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ON WELL-BEHAVED REPRESENTATIONS OF λ-DEFORMED CCR

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In memory of beloved professor M. L. Gorbachuk

Abstract. We study well-behaved $*$-representations of a λ-deformation of Wick analog of CCR algebra. Homogeneous Wick ideals of degrees two and three are described. Well-behaved irreducible $*$-representations of quotients by these ideals are classified up to unitary equivalence.

1. Introduction

In this paper we continue to study $*$-representations of certain type of conic commutation relations. Namely, we consider the Wick algebra, see [6], $\mathfrak{A}_\lambda$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, generated by a_i, a_i^*, $i = 1, 2$, satisfying the following relations:

$$a_i^* a_i = 1 + a_i a_i^*, \quad i = 1, 2; \quad a_1^* a_2 = \lambda a_2 a_1^*.$$

The case $\lambda = 1$ was studied in [8], where representations of the quotients of $\mathfrak{A}_1$ by the largest quadratic and cubic ideals were described. Representations of $\mathfrak{A}_\lambda/\mathcal{I}_2$ were classified in [12]. In particular, the definition of well-behaved representation of $\mathfrak{A}_\lambda/\mathcal{I}_2$ by unbounded operators was given. It was also proved that the Fock representation is a unique, up to unitary equivalence, irreducible well-behaved representation of $\mathfrak{A}_\lambda/\mathcal{I}_2$. We plan to discuss in full details well-behaved representations of $\mathfrak{A}_\lambda/\mathcal{I}_3$. For $d > 2$ we study the case where $\lambda = 1$.

2. Preliminaries

Recall the notion of analytic vectors for a linear operator on a Hilbert space, see [1]. The history of the subject can also be found in [3, 4, 7].

Definition 1. *Let A be a linear operator acting on a Hilbert space $\mathcal{H}$. A vector f is called analytic for A if $f \in \mathcal{D}(A^k)$, $k \in \mathbb{N}$, and the series*

$$\sum_{n=1}^{\infty} \frac{||A^n f||}{n!} s^n$$

converges for some $s > 0$.

Remark 1. *A vector $f \in \mathcal{D}$ is analytic for A iff there exist $C > 0$ and $M > 0$ such that*

$$||A^n f|| \le C \cdot M^n n!, \quad n \in \mathbb{N}.$$

Denote by Δ_n, $n \in \mathbb{N}$, the set of all words of length n in an alphabet $\mathcal{F} = \{A_j, \quad j = 1, \ldots, d\}$. In the following we identify any $\nu \in \Delta_n$ with the corresponding product of operators.

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Definition 2. *Let $\mathcal{D}$ be a linear domain of a Hilbert space $\mathcal{H}$ which is invariant with respect to the family $\mathcal{F} = \{A_j, \quad j = 1, \dots, d\}$ of closed operators on $\mathcal{H}$. We say that $f \in \mathcal{D}$ is jointly analytic with respect to $\mathcal{F}$ if the series*

$$\sum_{n=1}^{\infty} \frac{s^n}{n!} \sum_{\nu \in \Delta_n} ||\nu f||$$

converges for some $s > 0$.

We say that $f \in \mathcal{D}$ is jointly analytic in a strong sense with respect to $\mathcal{F}$ if the series

$$\sum_{n=1}^{\infty} \frac{s^n}{n!} \sum_{\nu \in \Delta_{2n}} ||\nu f||$$

converges for some $s > 0$.

Remark 2. *A vector $f \in \mathcal{D}$ is jointly analytic with respect to $\mathcal{F}$ iff there exist $C > 0$, $M > 0$ such that*

$$\sum_{\nu \in \Delta_n} ||\nu f|| \le C \cdot M^n n!, \quad n \in \mathbb{N}.$$

The condition of strong joint analyticity is equivalent to

$$\sum_{\nu \in \Delta_{2n}} ||\nu f|| \le C \cdot M^n n!, \quad n \in \mathbb{N}.$$

3. $*$-Algebra $\mathfrak{A}_\lambda$

In this section we study representations of the Wick algebra $\mathfrak{A}_\lambda$, generated by elements a_1, a_2 with the relations

$$a_i^* a_i = 1 + a_i a_i^*, \quad i = 1, 2,$$
$$a_1^* a_2 = \lambda a_1 a_2^*,$$

where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, is fixed.

We shall focus on irreducible well-behaved representations of quotients of $\mathfrak{A}_\lambda$ by the largest quadratic and cubic Wick ideals.

3.1. **Homogeneous Wick ideals in $\mathfrak{A}_\lambda$.** Recall that by a Wick ideal in $\mathfrak{A}_\lambda$ we mean a two-sided ideal $\mathcal{I}$ of the subalgebra $\mathbb{C}\langle a_1, a_2\rangle \subset \mathfrak{A}_\lambda$ having the following property:

$$a_i^* \mathcal{I} \subset \mathcal{I} + \mathcal{I} a_1^* + \mathcal{I} a_2^*, \quad i = 1, 2,$$

see [6] for definition of Wick ideals for general algebras allowing Wick ordering. If a Wick ideal $\mathcal{I}$ is generated by homogeneous polynomials in the generators a_1, a_2, it is called a **homogeneous** Wick ideal of the corresponding degree.

First we describe the largest quadratic and cubic Wick ideals of this algebra. To do so we consider the operator of coefficients T, see [6]. For $\mathfrak{A}_\lambda$ it has the following form:

$$T \colon \mathbb{H}^{\otimes 2} \to \mathbb{H}^{\otimes 2}, \quad \mathbb{H} = \mathbb{C}\left\langle e_1,\ e_2 \right\rangle,$$

$$T e_i \otimes e_i = e_i \otimes e_i, \quad T e_1 \otimes e_2 = \overline{\lambda} e_2 \otimes e_1, \quad T e_2 \otimes e_1 = \lambda e_1 \otimes e_2.$$

It is easy to see that T satisfies the Yang-Baxter equation and

$$\ker(\mathbf{1} + T) = \mathbb{C}\left\langle E_{12} = e_2 \otimes e_1 - \lambda e_1 \otimes e_2 \right\rangle.$$

Then, as it follows from results of [6, 10], the largest quadratic Wick ideal $\mathcal{I}_2$ of $\mathfrak{A}_\lambda$ is generated by the element $a_{12} = a_2 a_1 - \lambda a_1 a_2$. Here and below we identify the subalgebra, generated by a_i, $i = 1, 2$, with the full tensor algebra $\mathcal{T}(\mathbb{H})$.

Note also that $-\mathbf{1} \le T \le \mathbf{1}$ and the Fock representation of $\mathfrak{A}_\lambda$ is positive, see [2, 5]. Moreover, the kernel of π_F coincides with the two-sided $*$-ideal generated by $\mathcal{I}_2$. So, π_F is a faithful irreducible representation of $\mathfrak{A}_\lambda / \mathcal{I}_2$. In [12] the authors prove that

π_F is a unique irreducible well-behaved representation of this algebra. This result can be treated as an analog of the von Neumann Theorem on uniqueness of well-behaved irreducible representation of CCR with finite degrees of freedom.

