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A-REGULAR–A-SINGULAR FACTORIZATIONS OF GENERALIZED J-INNER MATRIX FUNCTIONS

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Dedicated to Eduard Tsekanovskii on the occasion of his 80th birthday

Abstract. Let J be an $m \times m$ signature matrix, i.e., $J = J^* = J^{-1}$. An $m \times m$ mvf (matrix valued function) $W(\lambda)$ that is meromorphic in the unit disk $\mathbb{D}$ is called J-inner if $W(\lambda)JW(\lambda)^* \le J$ for every λ from $\mathfrak{h}_W^+$, the domain of holomorphy of W, in $\mathbb{D}$, and $W(\mu)JW(\mu)^* = J$ for a.e. $\mu \in \mathbb{T} = \partial\mathbb{D}$. A J-inner mvf $W(\lambda)$ is called A-singular if it is outer and it is called right A-regular if it has no non-constant A-singular right divisors. As was shown by D. Arov [8] every J-inner mvf admits an essentially unique A-regular–A-singular factorization $W = W^{(1)}W^{(2)}$. In the present paper this factorization result is extended to the class $\mathcal{U}_\kappa^r(J)$ of right generalized J-inner mvf's introduced in [18]. The notion and criterion of A-regularity for right generalized J-inner mvf's are presented. The main result of the paper is that we find a criterion for existence of an A-regular–A-singular factorization for a rational generalized J-inner mvf.

1. Introduction

Let Ω_+ be equal to either $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ or $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : -i(\lambda - \bar{\lambda}) > 0\}$. Let us set

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\overline{\omega}, & \text{if } \Omega_+ = \mathbb{D}, \\ -2\pi i(\lambda - \overline{\omega}), & \text{if } \Omega_+ = \mathbb{C}_+, \end{cases}$$

and let $\Omega_- := \{\omega \in \mathbb{C} : \rho_\omega(\omega) < 0\}$. Then $\Omega_0 := \partial\Omega_+$ is either the unit circle $\mathbb{T}$, if $\Omega_+ = \mathbb{D}$, or the real axis $\mathbb{R}$, if $\Omega_+ = \mathbb{C}_+$.

The following basic classes of mvf's will be used in this paper:
H_r $(1 \le r \le \infty)$, the Hardy class with respect to Ω_+;
$H_r^{p\times q}$, the class of $p \times q$-mvf's with entries in H_r, $H_r^p := H_r^{p\times 1}$ $(1 \le r \le \infty)$;
$\mathcal{S}^{p\times q}$, the Schur class of contractive and holomorphic on Ω_+ $p \times q$-mvf's;
$\mathcal{S}_{out}^{p\times q} = \{s \in \mathcal{S}^{p\times q} : \overline{sH_2^q} = H_2^p\}$ $(\mathcal{S}_{in}^{p\times q})$, the class of outer (inner, resp.) mvf's from $\mathcal{S}^{p\times q}$.

In this paper we consider a signature matrix J of the following specific form:

$$J = j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad \text{where } p + q = m. \tag{1.1}$$

Definition 1.1. ([4, 18]). *An $m \times m$ mvf (matrix valued function) $W(\lambda)$ that is meromorphic in Ω_+ is said to belong to the class $\mathcal{U}_\kappa(j_{pq})$ of generalized j_{pq}-inner mvf's, if*

(i) *the kernel*

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)} \tag{1.2}$$

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has κ negative squares in $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$, where $\mathfrak{h}_W^+$ denotes the domain of holomorphy of W in Ω_+ and

(ii) $j_{pq} - W(\mu) j_{pq} W(\mu)^* = 0$ *a.e. on the boundary Ω_0 of Ω_+.*

The class $\mathcal{U}(j_{pq}) := \mathcal{U}_0(j_{pq})$ is contained in the class $\mathcal{P}(j_{pq})$ of j_{pq}-contractive meromorphic on Ω_+ mvf's. The class $\mathcal{P}(j_{pq})$ was introduced and studied by M. S. Livsič [25] in connection with the theory of characteristic functions of quasi-Hermitian operators, see also [31] for the case of unbounded operators. A complete factorization theory for mvf's from the class $\mathcal{P}(j_{pq})$ was developed by V. P. Potapov [28]. Mvf's from the class $\mathcal{U}(j_{pq})$ are called j_{pq}-inner. j_{pq}-inner mvf's appear in [22], [26], [14], [8], [21] as resolvent matrices of various interpolation problems.

A j_{pq}-inner mvf $W(\lambda)$ is called *A-singular*, if $W \in \mathcal{S}_{out}^{m\times m}$. A j_{pq}-inner mvf $W(\lambda)$ is called *right A-regular*, if it has no non-constant A-singular right divisors in the class $\mathcal{U}(j_{pq})$. In particular, the resolvent matrix of a bitangential problem belongs to the class $\mathcal{U}(j_{pq})$ and turns out to be a right A-regular j_{pq}-inner mvf, see [8], [10]. An important result of [8] claims that an arbitrary j_{pq}-inner mvf $W(\lambda)$ admits an essentially unique factorization

$$W(\lambda) = W^{(1)}(\lambda) W^{(2)}(\lambda), \tag{1.3}$$

where $W^{(1)}(\lambda)$ and $W^{(2)}(\lambda)$ are right A-regular and A-singular mvf's, respectively.

The class $\mathcal{U}_\kappa(j_{pq})$, $\kappa \in \mathbb{N}$, and a reproducing kernel Pontryagin space $\mathcal{K}(W)$ with the reproducing kernel $\mathsf{K}_\omega^W(\lambda)$ based on $W \in \mathcal{U}_\kappa(j_{pq})$ were studied in [4] and [2]. In [27], [14], [13], [17], [19], [20] mvf's $W \in \mathcal{U}_\kappa(j_{pq})$ appear as resolvent matrices of some indefinite interpolation problems. In most cases these resolvent matrices belong also to a subclass $\mathcal{U}_\kappa^r(j_{pq})$ of *right generalized j_{pq}-inner* mvf's introduced and studied in [18]. The class of right and left *A-singular* generalized j_{pq}-inner mvf's was introduced and characterized in [30].

In the present paper we introduce the notions of right and left *A-regular* generalized j_{pq}-inner mvf's and prove a criterion of A-regularity for rational generalized j_{pq}-inner mvf's. The main result of the paper contains a criterion of existence of A-regular–A-singular factorization (1.3) for a rational generalized j_{pq}-inner mvf. This criterion is formulated in terms of reproducing kernel Pontryagin spaces $\mathcal{K}(W)$ associated with $W(\lambda)$. An example of a right generalized j_{pq}-inner mvf $W(\lambda)$ is given such that $W(\lambda)$ does not admit an A-regular–A-singular factorization in the class of generalized j_{pq}-inner mvf's.

2. Preliminaries

2.1. **The generalized Schur class.** Let $\kappa \in \mathbb{Z}_+$. Recall [6] that a Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m\times m}$ is said to have κ negative squares, if for every positive integer n and every choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ $(j = 1, \dots, n)$ the matrix

$$(u_k^* \mathsf{K}_{\omega_j}(\omega_k) u_j)_{j,k=1}^n$$

has at most κ, and for some choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ exactly κ negative eigenvalues.

Denote by $\mathfrak{h}_s$ the domain of holomorphy of the mvf s and let us set $\mathfrak{h}_s^\pm = \mathfrak{h}_s \cap \Omega_\pm$.

Let $\mathcal{S}_\kappa^{q\times p}$ denote *the generalized Schur class* of $q \times p$ mvf's s that are meromorphic in Ω_+ and for which the kernel

$$\Lambda_\omega^s(\lambda) = \frac{I_p - s(\lambda) s(\omega)^*}{\rho_\omega(\lambda)} \tag{2.1}$$

has κ negative squares on $\mathfrak{h}_s^+ \times \mathfrak{h}_s^+$ (see [23]). In the case where $\kappa = 0$ the class $\mathcal{S}_0^{q\times p}$ coincides with the Schur class $\mathcal{S}^{q\times p}$ of contractive mvf's holomorphic in Ω_+.

Let $b_\omega(\lambda)$ be an elementary factor Blaschke

$$b_\omega(\lambda) = \begin{cases} (\lambda-\omega)/(1-\lambda\overline{\omega}), & \text{if } \Omega_+ = \mathbb{D}, \\ (\lambda-\omega)/(\lambda-\overline{\omega}), & \text{if } \Omega_+ = \mathbb{C}_+ \end{cases} \tag{2.2}$$

and let P be an orthogonal projection in $\mathbb{C}^p$. Then the mvf

$$B_\omega(\lambda) = I_m + (b_\omega(\lambda) - 1)P$$

belongs to the Schur class $\mathcal{S}^{p\times p}$ and is called *an elementary BP* (*Blaschke–Potapov*) *factor* and $B(\lambda)$ is called *primary* if *rank* $P = 1$. The product

$$B(\lambda) = \prod_{j=1}^{\overset{\kappa}{\curvearrowright}} B_{\omega_j}(\lambda),$$

where $B_{\omega_j}(\lambda)$ are primary BP-factors is called *a Blaschke–Potapov product* of degree κ.

