s-2m+\varepsilon)/(2b);\varphi}\big]_{1/2}. \tag{4.11}$$

Here, the number $\varepsilon\in(0,1/2)$ is arbitrarily chosen, and the right-hand side of the equality is the result of the interpolation of the written pair of Hilbert spaces with the parameter 1/2. We will recall the definition of the interpolation between Hilbert spaces in Section 5. The Hilbert space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ defined by formula (4.11) does not depend on the choice of ε up to equivalence of norms and is continuously embedded in $\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$. This will be shown in Remark 6.1 at the end of Section 6.

Now, we may formulate the result of the paper.

Theorem 4.1. *For arbitrary $s>\sigma_0$ and $\varphi\in\mathcal{M}$ the mapping* (4.1) *extends uniquely (by continuity) to an isomorphism*

$$\Lambda:\,H^{s,s/(2b);\varphi}(\Omega)\leftrightarrow\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}. \tag{4.12}$$

This Theorem is known in the Sobolev case where $\varphi\equiv1$. Namely, it's contained in Agranovich and Vishik's result [1, Theorem 12.1] in the case of $s,s/(2b)\in\mathbb{Z}$ and is covered by Zhitarashu result [46, Theorem 9.1]. Note that these results include the limiting case of $s=\sigma_0$. In the general situation, we will deduce Theorem 4.1 from the Sobolev case with the help of the interpolation with a function parameter between Hilbert spaces.

Note that the necessity to define the target space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ separately in the $s\in E$ case is caused by the following: if we defined this space for $s\in E$ in the way used in the $s\notin E$ case, then the isomorphism (4.12) would not be hold at least for $\varphi\equiv1$. This follows from a result by Solonnikov [42, Section 6], see also [17, Remark 6.4].

5. Interpolation with a function parameter between Hilbert spaces

This method of interpolation is a natural generalization of the classical interpolation method by S. Krein and J.-L. Lions to the case when a general enough function is used instead of a number as an interpolation parameter; see, e.g., monographs [14, Chapter IV, Section 1, Subsection 10] and [16, Chapter 1, Sections 2 and 5]. For our purposes, it is sufficient to restrict the discussion of the interpolation with a function parameter to the case of separable complex Hilbert spaces. We mainly follow the monograph [33, Section 1.1], which systematically expounds this interpolation (see also [30, Section 2]).

Let $X:=[X_0,X_1]$ be an ordered pair of separable complex Hilbert spaces such that $X_1\subseteq X_0$ and this embedding is continuous and dense. This pair is said to be admissible. For X, there is a positive-definite self-adjoint operator J on X_0 with the domain X_1 such that $\|Jv\|_{X_0}=\|v\|_{X_1}$ for every $v\in X_1$. This operator is uniquely determined by the pair X and is called a generating operator for X; see, e.g., [14, Chapter IV, Theorem 1.12]. The operator defines an isometric isomorphism $J:X_1\leftrightarrow X_0$.

Let $\mathcal{B}$ denote the set of all Borel measurable functions $\psi:(0,\infty)\to(0,\infty)$ such that ψ is bounded on each compact interval $[a,b]$, with $0<a<b<\infty$, and that $1/\psi$ is bounded on every semiaxis $[a,\infty)$, with $a>0$.

Choosing a function $\psi\in\mathcal{B}$ arbitrarily, we consider the (generally, unbounded) operator $\psi(J)$ defined on X_0 as the Borel function ψ of J. This operator is built with the help of Spectral Theorem applied to the self-adjoint operator J. Let $[X_0,X_1]_\psi$ or, simply, X_ψ denote the domain of $\psi(J)$ endowed with the inner product $(v_1,v_2)_{X_\psi}:=(\psi(J)v_1,\psi(J)v_2)_{X_0}$ and the corresponding norm $\|v\|_{X_\psi}:=\|\psi(J)v\|_{X_0}$. The linear space X_ψ is Hilbert and separable with respect to this norm.

A function $\psi\in\mathcal{B}$ is called an interpolation parameter if the following condition is satisfied for all admissible pairs $X=[X_0,X_1]$ and $Y=[Y_0,Y_1]$ of Hilbert spaces and for an arbitrary linear mapping T given on X_0: if the restriction of T to X_j is a bounded operator $T:X_j\to Y_j$ for each $j\in\{0,1\}$, then the restriction of T to X_ψ is also a bounded operator $T:X_\psi\to Y_\psi$.