To construct a largest cubic Wick ideal one has to use extensions of T to $\mathbb{H}^{\otimes 3}$, i.e.,

$$T_1 = T \otimes \mathbf{1}_{\mathbb{H}}, \quad T_2 = \mathbf{1}_{\mathbb{H}} \otimes T.$$

The largest cubic ideal $\mathcal{I}_3$, see [10, 8], corresponds to the subspace

$$(\mathbf{1} - T_1T_2)(\ker(\mathbf{1} + T) \otimes \mathbb{H}) = \mathbb{C}\langle E_{12} \otimes e_1 - \lambda e_1 \otimes E_{12},\ E_{12} \otimes e_2 - \overline{\lambda} e_2 \otimes E_{12}\rangle.$$

In an explicit form, we have

$$\mathcal{I}_3 = \left\langle a_{12}a_1 - \lambda a_1 a_{12},\ a_{12}a_2 - \overline{\lambda} a_2 a_{12} \right\rangle.$$

In the following we will need commutation relations between the generators a_1^*, a_2^* and a_{12}. Namely,

$$\begin{aligned} a_1^* a_{12} &= a_1^*(a_2a_1 - \lambda a_1 a_2) = \lambda a_2 a_1^* a_1 - \lambda a_1^* a_1 a_2 \\ &= \lambda a_2(1 + a_1 a_1^*) - \lambda(1 + a_1 a_1^*)a_2 = \lambda a_2 a_1 a_1^* - \lambda^2 a_1 a_2 a_1^* = \lambda a_{12} a_1^*. \end{aligned}$$

Similarly one can get

$$a_2^* a_{12} = \overline{\lambda} a_{12} a_2^*.$$

3.2. **Representations of $\mathfrak{A}_{\lambda,2}$.** Before a detailed study of well-behaved representations of $\mathcal{A}_{\lambda,3} = \mathfrak{A}_\lambda/\mathcal{I}_3$ we give a sketch of the situation with representations of $\mathfrak{A}_{\lambda,2} = \mathfrak{A}_\lambda/\mathcal{I}_2$, see [12] for more details,

$$\mathfrak{A}_{\lambda,2} = \mathbb{C}\langle a_1,\ a_2 \mid a_i^* a_i = 1 - a_i a_i^*,\ i = 1,2,\ a_1^* a_2 = \lambda a_2 a_1^*,\ a_2 a_1 = \lambda a_1 a_2\rangle.$$

First we recall a definition of well-behaved representations of $\mathfrak{A}_{\lambda,2}$. One can do it in terms of invariant domains, in a manner presented in [13, 9].

Definition 3. *We say that closed operators A_i, $i = 1, 2$, acting on a Hilbert space $\mathcal{H}$ determine a well-behaved representation of $\mathfrak{A}_{\lambda,2}$ if there exists a dense linear domain $\mathcal{D} \subset \mathcal{H}$, invariant with respect to A_i, A_i^*, $i = 1, 2$, and such that*

(1) *for any $f \in \mathcal{D}$ one has*

$$\begin{aligned} &A_i^* A_i f = (\mathbf{1} + A_i A_i^*)f, \quad i = 1, 2, \\ &A_1^* A_2 f = \lambda A_2 A_1^* x, \quad A_2 A_1 f = \lambda A_1 A_2 f; \end{aligned}$$

(2) *all vectors in $\mathcal{D}$ are analytic for $\Delta = A_1^* A_1 + A_2^* A_2$.*

The definition can also be given in terms of bounded operators. For a selfadjoint operator A let $E_A(\cdot)$ be the resolution of the identity of A.

Definition 4. *For closed operators A_1, A_2 on $\mathcal{H}$ let $A_i = S_i C_i$, where $C_i^2 = A_i^* A_i$, be the (left) polar decompositions of A_i, $i = 1, 2$, and let $D_i = S_i C_i S_i^*$, $i = 1, 2$. We say that A_1, A_2 determine a well-behaved representation of $\mathfrak{A}_{\lambda,2}$ if*

(1) *C_1 and C_2 strongly commute, i.e. $E_{C_1}(\Delta_1)E_{C_2}(\Delta_2) = E_{C_2}(\Delta_2)E_{C_1}(\Delta_1)$ for any Borel subsets $\Delta_i \subset \mathbb{R}$, $i = 1, 2$;*

(2) *S_1, S_2 satisfy the relations*

$$\begin{aligned} &S_i^* S_i = \mathbf{1}, \quad i = 1, 2, \\ &S_1^* S_2 = \lambda S_2 S_1^*, \quad S_2 S_1 = \lambda S_1 S_2; \end{aligned}$$

(3) *if $F(\cdot)$ is a real bounded Borel function, then*

$$F(D_i^2)S_i = S_i F(\mathbf{1} + D_i^2), \quad F(D_i^2)S_j = S_j F(D_i^2), \quad i, j = 1, 2, \quad i \neq j.$$

Remark 3.
Note that condition 3) is satisfied iff for any Borel $\delta \subset \mathbb{R}_+$

$$E_{D_j^2}(\delta)S_j = S_j E_{D_j^2}(\delta - 1), \quad E_{D_j^2}(\delta)S_i = S_i E_{D_j^2}(\delta). \quad i \neq j, \quad i, j = 1, 2.$$

The following proposition was proved in [12].

Proposition 1. Definition 3 *and* **Definition 4** *are equivalent.*

The next step is to define notions of irreducible representation and unitary equivalent representations of $\mathfrak{A}_{\lambda,2}$.

Definition 5. *A family of closed operators* $\{A_1, A_2\}$ *acting on a Hilbert space* $\mathcal{H}$ *determines an irreducible well-behaved representation of* $\mathfrak{A}_{\lambda,2}$*, if it satisfies the conditions of* **Definition 4** *and the following family of bounded operators*

$$\mathcal{B} = \{S_i,\ S_i^*, E_{D_j^2}(\delta_j), \quad i, j = 1, 2, \quad \delta_j \in \mathfrak{B}(\mathbb{R})\}$$

is irreducible on $\mathcal{H}$*. Here* $\mathfrak{B}(\mathbb{R})$ *denotes the Borel* σ*-algebra.*

Definition 6. *Irreducible representations of* $\mathfrak{A}_{\lambda,2}$ *determined by families* $\{A_1^{(i)}, A_2^{(i)}\}$, $i = 1, 2$*, are unitary equivalent iff the corresponding families of bounded operators* $\mathcal{B}^{(i)}$, $i = 1, 2$*, are unitary equivalent.*

The main result of [12] gives the following classification of irreducible well-behaved representations of $\mathfrak{A}_{\lambda,2}$.