Every mvf $s \in \mathcal{S}^{p\times p}$ of rank p admits an inner-outer factorization of F. Riesz

$$s = ba = a_* b_*, \quad \text{where} \quad b, b_* \in \mathcal{S}_{in}^{p\times p}, \quad a, a_* \in \mathcal{S}_{out}^{p\times p}. \tag{2.3}$$

If b and b_* in (2.3) are Blaschke–Potapov products of finite degree, then deg b = deg b_*. The notation $\mathcal{M}_\zeta(s, \Omega_+) :=$ deg b will be used for the degree of the factors b and b_*.

As was shown in [23] every mvf $s \in \mathcal{S}_\kappa^{q\times p}$ admits a factorization of the form

$$s(\lambda) = b_\ell(\lambda)^{-1} s_\ell(\lambda), \quad \lambda \in \mathfrak{h}_s^+, \tag{2.4}$$

where $b_\ell \in \mathcal{S}^{q\times q}$ is a $q\times q$ Blaschke–Potapov product of degree κ, $s_\ell \in \mathcal{S}^{q\times p}$ and

$$\operatorname{rank} \begin{bmatrix} b_\ell(\lambda) & s_\ell(\lambda) \end{bmatrix} = q \quad (\lambda \in \Omega_+). \tag{2.5}$$

The representation (2.4) is called a *left KL* (*Kreĭn–Langer*) *factorization*. Similarly, every generalized Schur function $s \in \mathcal{S}_\kappa^{q\times p}$ admits a *right KL-factorization*

$$s(\lambda) = s_r(\lambda) b_r(\lambda)^{-1} \quad \text{for} \quad \lambda \in \mathfrak{h}_s^+, \tag{2.6}$$

where $b_r \in \mathcal{S}^{p\times p}$ is a Blaschke–Potapov product of degree κ, $s_r \in \mathcal{S}^{q\times p}$ and

$$\operatorname{rank} \begin{bmatrix} b_r(\lambda)^* & s_r(\lambda)^* \end{bmatrix} = p \quad (\lambda \in \Omega_+). \tag{2.7}$$

The following generalization of the Rouche theorem was presented in [24]. The proof of this theorem was not complete and was fixed in [20]. Its scalar version was proved in [1].

Theorem 2.1. **(Generalized Rouche Theorem)** ([24]). *Let* $\varphi, \psi \in H_\infty^{q\times q}$, $\det(\varphi + \psi) \not\equiv 0$ *in* Ω_+, $M_\zeta(\varphi, \Omega_+) < \infty$,

$$\|\varphi(\mu)^{-1}\psi(\mu)\| \le 1 \quad \text{a.e. on } \Omega_0. \tag{2.8}$$

Then $M_\zeta(\varphi + \psi, \Omega_+) \le M_\zeta(\varphi, \Omega_+)$ *with equality if*

$$(\varphi + \psi)^{-1}\varphi|_{\Omega_0} \in \widetilde{L}_1^{q\times q}. \tag{2.9}$$

The coprimeness condition (2.5) for a right KL-factorization (2.4) can be reformulated as follows.

Lemma 2.2. ([18]). *A mvf* $s_\ell \in \mathcal{S}^{q\times p}$ *and a finite Blaschke–Potapov product* $b_\ell \in \mathcal{S}_{in}^{q\times q}$ *meet the rank condition (2.5) if and only if there exists a pair of mvf's* $c_\ell \in H_\infty^{q\times q}$ *and* $d_\ell \in H_\infty^{p\times q}$ *such that*

$$b_\ell(\lambda)c_\ell(\lambda) + s_\ell(\lambda)d_\ell(\lambda) = I_q \quad \textit{for} \quad \lambda \in \Omega_+. \tag{2.10}$$

2.2. **Generalized j_{pq}-inner mvf's.** Let us recall some facts concerning the PG (Potapov–Ginzburg) transform of generalized j_{pq}-inner mvf's. As is known [4, Theorem 6.8], for every $W \in \mathcal{U}_\kappa(j_{pq})$ the matrix $w_{22}(\lambda)$ is invertible for all $\lambda \in \mathfrak{h}_W^+$ except for at most κ point in Ω_+. Thus, the PG-transform S of W (see [2])

$$S(\lambda) = (PG(W))(\lambda) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1} \quad (\lambda \in \mathfrak{h}_S^+ \cap \mathfrak{h}_W^+) \tag{2.11}$$

is well defined for those $\lambda \in \mathfrak{h}_W^+$, for which $w_{22}(\lambda)$ is invertible. As is easily seen, $S(\lambda)$ belongs to the class $\mathcal{S}_\kappa^{m\times m}$ and $S(\mu)$ is unitary for a.e. $\mu \in \Omega_0$ (see [4], [18]).

The formula (2.11) can be rewritten as

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} w_{11} - w_{12}w_{22}^{-1}w_{21} & w_{12}w_{22}^{-1} \\ -w_{22}^{-1}w_{21} & w_{22}^{-1} \end{bmatrix}. \tag{2.12}$$

Since the mvf $S(\lambda)$ has unitary nontangential boundary limits a.e. on Ω_0, the pseudo-continuation of S to Ω_- can be defined by the formula $S(\lambda) = (S^{\#}(\lambda))^{-1}$, where the reflection function $S^{\#}(\lambda)$ is defined by

$$S^{\#}(\lambda) = S(\lambda^\circ)^*, \quad \lambda^\circ = \begin{cases} 1/\overline{\lambda} & : \text{ if } \Omega_+ = \mathbb{D},\ \lambda \ne 0, \\ \overline{\lambda} & : \text{ if } \Omega_+ = \mathbb{C}_+. \end{cases} \tag{2.13}$$

Formulas (2.13) and (2.12) lead to the dual formula for S:

$$S = \begin{bmatrix} w_{11}^{\#} & 0 \\ w_{12}^{\#} & I_q \end{bmatrix}^{-1} \begin{bmatrix} I_p & w_{21}^{\#} \\ 0 & w_{22}^{\#} \end{bmatrix} = \begin{bmatrix} w_{11}^{-\#} & w_{11}^{-\#}w_{21}^{\#} \\ -w_{12}^{\#}w_{11}^{-\#} & w_{22}^{\#} - w_{12}^{\#}w_{11}^{-\#}w_{21}^{\#} \end{bmatrix} \tag{2.14}$$

on $\mathfrak{h}_S^+ \cap \mathfrak{h}_{W^\#}^+$. Moreover, $s_{22}(\lambda)$ is invertible for all $\lambda \in \mathfrak{h}_W^+$, the PG-transform of $S(\lambda)$ makes sense, and $W = PG(S)$.

Let

$$T_W^r[\varepsilon] := (w_{11}(\lambda)\varepsilon(\lambda) + w_{12}(\lambda))(w_{21}(\lambda)\varepsilon(\lambda) + w_{22}(\lambda))^{-1} \tag{2.15}$$

denote the (right) linear fractional transformation of a mvf $\varepsilon \in \mathcal{S}_{\kappa_2}^{p\times q}$ ($\kappa_2 \in \mathbb{Z}_+$) based on the block decomposition

$$W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} \tag{2.16}$$

of a mvf $W \in \mathcal{U}_\kappa(j_{pq})$ with blocks $w_{11}(\lambda)$ and $w_{22}(\lambda)$ of sizes $p \times p$ and $q \times q$, respectively. Let

$$\Lambda = \{\lambda \in \mathfrak{h}_W^+ \cap \mathfrak{h}_\varepsilon^+ : \det(w_{21}(\lambda)\varepsilon(\lambda) + w_{22}(\lambda)) = 0\}. \tag{2.17}$$

The transformation $T_W^r[\varepsilon]$ is well defined for $\lambda \in (\mathfrak{h}_W^+ \cap \mathfrak{h}_\varepsilon^+) \setminus \Lambda$.

Lemma 2.3. *Let* W $\in \mathcal{U}_{\kappa_1}(j_{pq})$, $\varepsilon \in \mathcal{S}_{\kappa_2}^{p\times q}$. *Then* $T_W^r[\varepsilon] \in \mathcal{S}_{\kappa'}^{p\times q}$ *with* $\kappa' \le \kappa_2 + \kappa_1$.

2.3. **The class $\mathcal{U}_\kappa^r(j_{pq})$.**

Definition 2.4. ([18]). *An $m \times m$ mvf $W(\lambda) \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^r(j_{pq})$, if*

$$s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q\times p}. \tag{2.18}$$

Theorem 2.5. ([18]). *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let the BP-factors b_ℓ and b_r be defined by the KL-factorizations of s_{21}:*

$$s_{21}(\lambda) := b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1}, \quad \lambda \in \mathfrak{h}_{s_{21}}^+, \tag{2.19}$$

where $b_\ell \in \mathcal{S}_{in}^{q\times q}$, $b_r \in \mathcal{S}_{in}^{p\times p}$, $s_\ell, s_r \in \mathcal{S}^{q\times p}$. Then the mvf's $b_\ell s_{22}$ and $s_{11}b_r$ are holomorphic in Ω_+, and hence they admit the following inner-outer and outer-inner factorizations

$$s_{11}b_r = b_1 a_1, \quad b_\ell s_{22} = a_2 b_2, \tag{2.20}$$

where $b_1 \in \mathcal{S}_{in}^{p\times p}$, $b_2 \in \mathcal{S}_{in}^{q\times q}$, $a_1 \in \mathcal{S}_{out}^{p\times p}$, $a_2 \in \mathcal{S}_{out}^{q\times q}$.

