Methods of Functional Analysis and Topology
Vol. 23 (2017), no. 3, pp. 212–230

TRANSFORMATIONS OF NEVANLINNA OPERATOR-FUNCTIONS AND THEIR FIXED POINTS

YU. M. ARLINSKIĬ

To Eduard R. Tsekanovskiĭ on the occasion of his 80th birthday

Abstract. We give a new characterization of the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ of the operator-valued in the Hilbert space $\mathfrak{M}$ Nevanlinna functions that admit representations as compressed resolvents (m-functions) of selfadjoint contractions. We consider the automorphism $\mathbf{\Gamma}: M(\lambda)\mapsto M_{\mathbf{\Gamma}}(\lambda) := \left((\lambda^2-1)M(\lambda)\right)^{-1}$ of the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ and construct a realization of $M_{\mathbf{\Gamma}}(\lambda)$ as a compressed resolvent. The unique fixed point of $\mathbf{\Gamma}$ is the m-function of the block-operator Jacobi matrix related to the Chebyshev polynomials of the first kind. We study a transformation $\widehat{\mathbf{\Gamma}}: \mathcal{M}(\lambda)\mapsto \mathcal{M}_{\widehat{\mathbf{\Gamma}}}(\lambda) := -(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1}$ that maps the set of all Nevanlinna operator-valued functions into its subset. The unique fixed point $\mathcal{M}_0$ of $\widehat{\mathbf{\Gamma}}$ admits a realization as the compressed resolvent of the "free" discrete Schrödinger operator $\widehat{\mathbf{J}}_0$ in the Hilbert space $\mathbf{H}_0 = \ell^2(\mathbb{N}_0)\bigotimes\mathfrak{M}$. We prove that $\mathcal{M}_0$ is the uniform limit on compact sets of the open upper/lower half-plane in the operator norm topology of the iterations $\{\mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda)+\lambda I_{\mathfrak{M}})^{-1}\}$ of $\widehat{\mathbf{\Gamma}}$. We show that the pair $\{\mathbf{H}_0, \widehat{\mathbf{J}}_0\}$ is the inductive limit of the sequence of realizations $\{\widehat{\mathfrak{H}}_n, \widehat{A}_n\}$ of $\{\mathcal{M}_n\}$. In the scalar case ($\mathfrak{M}=\mathbb{C}$), applying the algorithm of I. S. Kac, a realization of iterates $\{\mathcal{M}_n(\lambda)\}$ as m-functions of canonical (Hamiltonian) systems is constructed.

1. Introduction and preliminaries

Notations. We use the symbols $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ for the domain, the range, and the null-subspace of a linear operator T. The closures of $\operatorname{dom} T$, $\operatorname{ran} T$ are denoted by $\overline{\operatorname{dom}}\, T$, $\overline{\operatorname{ran}}\, T$, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by I and sometimes by $I_{\mathfrak{H}}$. If $\mathfrak{L}$ is a subspace, i.e., a closed linear subset of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T{\upharpoonright}\mathfrak{L}$ means the restriction of a linear operator T on the set $\mathfrak{L}\subset \operatorname{dom} T$. The resolvent set of T is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathbf{B}(\mathfrak{H},\mathfrak{K})$ and the Banach algebra $\mathbf{B}(\mathfrak{H},\mathfrak{H})$ by $\mathbf{B}(\mathfrak{H})$. Throughout this paper we consider separable Hilbert spaces over the field $\mathbb{C}$ of complex numbers. $\mathbb{C}_+/\mathbb{C}_-$ denotes the open upper/lower half-plane of $\mathbb{C}$, $\mathbb{R}_+ := [0,+\infty)$, $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

Definition 1.1. *A $\mathbf{B}(\mathfrak{M})$-valued function M is called a Nevanlinna function (R-function* [15]*,* [20]*, Herglotz function* [12]*, Herglotz-Nevanlinna function* [1]*,* [3]*) if it is holomorphic outside the real axis, symmetric $M(\lambda)^* = M(\bar\lambda)$, and satisfies the inequality* $\operatorname{Im}\lambda \operatorname{Im} M(\lambda)\ge 0$ *for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$.*

This class is often denoted by $\mathcal{R}[\mathfrak{M}]$. A more general is the notion of Nevanlinna family, cf. [9].

2010 *Mathematics Subject Classification.* 47A06, 47A56, 47B25, 47B36.

Key words and phrases. Nevanlinna operator-valued function, compressed resolvent, fixed point, block-operator Jacobi matrix, canonical system.

Definition 1.2. *A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, in a Hilbert space $\mathfrak{M}$ is called a Nevanlinna family if:*

(1) $\mathcal{M}(\lambda)$ *is maximal dissipative for every* $\lambda \in \mathbb{C}_+$ (*resp. accumulative for every* $\lambda \in \mathbb{C}_-$);
(2) $\mathcal{M}(\lambda)^* = \mathcal{M}(\bar{\lambda})$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$;
(3) *for some, and hence for all,* $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ *the operator family* $(\mathcal{M}(\lambda)+\mu I_{\mathfrak{M}})^{-1}(\in \mathbf{B}(\mathfrak{M}))$ *is holomorphic on* $\mathbb{C}_+(\mathbb{C}_-)$.

The class of all Nevanlinna families in a Hilbert space $\mathfrak{M}$ is denoted by $\widetilde{R}(\mathfrak{M})$. Each Nevanlinna family $\mathcal{M} \in \widetilde{R}(\mathfrak{M})$ admits the following decomposition to the operator part $M_s(\lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and constant multi-valued part M_∞:

$$\mathcal{M}(\lambda) = M_s(\lambda) \oplus M_\infty, \quad M_\infty = \{0\} \times \operatorname{mul}\mathcal{M}(\lambda).$$

Here $M_s(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathfrak{M} \ominus \operatorname{mul}\mathcal{M}(\lambda)$.

A Nevanlinna $\mathbf{B}(\mathfrak{M})$-valued function admits the integral representation, see [15], [20],

$$M(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d\Sigma(t), \quad \int_{\mathbb{R}} \frac{d\Sigma(t)}{t^2+1} \in \mathbf{B}(\mathfrak{M}), \tag{1.1}$$

where $A = A^* \in \mathbf{B}(\mathfrak{M})$, $0 \leq B = B^* \in \mathbf{B}(\mathfrak{M})$, the $\mathbf{B}(\mathfrak{M})$-valued function $\Sigma(\cdot)$ is nondecreasing and $\Sigma(t) = \Sigma(t-0)$. The integral is uniformly convergent in the strong topology; cf. [8], [15]. The following condition is equivalent to the definition of a $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function $M(\lambda)$ holomorphic on $\mathbb{C}\backslash\mathbb{R}$: the function of two variables

$$K(\lambda,\mu) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}}$$

is a nonnegative kernel, i.e., $\sum\limits_{k,l=1}^{n} (K(\lambda_k,\lambda_l)f_l, f_k) \geq 0$ for an arbitrary set of points $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}_+/(\subset \mathbb{C}_-)$ and an arbitrary set of vectors $\{f_1, f_2, \ldots, f_n\} \subset \mathfrak{M}$.

It follows from (1.1) that

$$B = s - \lim_{y\uparrow\infty} \frac{M(iy)}{y} = s - \lim_{y\uparrow\infty} \frac{\operatorname{Im} M(iy)}{y},$$

$$\operatorname{Im} M(iy) = B\,y + \int_{\mathbb{R}} \frac{y}{t^2+y^2}\, d\Sigma(t),$$

and this implies that $\lim_{y\to\infty} y \operatorname{Im} M(iy)$ exists in the strong resolvent sense as a selfadjoint relation; see e.g. [5]. This limit is a bounded selfadjoint operator if and only if $B = 0$ and $\int_{\mathbb{R}} d\Sigma(t) \in \mathbf{B}(\mathfrak{M})$, in which case $s - \lim_{y\to\infty} y \operatorname{Im} M(iy) = \int_{\mathbb{R}} d\Sigma(t)$. In this case one can rewrite the integral representation (1.1) in the form

$$M(\lambda) = E + \int_{\mathbb{R}} \frac{1}{t-\lambda}\, d\Sigma(t), \quad \int_{\mathbb{R}} d\Sigma(t) \in \mathbf{B}(\mathfrak{M}), \tag{1.2}$$

and $E = \lim_{y\to\infty} M(iy)$ in $\mathbf{B}(\mathfrak{M})$.

The class of $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna functions M with the integral representation (1.2) with $E = 0$ is denoted by $\mathcal{R}_0[\mathfrak{M}]$. In this paper we will consider the following subclasses of the class $\mathcal{R}_0[\mathfrak{M}]$.

Definition 1.3. *A function N from the class $\mathcal{R}_0[\mathfrak{M}]$ is said to belong to the class*

(1) $\mathcal{N}[\mathfrak{M}]$ *if* $s - \lim_{y\to\infty} iyN(iy) = -I_{\mathfrak{M}}$,
(2) $\mathbf{N}^0_{\mathfrak{M}}$ *if* $N \in \mathcal{N}[\mathfrak{M}]$ *and N is holomorphic at infinity,*
(3) $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ *if* $N \in \mathbf{N}^0_{\mathfrak{M}}$ *and is holomorphic outside the interval* $[-1,1]$.

Thus, we have inclusions

$$\mathbf{N}^0_{\mathfrak{M}}[-1,1] \subset \mathbf{N}^0_{\mathfrak{M}} \subset \mathcal{N}[\mathfrak{M}] \subset \mathcal{R}_0[\mathfrak{M}] \subset \mathcal{R}[\mathfrak{M}] \subset \widetilde{R}(\mathfrak{M}).$$

A selfadjoint operator T in the Hilbert space $\mathfrak{H}$ is called $\mathfrak{M}$-*simple*, where $\mathfrak{M}$ is a subspace of $\mathfrak{H}$, if $\overline{\operatorname{span}}\{(T-\lambda I)^{-1}\mathfrak{M},\ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\} = \mathfrak{H}$. If T is bounded then the latter condition is equivalent to $\overline{\operatorname{span}}\{T^n\mathfrak{M},\ n \in \mathbb{N}_0\} = \mathfrak{H}$.

The next theorem follows from [8, Theorem 4.8] and the Naĭmark's dilation theorem [8, Theorem 1, Appendix I], see [2] and [3] for the case $M \in \mathbf{N}^0_{\mathfrak{M}}$.

Theorem 1.4. *1) If $M \in \mathcal{N}[\mathfrak{M}]$, then there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{M}$ as a subspace and a selfadjoint operator T in $\mathfrak{H}$ such that T is $\mathfrak{M}$-simple and*

$$M(\lambda) = P_{\mathfrak{M}}(T-\lambda I)^{-1}\upharpoonright\mathfrak{M}. \tag{1.3}$$

for λ in the domain of M. If $M \in \mathbf{N}^0_{\mathfrak{M}}$, then T is bounded and if $M \in \mathbf{N}^0_{\mathfrak{M}}[-1,1]$, then T is a selfadjoint contraction.

2) If T_1 and T_2 are selfadjoint operators in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, $\mathfrak{M}$ is a subspace in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, T_1 and T_2 are $\mathfrak{M}$-simple, and

$$M(\lambda) = P_{\mathfrak{M}}(T_1-\lambda I_{\mathfrak{H}_1})^{-1}\upharpoonright\mathfrak{M} = P_{\mathfrak{M}}(T_2-\lambda I_{\mathfrak{H}_2})^{-1}\upharpoonright\mathfrak{M}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

then there exists a unitary operator U mapping $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$U\upharpoonright\mathfrak{M} = I_{\mathfrak{M}} \quad \text{and} \quad UT_1 = T_2U.$$

The right hand side in (1.3) is often called *compressed resolvent/$\mathfrak{M}$-resolvent/the Weyl function/m-function*, [6], [11]. A representation $M \in \mathbf{N}^0_{\mathfrak{M}}$ in the form (1.3) will be called a *realization of M*.