Theorem 1. *There exists a unique, up to unitary equivalence, irreducible well-behaved representation of* $\mathfrak{A}_{\lambda,2}$*. Namely, the representation space is* $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ *and*

$$D_1 = D \otimes \mathbf{1}, \quad D_2 = \mathbf{1} \otimes D,$$
$$S_1 = S \otimes \mathbf{1}, \quad S_2 = d(\lambda) \otimes S,$$

where $D, S, d(\lambda)\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$ *are defined on the standard basis* e_n, $n \in \mathbb{Z}_+$*, as follows:*

$$De_n = \sqrt{n} e_n, \quad Se_n = e_{n+1}, \quad d(\lambda)e_n = \lambda^n e_n.$$

The operators A_1, A_2 *in this case are of the form*

$$A_1 = a \otimes \mathbf{1},$$
$$A_2 = d(\lambda) \otimes a,$$

where a *denotes the creation operator of the Fock representation of CCR with one generator given by*

$$ae_n = \sqrt{n+1} e_{n+1}, \quad n \in \mathbb{Z}_+.$$

Remark 4.
1. *The vector* $\Omega = e_0 \otimes e_0 \in \mathcal{H} = l_2(\mathbb{Z}_+)^{\otimes 2}$ *is cyclic for the constructed representation and*

$$A_1^* \Omega = 0, \quad A_2^* \Omega = 0.$$

So, the unique irreducible well-behaved representation coincides with the Fock representation of $\mathfrak{A}_{\lambda,2}$ *and we have an analog of J. von Neumann's result.*
2. *The proof of* **Theorem 1** *implies that any well-behaved representation of* $\mathfrak{A}_{2,\lambda}$ *is defined, up to unitary equivalence, by operators*

$$A_i\colon l_2(\mathbb{Z}_+)^{\otimes 2} \otimes \mathcal{K} \to l_2(\mathbb{Z}_+)^{\otimes 2} \otimes \mathcal{K},$$

having the following form:

$$A_1 = a \otimes \mathbf{1}_{l_2} \otimes \mathbf{1}_{\mathcal{K}}, \quad A_2 = d(\lambda) \otimes a \otimes \mathbf{1}_{\mathcal{K}},$$

$\mathcal{K}$ *being a Hilbert space.*

3.3. **Representations of $\mathfrak{A}_{\lambda,3}$.** In this Section we focus on well-behaved irreducible representations of the algebra

$$\mathfrak{A}_{\lambda,3} = \mathfrak{A}_\lambda/\mathcal{I}_3.$$

Note that the case $\lambda = 1$ was considered in [8], so here we generalize ideas presented there. The algebra $\mathfrak{A}_{\lambda,3}$ is generated by the elements a_1, a_2, a_{12} subject to the relations

$$\begin{aligned} a_i^* a_i &= 1 + a_i a_i^*, \quad i = 1, 2, \quad a_1^* a_2 = \lambda a_2 a_1^*, \\ a_{12} &= a_2 a_1 - \lambda a_1 a_2, \\ a_{12} a_1 &= \lambda a_1 a_{12}, \quad a_{12} a_2 = \overline{\lambda} a_2 a_{12}, \\ a_1^* a_{12} &= \lambda a_{12} a_1^*, \quad a_2^* a_{12} = \overline{\lambda} a_{12} a_2^*. \end{aligned}$$

Let us first observe that a_{12} is normal. Indeed,

$$\begin{aligned} a_{12}^* a_{12} &= (a_1^* a_2^* - \overline{\lambda} a_2^* a_1^*) a_{12} = \overline{\lambda} a_1^* a_{12} a_2^* - \overline{\lambda}\lambda a_2^* a_{12} a_1^* \\ &= \overline{\lambda}\lambda a_{12} a_1^* a_2^* - \overline{\lambda} a_{12} a_2^* a_1^* = a_{12} a_{12}^*. \end{aligned}$$

Moreover, $a_{12}^* a_{12}$ is contained in the center of the algebra

$$a_1^* a_{12}^* a_{12} = \overline{\lambda} a_{12}^* a_1^* a_{12} = \overline{\lambda}\lambda a_{12}^* a_{12} a_1^* = a_{12}^* a_{12} a_1^*,$$

taking the adjoint we get $a_{12}^* a_{12} a_1 = a_1 a_{12} a_{12}^*$; in the same way one can show that

$$a_2^* a_{12}^* a_{12} = a_{12}^* a_{12} a_2^* \quad \text{and} \quad a_{12}^* a_{12} a_2 = a_2 a_{12}^* a_{12}.$$

Further, we construct new generators of $\mathfrak{A}_{\lambda,3}$. Namely, put $b_2 \in \mathfrak{A}_{\lambda,3}$ to be

$$b_2 = a_2 - a_{12} a_1^*.$$

Obviously, $\mathfrak{A}_{\lambda,3}$ is generated as a $*$-algebra by the elements a_1, b_2 and a_{12}.

Lemma 1. *The following commutation relations hold:*

$$\begin{aligned} b_2^* b_2 - b_2 b_2^* &= 1 + a_{12}^* a_{12}, \\ a_1^* b_2 = \lambda b_2 a_1^*, &\quad b_2 a_1 = \lambda a_1 b_2, \\ a_{12}^* b_2 = \lambda b_2 a_{12}^*, &\quad b_2 a_{12} = \lambda a_{12} b_2. \end{aligned}$$

Proof.
1. We first show that $b_2^* b_2 - b_2 b_2^* = 1 + a_{12} a_{12}^*$

$$\begin{aligned} b_2^* b_2 - b_2 b_2^* &= (a_2^* - a_1 a_{12}^*)(a_2 - a_{12} a_1^*) - (a_2 - a_{12} a_1^*)(a_2^* - a_1 a_{12}^*) \\ &= a_2^* a_2 - a_2 a_2^* + (a_1 a_{12}^* a_{12} a_1^* - a_{12} a_1^* a_1 a_{12}^*) \\ &\quad + (a_2 a_1 a_{12}^* - a_1 a_{12}^* a_2) + (a_{12} a_1^* a_2^* - a_2^* a_{12} a_1^*) \\ &= 1 + (a_{12} a_{12}^* a_1 a_1^* - a_1^* a_1 a_{12} a_{12}^*) \\ &\quad + (a_2 a_1 a_{12}^* - \lambda a_1 a_2 a_{12}^*) + (a_{12} a_1^* a_2^* - \overline{\lambda} a_{12} a_2^* a_1^*) \\ &= 1 - a_{12} a_{12}^* + a_{12} a_{12}^* + a_{12} a_{12}^* = 1 + a_{12} a_{12}^*. \end{aligned}$$