The pair $\{b_1, b_2\}$ is called the *right associated pair* of the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ and is written as $\{b_1, b_2\} \in \mathrm{ap}^r(W)$. In the case $\kappa = 0$ this notion was introduced in [10].

As was shown in [18, Theorem 4.11] for every $W \in \mathcal{U}_\kappa^r(j_{pq})$ and c_ℓ and d_ℓ as in (2.10) the mvf

$$K = (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1}, \tag{2.21}$$

belongs to $H_\infty^{p\times q}$ and admits the representations

$$K = (-w_{11}d_\ell + w_{12}c_\ell)a_2 b_2, \tag{2.22}$$

where $\{b_1, b_2\} \in ap^r(W)$.

Let us set $K^\#(\lambda) = K(\bar{\lambda})^*$, $\lambda \in \mathbb{C}_-$. It is clear that $K^\# \in H_\infty^{q\times p}(\Omega_-)$.

Example 1. A j_{pq}-inner mvf $W(\lambda)$ is called *elementary* if it has no nontrivial factorization in the class of j_{pq}-inner mvf's. All elementary j_{pq}-inner mvf's are exhausted by the set of BP-factors of the following three types (see [22]):

(1) $U_\omega(\lambda) = U(I_m + (b_\omega(\lambda) - 1)P)$, $\omega \in \Omega_+$, $P = P^2$ and $Pj_{pq} \geq 0$;
(2) $U_\omega(\lambda) = U(I_m + (b_\omega(\lambda) - 1)P)$, $\omega \in \Omega_-$, $P = P^2$ and $Pj_{pq} \leq 0$;
(3) $U_\omega(\lambda) = U(I_m - c_\omega(\lambda)E)$, $\omega \in \Omega_0$, $E^2 = 0$ and $Ej_{pq} \geq 0$.

Here U are constant j_{pq}-unitary matrices, $b_\omega(\lambda)$ are elementary Blaschke factors of the form (2.2) and

$$c_\omega(\lambda) = \begin{cases} (\omega + \lambda)/(\omega - \lambda), & \text{if } \Omega_+ = \mathbb{D},\ \omega \in \Omega_0, \\ 1/(\pi i(\omega - \lambda)), & \text{if } \Omega_+ = \mathbb{C}_+,\ \omega \in \Omega_0. \end{cases}$$

If $\Omega_+ = \mathbb{C}_+$ then there exists one more type of BP-factors (of the fourth kind), corresponding to $\omega = \infty$,

$$U_\infty(\lambda) = U\exp(i\lambda E).$$

An elementary BP-factor is said to be primary, if $\operatorname{rank} P = 1$ or $\operatorname{rank} E = 1$. The preceding three types of primary BP-factors take the form

(1) $U_\omega(\lambda) = U(I_m + (b_\omega(\lambda) - 1)vv^* j_{pq})$, $\omega \in \Omega_+$, $v \in \mathbb{C}^m$ and $v^* j_{pq} v = 1$;
(2) $U_\omega(\lambda) = U(I_m - (b_\omega(\lambda) - 1)vv^* j_{pq})$, $\omega \in \Omega_-$, $v \in \mathbb{C}^m$ and $v^* j_{pq} v = -1$;
(3) $U_\omega(\lambda) = U(I_m - c_\omega(\lambda)vv^* j_{pq})$, $\omega \in \Omega_0$, $v \in \mathbb{C}^m$ and $v^* j_{pq} v = 0$.

Notice that by changing sign of $v^* j_{pq} v$ in the first two types of primary BP-factors one obtains generalized j_{pq}-inner mvf's which belong to the class $\mathcal{U}_1(j_{pq})$,

$$U_\omega(\lambda) = U(I_m - (b_\omega(\lambda) - 1)vv^* j_{pq}), \quad \omega \in \Omega_+, \quad v \in \mathbb{C}^m \quad \text{and} \quad v^* j_{pq} v = -1; \tag{2.23}$$

$$U_\omega(\lambda) = U(I_m + (b_\omega(\lambda) - 1)vv^* j_{pq}), \quad \omega \in \Omega_-, \quad v \in \mathbb{C}^m \quad \text{and} \quad v^* j_{pq} v = 1. \tag{2.24}$$

Moreover, the mvf $U_\omega(\lambda)$ in (2.23) and (2.24) belongs to the class $\mathcal{U}_1^r(j_{pq})$, if the vector $v = \mathrm{col}\{v_1, v_2\}$ satisfies the condition $v_2 v_1^* \neq 0$.

2.4. **The class $\mathcal{U}_\kappa^\ell(j_{pq})$.** The following definitions and statements concerning the dual class $\mathcal{U}_\kappa^\ell(j_{pq})$ are taken from [30].

Definition 2.6. *An $m \times m$ mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^\ell(j_{pq})$, if*

$$s_{12} := w_{12}w_{22}^{-1} \in \mathcal{S}_\kappa^{p\times q}. \tag{2.25}$$

If $W \in \mathcal{U}_\kappa(j_{pq})$ and the mvf $\widetilde{W}$ is defined by

$$\widetilde{W}(\lambda) = \begin{cases} W(\overline{\lambda})^*, & \text{if } \Omega_+ = \mathbb{D}, \\ W(-\overline{\lambda})^*, & \text{if } \Omega_+ = \mathbb{C}_+, \end{cases} \tag{2.26}$$

then, as was shown [30], the following equivalence holds:

$$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \Longleftrightarrow \widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq}), \tag{2.27}$$

and as a corollary of Theorem 2.5 one can get the following statement.

Theorem 2.7. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and let the BP-factors $\mathfrak{b}_\ell$ and $\mathfrak{b}_r$ be defined by the KL-factorizations* (2.4)*,* (2.6) *of s_{12},*

$$s_{12}(\lambda) = \mathfrak{b}_\ell(\lambda)^{-1}\mathfrak{s}_\ell(\lambda) = \mathfrak{s}_r(\lambda)\mathfrak{b}_r(\lambda)^{-1}, \quad (\lambda \in \mathfrak{h}_{s_{12}}^+), \tag{2.28}$$

where $\mathfrak{b}_\ell \in \mathcal{S}_{in}^{p\times p}$, $\mathfrak{b}_r \in \mathcal{S}_{in}^{q\times q}$, $\mathfrak{s}_\ell, \mathfrak{s}_r \in \mathcal{S}^{p\times q}$. Then

$$s_{22}\mathfrak{b}_r \in \mathcal{S}^{q\times q} \quad \text{and} \quad \mathfrak{b}_\ell s_{11} \in \mathcal{S}^{p\times p}. \tag{2.29}$$

Definition 2.8. *Consider inner-outer factorizations of $\mathfrak{b}_\ell s_{11}$ and $s_{22}\mathfrak{b}_r$*

$$\mathfrak{b}_\ell s_{11} = \mathfrak{a}_1\mathfrak{b}_1, \quad s_{22}\mathfrak{b}_r = \mathfrak{b}_2\mathfrak{a}_2, \tag{2.30}$$

where $\mathfrak{b}_1 \in \mathcal{S}_{in}^{p\times p}$, $\mathfrak{b}_2 \in \mathcal{S}_{in}^{q\times q}$, $\mathfrak{a}_1 \in \mathcal{S}_{out}^{p\times p}$, $\mathfrak{a}_2 \in \mathcal{S}_{out}^{q\times q}$. The pair $\mathfrak{b}_1, \mathfrak{b}_2$ of inner factors in the factorizations (2.30) *is called the left associated pair of the mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and is written as $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$, for short.*

The following example shows that the classes $\mathcal{U}_\kappa^r(j_{pq})$ and $\mathcal{U}_\kappa^\ell(j_{pq})$ do not coincide.

Example 2. Let $\Omega_+ = \mathbb{D}$ and $W = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & \lambda \\ 1 & 2\lambda \end{bmatrix}$. The kernel $\mathsf{K}_\omega^W(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ has 1 negative square, therefore $W \in \mathcal{U}_1(j_{11})$. The mvf $W(\lambda)$ belongs to the class $\mathcal{U}_1^r(j_{11})$, since $s_{21} = \frac{1}{2\lambda} \in \mathcal{S}_1$. On the other hand $W \notin \mathcal{U}_1^\ell(j_{11})$, since $s_{12} = \frac{1}{2} \notin \mathcal{S}_1$.

Similarly, one has $\widetilde{W} = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & 1 \\ \lambda & 2\lambda \end{bmatrix} \in \mathcal{U}_1^\ell(j_{11}) \setminus \mathcal{U}_1^r(j_{11})$.

Let $W \in \mathcal{U}_\kappa(j_{pq})$ be a mvf with the block decomposition (2.16) and let the left linear fractional transformation T_W^ℓ be defined by

$$T_W^\ell[\varepsilon] := (\varepsilon(\lambda)w_{12}(\lambda) + w_{22}(\lambda))^{-1}(\varepsilon(\lambda)w_{11}(\lambda) + w_{21}(\lambda)). \tag{2.31}$$

Then the left and the right linear fractional transformations are connected by the equality

$$T_W^\ell[\varepsilon] = (T_{\widetilde{W}}^r[\widetilde{\varepsilon}])^{\sim}. \tag{2.32}$$

The following statement is implied by (2.32) and Lemma 2.3.