We show in Section 2, that $M(\lambda) \in \mathbf{N}^0_{\mathfrak{M}}[-1,1] \iff (\lambda^2-1)^{-1}M(\lambda)^{-1} \in \mathbf{N}^0_{\mathfrak{M}}[-1,1]$. It follows that the transformation

$$\mathbf{N}^0_{\mathfrak{M}}[-1,1] \ni M(\lambda) \overset{\mathbf{\Gamma}}{\mapsto} M_{\mathbf{\Gamma}}(\lambda) := \frac{M(\lambda)^{-1}}{\lambda^2-1} \in \mathbf{N}^0_{\mathfrak{M}}[-1,1] \tag{1.4}$$

maps the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ onto itself and $\mathbf{\Gamma}^{-1} = \mathbf{\Gamma}$. In Theorem 2.6 we construct a realization of $(\lambda^2-1)^{-1}M(\lambda)^{-1}$ as a compressed resolvent by means of the contraction T that realizes M. The mapping $\mathbf{\Gamma}$ has the unique fixed point $M_0(\lambda) = -\dfrac{I_{\mathfrak{M}}}{\sqrt{\lambda^2-1}}$ that is compressed resolvent $P_{\mathfrak{M}_0}(\mathbf{J}_0-\lambda I)^{-1}\upharpoonright\mathfrak{M}_0$ of the block-operator Jacobi matrix

$$\mathbf{J}_0 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}I_{\mathfrak{M}} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{2}}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{1.5}$$

acting in the Hilbert space $\ell^2(\mathbb{N}_0)\bigotimes\mathfrak{M}$, and $\mathfrak{M}_0 = \mathfrak{M}\oplus\{0\}\oplus\cdots$, see Proposition 2.7.

A selfadjoint linear relation $\widetilde{A}$ in the orthogonal sum $\mathfrak{M}\oplus\mathcal{K}$ is called *minimal with respect to* $\mathfrak{M}$ (see [9, page 5366]) if

$$\mathfrak{M}\oplus\mathcal{K} = \overline{\operatorname{span}}\left\{\mathfrak{M} + (\widetilde{A}-\lambda I)^{-1}\mathfrak{M} : \lambda \in \rho(\widetilde{A})\right\}.$$

One of the statements obtained in [9] in the context of the Weyl family of a boundary relation is the following:

Theorem 1.5. *Let $\mathcal{M}$ be a Nevanlinna family in the Hilbert space $\mathfrak{M}$. Then there exists unique up to unitary equivalence a selfadjoint linear relation $\widetilde{A}$ in the Hilbert space $\mathfrak{M}\oplus\mathcal{K}$ such that $\widetilde{A}$ is minimal with respect to $\mathfrak{M}$ and the equality*

$$\mathcal{M}(\lambda) = -\left(P_{\mathfrak{M}}\left(\widetilde{A}-\lambda I\right)^{-1}\upharpoonright\mathfrak{M}\right)^{-1} - \lambda I_{\mathfrak{M}}, \quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{1.6}$$

holds.

The equivalent form of (1.6) is

$$P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{M} = -(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1}, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

The compressed resolvent $P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{M}$ belongs to the class $\mathcal{R}_0[\mathfrak{M}]$ and even to its more narrow subclass, see Corollary 2.4.

In Section 3 we consider the following mapping defined on the whole class $\widetilde{R}(\mathfrak{M})$ of Nevanlinna families:

$$\mathcal{M}(\lambda) \overset{\widehat{\mathbf{\Gamma}}}{\mapsto} \mathcal{M}_{\widehat{\mathbf{\Gamma}}}(\lambda) := -(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1}, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{1.7}$$

We prove (Theorem 3.1) that the mapping $\widehat{\mathbf{\Gamma}}$ and each its degree $\widehat{\mathbf{\Gamma}}^k$ has the unique fixed point

$$\mathcal{M}_0(\lambda) = \frac{-\lambda+\sqrt{\lambda^2-4}}{2} I_{\mathfrak{M}}$$

and the sequence of iterations

$$\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1}, \quad \mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda)+\lambda I_{\mathfrak{M}})^{-1}, \quad n\in\mathbb{N},$$

starting with an arbitrary Nevanlinna family $\mathcal{M}$, converges to $\mathcal{M}_0$ in the operator norm topology uniformly on compact sets lying in the open left/right half-plane of the complex plane. The function $\mathcal{M}_0(\lambda)$ can be realized by the free discrete Schrödinger operator given by the block-operator Jacobi matrix

$$\widehat{\mathbf{J}}_{\mathbf{0}} = \begin{bmatrix} 0 & I_{\mathfrak{M}} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ I_{\mathfrak{M}} & 0 & I_{\mathfrak{M}} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & I_{\mathfrak{M}} & 0 & I_{\mathfrak{M}} & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{1.8}$$

acting in the Hilbert space $\ell^2(\mathbb{N}_0)\bigotimes\mathfrak{M}$. Besides we construct a sequence $\{\widehat{\mathfrak{H}}_n, \widehat{A}_n\}$ of realizations of functions $\mathcal{M}_n$ ($\mathcal{M}_n(\lambda) = P_{\mathfrak{M}}(\widehat{A}_{n-1}-\lambda I)^{-1}\upharpoonright\mathfrak{M}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$) and show that the Hilbert space $\ell^2(\mathbb{N}_0)\bigotimes\mathfrak{M}$ and the block-operator Jacobi matrix $\widehat{\mathbf{J}}_{\mathbf{0}}$ are the inductive limits of $\{\widehat{\mathfrak{H}}_n\}$ and $\{\widehat{A}_n\}$, respectively. Observe that when $\mathfrak{M}=\mathbb{C}$, the Jacobi matrices $\mathbf{J}_0$ and $\frac{1}{2}\widehat{\mathbf{J}}_0$ are connected with Chebyshev polynomials of the first and second kinds, respectively [6].

Let $\mathcal{H}(t) = \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t)\end{bmatrix}$ be symmetric and nonnegative 2×2 matrix-function with scalar real-valued entries on $\mathbb{R}_+$. Assume that $\mathcal{H}(t)$ is locally integrable on $\mathbb{R}_+$ and is *trace-normed*, i.e., $\operatorname{tr}\mathcal{H}(t)=1$ a.e. on $\mathbb{R}_+$. Let $\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}$. The system of differential equations

$$\mathcal{J}\frac{d\vec{x}}{dt} = \lambda\mathcal{H}(t)\vec{x}(t), \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t)\end{bmatrix}, \quad t\in\mathbb{R}_+, \quad \lambda\in\mathbb{C}, \tag{1.9}$$

is called *the canonical system with the Hamiltonian $\mathcal{H}$* or *the Hamiltonian system.*

The *m-function* $m_{\mathcal{H}}$ of the canonical system (1.9) can be defined as follows:

$$m_{\mathcal{H}}(\lambda)=\frac{x_2(0,\lambda)}{x_1(0,\lambda)},\quad \lambda\in\mathbb{C}\backslash\mathbb{R},$$

where $\vec{x}(t,\lambda)$ is the solution of (1.9), satisfying

$$x_1(0,\lambda)\neq 0\quad\text{and}\quad \int\limits_{\mathbb{R}_+}\vec{x}(t,\lambda)^*\mathcal{H}(t)\vec{x}(t,\lambda)\,dt<\infty.$$

The m-function of a canonical system is a Nevanlinna function. As has been proved by L. de Branges [7], see also [22], for each Nevanlinna function m there exists a unique trace-normed canonical system such that its m-function $m_{\mathcal{H}}$ coincides with m. In the last Section 4, applying the algorithm suggested by I.S. Kac in [14], we construct a sequence of Hamiltonians $\{\mathcal{H}_n\}$ such that the m-functions of the corresponding canonical systems coincides with the sequence of the iterates $\{m_n\}$ of the mapping $\widehat{\mathbf{\Gamma}}$

$$m_1(\lambda)=-\frac{1}{m(\lambda)+\lambda},\ldots,\ m_{n+1}(\lambda)=-\frac{1}{m_n(\lambda)+\lambda},\ldots,\quad\lambda\in\mathbb{C}\backslash\mathbb{R},$$

where $m(\lambda)$ is a non-rational Nevanlinna function form the class $\mathbf{N}^0_{\mathbb{C}}$. This sequence $\{m_n\}$ converges locally uniformly on $\mathbb{C}_+/\mathbb{C}_-$ to the function $m_0(\lambda)=\dfrac{-\lambda+\sqrt{\lambda^2-4}}{2}$ that is the m-function of the canonical system with the Hamiltonian

$$\mathcal{H}_0(t)=\begin{bmatrix}\cos^2(j+1)\dfrac{\pi}{2} & 0\\ 0 & \sin^2(j+1)\dfrac{\pi}{2}\end{bmatrix},\quad t\in[j,j+1)\quad\forall j\in\mathbb{N}_0.$$

For the constructed Hamiltonian $\mathcal{H}_n$ the property $\mathcal{H}_n\restriction[0,n+1)=\mathcal{H}_0\restriction[0,n+1)$ is valid for each $n\in\mathbb{N}$. Moreover, our construction shows that for the Hamiltonian $\mathcal{H}$ such that the m-function $m_{\mathcal{H}}$ of the corresponding canonical system belongs to the class $\mathbf{N}^0_{\mathbb{C}}$, the Hamiltonian $\mathcal{H}_{\widehat{\mathbf{\Gamma}}}$ of the canonical system having $\widehat{\mathbf{\Gamma}}(m)$ as its m-function, is of the form

$$\mathcal{H}_{\widehat{\mathbf{\Gamma}}}(t)=\begin{cases}\mathcal{H}_0(t),\ t\in[0,2)\\ \begin{bmatrix}1&0\\0&1\end{bmatrix}-\mathcal{H}(t-1),\ t\in[2,+\infty)\end{cases}.$$

2. Characterizations of subclasses

2.1. The subclass $\mathcal{R}_0[\mathfrak{M}]$. The next proposition is well known, cf.[8].

Proposition 2.1. *Let $M(\lambda)$ be a $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function. Then the following statements are equivalent:*

(i) $M\in R_0[\mathfrak{M}]$;
(ii) *the function $y\|M(iy)\|$ is bounded on $[1,\infty)$,*
(iii) *there exists a strong limit $s-\lim\limits_{y\to+\infty}iyM(iy)=-C$, where C is a bounded selfadjoint nonnegative operator in $\mathfrak{M}$;*
(iv) *M admits a representation*

$$M(\lambda)=K^*(T-\lambda I)^{-1}K,\quad\lambda\in\mathbb{C}\backslash\mathbb{R},\tag{2.1}$$

where T is a selfadjoint operator in a Hilbert space $\mathcal{K}$ and $K\in\mathbf{B}(\mathfrak{M},\mathcal{K})$; here $\mathcal{K}$, T, and K can be selected such that T is $\overline{\operatorname{ran}}\,K$-simple, i.e.,

$$\overline{\operatorname{span}}\,\{(T-\lambda)^{-1}\operatorname{ran}K:\ \lambda\in\mathbb{C}\backslash\mathbb{R}\}=\mathcal{K}.$$

Proposition 2.2. ([9],Lemma 2.14, Example 6.6). *Let $\mathcal{K}$ and $\mathfrak{M}$ be Hilbert spaces, let $K \in \mathbf{B}(\mathfrak{M}, \mathcal{K})$ and let D and T be selfadjoint operators in $\mathfrak{M}$ and $\mathcal{K}$, respectively. Consider a selfadjoint operator $\widetilde{A}$ in the Hilbert space $\mathfrak{M} \oplus \mathcal{K}$ given by the block-operator matrix*

$$\widetilde{A} = \begin{bmatrix} D & K^* \\ K & T \end{bmatrix}, \quad \operatorname{dom} \widetilde{A} = \operatorname{dom} D \oplus \operatorname{dom} T.$$

Then $\widetilde{A}$ is $\mathfrak{M}$-minimal if and only if T is $\overline{\operatorname{ran}}\, K$-simple.