2. We next prove that $a_1^* b_2 = \lambda b_2 a_1^*$, and $b_2 a_1 = \lambda a_1 b_2$

$$a_1^* b_2 = a_1^*(a_2 - a_{12} a_1^*) = \lambda a_2 a_1^* - \lambda a_{12} {a_1^*}^2 = \lambda b_2 a_1^*$$

and

$$\begin{aligned} b_2 a_1 &= (a_2 - a_{12} a_1^*) a_1 = a_2 a_1 - a_{12} a_1^* a_1 \\ &= a_{12} + \lambda a_1 a_2 - a_{12}(1 + a_1 a_1^*) = a_{12} + \lambda a_1 a_2 - a_{12} - \lambda a_1 a_{12} a_1^* \\ &= \lambda a_1 (a_2 - a_{12} a_1^*) = \lambda a_1 b_2. \end{aligned}$$

3. We also have $a_{12}^*b_2 = \lambda b_2 a_{12}^*$ and $b_2 a_{12} = \lambda a_{12} b_2$. Indeed, multiplying the equality $a_2 = b_2 + a_{12}a_1^*$ by a_{12} from the left we obtain

$$a_{12}a_2 = a_{12}b_2 + a_{12}^2 a_1^* = a_{12}b_2 + \overline{\lambda} a_{12}a_1^* a_{12}$$

implying

$$a_{12}b_2 = a_{12}a_2 - \overline{\lambda} a_{12}a_1^* a_{12} = \overline{\lambda} a_2 a_{12} - \overline{\lambda} a_{12}a_1^* a_{12} = \overline{\lambda} b_2 a_{12}.$$

Similarly, multiply $a_2^* = b_2^* + a_1 a_{12}^*$ by a_{12} from the right to get

$$a_2^* a_{12} = b_2^* a_{12} + a_1 a_{12}^* a_{12}.$$

Further we use that $a_{12}^* a_{12} = a_{12}a_{12}^*$ and $a_1 a_{12} = \overline{\lambda} a_{12} a_1$ to get

$$\overline{\lambda} a_{12} a_2^* = b_2^* a_{12} + \overline{\lambda} a_{12} a_1 a_{12}^*$$

or

$$b_2^* a_{12} = \overline{\lambda} a_{12}(a_2^* - a_1 a_{12}^*) = \overline{\lambda} a_{12} b_2^*.$$

□

Conversely, consider the $*$-algebra $\mathfrak{B}_{\lambda,2}$, generated by c_1, c_2, c_{12}, satisfying relations of the form

$$\begin{aligned} c_2^*c_2 - c_2c_2^* &= 1 + c_{12}^*c_{12}, \quad c_1^*c_1 - c_1c_1^* = 1, \quad c_{12}^*c_{12} = c_{12}c_{12}^*, \\ c_1^*c_2 &= \lambda c_2 c_1^*, \qquad c_2 c_1 = \lambda c_1 c_2, \\ c_1^* c_{12} &= \lambda c_{12} c_1^*, \quad c_{12}c_1 = \lambda c_1 c_{12}, \\ c_{12}^* c_2 &= \lambda c_2 c_{12}^*, \quad c_2 c_{12} = \lambda c_{12} c_2. \end{aligned} \tag{1}$$

Note that relations (1) imply that c_{12} is normal and $c_{12}^*c_{12}$ is contained in the center of $\mathfrak{B}_{\lambda,2}$. Put $d_2 = c_2 + c_{12}c_1^*$.

Lemma 2. *The elements c_1, d_2, c_{12} generate $\mathfrak{B}_{\lambda,2}$ and satisfy the following commutation relations:*

$$\begin{aligned} d_2^*d_2 - d_2d_2^* &= 1, \\ c_1^*d_2 = \lambda d_2 c_1^*, &\quad d_2c_1 - \lambda c_1 d_2 = c_{12}, \\ d_2^* c_{12} = \overline{\lambda} c_{12} d_2^*, &\quad c_{12} d_2 = \overline{\lambda} d_2 c_{12}. \end{aligned}$$

Proof.

1. First we show that $d_2^*d_2 - d_2d_2^* = 1$:

$$\begin{aligned} d_2^*d_2 - d_2d_2^* &= (c_2^* + c_1c_{12}^*)(c_2 + c_{12}c_1^*) - (c_2 + c_{12}c_1^*)(c_2^* + c_1c_{12}^*) \\ &= c_2^*c_2 - c_2c_2^* + (c_1c_{12}^*c_{12}c_1^* - c_{12}c_1^*c_1c_{12}^*) \\ &\quad + (c_1c_{12}^*c_2 - c_2c_1c_{12}^*) + (c_2^*c_{12}c_1^* - c_{12}c_1^*c_2^*) \\ &= 1 + c_{12}^*c_{12} + (c_1c_1^* - c_1^*c_1)c_{12}^*c_{12} \\ &\quad + (\lambda c_1c_2c_{12}^* - \lambda c_1c_2c_{12}^*) + (\overline{\lambda} c_{12}c_2^*c_1^* - \overline{\lambda} c_{12}c_2^*c_1^*) \\ &= 1 + c_{12}^*c_{12} - c_{12}^*c_{12} = 1. \end{aligned}$$

2. We prove that $c_1^*d_2 = \lambda d_2 c_1^*$:

$$\begin{aligned} c_1^*d_2 - \lambda d_2 c_1^* &= c_1^*(c_2 + c_{12}c_1^*) - \lambda(c_2 + c_{12}c_1^*)c_1^* \\ &= c_1^*c_2 - \lambda c_2 c_1^* + (c_1^*c_{12} - \lambda c_{12}c_1^*)c_1^* = 0. \end{aligned}$$

3. Let us verify that $d_2c_1 - \lambda c_1 d_2 = c_{12}$:

$$\begin{aligned} d_2c_1 - \lambda c_1 d_2 &= (c_2 + c_{12}c_1^*)c_1 - \lambda c_1(c_2 + c_{12}c_1^*) \\ &= c_2c_1 - \lambda c_1c_2 + (c_{12}c_1^*c_1 - \lambda c_1c_{12}c_1^*) \\ &= c_{12}(c_1^*c_1 - c_1c_1^*) = c_{12}. \end{aligned}$$

4. We have also that $d_2^* c_{12} = \overline{\lambda} c_{12} d_2^*$:

$$\begin{aligned} d_2^* c_{12} &= (c_2^* + c_1 c_{12}^*) c_{12} = c_2^* c_{12} + c_1 c_{12}^* c_{12} = c_2^* c_{12} + c_1 c_{12} c_{12}^* \\ &= \overline{\lambda} c_{12} c_2^* + \overline{\lambda} c_{12} c_1 c_{12}^* = \overline{\lambda} c_{12} (c_2^* + c_1 c_{12}) = \overline{\lambda} c_{12} d_2^*. \end{aligned}$$

5. Finally we show that $c_{12} d_2 = \overline{\lambda} d_2 c_{12}$:

$$\begin{aligned} c_{12} d_2 &= c_{12}(c_2 + c_{12} c_1 *) = \overline{\lambda} c_2 c_{12} + \overline{\lambda} c_{12} c_1^* c_{12} \\ &= \overline{\lambda} (c_2 + c_{12} c_1^*) c_{12} = \overline{\lambda} d_2 c_{12}. \end{aligned}$$

□

The following statement is evident.