Lemma 2.9. *Let $W \in \mathcal{U}_{\kappa_1}(j_{pq})$, $\varepsilon \in \mathcal{S}_{\kappa_2}^{q\times p}$. Then $T_W^\ell[\varepsilon] \in \mathcal{S}_{\kappa'}^{q\times p}$ with $\kappa' \le \kappa_2 + \kappa_1$.*

2.5. **Reproducing kernel Pontryagin spaces.** In this subsection we review some facts and notation from [11, 16, 18] on the theory of indefinite inner product spaces for the convenience of the reader. A linear space $\mathcal{K}$ equipped with a sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ on $\mathcal{K}\times\mathcal{K}$ is called an indefinite inner product space. A subspace $\mathcal{F}$ of $\mathcal{K}$ is called positive (negative) if $\langle f, f\rangle_{\mathcal{K}} > 0\,(< 0)$ for all $f \in \mathcal{F}$, $f \ne 0$. If the full space $\mathcal{K}$ is positive and complete with respect to the norm $\|f\| = \langle f, f\rangle_{\mathcal{K}}^{1/2}$ then it is a Hilbert space.

An indefinite inner product space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ is called a Pontryagin space, if it can be decomposed as the orthogonal sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \tag{2.33}$$

of a positive subspace $\mathcal{K}_+$ which is a Hilbert space and a negative subspace $\mathcal{K}_-$ of finite dimension. The number $\operatorname{ind}_-\mathcal{K} := \dim\mathcal{K}_-$ is referred to as the negative index of $\mathcal{K}$. The convergence in a Pontryagin space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ is meant with respect to the Hilbert space norm

$$\|h\|^2 = \langle h_+, h_+\rangle_{\mathcal{K}} - \langle h_-, h_-\rangle_{\mathcal{K}}, \quad h = h_+ + h_-, \quad h_\pm \in \mathcal{K}_\pm. \tag{2.34}$$

It is easily seen that the convergence does not depend on a choice of the decomposition (2.33).

A Pontryagin space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ of $\mathbb{C}^m$-valued functions defined on a subset Ω of $\mathbb{C}$ is called *a RKPS* (*reproducing kernel Pontryagin space*), if there exists a Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m\times m}$, such that

(1) for every $\omega \in \Omega$ and every $u \in \mathbb{C}^m$ the vvf $\mathsf{K}_\omega(\lambda)u$ belongs to $\mathcal{K}$;

(2) for every $h \in \mathcal{K}$, $\omega \in \Omega$ and $u \in \mathbb{C}^m$ the following identity holds:

$$\langle h, \mathsf{K}_\omega u\rangle_{\mathcal{K}} = u^* f(\omega). \tag{2.35}$$

It is known (see [29]) that for every Hermitian kernel $\mathsf{K}_\omega(\lambda) : \Omega\times\Omega \to \mathbb{C}^{m\times m}$ with a finite number of negative squares on $\Omega\times\Omega$ there is a unique Pontryagin space $\mathcal{K}$ with reproducing kernel $\mathsf{K}_\omega(\lambda)$, and that $\operatorname{ind}_-\mathcal{K} = \operatorname{sq}_-\mathsf{K} = \kappa$. In the case $\kappa = 0$ this fact is due to Aronszajn [6].

If $W \in \mathcal{U}_\kappa(j_{pq})$, then assumption (ii) in the definition of $\mathcal{U}_\kappa(j_{pq})$ guarantees that $W(\lambda)$ is invertible in Ω_+ except for an isolated set of points. Define W in Ω_- by the formula (2.36)

$$W(\lambda) = j_{pq}W^{\#}(\lambda)^{-1}j_{pq} = j_{pq}W(\lambda^\circ)^{-*}j_{pq} \quad \text{if} \quad \lambda^\circ \in \mathfrak{h}_W^+ \quad \text{and} \quad \det W(\lambda^\circ) \neq 0.$$

Since W is of bounded type, the nontangential limits

$$W_\pm(\mu) = \angle\lim_{\lambda\to\mu}\{W(\lambda) : \lambda \in \Omega_\pm\}$$

exist a.e. on Ω_0; and assumption (ii) in the definition of $\mathcal{U}_\kappa(j_{pq})$ implies that the nontangential limits $W_+(\mu)$ and $W_-(\mu)$ coincide a.e. in Ω_0, that is, W in Ω_- is a pseudo-meromorphic extension of W in Ω_+. If $W(\lambda)$ is rational this extension is meromorphic on $\mathbb{C}$. The symbol $\mathfrak{h}_W$ will be used to denote the domain of holomorphy of W in $\mathbb{C}$. Formula (2.36) implies that $W(\lambda)$ is holomorphic and invertible in

$$\Omega_W := \mathfrak{h}_W \cap \mathfrak{h}_{W^{\#}}. \tag{2.37}$$

Let $W \in \mathcal{U}_\kappa(j_{pq})$ and let $\mathcal{K}(W)$ be the RKPS associated with the kernel $\mathsf{K}_\omega^W(\lambda)$. The kernel $\mathsf{K}_\omega^W(\lambda)$ extended to Ω_W by the equality (2.36) has the same number κ of negative squares [2, Theorem 2.5.2].

In the case where W belongs to the subclass $\mathcal{U}_\kappa^r(j_{pq})$ the subspaces

$$\mathcal{L}_W^+ := \mathcal{K}(W) \cap H_2^m, \quad \mathcal{L}_W^- := \mathcal{K}(W) \cap (H_2^m)^\perp, \quad \mathcal{L}_W := \mathcal{K}(W) \cap L_2^m \tag{2.38}$$

can be characterized by the following.

Theorem 2.10 ([18, Theorem 4.19]). *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$, $\{b_1, b_2\} \in ap^r(W)$, let K be defined by (2.22), let*

$$\mathcal{H}(b_1) = H_2^m \ominus b_1 H_2^m, \quad \mathcal{H}_*(b_2) = (H_2^m)^\perp \ominus b_2^*(H_2^m)^\perp, \tag{2.39}$$

and let

$$\Gamma_{11} : f \in H_2^q \longrightarrow P_{\mathcal{H}(b_1)}Kf, \quad \Gamma_{22} : f \in \mathcal{H}_*(b_2) \longrightarrow P_{(H_2^p)^\perp}Kf.$$

Then

$$\mathcal{L}_W^+ = \left\{ \begin{bmatrix} u_1 \\ \Gamma_{11}^* u_1 \end{bmatrix} : u_1 \in \mathcal{H}(b_1) \right\}, \tag{2.40}$$

$$\mathcal{L}_W^- = \left\{ \begin{bmatrix} \Gamma_{22} u_2 \\ u_2 \end{bmatrix} : u_2 \in \mathcal{H}_*(b_2) \right\}, \tag{2.41}$$

$$\mathcal{L}_W = \mathcal{L}_W^+ \dot{+} \mathcal{L}_W^-. \tag{2.42}$$

3. A-regular and A-singular generalized j_{pq}-inner mvf's

3.1. **A-singular generalized j_{pq}-inner mvf.** Let us recall the notations (see [10]):

$$\mathcal{N}_{\pm}^{p\times q} = \{f = h^{-1}g : g \in H_{\infty}^{p\times q}(\Omega_{\pm}), h \in \mathcal{S}_{out}^{1\times 1}(\Omega_{\pm})\},$$
$$\mathcal{N}_{out}^{p\times q} = \{f = h^{-1}g : g \in \mathcal{S}_{out}^{p\times q}, h \in \mathcal{S}_{out}^{1\times 1}\}.$$

A mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is called *A-singular*, if it is an outer mvf (see [7, 30]). The set of A-singular mvf's in $\mathcal{U}_\kappa(j_{pq})$ is denoted by $\mathcal{U}_\kappa^S(j_{pq})$.

We will be also using the following subclasses of the class $\mathcal{U}_\kappa^S(j_{pq})$:

$$\mathcal{U}_\kappa^{r,S}(j_{pq}) := \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_{out}^{m\times m}, \quad \mathcal{U}_\kappa^{\ell,S}(j_{pq}) := \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{N}_{out}^{m\times m}.$$

In the case $\kappa = 0$ the class $\mathcal{U}^S(j_{pq}) := \mathcal{U}_0^S(j_{pq})$ was introduced and characterized in terms of associated pairs by D. Arov in [9]. For $\kappa \neq 0$ a definition of A-singular generalized j_{pq}-inner mvf and its characterization in terms of associated pairs was given in [30].

Theorem 3.1 ([30])**.** *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and $\{b_1, b_2\} \in ap^r(W)$. Then*
1. *$W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+$ if and only if $b_2 \equiv \text{const}$;*
2. *$W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_-$ if and only if $b_1 \equiv \text{const}$;*
3. *$W \in \mathcal{U}_\kappa^{r,S}(j_{pq})$ if and only if $b_1 \equiv \text{const}$ and $b_2 \equiv \text{const}$.*

If $W \in \mathcal{U}_\kappa(j_{pq})$ and the mvf $\widetilde{W}$ is defined by (2.26) than as follows from (2.27)

$$W \in \mathcal{U}_\kappa^{\ell,S}(j_{pq}) \Longleftrightarrow \widetilde{W} \in \mathcal{U}_\kappa^{r,S}(j_{pq}). \tag{3.1}$$

As a corollary of Theorem 3.1 one get a similar characterization of the class $\mathcal{U}_\kappa^\ell(j_{pq})$.