Proof. Our proof is based on the Schur-Frobenius formula for the resolvent $(\widetilde{A} - \lambda I)^{-1}$

(2.2)
$$(\widetilde{A} - \lambda I)^{-1} = \begin{bmatrix} -V(\lambda)^{-1} & V(\lambda)^{-1} K^* (T - \lambda I)^{-1} \\ (T - \lambda I)^{-1} K V(\lambda)^{-1} & (T - \lambda I)^{-1} \left(I_{\mathcal{K}} - K V(\lambda)^{-1} K^* (T - \lambda I)^{-1} \right) \end{bmatrix},$$
$$V(\lambda) := \lambda I_{\mathfrak{M}} - D + K^* (T - \lambda I)^{-1} K,$$
$$\lambda \in \rho(T) \cap \rho(\widetilde{A}).$$

Actually, (2.2) implies the equivalences

$$\overline{\operatorname{span}} \left\{ \mathfrak{M} + (\widetilde{A} - \lambda I)^{-1} \mathfrak{M} : \lambda \in \mathbb{C} \backslash \mathbb{R} \right\} = \mathfrak{M} \oplus \mathcal{K}$$
$$\iff \mathcal{K} \bigcap_{\lambda \in \mathbb{C} \backslash \mathbb{R}} \ker \left(P_{\mathfrak{M}} (\widetilde{A} - \lambda I)^{-1} \right) = \{0\} \iff \bigcap_{\lambda \in \mathbb{C} \backslash \mathbb{R}} \ker \left(K^* (T - \lambda I)^{-1} \right) = \{0\}$$
$$\iff \overline{\operatorname{span}} \, \{ (T - \lambda)^{-1} \operatorname{ran} K : \ \lambda \in \mathbb{C} \backslash \mathbb{R} \} = \mathcal{K}.$$

□

In the sequel we will use the following consequence of (2.2):

(2.3) $$P_{\mathfrak{M}} (\widetilde{A} - \lambda I)^{-1} \upharpoonright \mathfrak{M} = - \left(-D + K^* (T - \lambda I_{\mathfrak{M}})^{-1} K + \lambda I_{\mathfrak{M}} \right)^{-1}, \quad \lambda \in \rho(T) \cap \rho(\widetilde{A}).$$

Proposition 2.3. (cf. [9], the proof of Theorem 3.9). *For a $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function M the following statements are equivalent:*

(i) *the limit value $C := -s - \lim_{y \to +\infty} iy M(iy)$ satisfies $0 \le C \le I_{\mathfrak{M}}$;*

(ii) *M admits a representation*

(2.4) $$M(\lambda) = P_{\mathfrak{M}} (\widetilde{A} - \lambda I)^{-1} \upharpoonright \mathfrak{M}, \quad \lambda \in \mathbb{C} \backslash \mathbb{R},$$

where $\widetilde{A}$ is a selfadjoint linear relation in a Hilbert space $\mathfrak{H} \supset \mathfrak{M}$ and $P_{\mathfrak{M}}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{M}$;

(iii) *M admits a representation* (2.1) *with a contraction $K \in \mathbf{B}(\mathfrak{M}, \widetilde{\mathfrak{H}})$;*

(iv) *the following inequality holds*

$$\frac{\operatorname{Im} M(\lambda)}{\operatorname{Im} \lambda} - M(\lambda) M(\lambda)^* \ge 0, \quad \lambda \in \mathbb{C} \backslash \mathbb{R}.$$

In (ii) $\mathfrak{H}$ and $\widetilde{A}$ can be selected such that $\widetilde{A}$ is minimal w.r.t. $\mathfrak{M}$. Moreover, $\widetilde{A}$ in (2.4) *can be taken to be a selfadjoint operator if and only if $C = I_{\mathfrak{M}}$. The operator K in (iii) is an isometry if and only if $C = I_{\mathfrak{M}}$.*

Proof. The equivalence (i)$\iff$ (iii) follows from Proposition 2.1.

(i)$\Longrightarrow$(iv). Since (2.1) holds, we get $C = K^* K$ and the inequality $0 \le C \le I_{\mathfrak{M}}$ implies $||K|| \le 1$ and, therefore holds the inequality.

$$\frac{\operatorname{Im} M(\lambda)}{\operatorname{Im} \lambda} - M(\lambda) M(\lambda)^* \ge 0, \quad \lambda \in \mathbb{C} \backslash \mathbb{R}.$$

(iv)$\Longrightarrow$(ii). Consider $-M(\lambda)^{-1}$. Then

$$\frac{\operatorname{Im}(-M(\lambda)^{-1}h-\lambda h,h)}{\operatorname{Im}\lambda}=\frac{\operatorname{Im}(-M(\lambda)^{-1}h,h)}{\operatorname{Im}\lambda}-||h||^2\geq 0,\quad h\in\mathfrak{M}.$$

Hence $\mathcal{M}(\lambda):=-M(\lambda)^{-1}-\lambda I_{\mathfrak{M}}$ is a Nevanlinna family. Due to Theorem 1.5 and (1.6) we have

$$-(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1}=P_{\mathfrak{M}}(\widetilde{A}-\lambda I_{\mathfrak{H}})^{-1}{\upharpoonright}\mathfrak{M},\quad \lambda\in\mathbb{C}\backslash\mathbb{R},$$

where $\widetilde{A}$ is a selfadjoint linear relation in some Hilbert space $\mathfrak{H}=\mathfrak{M}\oplus\mathcal{K}$.

(ii)$\Longrightarrow$(i). Let $\widehat{A}_0$ be the operator part of $\widetilde{A}$ acting in a subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. Decompose $\widetilde{A}$ as $H=\operatorname{Gr}\widehat{A}_0\oplus\{0,\mathfrak{H}\ominus\mathfrak{H}_0\}$. Then

$$P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}{\upharpoonright}\mathfrak{M}=P_{\mathfrak{M}}(\widehat{A}_0-\lambda I)^{-1}P_{\mathfrak{H}_0}{\upharpoonright}\mathfrak{M}=P_{\mathfrak{M}}P_{\mathfrak{H}_0}(\widehat{A}_0-\lambda I)^{-1}P_{\mathfrak{H}_0}{\upharpoonright}\mathfrak{M}.$$

Set $K=P_{\mathfrak{H}_0}{\upharpoonright}\mathfrak{M}:\mathfrak{M}\to\mathfrak{H}_0$. Then $K^*=P_{\mathfrak{M}}P_{\mathfrak{H}_0}$, $||K||\leq 1$,

$$M(\lambda)=K^*(\widehat{A}_0-\lambda I)^{-1}K,\quad \lambda\in\mathbb{C}\backslash\mathbb{R},$$

and

$$s-\lim_{x\to+\infty} iyM(iy)=-K^*K,\quad C=K^*K\in[0,I_{\mathfrak{M}}].$$

(iii)$\Longrightarrow$(ii). Since $||K||\leq 1$, $\mathcal{M}(\lambda)=-M^{-1}(\lambda)-\lambda I_{\mathfrak{M}}$ is a Nevanlinna family. By Theorem 1.5 there is a Hilbert space $\mathcal{K}$ and a selfadjoint linear relation $\widetilde{A}$ in $\mathfrak{M}\oplus\mathcal{K}$ minimal w.r.t. $\mathfrak{M}$ such that $\mathcal{M}(\lambda)=-\left(P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}{\upharpoonright}\mathfrak{M}\right)^{-1}-\lambda I_{\mathfrak{M}}$, $\lambda\in\mathbb{C}\backslash\mathbb{R}$. □

Corollary 2.4. *There is a one-to-one correspondence between all Nevanlinna families $\mathcal{M}$ in $\mathfrak{M}$ and all $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna functions M satisfying the condition (ii) in Proposition 2.1 with $C\in[0,I_{\mathfrak{M}}]$. This correspondence is given by the relations*

$$M(\lambda)=-(\mathcal{M}(\lambda)+\lambda I_{\mathfrak{M}})^{-1},\quad \mathcal{M}(\lambda)=-M(\lambda)^{-1}-\lambda I_{\mathfrak{M}},\quad \lambda\in\mathbb{C}\backslash\mathbb{R}.$$

Remark 2.5. *For the case $\mathfrak{M}=\mathbb{C}$ the statement of Corollary 2.4 can be found in* [6, Chapter VII, § 1, Lemma 1.7].

In [10] *(see also* [4]*) it is established that an $\mathbf{B}(\mathfrak{M})$-valued function $M(\lambda)$, $\lambda\in\mathcal{D}\subset\mathbb{C}_+/\mathbb{C}_-$ admits the representation* (2.4) *iff the kernel*

$$K(\lambda,\mu)=\frac{M(\lambda)-M(\mu)^*}{\lambda-\bar{\mu}}-M(\mu)^*M(\lambda)$$

is nonnegative on $\mathcal{D}$.

2.2. **The subclass $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$.** Notice, that if $M\in\mathbf{N}^0_{\mathfrak{M}}[-1,1]$, then

$$\begin{cases}(M(x)g,g)>0\ \forall g\in\mathfrak{M}\setminus\{0\},\ x<-1\\ (M(x)g,g)<0\ \forall g\in\mathfrak{M}\setminus\{0\},\ x>1\end{cases}.$$

Therefore, see [16, Appendix]

$$(1+\lambda)M(\lambda),\quad (1-\lambda)M(\lambda)\in\mathcal{R}[\mathfrak{M}].$$

Theorem 2.6. *1) A $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function M belongs to $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ if and only if the function*

$$\mathsf{L}(\lambda,\xi)=\frac{(1-\lambda^2)M(\lambda)-(1-\bar{\xi}^2)M(\xi)^*-(\lambda-\bar{\xi})I_{\mathfrak{M}}}{\lambda-\bar{\xi}},$$

with $\lambda,\xi\in\mathbb{C}\setminus[-1,1]$, $\lambda\neq\bar{\xi}$ is a nonnegative kernel.