Proposition 2. *The $*$-algebras $\mathfrak{A}_{\lambda,3}$ and $\mathfrak{B}_{\lambda,2}$ are isomorphic.*

Let us give definitions of well-behaved representations of $\mathfrak{B}_{\lambda,2}$ similar to those formulated above for the algebra $\mathfrak{A}_{\lambda,2}$.

Definition 7. *We say that closed operators C_1, C_2, C_{12} determine a well-behaved representation of $\mathfrak{B}_{\lambda,2}$ on a Hilbert space $\mathcal{H}$ if there exists a dense linear $\mathcal{D} \subset \mathcal{H}$ invariant with respect to C_i, C_i^*, C_{12}, C_{12}^*, $i = 1, 2$, and such that*

(1) *C_1, C_2, C_{12} satisfy relations (1) on $\mathcal{D}$;*
(2) *any $f \in \mathcal{D}$ is analytic for*

$$\Delta = C_1^* C_1 + C_2^* C_2 + C_{12}^* C_{12}.$$

Definition 8. *For closed operators C_1, C_2, C_{12} on a Hilbert space $\mathcal{H}$ let $C_i = S_i T_i$, $i = 1, 2$, $C_{12} = UT$ be the (left) polar decompositions of C_i, $i = 1, 2$, and C_{12}, and let $D_i = S_i T_i S_i^*$, $i = 1, 2$. We say that C_1, C_2, C_{12} determine a well-behaved representation of $\mathfrak{B}_{\lambda,2}$ if*

(1) *T_1, T_2 and T strongly commute, i.e., they commute in the sense of their resolutions of the identity;*
(2) *for any real bounded Borel function $F(\cdot)$ and $i, j = 1, 2$,*

$$\begin{aligned} &F(D_1^2) S_1 = S_1 F(\mathbf{1} + D_1^2), \quad F(D_2^2) S_2 = S_2 F(\mathbf{1} + T^2 + D_2^2), \\ &F(D_i^2) S_j = S_j F(D_i^2), \quad F(T) S_j = S_j F(T), \\ &F(D_i^2) U = U F(D_i^2), \quad F(T) U = U F(T). \end{aligned}$$

(3) *S_i, $i = 1, 2$, are isometries that together with the partial isometry U satisfy the following commutation relations:*

$$\begin{aligned} S_1^* S_2 &= \lambda S_2 S_1^*, \quad S_2 S_1 = \lambda S_1 S_2, \\ S_1^* U &= \lambda U S_1^*, \quad U S_1 = \lambda S_1 U, \\ S_2^* U &= \overline{\lambda} U S_2^*, \quad U S_2 = \overline{\lambda} S_2 U. \end{aligned}$$

Definitions 7 and 8 are equivalent, the idea of the proof is the same as in the case of the equivalence of **Definitions 3, 4**, see [12].

Further, definitions of irreducible well-behaved representation and unitary equivalent well-behaved representations of $\mathfrak{B}_{\lambda,2}$ can be given in a similar way as it was done for $\mathfrak{A}_{\lambda,2}$. Namely, denote by $E_j(\cdot)$ the resolutions of the identity of the operators D_j^2, $j = 1, 2$, and by $E(\cdot)$ the resolution of the identity of T^2.

Definition 9. *We say that a well-behaved representation of $\mathfrak{B}_{\lambda,2}$ is irreducible iff the family of bounded operators*

$$\mathcal{F} = \{S_i,\ S_i^*,\ U,\ U^*,\ E_j(\delta_j),\ E(\delta) \mid \delta, \delta_j \subset \mathfrak{B}(\mathbb{R}_+),\ j = 1, 2\}$$

is irreducible.

Remark 5. *A well-behaved representation of $\mathfrak{B}_{\lambda,2}$, determined by the operators C_1, C_2, C_{12} on $\mathcal{H}$ is irreducible iff there is no non-trivial subspace $\mathcal{K} \subset \mathcal{H}$ and a dense linear domain $\mathcal{D}_1 \subset \mathcal{K}$, satisfying the conditions of* **Definition 7**.

Definition 10. *Well-behaved representations corresponding to families $\mathcal{F}_i$, $i = 1, 2$ are unitary equivalent iff the families $\mathcal{F}_i$, $i = 1, 2$, are unitary equivalent.*

Below we give a classification of irreducible well-behaved representations of $\mathfrak{B}_{\lambda,2}$.

Theorem 2. *For any irreducible well-behaved representation of $\mathfrak{B}_{\lambda,2}$ one has $T = \rho\mathbf{1}$ for some $\rho \geq 0$. Moreover, any such representation is unitary equivalent to one presented below.*

(1) *If $\rho = 0$, then $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ and*

$$T = 0, \quad U = 0,$$
$$D_1 = D \otimes \mathbf{1}, \quad D_2 = \mathbf{1} \otimes D,$$
$$S_1 = S \otimes \mathbf{1}, \quad S_2 = d(\lambda) \otimes S.$$

(2) *If $\rho > 0$, then $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ and*

$$T = \rho \cdot \mathbf{1} \otimes \mathbf{1}, \quad D_1 = D \otimes \mathbf{1}, \quad D_2 = \sqrt{1 + \rho^2} \cdot (\mathbf{1} \otimes D),$$
$$U = e^{i\phi} \cdot d(\lambda) \otimes d(\overline{\lambda}), \quad S_1 = S \otimes \mathbf{1}, \quad S_2 = d(\lambda) \otimes \mathbf{1},$$

where, $\phi \in [0, 2\pi)$ and, as above, $D, S, d(\lambda) \colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$ are given by

$$Se_n = e_{n+1}, \quad De_n = \sqrt{n} e_n, \quad d(\lambda) e_n = \lambda^n e_n, \quad n \in \mathbb{Z}_+.$$

Representations corresponding to different values of parameters are non-equivalent.

Proof. We note first that **Definition 8** implies that the spectral projections of T commute with the spectral projections $E_j(\delta_j)$, $\delta_j \subset \mathfrak{B}(\mathbb{R})$, and the operators S_j, S_j^*, $j = 1, 2$, U, U^*. Then using Schur's Lemma we state that $E(\delta)$ is a scalar operator for any $\delta \subset \mathfrak{B}(\mathbb{R})$. Hence T is a scalar operator, $T = \rho\mathbf{1}$ for some $\rho \geq 0$.

If $\rho = 0$ we get $T = 0$, $U = 0$ and the operators S_i, D_i, $i = 1, 2$, determine an irreducible well-behaved representation of $\mathfrak{A}_{\lambda,2}$ considered in **Theorem 1**.