If ψ is an interpolation parameter, then we say that the Hilbert space X_ψ is obtained by the interpolation with the function parameter ψ of the pair $X=[X_0,X_1]$ or, otherwise speaking, between the spaces X_0 and X_1. In this case, the dense and continuous embeddings $X_1\hookrightarrow X_\psi\hookrightarrow X_0$ hold.

The class of all interpolation parameters (in the sense of the given definition) admits a constructive description. Namely, a function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if ψ is pseudoconcave in a neighborhood of infinity. The latter property means that there exists a concave positive function $\psi_1(r)$ of $r \gg 1$ that both the functions ψ/ψ_1 and ψ_1/ψ are bounded in some neighborhood of infinity. This criterion follows from Peetre's description of all interpolation functions for the weighted Lebesgue spaces [38, 39] (this result of Peetre is set forth in the monograph [3, Theorem 5.4.4]). The proof of the criterion is given in [33, Section 1.1.9].

An application of this criterion to power functions gives the classical result by Lions and S. Krein. Namely, the function $\psi(r) \equiv r^\theta$ is an interpolation parameter whenever $0 \le \theta \le 1$. In this case, the exponent θ serves as a number parameter of the interpolation, and the interpolation space X_ψ is also denoted by X_θ. This interpolation was used in formula (4.11) in the special case of $\theta = 1/2$.

Let us formulate some general properties of interpolation with a function parameter; they will be used in our proofs. The first of these properties enables us to reduce the interpolation of subspaces to the interpolation of the whole spaces (see [33, Theorem 1.6] or [43, Section 1.17.1, Theorem 1]). As usual, subspaces of normed spaces are assumed to be closed. Generally, we consider nonorthogonal projectors onto subspaces of a Hilbert space.

Proposition 5.1. *Let $X = [X_0, X_1]$ be an admissible pair of Hilbert spaces, and let Y_0 be a subspace of X_0. Then $Y_1 := X_1 \cap Y_0$ is a subspace of X_1. Suppose that there exists a linear mapping $P : X_0 \to X_0$ such that P is a projector of the space X_j onto its subspace Y_j for each $j \in \{0, 1\}$. Then the pair $[Y_0, Y_1]$ is admissible, and $[Y_0, Y_1]_\psi = X_\psi \cap Y_0$ with equivalence of norms for an arbitrary interpolation parameter $\psi \in \mathcal{B}$. Here, $X_\psi \cap Y_0$ is a subspace of X_ψ.*

The second property reduces the interpolation of orthogonal sums of Hilbert spaces to the interpolation of their summands (see [33, Theorem 1.8].

Proposition 5.2. *Let $[X_0^{(j)}, X_1^{(j)}]$, with $j = 1, \dots, q$, be a finite collection of admissible pairs of Hilbert spaces. Then*

$$\Big[\bigoplus_{j=1}^{q} X_0^{(j)}, \bigoplus_{j=1}^{q} X_1^{(j)}\Big]_\psi = \bigoplus_{j=1}^{q} [X_0^{(j)}, X_1^{(j)}]_\psi$$

with equality of norms for every function $\psi \in \mathcal{B}$.

Our proof of Theorem 4.1 is based on the key fact that the interpolation with an appropriate function parameter between margin Sobolev spaces in (3.11) and (3.12) gives the intermediate Hörmander spaces $H^{s,s\gamma;\varphi}(\cdot)$ and $H^{s;\varphi}(\cdot)$ respectively. Let us formulate this property separately for isotropic and for anisotropic spaces.

Proposition 5.3. *Let real numbers s_0, s, and s_1 satisfy the inequalities $s_0 < s < s_1$, and let $\varphi \in \mathcal{M}$. Put*

$$\psi(r) := \begin{cases} r^{(s-s_0)/(s_1-s_0)}\,\varphi(r^{1/(s_1-s_0)}) & \text{if} \quad r \ge 1, \\ \varphi(1) & \text{if} \quad 0 < r < 1. \end{cases} \tag{5.1}$$

Then the function $\psi \in \mathcal{B}$ is an interpolation parameter, and the equality of spaces

$$H^{s-\lambda;\varphi}(0,d) = \big[H^{s_0-\lambda}(0,d), H^{s_1-\lambda}(0,d)\big]_\psi \tag{5.2}$$

holds true up to equivalence of norms for arbitrary $\lambda \in \mathbb{R}$.