2) If $M\in\mathbf{N}^0_{\mathfrak{M}}[-1,1]$, then the function

$$\frac{M(\lambda)^{-1}}{\lambda^2-1},\quad \lambda\in\mathbb{C}\setminus[-1,1]$$

belongs to $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ *as well.*

3) If a selfadjoint contraction T *in the Hilbert space* $\mathfrak{H}$*, containing* $\mathfrak{M}$ *as a subspace, realizes* M*, i.e.,* $M(\lambda)=P_{\mathfrak{M}}(T-\lambda I)^{-1}\upharpoonright\mathfrak{M}$*, for all* $\lambda\in\mathbb{C}\setminus[-1,1]$*, then*

$$\frac{M(\lambda)^{-1}}{\lambda^2-1}=P_{\mathfrak{M}}(\mathbf{T}-\lambda I)^{-1}\upharpoonright\mathfrak{M},\quad \lambda\in\mathbb{C}\setminus[-1,1],$$

where a selfadjoint contraction $\mathbf{T}$ *is given by*

$$\mathbf{T}:=\begin{bmatrix}-P_{\mathfrak{M}}T\upharpoonright\mathfrak{M} & P_{\mathfrak{M}}D_T\\ D_T\upharpoonright\mathfrak{M} & T\end{bmatrix}:\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_T\end{array}\to\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_T\end{array},\tag{2.5}$$

and $D_T:=(I-T^2)^{1/2}$*,* $\mathfrak{D}_T:=\overline{\operatorname{ran}}\,D_T$*. Moreover, if* T *is* $\mathfrak{M}$*-simple, then* $\mathbf{T}$ *is* $\mathfrak{M}$*-simple as well and the operator* $\mathbf{T}\upharpoonright\mathfrak{D}_{\mathbf{T}}$ *is unitarily equivalent to the operator* $P_{\mathfrak{M}^\perp}T\upharpoonright\mathfrak{M}^\perp$*.*

Proof. The statement in 1) follows from [2, Theorem 6.1]. Observe that if $M(\lambda)=P_{\mathfrak{M}}(T-\lambda I)^{-1}\upharpoonright\mathfrak{M}\ \forall\lambda\in\mathbb{C}\setminus[-1,1]$, where T is a selfadjoint contraction, then

$$\begin{aligned}\mathsf{L}(\lambda,\xi)&=\frac{(1-\lambda^2)M(\lambda)-(1-\bar\xi^2)M(\xi)^*-(\lambda-\bar\xi)I_{\mathfrak{M}}}{\lambda-\bar\xi}\\&=P_{\mathfrak{M}}(T-\lambda I)^{-1}(I-T^2)(T-\bar\xi I)^{-1}\upharpoonright\mathfrak{M},\quad\lambda,\xi\in\mathbb{C}\setminus[-1,1],\quad\lambda\neq\bar\xi.\end{aligned}\tag{2.6}$$

2) Let $\lambda\in\mathbb{C}\setminus[-1,1]$, then

$$|((T-\lambda I)h,h)|\ge d(\lambda)||h||^2\quad\forall h\in\mathfrak{H},$$

where $d(\lambda)=\operatorname{dist}(\lambda,[-1,1])$. Set $h=(T-\lambda I)^{-1}f$, $f\in\mathfrak{M}$. Then

$$\begin{aligned}||M(\lambda)f||||f||&\ge|(f,M(\lambda)f)|=\left|\left(f,(T-\lambda I)^{-1}f\right)\right|\\&=|(h,(T-\lambda I)h)|\ge d(\lambda)||h||^2\ge c(\lambda)||f||^2,\quad c(\lambda)>0.\end{aligned}$$

Hence, $||M(\lambda)f||\ge c(\lambda)||f||$ and since $M(\bar\lambda)=M(\lambda)^*$, we get $||M(\lambda)^*f||\ge c(\bar\lambda)||f||$. It follows that $M(\lambda)^{-1}\in\mathbf{B}(\mathfrak{M})$ for all $\lambda\in\mathbb{C}\setminus[-1,1]$.

Set

$$L(\lambda):=(1-\lambda^2)M(\lambda)-\lambda I_{\mathfrak{M}},\quad\lambda\in\mathbb{C}\setminus[-1,1].$$

Then from (2.6) we get

$$\begin{aligned}L(\lambda)-L(\lambda)^*&=(1-\lambda^2)M(\lambda)-(1-\bar\lambda^2)(M(\lambda)^*-(\lambda-\bar\lambda)I_{\mathfrak{M}}\\&=(\lambda-\bar\lambda)P_{\mathfrak{M}}(T-\lambda I)^{-1}(I-T^2)(T-\bar\lambda I)^{-1}\upharpoonright\mathfrak{M}.\end{aligned}$$

It follows that $L(\lambda)$ and the functions

$$(1-\lambda^2)M(\lambda)=L(\lambda)+\lambda I_{\mathfrak{M}},\quad\lambda\in\mathbb{C}\setminus[-1,1]$$

and

$$-\left((1-\lambda^2)M(\lambda)\right)^{-1}=\frac{M(\lambda)^{-1}}{\lambda^2-1},\quad\lambda\in\mathbb{C}\setminus[-1,1]$$

are Nevanlinna functions. Then from the equality $M(\lambda)=-\lambda^{-1}+o(\lambda^{-1})$, $\lambda\to\infty$, we get that also

$$\frac{M(\lambda)^{-1}}{\lambda^2-1}=-\lambda^{-1}+o(\lambda^{-1}),\quad\lambda\to\infty,$$

i.e.,

$$\frac{M(\lambda)^{-1}}{\lambda^2-1}\in\mathbf{N}^0_{\mathfrak{M}}[-1,1].$$

3) Observe that the subspace $\mathfrak{D}_T$ is contained in the Hilbert space $\mathfrak{H}$. Let $\mathbf{H} := \mathfrak{M} \oplus \mathfrak{D}_T$ and let $\mathbf{T}$ be given by (2.5). Since T is a selfadjoint contraction in $\mathfrak{H}$, we get for an arbitrary $\varphi \in \mathfrak{M}$ and $f \in \mathfrak{D}_T$ the equalities

$$\left(\begin{bmatrix}\varphi\\ f\end{bmatrix}, \begin{bmatrix}\varphi\\ f\end{bmatrix}\right) \pm \left(\begin{bmatrix}\varphi\\ f\end{bmatrix}, \mathbf{T}\begin{bmatrix}\varphi\\ f\end{bmatrix}\right) = \left\|(I \mp T)^{1/2}\varphi \pm (I \pm T)^{1/2} f\right\|^2.$$

Therefore $\mathbf{T}$ is a selfadjoint contraction in the Hilbert space $\mathbf{H}$.

Applying (2.3) we obtain

$$\begin{aligned} P_{\mathfrak{M}}(\mathbf{T} - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M} &= -\left(\lambda I + P_{\mathfrak{M}} T{\upharpoonright}\,\mathfrak{M} + P_{\mathfrak{M}} D_T (T - \lambda I)^{-1} D_T{\upharpoonright}\,\mathfrak{M}\right)^{-1} \\ &= -\left(\lambda I + P_{\mathfrak{M}}\left(T(T - \lambda I) + I - T^2\right)(T - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M}\right)^{-1} \\ &= -\left(\lambda I + P_{\mathfrak{M}}(I - \lambda T)(T - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M}\right)^{-1} \\ &= -\left((1 - \lambda^2) P_{\mathfrak{M}}(T - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M}\right)^{-1} = \frac{M^{-1}(\lambda)}{\lambda^2 - 1}, \quad \lambda \in \mathbb{C} \setminus [-1, 1]. \end{aligned}$$

Suppose that T is $\mathfrak{M}$-simple, i.e.,

$$\overline{\operatorname{span}}\,\{T^n \mathfrak{M},\ n \in \mathbb{N}_0\} = \mathfrak{M} \oplus \mathcal{K} \Longleftrightarrow \bigcap_{n=0}^{\infty} \ker(P_{\mathfrak{M}} T^n) = \{0\}.$$

Hence, since

$$\mathfrak{D}_T \ominus \{\overline{\operatorname{span}}\,\{T^n D_T \mathfrak{M},\ n \in \mathbb{N}_0\}\} = \bigcap_{n=0}^{\infty} \ker(P_{\mathfrak{M}} T^n D_T),$$

we get $\overline{\operatorname{span}}\,\{T^n D_T \mathfrak{M},\ n \in \mathbb{N}_0\} = \mathfrak{D}_T$. This means that the operator $\mathbf{T}$ is $\mathfrak{M}$-simple.

Let

$$\mathbb{T} = \begin{bmatrix} -P_{\mathfrak{M}}\mathbf{T}{\upharpoonright}\,\mathfrak{M} & P_{\mathfrak{M}} D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \\ D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{M} & \mathbf{T}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} P_{\mathfrak{M}} T{\upharpoonright}\,\mathfrak{M} & P_{\mathfrak{M}} D_{\mathbf{T}} \\ D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{M} & \mathbf{T}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_{\mathbf{T}}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_{\mathbf{T}}\end{matrix}.$$

As has been proved above because the selfadjoint contraction $\mathbf{T}$ realizes the function $Q(\lambda) := (\lambda^2 - 1)^{-1} M(\lambda)^{-1}$, i.e.,

$$P_{\mathfrak{M}}(\mathbf{T} - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M} = Q(\lambda) = \frac{M(\lambda)^{-1}}{\lambda^2 - 1}, \quad \lambda \in \mathbb{C} \setminus [-1, 1],$$

the selfadjoint contraction $\mathbb{T}$ realizes the function $(\lambda^2 - 1)^{-1} Q(\lambda)^{-1} = M(\lambda)$. In addition, if T is $\mathfrak{M}$-simple, then $\mathbf{T}$ and therefore $\mathbb{T}$ are $\mathfrak{M}$-simple. Since

$$P_{\mathfrak{M}}(\mathbb{T} - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M} = P_{\mathfrak{M}}(T - \lambda I)^{-1}{\upharpoonright}\,\mathfrak{M} = M(\lambda), \quad |\lambda| > 1,$$

the operators $\mathbb{T}$ and T are unitarily equivalent and, moreover, see Theorem 1.4, there exists a unitary operator $\mathbb{U}$ of the form

$$\mathbb{U} = \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & U \end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_{\mathbf{T}}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix},$$

where $\mathcal{K} := \mathfrak{H} \ominus \mathfrak{M}$ and U is a unitary operator from $\mathfrak{D}_T$ onto $\mathcal{K}$ such that

$$\begin{aligned} T\mathbb{U} = \mathbb{U}\mathbb{T} &\Longleftrightarrow \begin{bmatrix} P_{\mathfrak{M}} T{\upharpoonright}\,\mathfrak{M} & P_{\mathfrak{M}} T{\upharpoonright}\,\mathcal{K} \\ P_{\mathcal{K}} T{\upharpoonright}\,\mathfrak{M} & P_{\mathcal{K}} T{\upharpoonright}\,\mathcal{K} \end{bmatrix} \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} P_{\mathfrak{M}} T{\upharpoonright}\,\mathfrak{M} & P_{\mathfrak{M}} D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \\ D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{M} & \mathbf{T} \end{bmatrix} \\ &\Longleftrightarrow \left\{ \begin{array}{l} (P_{\mathfrak{M}} T{\upharpoonright}\,\mathcal{K})\, U = P_{\mathfrak{M}} D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \\ P_{\mathcal{K}} T{\upharpoonright}\,\mathfrak{M} = U D_{\mathbf{T}}{\upharpoonright}\,\mathfrak{M} \\ (P_{\mathcal{K}} T{\upharpoonright}\,\mathcal{K})\, U = U \mathbf{T}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}} \end{array} \right. . \end{aligned}$$

In particular $P_{\mathcal{K}} T{\upharpoonright}\,\mathcal{K}$ and $\mathbf{T}{\upharpoonright}\,\mathfrak{D}_{\mathbf{T}}$ are unitarily equivalent. □

Observe that for a bounded selfadjoint T the equality $M(\lambda) = P_{\mathfrak{M}}(T-\lambda I)^{-1}\upharpoonright\mathfrak{M}$ yields the following relation for $\lambda\in\mathbb{C}\setminus\mathbb{R}$:

$$\frac{1-|\lambda|^2}{\operatorname{Im}\lambda}\operatorname{Im}M(\lambda)-2\operatorname{Re}\,(\lambda M(\lambda))-I_{\mathfrak{M}}=P_{\mathfrak{M}}(T-\lambda I)^{-1}(I-T^2)(T-\bar\lambda I)^{-1}\upharpoonright\mathfrak{M}.$$