Let $\rho \neq 0$. Then U is unitary and S_i, S_i^*, $i = 1, 2$, D_1, $(1 + \rho^2)^{-\frac{1}{2}} D_2$ determine a well-behaved representation of $\mathfrak{A}_{\lambda,2}$. In particular, S_1, S_2 are pure isometries and the spectral decompositions of D_1, D_2 have the following form:

$$D_1 = \sum_{n=0}^{\infty} \sqrt{n} (S_1^n S_1^{*n} - S_1^{n+1} S_1^{*n+1}),$$
$$D_2 = (1 + \rho^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sqrt{n} (S_2^n S_2^{*n} - S_2^{n+1} S_2^{*n+1}).$$

Evidently, the representation is irreducible iff such is the family

$$\mathcal{S} = \{S_i, S_i^*, U, U^*, \ i = 1, 2\}.$$

Similarly, two such representations are unitary equivalent iff such are the corresponding families $\mathcal{S}$.

So it remains to classify irreducible families $\mathcal{S} = \{S_i, S_i^*, U, \ i = 1, 2\}$, satisfying relations of the form

$$S_i^* S_i = \mathbf{1}, \quad S_1^* S_2 = \lambda S_2 S_1^*, \quad i = 1, 2,$$
$$S_1^* U = \lambda U S_1^*, \quad U S_1 = \lambda S_1 U, \quad S_2^* U = \overline{\lambda} U S_2^*, \quad U S_2 = \overline{\lambda} S_2 U,$$

here U is unitary and S_1, S_2 are pure isometries. It follows from results of [11] that, up to unitary equivalence, any such family acts on $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ as follows:

$$S_1 = S \otimes \mathbf{1}, \quad S_2 = d(\lambda) \otimes S, \quad U = e^{i\phi} \cdot d(\lambda) \otimes d(\overline{\lambda}),$$

where $\phi \in [0, 2\pi)$.

Finally, from spectral decomposition formulas we get that

$$D_1 = D \otimes \mathbf{1}, \quad D_2 = (1 + \rho^2)^{\frac{1}{2}} \mathbf{1} \otimes D.$$

Obviously, representations corresponding to different pairs (ρ, ϕ), $\rho > 0$, $\phi \in [0, 2\pi)$ are non-equivalent. □

We return now to the algebra $\mathfrak{A}_{\lambda,3}$.

The following definition is a natural generalization of a well-behaved representation of $\mathfrak{A}_{1,3}$ presented in [8].

Definition 11. *We say that closed operators A_1, A_2, A_{12} determine a well-behaved representation of $\mathfrak{A}_{\lambda,3}$ iff $C_1 = A_1$, $C_2 = B_2 = A_2 - A_{12}A_1^*$, $C_{12} = A_{12}$ determine a well-behaved representation of $\mathfrak{B}_{\lambda,2}$.*

We also call a representation of $\mathfrak{A}_{\lambda,3}$ irreducible iff such is the corresponding representation of $\mathfrak{B}_{\lambda,2}$. The equivalence of representations is defined in the same manner.

Applying **Theorem 2** we get immediately the following result.

Theorem 3. *Let operators A_1, A_2, A_{12} determine an irreducible well-behaved representation of $\mathfrak{A}_{\lambda,3}$ on a Hilbert space $\mathcal{H}$. Then, up to unitary equivalence, $\mathcal{H} = l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}_+)$ and*

$$\begin{aligned} A_1 &= a \otimes \mathbf{1}, \\ A_2 &= \sqrt{1+\rho^2} \cdot d(\lambda) \otimes a + \rho e^{i\phi} \cdot d(\lambda) a^* \otimes d(\overline{\lambda}), \\ A_{12} &= \rho e^{i\phi} \cdot d(\lambda) \otimes d(\overline{\lambda}). \end{aligned} \tag{2}$$

Representations corresponding to $(\rho_1, \phi_1) \neq (\rho_2, \phi_2)$, where $\rho_1 > 0$, are non-equivalent. Representations corresponding to $(0, \phi)$ are unitary equivalent for any $\phi \in [0, 2\pi)$.

Proof. To prove the theorem we only note that due to **Theorem 2** we have

$$A_1 = a \otimes \mathbf{1}, \quad B_2 = d(\lambda) \otimes a, \quad A_{12} = \rho e^{i\phi} \cdot d(\lambda) \otimes d(\overline{\lambda}),$$

for some $\rho \geq 0$ and $\phi \in [0, 2\pi)$. □

Fix $\rho > 0$ and consider the $*$-algebra $\mathfrak{A}_{\lambda,3,\rho}$ generated by elements a_1, a_2, u, subject to the relations

$$\begin{aligned} a_i^* a_i - a_i a_i^* = 1, \quad i = 1, 2, \quad a_1^* a_2 = \lambda a_2 a_1, \\ a_2 a_1 - \lambda a_1 a_2 = \rho u, \quad u^* u = 1, \\ a_1^* u = \lambda u a_1^*, \quad a_2^* u = \overline{\lambda} u a_2^*. \end{aligned} \tag{3}$$

Let us give a definition of well-behaved representations of $\mathfrak{A}_{\lambda,3,\rho}$ in terms of a_i, $i = 1, 2$, and u.

Definition 12. *We call a representation of $\mathfrak{A}_{\lambda,3,\rho}$, determined by closed operators A_1, A_2 and unitary U, well-behaved if there exists a dense linear domain $\mathcal{D} \subset \mathcal{H}$ such that $\mathcal{D}$ is invariant with respect to U, A_i, A_i^*, $i = 1, 2$, any $f \in \mathcal{D}$ is jointly analytic in a strong sense for the family $\{A_i, A_i^*,\ i = 1, 2\}$ and the operators A_1, A_2, U satisfy (3) on $\mathcal{D}$.*

Definition 13. *Say that a well-behaved representation of $\mathfrak{A}_{\lambda,3,\rho}$ is irreducible if there is no closed subspace $\mathcal{K} \subset \mathcal{H}$ and a dense linear domain $\mathcal{D}_1 \subset \mathcal{K}$ satisfying conditions of* **Definition 12**.

Theorem 4. *Let $\rho > 0$. Any irreducible well-behaved representation of $\mathfrak{A}_{\lambda,3,\rho}$ is determined, up to unitary equivalence, by $\{A_1, A_2, U := \rho^{-1}A_{12}\}$, where A_1, A_2, A_{12} are given by (2) and $\phi \in [0, 2\pi)$.*

Proof.
1. We show first that if $\{A_1, A_2, U\}$ satisfy the conditions of **Definition 12** then $\{A_1, A_2, \rho U\}$ determine a well-behaved representation of $\mathfrak{A}_{\lambda,3}$ (in the sense of **Definition 11**). Indeed, evidently $\mathcal{D}$ is invariant with respect to B_2, B_2^*, where $B_2 = A_2 - \rho U A_1^*$. Next we see that any vector of $\mathcal{D}$, which is jointly analytic in a strong sense for the family $\{A_1, A_1^*, B_2, B_2^*\}$, is analytic for the operator

$$\Delta = A_1^*A_1 + B_2^*B_2 + \rho^2 U^*U = \rho^2\mathbf{1} + A_1^*A_1 + B_2^*B_2.$$

Since

$$UA_1 = \lambda A_1 U, \quad A_1^*U = \lambda U A_1^*, \quad A_2^*U = \overline{\lambda} U A_2^*, \quad UA_2 = \overline{\lambda} A_2 U$$

and $|\lambda| = 1$, for any product ν of the elements A_i^*, A_i, U, U^*, $i = 1, 2$ and $f \in \mathcal{D}$ we get

$$||\nu f|| = ||\widehat{\nu} f||,$$

where the product $\widehat{\nu}$ is obtained from ν by dropping all the factors equal to U or U^*; the empty product is assumed to be equal to the identity operator.