This result is due to [28, Theorem 3.1]; see also monograph [33, Theorems 1.14 and 3.2].

Proposition 5.4. *Let real numbers s_0, s, and s_1 satisfy the inequalities $0 \leq s_0 < s < s_1$, and let $\varphi \in \mathcal{M}$. Define an interpolation parameter $\psi \in \mathcal{B}$ by formula* (5.1). *Then the equality of spaces*

$$H^{s-\lambda,(s-\lambda)/(2b);\varphi}(\Omega) = \bigl[H^{s_0-\lambda,(s_0-\lambda)/(2b)}(\Omega), H^{s_1-\lambda,(s_1-\lambda)/(2b)}(\Omega)\bigr]_{\psi} \tag{5.3}$$

holds true up to equivalence of norms for arbitrary real $\lambda \leq s_0$.

The proof of the result is the same as the proof of its analog for a strip [20, Lemma 2]. To prove Theorem 4.1 in the case $s \in E$ we need the following proposition.

Proposition 5.5. *Let real numbers s and ε such, that $s > \varepsilon > 0$, and let $\varphi \in \mathcal{M}$. Then the equality of spaces*

$$H^{s,s/(2b);\varphi}(\Omega) = \bigl[H^{s-\varepsilon,(s-\varepsilon)/(2b);\varphi}(\Omega), H^{s+\varepsilon,(s+\varepsilon)/(2b);\varphi}(\Omega)\bigr]_{1/2} \tag{5.4}$$

holds true up to equivalence of norms.

The proof of this result is the same as the proof of its analog for isotropic space $H^{s;\varphi}(\cdot)$ (see. [33, Lemma 4.3]).

6. Proof of the main result

To deduce Theorem 4.1 from its known counterpart in the Sobolev case, we need to prove a version of Proposition 5.4 (with $\lambda = 0$) for the target spaces of isomorphism (4.12). From definition of these spaces follows that the interpolation formula need for them in the case where $s \notin E$. So consider these intervals

$$J_0 = (\sigma_0, \sigma_0 + 1/2), \quad J_r = (\sigma_0 + r - 1/2, \sigma_0 + r + 1/2) \quad \text{with} \quad 1 \leq r \in \mathbb{Z}$$

of the varying of s.

Lemma 6.1. *Let $1 \leq r \in \mathbb{Z}$. Suppose that real numbers $s_0, s, s_1 \in J_{r-1}$ satisfy the inequality $s_0 < s < s_1$ and that $\varphi \in \mathcal{M}$. Define an interpolation parameter $\psi \in \mathcal{B}$ by formula* (5.1). *Then the equality of spaces*

$$\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi} = \bigl[\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)}, \mathcal{Q}^{s_1-2m,(s_1-2m)/(2b)}\bigr]_{\psi} \tag{6.1}$$

holds true up to equivalence of norms.

Proof. According to Propositions 5.2, 5.3, and 5.4 we obtain the following:

$$\begin{aligned}
&\big[\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)},\mathcal{H}^{s_1-2m,(s_1-2m)/(2b)}\big]_\psi\\
&\quad=\Big[H^{s_0-2m,(s_0-2m)/(2b)}(\Omega)\oplus\bigoplus_{j=1}^{m}\big(H^{(s_0-m_j-1/2)/(2b)}(0,\tau)\big)^2\oplus\bigoplus_{k=0}^{\varkappa-1}H^{s_0-2bk-b}(0,l),\\
&H^{s_1-2m,(s_1-2m)/(2b)}(\Omega)\oplus\bigoplus_{j=1}^{m}\big(H^{(s_1-m_j-1/2)/(2b)}(0,\tau)\big)^2\oplus\bigoplus_{k=0}^{\varkappa-1}H^{s_1-2bk-b}(0,l)\Big]_\psi\\
&\quad=\big[H^{s_0-2m,(s_0-2m)/(2b)}(\Omega),H^{s_1-2m,(s_1-2m)/(2b)}(\Omega)\big]_\psi\\
&\qquad\oplus\bigoplus_{j=1}^{m}\Big(\big[H^{(s_0-m_j-1/2)/(2b)}(0,\tau),H^{(s_1-m_j-1/2)/(2b)}(0,\tau)\big]_\psi\Big)^2\\
&\qquad\oplus\bigoplus_{k=0}^{\varkappa-1}\big[H^{s_0-2bk-b}(0,l),H^{s_1-2bk-b}(0,l)\big]_\psi\\
&\quad=H^{s-2m,(s-2m)/(2b);\varphi}(\Omega)\oplus\bigoplus_{j=1}^{m}\big(H^{(s-m_j-1/2)/(2b);\varphi}(0,\tau)\big)^2\oplus\bigoplus_{k=0}^{\varkappa-1}H^{s-2bk-b;\varphi}(0,l)\\
&\quad=\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}.
\end{aligned}$$

Thus,

$$\big[\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)},\mathcal{H}^{s_1-2m,(s_1-2m)/(2b)}\big]_\psi=\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi} \tag{6.2}$$

up to equivalence of norms.