Hence for $M(\lambda)\in\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ we get

$$\frac{1-|\lambda|^2}{\operatorname{Im}\lambda}\operatorname{Im}M(\lambda)-2\operatorname{Re}\,(\lambda M(\lambda))-I_{\mathfrak{M}}=\frac{\operatorname{Im}\,\big((1-\lambda^2)M(\lambda)-\lambda\big)}{\operatorname{Im}\lambda}\geq 0,\quad \operatorname{Im}\lambda\neq 0.$$

2.3. **The fixed point of the mapping $\mathbf{\Gamma}$.**

Proposition 2.7. *Let $\mathfrak{M}$ be a Hilbert space. Then the mapping $\mathbf{\Gamma}$ (1.4) has a unique fixed point*

$$M_0(\lambda)=-\frac{I_{\mathfrak{M}}}{\sqrt{\lambda^2-1}}\quad (\operatorname{Im}\sqrt{\lambda^2-1}>0\quad \textit{for}\quad \operatorname{Im}\lambda>0). \tag{2.7}$$

Define the weight $\rho_0(t)$ and the weighted Hilbert space $\mathfrak{H}_0$ as follows

$$\begin{aligned}\rho_0(t)=&\frac{1}{\pi}\frac{1}{\sqrt{1-t^2}},\quad t\in(-1,1),\\ \mathfrak{H}_0:=&L_2([-1,1],\mathfrak{M},\rho_0(t))=L_2\left([-1,1],\rho_0(t)\right)\bigotimes\mathfrak{M}\\ =&\left\{f(t):\int\limits_{-1}^{1}\frac{||f(t)||^2_{\mathfrak{M}}}{\sqrt{1-t^2}}dt<\infty\right\}.\end{aligned}\tag{2.8}$$

Then $\mathfrak{H}_0$ is the Hilbert space with the inner product

$$(f(t),g(t))_{\mathfrak{H}_0}=\frac{1}{\pi}\int\limits_{-1}^{1}(f(t),g(t))_{\mathfrak{M}}\,\rho_0(t)\,dt=\frac{1}{\pi}\int\limits_{-1}^{1}\frac{(f(t),g(t))_{\mathfrak{M}}}{\sqrt{1-t^2}}dt.$$

Identify $\mathfrak{M}$ with a subspace of $\mathfrak{H}_0$ of constant vector-functions $\{f(t)\equiv f,\ f\in\mathfrak{M}\}$. Define in $\mathfrak{H}_0$ the multiplication operator

$$(T_0f)(t)=tf(t),\quad f\in\mathfrak{H}_0. \tag{2.9}$$

Then

$$M_0(\lambda)=P_{\mathfrak{M}}(T_0-\lambda I)^{-1}\upharpoonright\mathfrak{M}.$$

Let $\mathbf{H}_0=\bigoplus\limits_{j=0}^{\infty}\mathfrak{M}=\ell^2(\mathbb{N}_0)\bigotimes\mathfrak{M}$ and let $\mathbf{J_0}$ be the operator in $\mathbf{H}_0$ given by the block-operator Jacobi matrix of the form (1.5). Set $\mathfrak{M}_0:=\mathfrak{M}\bigoplus\{0\}\bigoplus\{0\}\bigoplus\cdots$. Then

$$M_0(\lambda)=P_{\mathfrak{M}_0}(\mathbf{J}_0-\lambda I)^{-1}\upharpoonright\mathfrak{M}_0.$$

Proof. Let $M_0(\lambda)$ be a fixed point of the mapping $\mathbf{\Gamma}$, i.e.,

$$M_0(\lambda)=\frac{M_0(\lambda)^{-1}}{\lambda^2-1}\Longleftrightarrow M_0(\lambda)^2=\frac{1}{\lambda^2-1}I_{\mathfrak{M}},\quad \lambda\in\mathbb{C}\setminus[-1,1].$$

Since $M_0(\lambda)$ is Nevanlinna function, we get (2.7).

For each $h\in\mathfrak{M}$ calculations give the equality, see [6, pages 545–546], [18],

$$-\frac{h}{\sqrt{\lambda^2-1}}=\frac{1}{\pi}\int\limits_{-1}^{1}\frac{h}{t-\lambda}\frac{1}{\sqrt{1-t^2}}dt,\quad \lambda\in\mathbb{C}\setminus[-1,1].$$

Therefore, if T_0 is the operator of the form (2.9), then

$$M_0(\lambda)=P_{\mathfrak{M}}(T_0-\lambda I)^{-1}\upharpoonright\mathfrak{M},\quad \lambda\in\mathbb{C}\setminus[-1,1].$$

As it is well known the Chebyshev polynomials of the first kind

$$\widehat{T}_0(t) = 1,\ \widehat{T}_n(t) := \sqrt{2}\cos(n \arccos t), \quad n \geq 1$$

form an orthonormal basis of the space $L_2([-1,1], \rho_0(t))$, where $\rho_0(t)$ is given by (2.8). This polynomials satisfy the recurrence relations

$$t\widehat{T}_0(t) = \frac{1}{\sqrt{2}}\widehat{T}_1(t), \quad t\widehat{T}_1(t) = \frac{1}{\sqrt{2}}\widehat{T}_0(t) + \frac{1}{2}\widehat{T}_2(t),$$

$$t\widehat{T}_n(t) = \frac{1}{2}\widehat{T}_{n-1}(t) + \frac{1}{2}\widehat{T}_{n+1}(t), \quad n \geq 2.$$

Hence the matrix of the operator $\mathfrak{T}_0$ of multiplication on the independent variable in the Hilbert space $L_2([-1,1], \rho_0(t))$ w.r.t. the basis $\{\widehat{T}_n(t)\}_{n=0}^{\infty}$ (the Jacobi matrix) takes the form (1.5) when $\mathfrak{M} = \mathbb{C}$. Besides $m_0(\lambda) := ((\mathbf{J}_0 - \lambda I)^{-1}\delta_0, \delta_0) = -\dfrac{1}{\sqrt{\lambda^2 - 1}}$, where $\delta_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}^T$ [6]. Since $T_0 = \mathfrak{T}_0 \bigotimes I_{\mathfrak{M}}$ we get that T_0 is unitarily equivalent to $\mathbf{J_0} = J_0 \bigotimes I_{\mathfrak{M}}$ and $M_0(\lambda) = P_{\mathfrak{M}_0}(\mathbf{J}_0 - \lambda I)^{-1}\upharpoonright \mathfrak{M}_0$. □

Observe that $\mathfrak{M}$-valued holomorphic in $\mathbb{C} \setminus [-1,1]$ function

$$M_1(\lambda) := 2(-\lambda I_{\mathfrak{M}} - M_0^{-1}(\lambda)) = 2(-\lambda + \sqrt{\lambda^2 - 1})I_{\mathfrak{M}}$$

belongs to the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$.

3. The fixed point of the mapping $\widehat{\mathbf{\Gamma}}$

Now we will study the mapping $\widehat{\mathbf{\Gamma}}$ (1.7). Let $\mathcal{M}$ be a Nevanlinna family in the Hilbert space $\mathfrak{M}$. Then since

$$|\mathrm{Im}\,((\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})f, f)| \geq |\mathrm{Im}\,\lambda| ||f||^2, \quad \mathrm{Im}\,\lambda \neq 0, \quad f \in \mathrm{dom}\,\mathcal{M}(\lambda),$$

the estimate

$$||(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}|| \leq \frac{1}{|\mathrm{Im}\,\lambda|},\ \mathrm{Im}\,\lambda \neq 0 \tag{3.1}$$

holds true. It follows that $\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}$ is $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function from the class $\mathcal{R}_0[\mathfrak{M}]$ and, moreover, $\mathcal{M}_1(\lambda) = K^*(\widetilde{T} - \lambda I)^{-1}K$, $\mathrm{Im}\,\lambda \neq 0$, where $\widetilde{T}$ is a selfadjoint operator in a Hilbert space $\widetilde{\mathfrak{H}}$ and $K \in \mathbf{B}(\mathfrak{M}, \widetilde{\mathfrak{H}})$ is a contraction, see Corollary 2.4 and Proposition 2.1. For $\mathcal{M}_2(\lambda) = -(\mathcal{M}_1(\lambda) + \lambda I_{\mathfrak{M}})^{-1}$ one has

$$\lim_{y \to \pm\infty} ||iy\mathcal{M}_2(iy) + I_{\mathfrak{M}}|| = 0,$$

i.e., $\mathcal{M}_2(\lambda) \in \mathcal{N}[\mathfrak{M}]$. Thus, see Corollary 2.4,

$$\mathrm{ran}\,\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}(\widetilde{R}[\mathfrak{M}]) = \left\{ M(\lambda) \in \mathcal{R}_0[\mathfrak{M}] : s - \lim_{y \to +\infty} (-iyM(iy)) \in [0, I_{\mathfrak{M}}] \right\},$$

$$\mathrm{ran}\,\widehat{\mathbf{\Gamma}}^k \subset \mathcal{N}[\mathfrak{M}], \quad k \geq 2.$$

Theorem 3.1. *Let $\mathfrak{M}$ be a Hilbert space. Then*

(1) *the function*

$$\mathcal{M}_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{\mathfrak{M}}, \quad \mathrm{Im}\,\lambda \neq 0, \quad \mathcal{M}_0(\infty) = 0 \tag{3.2}$$

is a unique fixed point of the mapping $\widehat{\mathbf{\Gamma}}$ (1.7);

(2) *if $\widehat{\mathbf{\Gamma}}(\mathcal{M}) = \mathcal{M}_0$, then $\mathcal{M}(\lambda) = \mathcal{M}_0(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$;*

(3) *for every sequence of iterations of the form*

$$\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad \mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad n = 1, 2 \ldots,$$

where $\mathcal{M}(\lambda)$ is an arbitrary Nevanlinna function, the relation

$$\lim_{n\to\infty} ||\mathcal{M}_n(\lambda) - \mathcal{M}_0(\lambda)|| = 0$$

holds uniformly on each compact subsets of the open upper/lower half-plane of the complex plane $\mathbb{C}$;

(4) *the function $\mathcal{M}_0(\lambda)$ is a unique fixed point for each degree of $\widehat{\mathbf{\Gamma}}$.*

Proof. (1) Since

$$\mathcal{M}(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} \Longleftrightarrow \mathcal{M}^2(\lambda) + \lambda\mathcal{M}(\lambda) + I_{\mathfrak{M}} = 0,$$

and $\mathcal{M}$ is a Nevanlinna family, we get that $\mathcal{M}_0$ given by (3.2) is a unique solution.