Denote by $\Delta_k^{(1)}$ (resp. $\Delta_k^{(2)}$) the set of all products of the operators A_i, A_i^*, $i = 1, 2$, (resp. A_1, A_1^*, B_2, B_2^*) of length k.

Note that, for any $\mu \in \Delta_k^{(2)}$,

$$||\mu f|| \le \sum_{\nu \in \Delta_k^{(1)}} (1 + \rho)^k ||\nu f||, \quad f \in \mathcal{D},$$

and hence

$$\sum_{\mu \in \Delta_{2n}^{(2)}} ||\mu f|| \le \sum_{\mu \in \Delta_{2n}^{(2)}} \sum_{\nu \in \Delta_{2n}^{(1)}} (1 + \rho)^{2n} ||\nu f||, \quad f \in \mathcal{D}.$$

As f is jointly analytic in a strong sense for the family $\{A_i, A_i^*, i = 1, 2\}$ we get

$$\sum_{\nu \in \Delta_{2n}^{(1)}} (1 + \rho)^{2n} ||\nu f|| \le M^n n!$$

for some $M > 0$. Therefore,

$$\sum_{\mu \in \Delta_{2n}^{(2)}} ||\mu f|| \le 4^{2n} M^n n!$$

and hence f is jointly analytic in a strong sense for $\{A_1, A_1^*, B_2, B_2^*\}$.

Obviously, a well-behaved representation of $\mathfrak{A}_{\lambda,3,\rho}$ is irreducible iff the corresponding representation of $\mathfrak{B}_{\lambda,2}$ is irreducible, hence one can apply formulas (2).
2. Next we prove that if A_1, A_2, A_{12}, are given by (2), then A_1, A_2, $U = \rho^{-1}A_{12}$ satisfy the conditions of **Definition 12** on the domain $\mathcal{D} = \mathbb{C}\langle e_r \otimes e_s,\ r, s \in \mathbb{Z}_+\rangle$.

Put $R_2 = \sqrt{1 + \rho^2} \cdot d(\lambda) \otimes a$ and $Q_2 = \rho e^{i\phi} d(\lambda) a^* \otimes d(\lambda)$. We show that the vectors of $\mathcal{D}$ are jointly analytic in a strong sense for the family $\mathcal{F}_3 = \{A_1, A_1^*, R_2, R_2^*, Q_2, Q_2^*\}$. Below by $\Delta_k^{(3)}$ we denote the set of all products of the elements of $\mathcal{F}_3$ of length k. Recall that

$$ae_n = \sqrt{n+1} e_{n+1}, \quad a^*e_n = \sqrt{n} e_{n-1}, \quad a^*e_0 = 0.$$

Then for any product μ of operators a, a^* of length k and any $r \in \mathbb{Z}_+$ one has

$$||\mu e_r|| \le ||a^k e_r||.$$

Further, since $d(\lambda)$ is unitary and $a^*d(\lambda) = \lambda d(\lambda)a^*$, $d(\lambda)a = \lambda a d(\lambda)$, for any product $\nu \in \Delta_{2n}^{(3)}$ and $r, s \in \mathbb{Z}_+$, $g \in \mathcal{D}$, $||g|| = 1$, one has

$$||\nu e_r \otimes e_s|| \leq (1+\rho^2)^n(||(a^{2n}e_r) \otimes e_s|| + ||e_r \otimes (a^{2n}e_s)||) \leq (1+\rho^2)^n 2 \prod_{k=1}^{2n} \sqrt{l+k},$$

where $l = \max\{r, s\}$.

Denote by $\alpha_n = \dfrac{\prod_{k=1}^{2n} \sqrt{l+k}}{n!}$. Then

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{\sqrt{(l+2n+1)(l+2n+2)}}{n+1} \to 2, \quad n \to \infty.$$

Hence, there exists $M > 1$ such that $\frac{\alpha_{n+1}}{\alpha_n} \leq M$, $n \geq 1$, and

$$||\nu e_r \otimes e_s|| \leq (1+\rho^2)^n \alpha_n \cdot n! \leq 2\alpha_1(1+\rho^2)^n M^n n!.$$

Finally,

$$\sum_{\nu \in \Delta_{2n}} ||\nu e_r \otimes e_s|| \leq \sum_{\nu \in \Delta_{2n}} \alpha_n \cdot n! \leq 4^{2n} \cdot 2\alpha_1(1+\rho^2)^n M^n n!.$$

Since $A_2 = R_2 + Q_2$ for any product μ of length k of the operators A_i, A_i^*, $i = 1, 2$, and any $f \in \mathcal{D}$ we get

$$||\mu f|| \leq \sum_{\nu \in \Delta_k^{(3)}} ||\nu f||.$$

Then

$$\sum_{\mu \in \Delta_{2n}^{(1)}} ||\mu f|| \leq \sum_{\mu \in \Delta_{2n}^{(1)}} \sum_{\nu \in \Delta_{2n}^{(3)}} ||\nu f|| \leq 4^{2n} \sum_{\nu \in \Delta_{2n}^{(3)}} ||\nu f||.$$

Thus f is jointly analytic in a strong sense for the family $\{A_i, A_i^*,\ i = 1, 2\}$. □

4. The case $\lambda = 1$

In this section we study representations of $\mathfrak{A}_{\lambda,3}^{(d)}$ with $\lambda = 1$. In this case the elements $y_{ij} = a_j a_i - a_i a_j$ belong to the center of the algebra. So, keeping in mind irreducible representations, we shall assume that $y_{ij} \in \mathbb{C}$, $i \neq j$, i.e., we shall consider the $*$-algebra $\mathfrak{A}_{1,3}^{(d)}(\{y_{ij}\})$ generated by the following set of relations:

$$a_i^* a_i - a_i a_i^* = 1, \quad a_j^* a_i - a_i a_j^* = 0, \quad i, j = 1, \dots, d, \quad i \neq j,$$
$$a_j a_i - a_i a_j = y_{j,i}\mathbf{1}, \quad j > i.$$