Corollary 3.2 ([30])**.** *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in ap^\ell(W)$. Then*
1. *$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{N}_+$ if and only if $\mathfrak{b}_2 \equiv \text{const}$;*
2. *$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{N}_-$ if and only if $\mathfrak{b}_1 \equiv \text{const}$;*
3. *$W \in \mathcal{U}_\kappa^{\ell,S}(j_{pq})$ if and only if $\mathfrak{b}_1 \equiv \text{const}$ and $\mathfrak{b}_2 \equiv \text{const}$.*

Next we will present a characterization of A-singular mvf's W in terms of reproducing kernel spaces $\mathcal{K}(W)$ and its subspaces $\mathcal{L}_+(W)$ and $\mathcal{L}_-(W)$ and $\mathcal{L}_W$, introduced in (2.38).

Theorem 3.3. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$, $\{b_1, b_2\} \in ap^r(W)$. Then*
1. *$W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+$ if and only if $\mathcal{L}_W^- = \{0\}$;*
2. *$W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_-$ if and only if $\mathcal{L}_W^+ = \{0\}$;*
3. *$W \in \mathcal{U}_\kappa^{r,S}(j_{pq})$ if and only if $\mathcal{L}_W = \{0\}$.*

Proof. Assume that $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+$. Then by Theorem 3.1 (1) $b_2 \equiv \text{const}$. Therefore, $\mathcal{H}_*(b_2) = (H_2^m)^\perp \ominus b_2^*(H_2^m)^\perp = \{0\}$ and by Theorem 2.10 one obtains

$$\mathcal{L}_W^- = \{0\}.$$

Conversely, if $\mathcal{L}_W^- = \{0\}$ then by formula (2.41)

$$\begin{bmatrix} \Gamma_{22} \\ I \end{bmatrix} \mathcal{H}^*(b_2) = \{0\},$$

and hence $\mathcal{H}_*(b_2) = \{0\}$. Therefore, $b_2 \equiv \text{const}$, and, consequently, $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+$.

Similarly, the equivalence (2) is implied by Theorem 3.1 (1) and (2.40), and the equivalence (3) is implied by (1), (2) and (2.42). □

Corollary 3.4. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$. Then*
1. *$W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{N}_+$ if and only if $\mathcal{L}_{\widetilde{W}}^+ = \{0\}$;*

(2) $W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{N}_-$ *if and only if* $\mathcal{L}_{\widetilde{W}}^- = \{0\}$;

(3) $W \in \mathcal{U}_\kappa^{\ell,S}(j_{pq})$ *if and only if* $\mathcal{L}_{\widetilde{W}} = \{0\}$.

Proof. Since $W \in \mathcal{U}_\kappa^{\ell,S}(j_{pq})$, then $\widetilde{W} \in \mathcal{U}_\kappa^{r,S}(j_{pq})$, and by Theorem 3.3 it is possible if and only if $\mathcal{L}_{\widetilde{W}} = \{0\}$. □

Remark 3.5. In the case $\kappa = 0$ descriptions of linear manifolds $\mathcal{L}_W^\pm$, $\mathcal{L}_W$ in the form of (2.40) and a criterion of A-singularity of mvf $W \in \mathcal{U}_\kappa(j_{pq})$ in terms of $\mathcal{L}_W$ was presented in [9].

3.2. **Factorization of generalized j_{pq}-inner mvf's and associated pairs.** If $W \in \mathcal{U}(j_{pq})$ admits a representation $W = W^{(1)}W^{(2)}$ with $W^{(1)}, W^{(2)} \in \mathcal{U}(j_{pq})$ and $\{b_1, b_2\} \in \mathrm{ap}(W)$ and $\{b_1^{(1)}, b_2^{(1)}\} \in \mathrm{ap}(W^{(1)})$ then $b_1^{(1)}$ is a left divisor of b_1 and $b_2^{(1)}$ is a right divisor of b_2, see [8], [10, Lemma 4.28]. In this section an analog of this statement is proved for right and left generalized j_{pq}-inner mvf's. Relations between RKPS's corresponding to W, $W^{(1)}$ and $W^{(2)}$ are presented in the following theorem.

Theorem 3.6 ([2, Theorem 4.11])**.** *Let a mvf* $W(\lambda)$ *admit a factorization*

$$W = W^{(1)}W^{(2)}, \quad W^{(1)} \in \mathcal{U}_{\kappa_1}(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq}). \tag{3.2}$$

Then $W \in \mathcal{U}_\kappa(j_{pq})$ *with* $\kappa \le \kappa_1 + \kappa_2$ *and*

$$\mathcal{K}(W) \subseteq \mathcal{K}(W^{(1)}) + W^{(1)}\mathcal{K}(W^{(2)}), \tag{3.3}$$

where $\mathcal{K}(W)$, $\mathcal{K}(W^{(1)})$ *and* $\mathcal{K}(W^{(2)})$ *are RKPS's with reproducing kernels* $\mathsf{K}_\omega^W(\lambda)$, $\mathsf{K}_\omega^{W^{(1)}}(\lambda)$ *and* $\mathsf{K}_\omega^{W^{(2)}}(\lambda)$, *respectively. The following conditions are equivalent:*

(1) $\kappa = \kappa_1 + \kappa_2$,

(2) $\mathcal{K}(W^{(1)})$ *is contained contractively in* $\mathcal{K}(W)$,

(3) $\mathcal{K}(W^{(1)}) \cap W^{(1)}\mathcal{K}(W^{(2)})$ *is a Hilbert subspace of* $\mathcal{K}(W)$,

and in this case the equality in (3.3) *prevails. Moreover,* $\mathcal{K}(W^{(1)})$ *sits isometrically in* $\mathcal{K}(W)$ *if and only if* $\mathcal{K}(W^{(1)}) \cap W^{(1)}\mathcal{K}(W^{(2)}) = \{0\}$ *and in this case the decomposition* (3.3) *becomes orthogonal*

$$\mathcal{K}(W) = \mathcal{K}(W^{(1)})[+]W^{(1)}\mathcal{K}(W^{(2)}). \tag{3.4}$$

The importance of the condition (1) in Theorem 3.6 is illustrated by the following

Example 3. Let $\Omega_+ = \mathbb{D}$ and let mvf's $U^{(1)}(\lambda)$ and $U^{(2)}(\lambda)$ be given by

$$U^{(1)}(\lambda) = \frac{1}{2(1-\lambda)}\begin{bmatrix} 3-\lambda & -\lambda-1 \\ 1+\lambda & 1-3\lambda \end{bmatrix}, \quad U^{(2)}(\lambda) = \frac{1}{2(1-\lambda)}\begin{bmatrix} 1-3\lambda & \lambda+1 \\ -1-\lambda & 3-\lambda \end{bmatrix}.$$

Then

$$\mathsf{K}_\omega^{U^{(1)}}(\lambda) = \frac{-1}{(1-\lambda)(1-\bar\omega)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathsf{K}_\omega^{U^{(2)}}(\lambda) = \frac{1}{(1-\lambda)(1-\bar\omega)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, $U^{(1)} \in \mathcal{U}_1^{r,S}(j_{11})$, $U^{(2)} \in \mathcal{U}(j_{11})$ and

$$\mathcal{K}(U^{(1)}) = \mathcal{K}(U^{(2)}) = U^{(1)}\mathcal{K}(U^{(2)}) = \mathrm{span}\left\{\frac{1}{1-\lambda}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

But $U(\lambda) = U^{(1)}(\lambda)U^{(2)}(\lambda) \equiv I$ and hence $\mathcal{K}(U) = \{0\} \ne \mathcal{K}(U^{(1)}) + U^{(1)}\mathcal{K}(U^{(2)})$. In this example all the assumptions of Theorem 3.6 hold except of (1).

Lemma 3.7. *Let a mvf* $W \in \mathcal{U}_\kappa^r(j_{pq})$ *admit a factorization* (3.2), *where* $\kappa_1 + \kappa_2 = \kappa$. *Then*

(i) $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$.

(ii) *For* $\{b_1, b_2\} \in \mathrm{ap}^r(W)$ *and* $\{b_1^{(1)}, b_2^{(1)}\} \in \mathrm{ap}^r(W^{(1)})$ *one has*

$$\theta_1 := (b_1^{(1)})^{-1} b_1 \in S_{in}^{p\times p}, \quad \theta_2 := b_2 (b_2^{(1)})^{-1} \in S_{in}^{q\times q}. \tag{3.5}$$

Proof. The proof is divided into steps.