(2) Suppose $\widehat{\mathbf{\Gamma}}(\mathcal{M}) = \mathcal{M}_0$, i.e.,

$$-(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{\mathfrak{M}}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Then

$$\mathcal{M}(\lambda) = \left(-\frac{2}{-\lambda + \sqrt{\lambda^2 - 4}} - \lambda\right) I_{\mathfrak{M}} = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{\mathfrak{M}} = \mathcal{M}_0(\lambda).$$

(3) Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna functions. Set

$$\widehat{F}(\lambda) = -(\mathcal{F}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad \widehat{G}(\lambda) = -(\mathcal{G}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Then $\widehat{F}$ and $\widehat{G}$ are $\mathbf{B}(\mathfrak{M})$-valued and

$$\widehat{F}(\lambda) - \widehat{G}(\lambda) = (\mathcal{F}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} (\mathcal{F}(\lambda) - \mathcal{G}(\lambda)) (\mathcal{G}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}.$$

From (3.1) we get

$$||(\widehat{F}(\lambda) - \widehat{G}(\lambda))|| \le \frac{1}{|\operatorname{Im}\lambda|^2} ||\mathcal{F}(\lambda) - \mathcal{G}(\lambda)||.$$

Hence for the sequence of iterations $\{\mathcal{M}_n(\lambda)\}$ one has

$$||(\mathcal{M}_n(\lambda) - \mathcal{M}_m(\lambda))|| \le \frac{1}{(|\operatorname{Im}\lambda|^2)^{m-1}} ||\mathcal{M}_{n-m+1}(\lambda) - \mathcal{M}_1(\lambda)||, \quad n > m.$$

It follows that if $|\operatorname{Im}\lambda| > 1$, then

$$||(\mathcal{M}_n(\lambda) - \mathcal{M}_m(\lambda))|| \le \frac{(|\operatorname{Im}\lambda|^2)^{-m+1}}{1 - (|\operatorname{Im}\lambda|)^{-2}} ||\mathcal{M}_2(\lambda) - \mathcal{M}_1(\lambda)||, \quad n > m.$$

Therefore, the sequence of linear operators $\{\mathcal{M}_n(\lambda)\}_{n=1}^\infty$ convergence in the operator norm topology, and the limit satisfies the equality $\mathcal{M}(\lambda) = -(\mathcal{M}(\lambda) + \lambda I)^{-1}$, i.e., is the fixed point of the mapping $\widehat{\mathbf{\Gamma}}$. In addition due to the inequality

$$||(\mathcal{M}_n(\lambda) - \mathcal{M}_m(\lambda))|| \le \frac{1}{R^{m-1}} ||\mathcal{M}_{n-m+1}(\lambda) - \mathcal{M}_1(\lambda)||, \quad n > m, \quad |\operatorname{Im}\lambda| \ge R, \quad R > 1$$

we get that the convergence is uniform on λ on the domain $\{\lambda : |\operatorname{Im}\lambda| \ge R\}$, $R > 1$.

Note that from

$$||\mathcal{M}_n(\lambda)|| = ||(\mathcal{M}_{n-1}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}|| \le \frac{1}{|\operatorname{Im}\lambda|}, \quad \operatorname{Im}\lambda \ne 0$$

it follows that the sequence of operator-valued functions $\{\mathcal{M}_n(\lambda)\}_{n=1}^\infty$ is uniformly bounded on λ on each domain $|\operatorname{Im}\lambda| > r$, $r > 0$. Thus, the sequence $\{\mathcal{M}_n\}_{n=1}^\infty$ is locally uniformly bounded in the upper and lower open half-planes and, in addition, $\{\mathcal{M}_n\}$ uniformly converges in the operator-norm topology on the domains $\{\lambda : |\mathrm{Im}\,\lambda| \ge R\}$, $R > 1$. By the Vitali-Porter theorem [19] the relation

$$\lim_{n\to\infty} ||\mathcal{M}_n(\lambda) - \mathcal{M}_0(\lambda)|| = 0$$

holds uniformly on λ on each compact subset of the open upper/lower half-plane of the complex plane $\mathbb{C}$.

(4) The function $\mathcal{M}_0$ is a fixed point for each degree of $\widehat{\mathbf{\Gamma}}$. Suppose that the mapping $\widehat{\mathbf{\Gamma}}^{l_0}$, $l_0 \ge 2$ has one more fixed point $\mathcal{L}_0(\lambda)$. Then arguing as above, we get

$$||\mathcal{M}_0(\lambda) - \mathcal{L}_0(\lambda)|| \le |\mathrm{Im}\,\lambda|^{-2l_0} ||\mathcal{M}_0(\lambda) - \mathcal{L}_0(\lambda)|| \quad \forall \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

It follows that $\mathcal{L}_0(\lambda) \equiv \mathcal{M}_0(\lambda)$. □

The scalar case ($\mathfrak{M} = \mathbb{C}$) of the next Proposition can be found in [6, pages 544–545], [18].

Proposition 3.2. *Let $\mathfrak{M}$ be a Hilbert space.*

(1) *Consider the weighted Hilbert space*

$$\mathfrak{L}_0 := L_2\left([-2,2],\ \frac{1}{2\pi}\sqrt{4-t^2}\right) \otimes \mathfrak{M}$$

and the operator

$$(\mathcal{T}_0 f)(t) = t f(t), \quad f(t) \in \mathfrak{L}.$$

Identify $\mathfrak{M}$ with a subspace of $\mathfrak{L}_0$ of constant vector-functions $\{f(t) \equiv f,\ f \in \mathfrak{M}\}$. Then

$$\mathcal{M}_0(\lambda) = P_{\mathfrak{M}}(\mathcal{T}_0 - \lambda I)^{-1} \upharpoonright \mathfrak{M}, \quad \lambda \in \mathbb{C} \setminus [-2,2],$$

where $\mathcal{M}_0(\lambda)$ is given by (3.2)*.*

(2) *Let $\mathbf{H}_0 = \bigoplus_{j=0}^{\infty} \mathfrak{M} = \ell^2(\mathbb{N}_0) \bigotimes \mathfrak{M}$ and let $\widehat{\mathbf{J}}_{\mathbf{0}}$ be the operator in $\mathbf{H}_0$ given by the block-operator Jacobi matrix of the form* (1.8)*.*

Set $\mathfrak{M}_0 := \mathfrak{M} \bigoplus \{0\} \bigoplus \{0\} \bigoplus \cdots$. Then

$$\mathcal{M}_0(\lambda) = P_{\mathfrak{M}_0}(\widehat{\mathbf{J}}_0 - \lambda I)^{-1} \upharpoonright \mathfrak{M}_0, \quad \lambda \in \mathbb{C} \setminus [-2,2].$$

In the next statement we show that one can construct a sequence $\{\widehat{\mathfrak{H}}_n, \widehat{A}_n\}$ of realizations for the iterates $\{\mathcal{M}_{n+1} = \widehat{\mathbf{\Gamma}}(\mathcal{M}_n)\}_{n=1}^{\infty}$ that inductively converges to $\{\mathbf{H}_{\mathbf{0}}, \widehat{\mathbf{J}}_0\}$.

Theorem 3.3. *Let $\mathcal{M}(\lambda)$ be an arbitrary Nevanlinna family in $\mathfrak{M}$. Define the iterations of the mapping $\widehat{\mathbf{\Gamma}}$* (1.7)*:*

$$\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1},\ \mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad n = 1, 2 \ldots, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Let $\mathcal{M}_1(\lambda) = K^*(\widehat{T} - \lambda I)^{-1}K$, $\mathrm{Im}\,\lambda \ne 0$ be a realization of $\mathcal{M}_1(\lambda)$, where $\widehat{T}$ is a self-adjoint operator in the Hilbert space $\widehat{\mathfrak{H}}$ and $K \in \mathbf{B}(\mathfrak{M}, \widehat{\mathfrak{H}})$ is a contraction. Further, set

$$\widehat{\mathfrak{H}}_1 = \mathfrak{M} \oplus \widehat{\mathfrak{H}},\ \widehat{\mathfrak{H}}_2 = \mathfrak{M} \oplus \widehat{\mathfrak{H}}_1 = \mathfrak{M} \oplus \mathfrak{M} \oplus \widehat{\mathfrak{H}}, \quad \widehat{\mathfrak{H}}_{n+1} = \mathfrak{M} \oplus \mathfrak{H}_n = \underbrace{\mathfrak{M} \oplus \mathfrak{M} \oplus \cdots \oplus \mathfrak{M}}_{n+1} \oplus \widehat{\mathfrak{H}}, \ldots \tag{3.3}$$

and define the following linear operators for each $n \in \mathbb{N}$:

$$\mathfrak{M} \ni x \mapsto \mathbb{I}_{\mathfrak{M}}^{(n)} x = [x, \underbrace{0, 0, \dots, 0}_{n}]^T \in \widehat{\mathfrak{H}}_n,$$

$$\widehat{\mathfrak{H}}_n \ni \begin{bmatrix} x \\ h \end{bmatrix} \mapsto P_{\mathfrak{M}}^{(0,n)} \begin{bmatrix} x \\ h \end{bmatrix} = x \in \mathfrak{M}(\perp \widehat{\mathfrak{H}}_n) \quad \forall x \in \mathfrak{M}, \quad \forall h \in \widehat{\mathfrak{H}}_n.$$

Define selfadjoint operators in the Hilbert spaces $\widehat{\mathfrak{H}}_n$ for $n \in \mathbb{N}$:

$$(3.4) \quad \widehat{A}_1 = \begin{bmatrix} 0 & K^* \\ K & \widehat{T} \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}} \end{matrix}, \quad \operatorname{dom} \widehat{A}_1 = \mathfrak{M} \oplus \operatorname{dom} \widehat{T},$$

$$\operatorname{dom} \widehat{T} \to \widehat{\mathfrak{H}}_1,$$

$$\widehat{A}_2 = \begin{bmatrix} 0 & P_{\mathfrak{M}}^{(0,1)} \\ \mathbb{I}_{\mathfrak{M}}^{(1)} & \widehat{A}_1 \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}}_1 \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}}_1 \end{matrix}, \quad \operatorname{dom} \widehat{A}_2 = \mathfrak{M} \oplus \operatorname{dom} \widehat{A}_1,$$

$$\widehat{A}_{n+1} = \begin{bmatrix} 0 & P_{\mathfrak{M}}^{(0,n)} \\ \mathbb{I}_{\mathfrak{M}}^{(n)} & \widehat{A}_n \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}}_n \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \widehat{\mathfrak{H}}_n \end{matrix}, \quad \operatorname{dom} \widehat{A}_{n+1} = \mathfrak{M} \oplus \operatorname{dom} \widehat{A}_n.$$

Then $\widehat{A}_n$ is a realization of $\mathcal{M}_{n+1}$ for each n, i.e.,

$$(3.5) \qquad \mathcal{M}_{n+1}(\lambda) = P_{\mathfrak{M}}(\widehat{A}_n - \lambda I)^{-1} \upharpoonright \mathfrak{M}, \quad n = 1, 2 \dots, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

If $\widehat{T}$ is $\overline{\operatorname{ran}}\, K$-simple, i.e., $\overline{\operatorname{span}}\,\{(\widehat{T} - \lambda)^{-1} \operatorname{ran} K : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{K}$, then $\widehat{A}_n$ is $\mathfrak{M}$-minimal for each $n \in \mathbb{N}$. Moreover, the Hilbert space $\mathbf{H}_0$ and the block-operator Jacobi matrix (1.8) *are the inductive limits $\mathbf{H}_0 = \lim\limits_{\rightarrow} \widehat{\mathfrak{H}}_n$ and $\widehat{\mathbf{J}}_0 = \lim\limits_{\rightarrow} \widehat{A}_n$, of the chains $\{\widehat{\mathfrak{H}}_n\}$ and $\{\widehat{A}_n\}$, respectively.*

Proof. Relations in (3.5) follow by induction from (2.3).