Let us introduce a new family of generators of $\mathfrak{A}_{1,3}^{(d)}(\{y_{ij}\})$,

$$b_1 = a_1, \quad b_j = a_j - y_{j,1} a_1^*, \quad j = 2, \dots, d.$$

Lemma 3. *The elements b_j, $j = 2, \dots, d$, commute with b_1 and b_1^* and satisfy the relations*

$$(4) \qquad \begin{aligned} b_i^* b_j - b_j b_i^* &= \overline{y_{i,1}} y_{j,1}\mathbf{1} \quad j \neq i, \\ b_i^* b_i - b_i b_i^* &= 1 + |y_{i,1}|^2, \\ b_j b_i - b_i b_j &= y_{j,i}\mathbf{1}. \end{aligned}$$

Proof. Evidently, $a_1^* b_j - b_j a_1^* = 0$ and

$$a_1 b_j - b_j a_1 = (a_1 a_j - y_{j,1} a_1 a_1^*) - (a_j a_1 - y_{j,1} a_1^* a_1) = (-y_{j,1} + y_{j,1})\mathbf{1} = 0.$$

Further, for $i \neq j$ one has

$$\begin{aligned} b_i^* b_j - b_j b_i^* =& (a_i^* - \bar{y_{i,1}} a_1)(a_j - y_{j,1} a_1^*) - (a_j - y_{j,1} a_1^*)(a_i^* - \bar{y_{i,1}} a_1) \\ =& (a_i^* a_j - a_j a_i^*) - \overline{y_{i,1}}(a_1 a_j - a_j a_1) - y_{j,1}(a_i^* a_1^* - a_1^* a_i^*) \\ &+ \overline{y_{i,1}} y_{j,1}(a_1 a_1^* - a_1^* a_1) = \overline{y_{i,1}} y_{j,1} \mathbf{1}. \end{aligned}$$

Analogously, one gets

$$b_i^* b_i - b_i b_i^* = 1 + |y_{i,1}|^2.$$

Finally, for $i \neq j$,

$$b_j b_i - b_i b_j = (a_j - y_{j,1} a_1^*)(a_i - y_{i,1} a_1^*) - (a_i - y_{i,1} a_1^*)(a_j - y_{j,1} a_1^*) = y_{ji} \mathbf{1}.$$

□

Let $\alpha_{i,j} = \overline{y_{i,1}} y_{j,1}$. At the next step of reduction, we put $c_i = b_i$, $i = 1, 2$, and

$$c_k = b_k - \frac{\overline{y_{2,1}} y_{k,1}}{1 + |y_{2,1}|^2} b_2 + \frac{y_{2,k}}{1 + |y_{2,1}|^2} b_2^* = b_k - \frac{\alpha_{2,k}}{1 + \alpha_{2,2}} b_2 + \frac{y_{2,k}}{1 + \alpha_{2,2}} b_2^*, \quad k = 3, \ldots, d.$$

Lemma 4.

The elements c_i and c_i^*, $i = 1, 2$, commute with all c_k, $k > 2$, and

$$\begin{aligned} c_j c_i - c_i c_j &= (y_{j,i} - \frac{\alpha_{2,i} y_{j,2} + \alpha_{2,j} y_{2,i}}{1 + \alpha_{2,2}}) \mathbf{1}, \quad j > i, \\ c_i^* c_j - c_j c_i^* &= \frac{\alpha_{i,j} + \overline{y_{2,i}} y_{2,j}}{1 + \alpha_{2,2}}, \quad i \neq j, \\ c_i^* c_i - c_i c_i^* &= 1 + \frac{\alpha_{i,i} + |y_{2,i}|^2}{1 + \alpha_{2,2}}. \end{aligned}$$

So, for the generators c_i, $i > 2$, we have obtained relations similar to (4) and can continue the process inductively.

If all $y_{j,i} = y$, $j > i$, then the reduction looks very simple,

$$c_k = b_k - \frac{\alpha}{1 + \alpha} b_2 - \frac{y}{1 + \alpha} b_2^*,$$

where $\alpha = |y|^2$, and the relations between b_j, $j = 1, \ldots, d$,

$$\begin{aligned} b_j b_1 - b_1 b_j &= 0, \quad b_j b_1^* - b_1^* b_j = 0, \quad j > 1, \\ b_j b_i - b_i b_j &= y, \quad b_i^* b_j - b_j b_i^* = \alpha, \quad i \neq j, \quad i, j > 1, \\ b_1^* b_1 - b_1 b_1^* &= 1, \quad b_i^* b_i - b_i b_i^* = 1 + \alpha, \quad i > 1, \end{aligned}$$

transform into

$$\begin{aligned} c_j c_i - c_i c_j &= 0, \quad c_i^* c_j - c_j c_i^* = 0, \quad i = 1, 2, \quad j > 2, \\ c_j c_i - c_i c_j &= y, \quad c_i^* c_j - c_j c_i^* = \frac{\alpha + |y|^2}{1 + \alpha}, \quad i \neq j, \quad i, j > 2, \\ c_1^* c_1 - c_1 c_1^* &= 1, \quad c_2^* c_2 - c_2 c_2^* = 1 + \alpha, \\ c_i^* c_i - c_i c_i^* &= 1 + \frac{\alpha + |y|^2}{1 + \alpha}, \quad i > 2. \end{aligned}$$

We will get at the end that the algebra $\mathfrak{A}_{1,3}^{(d)}(y)$ is isomorphic to the algebra generated by d_k, d_k^*, $k = 1, \ldots, d$, such that

$$[d_i^*, d_j] = 0, \quad [d_i, d_j] = 0, \quad i \neq j.$$

and

$$d_i^* d_i - d_i^* d_i = 1 + \alpha_i, \quad i = 1, \dots, d,$$

where α_i are defined recursively,

$$\alpha_1 = |y|^2, \quad \alpha_{k+1} = \frac{\alpha_k + |y|^2}{1 + \alpha_k}.$$

The isomorphism is given by

$$\begin{aligned}
d_1 &= a_1, \\
d_2 &= a_2 - y d_1^*, \\
d_3 &= a_3 - y d_1^* - \frac{\alpha_1}{1+\alpha_1} d_2 - \frac{y}{1+\alpha_1} d_2^*, \\
&\dots \\
d_k &= a_k - y d_1^* - \frac{\alpha_1}{1+\alpha_1} d_2 - \frac{y}{1+\alpha_1} d_2^* - \dots - \frac{\alpha_{k-2}}{1+\alpha_{k-2}} d_{k-1} - \frac{y}{1+\alpha_{k-2}} d_{k-1}^*.
\end{aligned}$$

The theory of irreducible well-behaved representations of the algebra generated by d_i, $i = 1, \dots, d$, satisfying the above relations is well-understood: there is the only one irreducible well-behaved representation given on $\mathcal{H} = \ell^2(\mathbb{Z}_+)^{\otimes d}$ by

$$d_i = \sqrt{1+\alpha_i} 1 \otimes \dots \otimes 1 \otimes \underbrace{a}_{i} \otimes 1 \dots \otimes 1, \; i = 1, \dots, d,$$

where a is the creation operator $a e_n = \sqrt{n+1} e_{n+1}$ on $\ell^2(\mathbb{Z}_+)$.

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