We will deduce the required formula (6.1) from (6.2) with the help of Proposition 5.1. To this end, we need to present a linear mapping P on $\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)}$ such that P is a projector of the space $\mathcal{H}^{s_j-2m,(s_j-2m)/(2b)}$ onto its subspace $\mathcal{Q}^{s_j-2m,(s_j-2m)/(2b)}$ for each $j\in\{0,1\}$. If we have this mapping, we will get

$$\begin{aligned}
&\big[\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)},\mathcal{Q}^{s_1-2m,(s_1-2m)/(2b)}\big]_\psi\\
&\quad=\big[\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)},\mathcal{H}^{s_1-2m,(s_1-2m)/(2b)}\big]_\psi\cap\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)}\\
&\quad=\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}\cap\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)}\\
&\quad=\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}
\end{aligned}$$

due to Proposition 5.1, formula (6.2), and the conditions $s_0,s\in J_{r-1}$ and $s_0<s$. Note that these conditions imply the last equality because the elements of the spaces $\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)}$ and $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$ satisfy the same compatibility conditions and because $\mathcal{H}^{s-2m,(s-2m)/(2b);\varphi}$ is embedded continuously in $\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)}$.

We will build the above-mentioned mapping P in the following way.

For any $1\le n\in\mathbb{Z}$, $s\in\mathbb{R}$, and $\varphi\in\mathcal{M}$ the mapping

$$\{z_0,\dots,z_{n-1}\}\mapsto w(t)=\sum_{k=0}^{n-1}\frac{z_kt^k}{k!},\quad\text{with}\quad z_0,\dots,z_{n-1}\in\mathbb{C}$$

defines a bounded operator

$$T:\mathbb{C}^n\to H^{s;\varphi}(0,\tau). \tag{6.3}$$

Besides, if $w=T(z_0,\dots,z_{n-1})$ then $\partial_t^k w(0)=z_k$ for each $k\in\{0,\dots,n-1\}$. For all $j\in\{1,\dots,m\}$ we put

$$q_{r,j}:=\left[\frac{\sigma_0+r-m_j-1-b}{2b}\right]=\left[\frac{s-m_j-1/2-b}{2b}\right].$$

Given

$$F:=(f,g_{1,0},g_{1,1},\ldots,g_{m,0},g_{m,1},h_0,\ldots,h_{\varkappa-1})\in\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)},$$

we put

$$g^*_{j,\lambda}:=g_{j,\lambda}\quad\text{if}\quad q_{r,j}<0;$$
$$g^*_{j,\lambda}:=g_{j,\lambda}+T(z_{j,\lambda,0},\ldots,z_{j,\lambda,q_{r,j}})\quad\text{if}\quad q_{r,j}\geq 0 \tag{6.4}$$

for all $j\in\{1,\ldots,m\}$ and $\lambda\in\{0,1\}$. Here

$$z_{j,\lambda,0}=B_{j,\lambda,(0)}[v_0,\ldots,v_{[m_j/(2b)]}]-g_{j,\lambda}|_{t=0},$$
$$\ldots$$
$$z_{j,\lambda,q_{r,j}}=B_{j,\lambda,(q_{r,j})}[v_0,\ldots,v_{[m_j/(2b)]+q_{r,j}}]-\partial_t^{q_{r,j}}g_{j,\lambda}|_{t=0},$$

and, the functions $v_k\in H^{s_0-b-2bk}(0,l)$, with $k=0,\ldots,\max\{[m_j/(2b)]+q_{r,j}\}$, are defined by the recurrent formula (4.8). The linear mapping $P:F\mapsto F^*$, with

$$F^*:=(f,g^*_{1,0},g^*_{1,1},\ldots,g^*_{m,0},g^*_{m,1},h_0,\ldots,h_{\varkappa-1}),$$

defined on all vectors $F\in\mathcal{H}^{s_0-2m,(s_0-2m)/(2b)}$ is required. Indeed, its restriction to each space $\mathcal{H}^{s_j-2m,(s_j-2m)/(2b)}$, with $j\in\{0,\,1\}$, is a bounded operator on this space. This follows directly from (6.3). Moreover, if $F\in\mathcal{Q}^{s_j-2m,(s_j-2m)/(2b)}$, then $PF=F$ due to the compatibility conditions (4.7). □