Note that the operator $\widehat{A}_n$ can be represented by the block-operator matrix

$$(3.6) \quad \widehat{A}_n = \begin{bmatrix} 0 & I_{\mathfrak{M}} & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ I_{\mathfrak{M}} & 0 & I_{\mathfrak{M}} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I_{\mathfrak{M}} & 0 & I_{\mathfrak{M}} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I_{\mathfrak{M}} & 0 & I_{\mathfrak{M}} & 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & I_{\mathfrak{M}} & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & I_{\mathfrak{M}} & 0 & K^* \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & K & \widehat{T} \end{bmatrix} : \begin{matrix} n\left\{\begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M} \\ \oplus \\ \vdots \\ \oplus \\ \mathfrak{M} \end{matrix}\right. \\ \oplus \\ \widehat{\mathfrak{H}} \end{matrix} \longrightarrow \begin{matrix} n\left\{\begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M} \\ \oplus \\ \vdots \\ \oplus \\ \mathfrak{M} \end{matrix}\right. \\ \oplus \\ \widehat{\mathfrak{H}} \end{matrix}.$$

Besides, if $\widehat{T}$ is bounded, then all operators $\{\widehat{A}_n\}_{n \geq 1}$ are bounded and each $\mathcal{M}_n(\lambda)$ belongs to the class $\mathbf{N}_{\mathfrak{M}}^0$ for $n \geq 2$.

Define the linear operators $\gamma_k^l : \widehat{\mathfrak{H}}_k \to \widehat{\mathfrak{H}}_l$, $l \geq k$, $\gamma_k : \widehat{\mathfrak{H}}_k \to \mathbf{H_0}$, $k \in \mathbb{N}$ as follows

$$(3.7) \quad \gamma_k^l[f_1, f_2, \dots, f_k, \varphi] = [f_1, f_2, \dots, f_k, \underbrace{0, 0, \dots, 0}_{l-k}, \varphi],$$

$$\gamma_k[f_1, f_2, \dots, f_k, \varphi] = [f_1, f_2, \dots, f_k, 0, 0, \dots],$$

$$\{f_i\}_{i=1}^k \subset \mathfrak{M}, \quad \varphi \in \widehat{\mathfrak{H}}.$$

Then

(1) γ_k^k is the identity on $\widetilde{\mathfrak{H}}_k$ for each $k \in \mathbb{N}$,

(2) $\gamma_k^m = \gamma_l^m \circ \gamma_k^l$ if $k \le l \le m$,
(3) $\gamma_k = \gamma_l \circ \gamma_k^l$, $l \ge k$, $k \in \mathbb{N}$,
(4) $\mathbf{H}_0 = \overline{\operatorname{span}}\,\{\gamma_k \widehat{\mathfrak{H}}_k,\ k \ge 1\}$.

Note that the operators $\{\gamma_k^l\}$ are isometries and the operators $\{\gamma_k\}$ are partial isometries and $\ker \gamma_k = \widetilde{\mathfrak{H}}$ for all k. The family $\{\widehat{\mathfrak{H}}_k, \gamma_k^l, \gamma_k\}$ forms the inductive isometric chain [17] and the Hilbert space $\mathbf{H}_0$ is the inductive limit of the Hilbert spaces $\{\widehat{\mathfrak{H}}_n\}$ (3.3): $\mathbf{H}_0 = \lim\limits_{\rightarrow} \widehat{\mathfrak{H}}_n$.

Define following [17] on $\mathcal{D}_\infty := \bigcup\limits_{n=1}^{\infty} \gamma_n \operatorname{dom} \widehat{A}_n$ a linear operator in $\mathbf{H}_0$:

$$\widehat{A}_\infty h := \lim_{m \to \infty} \gamma_m \widehat{A}_m \gamma_k^m h_k, \quad h = \gamma_k h_k, \quad h_k \in \widehat{\mathfrak{H}}_k \ominus \widehat{\mathfrak{H}},$$

where $\{\widehat{A}_n\}$ are defined in (3.4). Due to (3.7) and (3.6) the operator $\widehat{A}_\infty$ exists, densely defined and its closure is bounded selfadjoint operator in $\mathbf{H}_0$ given by the block-operator matrix $\widehat{\mathbf{J}}_\mathbf{0}$ of the form (1.8). □

Note that the operator $\widehat{\mathbf{J}}_\mathbf{0}$ is called the free discrete Schrödinger operator [18]. Observe also that the function

$$M_1(\lambda) = \frac{1}{2}\mathcal{M}_0\left(\frac{\lambda}{2}\right) = 2(-\lambda + \sqrt{\lambda^2 - 1}) I_{\mathfrak{M}}, \quad \lambda \in \mathbb{C} \setminus [-1, 1],$$

where $\mathcal{M}_0(\lambda)$ is given by (3.2), belongs to the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$. Besides, for all $\lambda \in \mathbb{C} \setminus [-1,1]$ the equality $M_1(\lambda) = P_{\mathfrak{M}}(\mathcal{T}_1 - \lambda I)^{-1} \restriction \mathfrak{M}$ holds, where $\mathcal{T}_1$ is the multiplication operator $(\mathcal{T}_1 f)(t) = t f(t)$ in the weighted Hilbert space

$$L_2\left([-1,1],\ \frac{2}{\pi}\sqrt{1-t^2}\right) \otimes \mathfrak{M}.$$

If $\mathfrak{M} = \mathbb{C}$, then the matrix of the corresponding operator $\mathcal{T}_1$ in the orthonormal basis of the Chebyshev polynomials of the second kind

$$U_n(t) = \frac{\sin[(n+1)\arccos t]}{\sqrt{1-t^2}}, \quad n = 0, 1, \ldots$$

is of the form $\frac{1}{2}\widehat{\mathbf{J}}_0$ [6].

4. Canonical systems and the mapping $\widehat{\mathbf{\Gamma}}$

Let $m \in \mathbf{N}^0_{\mathbb{C}}$. Then, see [6, Chapter VII, § 1, Theorem 1.11], [11], [18], the function m is the compressed resolvent $(m(\lambda) = \left((J - \lambda I)^{-1}\delta_0, \delta_0\right))$ of a unique finite or semi-infinite Jacobi matrix $J = J(\{a_k\}, \{b_k\})$ with real diagonal entries $\{a_k\}$ and positive off-diagonal entries $\{b_k\}$ and in the semi-infinite case one has $\{a_k\}, \{b_k\} \in \ell^\infty(\mathbb{N}_0)$. Observe that the entries of J can be found using the continued fraction (J-fraction) expansion of $m(\lambda)$ [11], [21]

$$m(\lambda) = \frac{-1}{\lambda - a_0} \genfrac{}{}{0pt}{}{}{+} \frac{-b_0^2}{\lambda - a_1} \genfrac{}{}{0pt}{}{}{+} \frac{-b_1^2}{\lambda - a_2} \genfrac{}{}{0pt}{}{}{+ \ldots +} \frac{-b_{n-1}^2}{\lambda - a_n} \genfrac{}{}{0pt}{}{}{+ \ldots}.$$

On the other hand the algorithm of I. S. Kac [14] enables to construct for given $J(\{a_k\}, \{b_k\})$ the Hamiltonian $\mathcal{H}(t)$ such that the m-function of $J(\{a_k\}, \{b_k\})$ is the m-function of the corresponding canonical system of the form (1.9).

Below we give the algorithm of Kac. Let J be a semi-infinite Jacobi matrix

$$J = J(\{a_k\},\{b_k\}) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ b_0 & a_1 & b_1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & b_1 & a_2 & b_2 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{4.1}$$

The condition $\{a_k\},\{b_k\} \in \ell^\infty(\mathbb{N}_0)$ is necessary and sufficient for the boundedness of the corresponding selfadjoint operator in the Hilbert space $\ell^2(\mathbb{N}_0)$.

Put

$$l_{-1} = 1, \quad l_0 = 1, \quad \theta_{-1} = 0, \quad \theta_0 = \frac{\pi}{2}. \tag{4.2}$$

Then calculate

$$\theta_1 = \arctan a_0 + \pi, \quad l_1 = \frac{1}{l_0 b_0^2 \sin^2(\theta_1 - \theta_0)}. \tag{4.3}$$

Find θ_2 from the system

$$\begin{cases} \cot(\theta_2 - \theta_1) = -a_1 l_1 - \cot(\theta_1 - \theta_0) \\ \theta_2 \in (\theta_1, \theta_1 + \pi) \end{cases}. \tag{4.4}$$

Find successively l_j and θ_{j+1}, $j = 2, 3, \ldots$

$$\begin{array}{l} l_j = \dfrac{1}{l_{j-1} b_{j-1}^2 \sin^2(\theta_j - \theta_{j-1})}, \\ \begin{cases} \cot(\theta_{j+1} - \theta_j) = -a_j l_j - \cot(\theta_j - \theta_{j-1}) \\ \theta_{j+1} \in (\theta_j, \theta_j + \pi) \end{cases} \end{array}. \tag{4.5}$$

Define intervals $[t_j, t_{j+1})$ as follows

$$t_{-1} = -1, \quad t_0 = t_{-1} + l_{-1} = 0, \quad t_1 = t_0 + l_0 = 1, \tag{4.6}$$
$$t_{j+1} = t_j + l_j = 1 + \sum_{k=1}^{j} l_k, \quad j \in \mathbb{N}.$$

Then necessarily, [14], we get that $\lim_{j\to\infty} t_j = +\infty$. Finally define the right continuous increasing step-function

$$\theta(t) := \begin{cases} \theta_0 = \dfrac{\pi}{2},\ t \in (t_0, t_1) = (0,1) \\ \theta_j,\ t \in [t_j, t_{j+1}),\ j \in \mathbb{N} \end{cases} \tag{4.7}$$

and the Hamiltonian $\mathcal{H}(t)$ on $\mathbb{R}_+$

$$\begin{aligned} \mathcal{H}(t) &:= \begin{bmatrix} \cos\theta(t) \\ \sin\theta(t) \end{bmatrix} \begin{bmatrix} \cos\theta(t) & \sin\theta(t) \end{bmatrix} = \begin{bmatrix} \cos^2\theta(t) & \cos\theta(t)\sin\theta(t) \\ \cos\theta(t)\sin\theta(t) & \sin^2\theta(t) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \cos 2\theta(t) & \sin 2\theta(t) \\ \sin 2\theta(t) & -\cos 2\theta(t) \end{bmatrix}. \end{aligned} \tag{4.8}$$

Then the Nevanlinna function $m(\lambda) = ((J - \lambda I)^{-1}\delta_0, \delta_0)$ coincides with m-function of the corresponding canonical system of the form (1.9). Observe that the algorithm shows that

$$\mathcal{H}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad t \in [0,1). \tag{4.9}$$

Using (4.2)–(4.8) for the Jacobi matrix $\widehat{J}_0$

$$\widehat{J}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

we get

$$l_j^0 = 1, \quad \theta_j^0 = (j+1)\frac{\pi}{2} \quad \forall j \in \mathbb{N}_0,$$

$$\theta^0(t) = (j+1)\frac{\pi}{2}, \quad t \in [j, j+1) \quad \forall j \in \mathbb{N}_0,$$

$$(4.10) \quad \mathcal{H}_0(t) = \begin{bmatrix} \cos^2(j+1)\frac{\pi}{2} & 0 \\ 0 & \sin^2(j+1)\frac{\pi}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-(-1)^j & 0 \\ 0 & 1+(-1)^j \end{bmatrix}, \quad t \in [j, j+1) \quad \forall j \in \mathbb{N}_0.$$

Proposition 4.1. *Let the scalar non-rational Nevanlinna function m belong to the class $\mathbf{N}_{\mathbb{C}}^0$. Define the functions*

$$m_1(\lambda) = -\frac{1}{m(\lambda)+\lambda}, \ldots, m_{n+1}(\lambda) = -\frac{1}{m_n(\lambda)+\lambda}, \ldots, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Let J be the Jacobi matrix with the m-function m, i.e., $m(\lambda) = \left((J-\lambda I)^{-1}\delta_0, \delta_0\right), \forall \lambda \in \mathbb{C}\backslash\mathbb{R}$. Assume that $\mathcal{H}(t)$ is the Hamiltonian such that the m-function of the corresponding canonical system coincides with m. Then the Hamiltonian $\mathcal{H}_n(t)$ of the canonical system whose m-function coincides with m_n, takes the form

$$(4.11) \quad \mathcal{H}_n(t) = \begin{cases} \mathcal{H}_0(t), \; t \in [0, n+1), \\ (-1)^n\mathcal{H}(t-n) + \frac{1}{2}\begin{bmatrix} 1-(-1)^n & 0 \\ 0 & 1-(-1)^n \end{bmatrix}, \quad t \in [n+1, \infty) \end{cases}$$

$$= \begin{cases} \mathcal{H}_0(t), \; t \in [0, n+1), \\ \begin{bmatrix} \cos^2\left(\theta_j + n\frac{\pi}{2}\right) & \frac{(-1)^n}{2}\sin 2\theta_j \\ \frac{(-1)^n}{2}\sin 2\theta_j & \sin^2\left(\theta_j + n\frac{\pi}{2}\right) \end{bmatrix}, \quad t \in [t_j+n, t_{j+1}+n), \quad j \in \mathbb{N} \end{cases},$$

where $\{t_j, \theta_j\}_{j\ge 1}$ are parameters of the Hamiltonian $\mathcal{H}(t)$.