1. *Verification of (i):* Let the mvf's W, $W^{(k)}$ and their PG-transforms S, $S^{(k)}$ $(k = 1, 2)$ defined by (2.11) have the block matrix representations:

(3.6)

$$W = (w_{ij})_{i,j=1}^2, \quad W^{(k)} = (w_{ij}^{(k)})_{i,j=1}^2, \quad S = (s_{ij})_{i,j=1}^2, \quad S^{(k)} = (s_{ij}^{(k)})_{i,j=1}^2, \quad k = 1, 2,$$

corresponding to the decomposition (1.1) of j_{pq}. It follows from the equality $W = W^{(1)} W^{(2)}$ that

$$w_{21} = w_{21}^{(1)} w_{11}^{(2)} + w_{22}^{(1)} w_{21}^{(2)}, \quad w_{22} = w_{21}^{(1)} w_{12}^{(2)} + w_{22}^{(1)} w_{22}^{(2)}. \tag{3.7}$$

Since $W \in \mathcal{U}_\kappa^r(j_{pq})$ and $W^{(1)} \in \mathcal{U}_{\kappa_1}(j_{pq})$, then the matrices $w_{22}(\lambda)$ (see Section 2.2) and $w_{22}^{(1)}(\lambda)$ are invertible for every $\lambda \in (\mathfrak{h}_W^+ \cap \mathfrak{h}_{W^{(1)}}^+)$ except a finite number of points and

$$s_{21} = -w_{22}^{-1} w_{21} \in S_\kappa^{q\times p}, \quad s_{21}^{(1)} = -(w_{22}^{(1)})^{-1} w_{21}^{(1)} \in S_{\kappa'}^{q\times p} \quad \text{with} \quad \kappa' \le \kappa_1. \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned} w_{22}^{-1} w_{21} &= (w_{21}^{(1)} w_{12}^{(2)} + w_{22}^{(1)} w_{22}^{(2)})^{-1} (w_{21}^{(1)} w_{11}^{(2)} + w_{22}^{(1)} w_{21}^{(2)}) \\ &= (-s_{21}^{(1)} w_{12}^{(2)} + w_{22}^{(2)})^{-1} (-s_{21}^{(1)} w_{11}^{(2)} + w_{21}^{(2)}). \end{aligned} \tag{3.9}$$

Since $W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq})$, then by Lemma 2.9

$$w_{22}^{-1} w_{21} = T_{W^{(2)}}^\ell [-s_{21}^{(1)}] \in S_{\kappa''}^{q\times p}, \quad \text{where} \quad \kappa'' \le \kappa' + \kappa_2. \tag{3.10}$$

On the other hand $w_{22}^{-1} w_{21} \in S_\kappa^{q\times p}$ by the assumption $W \in \mathcal{U}_\kappa^r(j_{pq})$. Comparing the equality $\kappa = \kappa''$ with (3.10) one obtains

$$\kappa = \kappa'' \le \kappa' + \kappa_2 \le \kappa_1 + \kappa_2 = \kappa$$

and hence $\kappa'' = \kappa$, $\kappa' = \kappa_1$. Therefore, $s_{21}^{(1)} \in S_{\kappa_1}^{q\times p}$. This proves the inclusion $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$.

2. *Verification of (ii):* Let $\mathcal{K}(W)$ and $\mathcal{K}(W^{(j)})$ $(j = 1, 2)$ be reproducing kernel spaces with the kernels (1.2) and

$$\mathsf{K}_\omega^{W^{(j)}}(\lambda) = \frac{j_{pq} - W^{(j)}(\lambda) j_{pq} W^{(j)}(\omega)^*}{\rho_\omega(\lambda)} \quad (j = 1, 2).$$

It follows from Theorem 3.6 that

$$\mathcal{K}(W) \cap H_2^m \supset \mathcal{K}(W^{(1)}) \cap H_2^m, \quad \mathcal{K}(W) \cap (H_2^m)^\perp \supset \mathcal{K}(W^{(1)}) \cap (H_2^m)^\perp.$$

Using the formulas for $\mathcal{K}(W) \cap H_2^m$ and $\mathcal{K}(W) \cap (H_2^m)^\perp$ from Theorem 2.10 one obtains

$$\mathcal{H}(b_1) \supseteq \mathcal{H}(b_1^{(1)}), \quad \mathcal{H}_*(b_2) \supseteq \mathcal{H}_*(b_2^{(1)}). \tag{3.11}$$

The inclusions (3.11) are equivalent to the relations (3.5). □

As shows the following example the assumption $W \in \mathcal{U}_\kappa^r(j_{pq})$ in Lemma 3.7 is essential.

Example 4. Let $\Omega_+ = \mathbb{D}$. Consider the mvf's

$$W^{(1)}(\lambda) = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & \lambda \\ 1 & 2\lambda \end{bmatrix} \in \mathcal{U}_1^r(j_{11}), \quad W^{(2)}(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \in \mathcal{U}_1(j_{11}) \setminus \mathcal{U}_1^\ell(j_{11}),$$

and let $W(\lambda) = W^{(1)}(\lambda) W^{(2)}(\lambda)$ be the product of these mvf's

$$W(\lambda) = W^{(1)}(\lambda) W^{(2)}(\lambda) = \frac{1}{\sqrt{3}} \begin{bmatrix} 2\lambda & \lambda^2 \\ \lambda & 2\lambda^2 \end{bmatrix}.$$

The kernel

$$\mathsf{K}_\omega^W(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{\lambda\overline{\omega}}{3} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

has 2 negative square, therefore, $W \in \mathcal{U}_2(j_{11})$. However, $W \notin \mathcal{U}_2^r(j_{11})$, since $s_{21} = -\frac{1}{2\lambda} \in \mathcal{S}_1$. This shows that the converse statement to Lemma 3.7 (i) is not true.

The next statement is a dual version of Lemma 3.7.

Lemma 3.8. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ admit the factorization* (3.2), *where $\kappa_1 + \kappa_2 = \kappa$. Then*

(i) $W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq})$.

(ii) *For $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in \mathrm{ap}^\ell(W)$ and $\{\mathfrak{b}_1^{(2)}, \mathfrak{b}_2^{(2)}\} \in \mathrm{ap}^\ell(W^{(2)})$ one has*

$$\vartheta_1 := \mathfrak{b}_1(\mathfrak{b}_1^{(2)})^{-1} \in S_{in}^{p\times p}, \quad \vartheta_2 := (\mathfrak{b}_2^{(2)})^{-1}\mathfrak{b}_2 \in S_{in}^{q\times q}. \tag{3.12}$$

Proof. If $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in \mathrm{ap}^\ell(W)$, then as was shown in [30, Proposition 3.7 and Theorem 3.8] $\{\widetilde{\mathfrak{b}}_1, \widetilde{\mathfrak{b}}_2\} \in \mathrm{ap}^r(\widetilde{W})$ and $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq})$ by (2.27). Due to Lemma 3.7 $\widetilde{W} = \widetilde{W}^{(2)}\widetilde{W}^{(1)}$, where $\widetilde{W}^{(2)} \in \mathcal{U}_{\kappa_2}^r(j_{pq})$. Applying again (2.27) one obtains the statement (i).

Next, if $\{\mathfrak{b}_1^{(2)}, \mathfrak{b}_2^{(2)}\} \in \mathrm{ap}^\ell(W^{(2)})$, then $\{\widetilde{\mathfrak{b}}_1^{(2)}, \widetilde{\mathfrak{b}}_2^{(2)}\} \in \mathrm{ap}^r(\widetilde{W}^{(2)})$ and by Lemma 3.7

$$(\widetilde{\mathfrak{b}}_1^{(2)})^{-1}\widetilde{\mathfrak{b}}_1 \in S_{in}^{p\times p}, \quad \widetilde{\mathfrak{b}}_2(\widetilde{\mathfrak{b}}_2^{(2)})^{-1} \in S_{in}^{q\times q}. \tag{3.13}$$

These inclusions are equivalent to (3.12). □

Corollary 3.9. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ admit the factorization* (3.2), *with $\kappa_1 = \kappa$, $\kappa_2 = 0$. Then $W^{(1)} \in \mathcal{U}_\kappa^r(j_{pq})$ and if $\{b_1, b_2\} \in \mathrm{ap}^r(W)$ and $\{b_1^{(1)}, b_2^{(1)}\} \in \mathrm{ap}^r(W^{(1)})$, then* (3.5) *holds.*

Corollary 3.10. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ admit the factorization* (3.2), *with $\kappa_1 = 0$, $\kappa_2 = \kappa$. Then $W^{(2)} \in \mathcal{U}_\kappa^\ell(j_{pq})$ and if $\{\mathfrak{b}_1, \mathfrak{b}_2\} \in \mathrm{ap}^\ell(W)$ and $\{\mathfrak{b}_1^{(1)}, \mathfrak{b}_2^{(1)}\} \in \mathrm{ap}^\ell(W^{(2)})$, then* (3.12) *holds.*

Lemma 3.11. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$admit the factorization* (3.2), *where*

$$W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq}), \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq}), \quad \kappa = \kappa_1 + \kappa_2,$$

and let $\{b_1, b_2\} \in \mathrm{ap}^r(W)$, $\{b_1^{(1)}, b_2^{(1)}\} \in \mathrm{ap}^r(W^{(1)})$, $\{\mathfrak{b}_1^{(2)}, \mathfrak{b}_2^{(2)}\} \in \mathrm{ap}^\ell(W^{(2)})$. Then

$$\deg b_1 \geq \deg b_1^{(1)} + \deg \mathfrak{b}_1^{(2)}, \quad \deg b_2 \geq \deg b_2^{(1)} + \deg \mathfrak{b}_2^{(2)}. \tag{3.14}$$