The proof of Theorem 4.1. Let $s>\sigma_0$ and $\varphi\in\mathcal{M}$. We first consider the case where $s\notin E$. Then $s\in J_{r-1}$ for a certain integer r. Choose numbers $s_0,s_1\in J_{r-1}$ such that $s_0/(2b)+1/2\notin\mathbb{Z}$, $s_1/(2b)+1/2\notin\mathbb{Z}$ and $s_0<s<s_1$. According to N. V. Zhitarashu [46, Theorem 9.1], the mapping (4.1) extends uniquely (by continuity) to an isomorphism

$$\Lambda:\,H^{s_j,s_j/(2b)}(\Omega)\leftrightarrow\mathcal{Q}^{s_j-2m,(s_j-2m)/(2b)}\quad\text{for each}\quad j\in\{0,1\}. \tag{6.5}$$

Let ψ be the interpolation parameter from Proposition 5.3. Then the restriction of the operator (6.5) with $j=0$ to the space

$$\bigl[H^{s_0,s_0/(2b)}(\Omega),H^{s_1,s_1/(2b)}(\Omega)\bigr]_\psi=H^{s,s/(2b);\varphi}(\Omega)$$

is an isomorphism

$$\Lambda:\,H^{s,s/(2b);\varphi}(\Omega)\leftrightarrow\bigl[\mathcal{Q}^{s_0-2m,(s_0-2m)/(2b)},\mathcal{Q}^{s_1-2m,(s_1-2m)/(2b)}\bigr]_\psi =\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}. \tag{6.6}$$

Here, the equalities of spaces hold true up to equivalence of norms due to Proposition 5.4 and Lemma 6.1. The operator (6.6) is an extension by continuity of the mapping (4.1) because $C^\infty(\overline{\Omega})$ is dense in $H^{s,s/(2b);\varphi}(\Omega)$. Thus, Theorem 4.1 is proved in the case considered.

Consider now the case where $s\in E$. Choose $\varepsilon\in(0,1/2)$ arbitrarily. Since $s\pm\varepsilon\notin E$ and $s-\varepsilon>\sigma_0$, we have the isomorphisms

$$\Lambda:H^{s\pm\varepsilon,(s\pm\varepsilon)/(2b);\varphi}(\Omega)\leftrightarrow\mathcal{Q}^{s\pm\varepsilon-2m,(s\pm\varepsilon-2m)/(2b);\varphi}. \tag{6.7}$$

They imply that the mapping (4.1) extends uniquely (by continuity) to an isomorphism

$$\Lambda:\bigl[H^{s-\varepsilon,(s-\varepsilon)/(2b);\varphi}(\Omega),H^{s+\varepsilon,(s+\varepsilon)/(2b);\varphi}(\Omega)\bigr]_{1/2} \leftrightarrow\bigl[\mathcal{Q}^{s-\varepsilon-2m,(s-\varepsilon-2m)/(2b);\varphi},\mathcal{Q}^{s+\varepsilon-2m,(s+\varepsilon-2m)/(2b);\varphi}\bigr]_{1/2}=\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}. \tag{6.8}$$

Recall that the last equality is the definition of the space $\mathcal{Q}^{s-2m,(s-2m)/(2b);\varphi}$. To complete the proof it remains to apply in (6.8) Proposition 5.5. □

Remark 6.1. The space defined by formula (4.11) is independent of the choice of the number $\varepsilon \in (0,1/2)$ up to equivalence of norms. Indeed, let $s \in E$; then according to Theorem 4.1 we have the isomorphism

$$\Lambda : H^{s,s/(2b);\varphi}(\Omega) \leftrightarrow \big[\mathcal{Q}^{s-2m-\varepsilon,(s-2m-\varepsilon)/(2b);\varphi}, \mathcal{Q}^{s-2m+\varepsilon,(s-2m+\varepsilon)/(2b);\varphi}\big]_{1/2}$$

whenever $0<\varepsilon<1/2$. This means the required independence.

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