Proof. Set

$$(4.12) \quad J_1 = \left[\begin{array}{c|c} 0 & 1 \;\; 0 \;\; 0 \;\; \ldots \\ \hline 1 & \\ 0 & J \\ \vdots & \end{array}\right], \ldots, \quad J_n = \left[\begin{array}{c|c} 0 & 1 \;\; 0 \;\; 0 \;\; \ldots \\ \hline 1 & \\ 0 & J_{n-1} \\ \vdots & \end{array}\right], \ldots$$

Then (2.3) and induction yield the equalities

$$\left((J_1-\lambda I)^{-1}\delta_0, \delta_0\right) = -(m(\lambda)+\lambda)^{-1} = m_1(\lambda), \ldots,$$
$$\left((J_n-\lambda I)^{-1}\delta_0, \delta_0\right) = -(m_{n-1}(\lambda)+\lambda)^{-1} = m_n(\lambda), \ldots,$$
$$\lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Let $J = J(\{a_k\}_{k=0}^{\infty}, \{b_k\}_{k=0}^{\infty})$ be of the form (4.1). Then from (4.12) it follows that for the entries of $J_n = J_n\left(\{a_k^{(n)}\}_{k=0}^{\infty}, \{b_k^{(n)}\}_{k=0}^{\infty}\right)$, $n \in \mathbb{N}$, we have the equalities

$$(4.13)\qquad \begin{cases} a_0^{(n)} = a_1^{(n)} = \cdots = a_{n-1}^{(n)} = 0 \\ a_k^{(n)} = a_{k-n},\ k \geq n \end{cases}, \quad \begin{cases} b_0^{(n)} = b_1^{(n)} = \cdots = b_{n-1}^{(n)} = 1 \\ b_k^{(n)} = b_{k-n},\ k \geq n \end{cases}.$$

In order to find an explicit form of the Hamiltonian corresponding to the Nevanlinna function m_n we apply the algorithm of Kac described by (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8). Then we obtain

$$\begin{aligned} &l_{-1}^{(n)} = l_0^{(n)} = l_1^{(n)} = \cdots = l_n^{(n)} = 1, \\ &\theta_{-1}^{(n)} = 0,\ \theta_0^{(n)} = \frac{\pi}{2},\ \theta_1^{(n)} = \pi, \ldots, \theta_n^{(n)} = (n+1)\frac{\pi}{2}, \\ &l_{n+j}^{(n)} = l_j, \quad \theta_{n+j}^{(n)} = \theta_j + (n+2)\frac{\pi}{2}, \quad j \in \mathbb{N}. \end{aligned}$$

Hence (4.8) and (4.10) yield (4.11). □

By Theorem 3.1 the sequence $\{m_n\}$ of Nevanlinna functions converges uniformly on each compact subset of $\mathbb{C}_+/\mathbb{C}_-$ to the Nevanlinna function

$$m_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

This function is the m-function of the Jacobi matrix $\widehat{J}_0$ and the m-function of the canonical system with the Hamiltonian $\mathcal{H}_0$. From (4.12) we see that for the sequence of selfadjoint Jacobi operators $\{J_n\}$ in $\ell^2(\mathbb{N}_0)$ the relations

$$P_n J_{n+1} P_n = P_n J_0 P_n \quad \forall n \in \mathbb{N}_0$$

hold, where P_n is the orthogonal projection in $\ell^2(\mathbb{N}_0)$ on the subspace

$$E_n = \operatorname{span}\{\delta_0, \delta_1, \ldots, \delta_{n-1}\}.$$

It follows that

$$s - \lim_{n\to\infty} P_n J_{n+1} P_n = \widehat{J}_0.$$

For the sequence (4.11) of $\{\mathcal{H}_n\}$ one has

$$(4.14)\qquad \mathcal{H}_n \restriction [0, n+1) = \mathcal{H}_0 \restriction [0, n+1) \quad \forall n.$$

From (4.14) it follows that if $\vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$ is a continuous function on $\mathbb{R}_+$ with a compact support, then there exists $n_0 \in \mathbb{N}$ such that $\int_0^{\infty} \vec{f}(t)^* \mathcal{H}_n(t) \vec{f}(t)\, dt = \int_0^{\infty} \vec{f}(t)^* \mathcal{H}_0(t) \vec{f}(t)\, dt$ for all $n \geq n_0$.

It is proved in [13, Proposition 5.1] that for a sequence of canonical systems with Hamiltonians $\{H_n\}$ and H the convergence $m_{H_n}(\lambda) \to m_H(\lambda)$, $n \to \infty$ of m-functions holds locally uniformly on $\mathbb{C}_+/\mathbb{C}_-$ if and only if $\int_0^{\infty} \vec{f}(t)^* H_n(t) \vec{f}(t)\, dt \to \int_0^{\infty} \vec{f}(t)^* H(t) \vec{f}(t)\, dt$ for all continuous functions $\vec{f}(t)$ with compact support on $\mathbb{R}_+$.

In conclusion we note that the equalities (4.9), (4.10), and (4.11) (for $n = 1$) show that for the transformation $\widehat{\mathbf{\Gamma}}$ one has the following scheme:

$$\mathbf{N}_{\mathbb{C}}^0 \ni m \text{ (non-rational)} \longrightarrow \mathcal{H}(t) \Longrightarrow$$

$$\mathcal{H}_{\widehat{\mathbf{\Gamma}}}(t) = \begin{cases} \mathcal{H}_0(t),\ t \in [0, 2) \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathcal{H}(t-1),\ t \in [2, +\infty) \end{cases} \longleftarrow \widehat{\mathbf{\Gamma}}(m).$$

References

1. Yu. Arlinskiĭ, S. Belyi, and E. Tsekanovskiĭ, *Conservative Realizations of Herglotz-Nevanlinna Functions*, Oper. Theory Adv. Appl., vol. 217, Birkhäuser Verlag, Basel, 2011.
2. Yu. M. Arlinskiĭ, S. Hassi, and H.S.V. de Snoo, *Q-functions of quasi-selfadjoint contractions*, Oper. Theory Adv. Appl. **163** (2006), 23–54.
3. Yu. Arlinskiĭ and L. Klotz, *Weyl functions of bounded quasi-selfadjoint operators and block operator Jacobi matrices*, Acta Sci. Math. (Szeged) **76** (2010), no. 3–4, 585–626.
4. T. Ya. Azizov, A. Dijksma, and G. Wanjala, *Compressions of maximal dissipative and self-adjoint linear relations and of dilations*, Linear Algebra Appl. **439** (2013), 771–792.
5. Ju. Behrndt, S. Hassi, H. de Snoo, and R. Wietsma, *Monotone convergence theorems for semi-bounded operators and forms with applications*, Proc. Royal Soc. of Edinburgh **140 A** (2010), 927–951.
6. Yu. M. Berezanskii, *Expansion in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
7. L. de Branges, *Some Hilbert spaces of entire functions.* II, Trans. Amer. Math. Soc. **99** (1961), 118-152.
8. M. S. Brodskiĭ, *Triangular and Jordan Representations of Linear Operators*, Nauka, Moscow, 1969. (Russian); English transl. Transl. Math. Monographs, Vol. 32. Amer. Math. Soc., Providence, RI, 1971.
9. V. Derkach, S. Hassi, M. M. Malamud, and H.S.V. de Snoo, *Boundary relations and their Weyl families*, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5351–5400.
10. A. Dijksma, H. Langer, and H.S.V. de Snoo, *Selfadjoint* Π_κ *-extensions of symmetric subspaces: an abstract approach to boundary problems with spectral parameter in the boundary conditions*, Integr. Equ. Oper. Theory **7** (1984), no. 4, 459-515. (Addendum, Integr. Equ. Oper. Theory **7** (1984), no. 6, 905)
11. F. Gesztesy and B. Simon, *M-functions and inverse spectral analysis for finite and semifinite Jacobi matrices*, J. Anal. Math. **73** (1997), 267–297.
12. F. Gesztesy and E. R. Tsekanovskiĭ, *On matrix-valued Herglotz functions*, Math. Nachr. **218** (2000), 61–138.
13. I. Hur, *Density of Schrödinger Weyl-Titchmarsh m-functions on Herglotz functions*, J. Differ. Equations **260** (2016), no. 11, 8137–8159.
14. I. S. Kac, *Inclusion of Hamburger's power moment problem in the spectral theory of canonical systems*, Zap. Nauchn. Semin. POMI **262** (1999), 147–171. (Russian); English transl. J. Math. Sci. (New York) **110** (2002), no. 5, 2991–3004.
15. I. S. Kac and M. G. Kreĭn, *R-functions – analytic functions mapping the upper halfplane into itself*, Supplement to the Russian edition of F. V. Atkinson, Discrete and Continuous Boundary Problems, Mir, Moscow, (1968). (Russian); English transl. Amer. Math. Soc. Transl. Ser. 2 **103** (1974), 1–18.
16. M. G. Kreĭn and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems*, Nauka, Moscow, 1973. (Russian); English transl. Transl. Math. Monographs, Vol. 50. Amer. Math. Soc., Providence, RI, 1977.
17. A. V. Marchenko, *Selfadjoint differential operators with an infinite number of independent variables*, Mat. Sbornik **96** (1975), no. 2, 276–293. (Russian); English transl. Mathematics of the USSR–Sbornik **25** (1975), no. 2, 259–275.
18. C. Remling, *Lecture Notes for Functional Analysis*, Chapter 13, www2.math.ou.edu/ cremling /teaching/lecturenotes/fa-new/ln13.pdf.
19. Joel L. Schiff, *Normal Families*, Springer-Verlag, 1993.
20. Yu. L. Shmul'yan, *On operator-valued R-functions*, Sibirsk. Mat. Zh. **12** (1971), no. 2, 442–452. (Russian); English transl. Siberian Math. J. **12** (1971), no. 2, 315–322.
21. H. S. Wall, *Analytic Theory of Continued Fractions*, Chelsea, Bronx, NY, 1973.
22. H. Winkler, *The inverse spectral problem for canonical systems*, Integr. Equ. Oper. Theory **22** (1995), no. 3, 360-374.

Department of Mathematics, Dragomanov National Pedagogical University, 9 Pirogova, Kyiv, 01601, Ukraine

E-mail address: yury.arlinskii@gmail.com

Received 19/04/2017; Revised 30/05/2017