If, in addition, $W^{(1)} \in \widetilde{L}_2^m$ then the following equalities hold:

$$\deg b_1 = \deg b_1^{(1)} + \deg \mathfrak{b}_1^{(2)}, \quad \deg b_2 = \deg b_2^{(1)} + \deg \mathfrak{b}_2^{(2)}. \tag{3.15}$$

Proof. **1.** *Two formulas for the blocks s_{11} and s_{22} of the PG-transform S of the mvf W will be established.* Let the mvf's W, $W^{(k)}$ and their PG-transforms S, $S^{(k)}$ $(k = 1, 2)$ defined by (2.11) have the block matrix representations (3.6). Using the equality

$$w_{11} = w_{11}^{(1)}w_{11}^{(2)} + w_{12}^{(1)}w_{21}^{(2)} \tag{3.16}$$

one obtains from (2.14) that the following equalities are valid on $\mathfrak{h}_S^+ \cap \mathfrak{h}_{W^\#}^+$:

$$\begin{aligned} s_{11} = w_{11}^{-\#} &= \left((w_{11}^{(2)})^\#(w_{11}^{(1)})^\# + (w_{21}^{(2)})^\#(w_{12}^{(1)})^\#\right)^{-1} \\ &= (w_{11}^{(1)})^{-\#}\left(I_p + (w_{11}^{(2)})^{-\#}(w_{21}^{(2)})^\#(w_{12}^{(1)})^\#(w_{11}^{(1)})^{-\#}\right)^{-1}(w_{11}^{(2)})^{-\#} \\ &= s_{11}^{(1)}(I_p - s_{12}^{(2)}s_{21}^{(1)})^{-1}s_{11}^{(2)}. \end{aligned} \tag{3.17}$$

Similarly, it follows from (3.7) and (2.12) that

$$w_{22} = w_{22}^{(1)}(I_q - s_{21}^{(1)}s_{12}^{(2)})w_{22}^{(2)}, \tag{3.18}$$

(3.19) $$s_{22} = w_{22}^{-1} = s_{22}^{(2)}(I_q - s_{21}^{(1)}s_{12}^{(2)})^{-1}s_{22}^{(1)}.$$

2. *Further factorizations in* (3.17) *and* (3.19) *is given in terms of associated pairs of W, $W^{(1)}$ and $W^{(2)}$.*

Since $W \in \mathcal{U}_\kappa^r(j_{pq})$, $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$ and $W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq})$, then

$$s_{21} \in S_\kappa^{q\times p}, \quad s_{21}^{(1)} \in S_{\kappa_1}^{q\times p}, \quad s_{12}^{(2)} \in S_{\kappa_2}^{p\times q}.$$

Let b_ℓ, b_r, $b_\ell^{(1)}$, $b_r^{(1)}$, $\mathfrak{b}_\ell^{(2)}$ and $\mathfrak{b}_r^{(2)}$ be inner factors determined by the KL-factorizations of mvf's s_{21}, $s_{21}^{(1)}$, $s_{12}^{(2)}$

$$\begin{aligned} s_{21} &= b_\ell^{-1}s_\ell = s_r b_r^{-1}, \\ s_{21}^{(1)} &= (b_\ell^{(1)})^{-1}s_\ell^{(1)} = s_r^{(1)}(b_r^{(1)})^{-1}, \\ s_{12}^{(2)} &= (\mathfrak{b}_\ell^{(2)})^{-1}\mathfrak{s}_\ell^{(2)} = \mathfrak{s}_r(\mathfrak{b}_r^{(2)})^{-1}. \end{aligned}$$

Then as follows from [18, Theorem 4.6] (see (2.20)) and [30, Theorem 3.8]

$$b_\ell s_{22},\ b_\ell^{(1)}s_{22}^{(1)},\ s_{22}^{(2)}\mathfrak{b}_r^{(2)} \in \mathcal{S}^{q\times q}, \quad s_{11}b_r,\ s_{11}^{(1)}b_r^{(1)},\ \mathfrak{b}_\ell^{(2)}s_{11}^{(2)} \in \mathcal{S}^{p\times p}.$$

Consider inner-outer (and outer-inner, resp.) factorizations for these mvf's

(3.20) $$s_{11}b_r = b_1a_1, \quad b_\ell s_{22} = a_2b_2,$$

(3.21) $$s_{11}^{(1)}b_r^{(1)} = b_1^{(1)}a_1^{(1)}, \quad b_\ell^{(1)}s_{22}^{(1)} = a_2^{(1)}b_2^{(1)},$$

(3.22) $$\mathfrak{b}_\ell^{(2)}s_{11}^{(2)} = \mathfrak{a}_1^{(2)}\mathfrak{b}_1^{(2)}, \quad s_{22}^{(2)}\mathfrak{b}_r^{(2)} = \mathfrak{b}_2^{(2)}\mathfrak{a}_2^{(2)},$$

where $b_1, b_1^{(1)}, \mathfrak{b}_1^{(2)} \in S_{in}^{p\times p}$, $b_2, b_2^{(1)}, \mathfrak{b}_2^{(2)} \in S_{in}^{q\times q}$, $a_1, a_1^{(1)}, \mathfrak{a}_1^{(2)} \in S_{out}^{p\times p}$, a_2, $a_2^{(1)}$, $\mathfrak{a}_2^{(2)} \in S_{out}^{q\times q}$.

Multiplying (3.17) by b_r from the right and using (3.20)–(3.22) one obtains

(3.23) $$\begin{aligned} b_1a_1 &= s_{11}^{(1)}(I_p - (\mathfrak{b}_\ell^{(2)})^{-1}\mathfrak{s}_\ell^{(2)}s_r^{(1)}(b_r^{(1)})^{-1})^{-1}s_{11}^{(2)}b_r \\ &= b_1^{(1)}a_1^{(1)}(\mathfrak{b}_\ell^{(2)}b_r^{(1)} - \mathfrak{s}_\ell^{(2)}s_r^{(1)})^{-1}\mathfrak{a}_1^{(2)}\mathfrak{b}_1^{(2)}b_r. \end{aligned}$$

Similarly, multiplying (3.19) by b_ℓ from the left and using (3.20)–(3.22), one obtains

(3.24) $$\begin{aligned} a_2b_2 &= b_\ell s_{22}^{(2)}(I_q - (b_\ell^{(1)})^{-1}s_\ell^{(1)}\mathfrak{s}_r^{(2)}(\mathfrak{b}_r^{(2)})^{-1})^{-1}\left(b_\ell^{(1)}\right)^{-1}a_2^{(1)}b_2^{(1)} \\ &= b_\ell\mathfrak{b}_2^{(2)}\mathfrak{a}_2^{(2)}(b_\ell^{(1)}\mathfrak{b}_r^{(2)} - s_\ell^{(1)}\mathfrak{s}_r^{(2)})^{-1}a_2^{(1)}b_2^{(1)}. \end{aligned}$$

3. *Verification of* (3.14)*:* Let θ_1, θ_2 be mvf's defined by (3.5). Then it follows from (3.23) and (3.24) that

(3.25) $$\theta_1a_1 = a_1^{(1)}(\mathfrak{b}_\ell^{(2)}b_r^{(1)} - \mathfrak{s}_\ell^{(2)}s_r^{(1)})^{-1}\mathfrak{a}_1^{(2)}\mathfrak{b}_1^{(2)}b_r,$$

(3.26) $$(\mathfrak{b}_\ell^{(2)}b_r^{(1)} - \mathfrak{s}_\ell^{(2)}s_r^{(1)})(a_1^{(1)})^{-1}\theta_1a_1 = \mathfrak{a}_1^{(2)}\mathfrak{b}_1^{(2)}b_r.$$

By the generalized Rouche Theorem (Theorem 2.1)

(3.27) $$\mathcal{M}_\zeta(\mathfrak{b}_\ell^{(2)}b_r^{(1)} - \mathfrak{s}_\ell^{(2)}s_r^{(1)}, \Omega_+) \le \kappa.$$

On the other hand,

(3.28) $$\mathcal{M}_\zeta(\mathfrak{a}_1^{(2)}\mathfrak{b}_1^{(2)}b_r, \Omega_+) = \deg b_r + \deg \mathfrak{b}_1^{(2)} = \kappa + \deg \mathfrak{b}_1^{(2)}.$$

Now (3.27), (3.28) imply the inequality

(3.29) $$\kappa + \deg \mathfrak{b}_1^{(2)} \le \kappa + \deg\theta = \kappa + \deg b_1 - \deg b_1^{(1)},$$

which coincides with the first inequality in (3.14).

Similarly, it follows from (3.24) that

$$a_2\theta_2(a_2^{(1)})^{-1}(b_\ell^{(1)}\mathfrak{b}_r^{(2)} - s_\ell^{(1)}\mathfrak{s}_r^{(2)}) = b_\ell \mathfrak{b}_2^{(2)}\mathfrak{a}_2^{(2)}. \tag{3.30}$$

When comparing zero multiplicities of both parts of (3.30) and applying Theorem 2.1 one obtains

$$\begin{aligned} \deg \mathfrak{b}_2^{(2)} + \kappa = \mathcal{M}_\zeta(b_\ell \mathfrak{b}_2^{(2)}\mathfrak{a}_2^{(2)}, \Omega_+) &= \mathcal{M}_\zeta(\theta_2(a_2^{(1)})^{-1}(b_\ell^{(1)}\mathfrak{b}_r^{(2)} - s_\ell^{(1)}\mathfrak{s}_r^{(2)}), \Omega_+) \\ &\le \kappa + \deg b_2 - \deg b_2^{(1)}, \end{aligned} \tag{3.31}$$

which coincides with the second inequality in (3.14).

4. *Verification of* (3.15): By [18, Lemma 4.22] the assumption $W^{(1)} \in \widetilde{L}_2^{m\times m}$ implies

$$(I_p - \varepsilon s_{21}^{(1)})^{-1} \in \widetilde{L}_1^{p\times p} \quad \text{and} \quad (I_p - s_{21}^{(1)}\varepsilon)^{-1} \in \widetilde{L}_1^{p\times p}$$

for all $\varepsilon \in S^{p\times q}$. Hence, by generalized Rouche Theorem (Theorem 2.1) one obtains

$$\mathcal{M}_\zeta(\mathfrak{b}_\ell^{(2)} b_r^{(1)} - \mathfrak{s}_\ell^{(2)} s_r^{(1)}, \Omega_+) = \mathcal{M}_\zeta(b_\ell^{(1)}\mathfrak{b}_r^{(2)} - s_\ell^{(1)}\mathfrak{s}_r^{(2)}, \Omega_+) = \kappa. \tag{3.32}$$

Therefore, the inequalities (3.29), and (3.31) will transform into equalities (3.15). □

Lemma 3.12. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let $W = W^{(1)}W^{(2)}$, where $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$, $W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq})$ and $\kappa = \kappa_1 + \kappa_2$. Then the following implication holds:*

$$\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W) \Rightarrow W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}). \tag{3.33}$$

If, in addition, $W^{(1)} \in \widetilde{L}_2^m$ then the converse is also true and thus the following equivalence holds

$$\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W) \Longleftrightarrow W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}). \tag{3.34}$$

Proof. Assume that $\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W)$, i.e.

$$b_1 = b_1^{(1)}\theta_1, \quad b_2 = \theta_2 b_2^{(1)} \tag{3.35}$$

for some constant unitary matrices $\theta_1\,\theta_2$. Then, by Lemma 3.11 $\deg \mathfrak{b}_1^{(2)} = 0$ and $\deg \mathfrak{b}_2^{(2)} = 0$. In view of Theorem 3.1 this implies, that $W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq})$.

Conversely, if $W^{(1)} \in \widetilde{L}_2^{m\times m}$ and $W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq})$, then by Theorem 3.1 $\deg \mathfrak{b}_1^{(2)} = 0$ and $\deg \mathfrak{b}_2^{(2)} = 0$. Now the second statement of Lemma 3.11 yields the equality $\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W)$. □

In the case $\kappa_2 = 0$ the previous statement takes the form.

Corollary 3.13. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let $W = W^{(1)}W^{(2)}$, where $W^{(1)} \in \mathcal{U}_\kappa^r(j_{pq})$, $W^{(2)} \in \mathcal{U}(j_{pq})$. Then the following implication holds:*

$$\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W) \Rightarrow W^{(2)} \in \mathcal{U}^S(j_{pq}). \tag{3.36}$$

If in addition, $W^{(1)} \in \widetilde{L}_2^m$ then the converse is also true and thus the following equivalence holds:

$$\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W) \Longleftrightarrow W^{(2)} \in \mathcal{U}^S(j_{pq}). \tag{3.37}$$

3.3. **A-regular generalized j_{pq}-inner mvf's.** Recall (see [7]), that a mvf $W\in\mathcal{U}(j_{pq})$ is called *right A-regular* (*left A-regular*), if for any factorization $W=W^{(1)}W^{(2)}$ with $W^{(1)},W^{(2)}\in\mathcal{U}(j_{pq})$ the assumption $W_2\in\mathcal{U}^S(j_{pq})$ $(W^{(1)}\in\mathcal{U}^S(j_{pq}))$ implies $W^{(2)}(\lambda)\equiv$ const $(W^{(1)}(\lambda)\equiv \text{const})$. The set of right A-regular and left A-regular mvf's in $\mathcal{U}(j_{pq})$ is denoted by $\mathcal{U}^{r,R}(j_{pq})$ and $\mathcal{U}^{\ell,R}(j_{pq})$.

Definition 3.14. *A mvf $W\in\mathcal{U}_\kappa^r(j_{pq})$ is called right A-regular, if for any factorization*

$$W=W^{(1)}W^{(2)},\quad W^{(1)}\in\mathcal{U}_{\kappa_1}^r(j_{pq}),\quad W^{(2)}\in\mathcal{U}_{\kappa_2}^\ell(j_{pq}),\tag{3.38}$$

with $\kappa_1+\kappa_2=\kappa$ the assumption $W^{(2)}\in\mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq})$ implies $W^{(2)}(\lambda)\equiv$ const.

Similarly, a mvf $W\in\mathcal{U}_\kappa^\ell(j_{pq})$ is called left A-regular, if for any factorization (3.38) *with $\kappa_1+\kappa_2=\kappa$ the assumption $W^{(1)}\in\mathcal{U}_{\kappa_1}^S(j_{pq})$ implies $W^{(1)}(\lambda)\equiv$ const.*

In order to prove the next result we will need the following two theorems from [5, Theorems 4.1 and 4.2] and [3, Theorem 8]. The first theorem was formulated in terms of the resolvent operator R_α acting in a RKPS $\mathcal{K}(W)$ $(W\in\mathcal{U}_\kappa(j_{p,q}))$ by the formula

$$(R_\alpha f)(\omega)=\frac{f(\lambda)-f(\omega)}{\lambda-\omega},\quad f\in\mathcal{K}(W),\quad \lambda,\omega\in\mathfrak{h}_W.$$

Recall, that $\mathcal{K}(W)$ denotes the RKPS with the reproducing kernel $\mathsf{K}_\omega^W(\lambda)$, see (1.2).

Theorem 3.15. ([5], Theorems 4.1 and 4.2). *A RKPS $\mathcal{K}$ of $\mathbb{C}^m$-valued vvf's holomorphic on a domain $\mathfrak{h}_\mathcal{K}$ with negative index $\kappa\in\mathbb{N}\cup\{0\}$ is a $\mathcal{K}(W)$ space for some $W\in\mathcal{U}_\kappa(j_{pq})$, if and only if the following three conditions hold:*

(1) *$\mathcal{K}$ is invariant with respect to R_α for all $\alpha\in\mathfrak{h}_\mathcal{K}$;*
(2) *for all $\alpha,\beta\in\mathfrak{h}_\mathcal{K}$ and $f,g\in\mathcal{K}$ one of the following equalities holds:*

$$[f,g]_\mathcal{K}+\alpha[R_\alpha f,g]_\mathcal{K}+\overline{\beta}[f,R_\beta g]_\mathcal{K}-(1-\alpha\overline{\beta})[R_\alpha f,R_\beta g]_\mathcal{K}=g(\beta)^*j_{pq}f(\alpha),\quad \text{if } \Omega_+=\mathbb{D},\tag{3.39}$$

$$\text{or}\quad [R_\alpha f,g]_\mathcal{K}-[f,R_\beta g]_\mathcal{K}-(\alpha-\overline{\beta})[R_\alpha f,R_\beta g]_\mathcal{K}=2\pi i g(\beta)^*j_{pq}f(\alpha),\quad \text{if } \Omega_+=\mathbb{C}_+;\tag{3.40}$$

(3) *$\mathfrak{h}_\mathcal{K}\cap\Omega_0\neq\emptyset$.*

Recall, that reproducing kernel Hilbert spaces $\mathcal{K}(W)$ were first characterized by L. de Branges [15] for the case $\Omega_+=\mathbb{C}_+$, the disc version is due to J. Ball [12]; a unified version of both that is applicable to Kreĭn spaces is presented in [5].

Another theorem gives a generalization of Leech's criterion for the existence of a factorization of operator valued functions in terms of the nonnegativity of certain kernel. We will adapt below Theorem 8 from [3] to our notations.

Theorem 3.16. *Suppose $W\in\mathcal{U}_\kappa(j_{pq})$ and $W^{(1)}\in\mathcal{U}_{\kappa_1}(j_{pq})$, where $0\le\kappa_1\le\kappa$. Put $\kappa_2=\kappa-\kappa_1$. The following are equivalent:*

(i) *$W(\lambda)$ admits a factorization $W(\lambda)=W^{(1)}(\lambda)W^{(2)}(\lambda)$ for some $W^{(2)}\in\mathcal{U}_{\kappa_2}(j_{pq})$;*
(ii) *the kernel $\dfrac{W^{(1)}(\lambda)j_{pq}W^{(1)}(\omega)^*-W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)}$ has κ_2 negative squares.*

The following theorem ensures the existence of some specific factorization of the form (3.2). In this section we present some sufficient conditions for a generalized j_{pq}-inner mvf $W\in\mathcal{U}_\kappa^r(j_{pq})$ $(W\in\mathcal{U}_\kappa^\ell(j_{pq}))$ to admit such a factorization.

Theorem 3.17. *Let $W\in\mathcal{U}_\kappa^r(j_{pq})$, let $\mathcal{K}(W)$ be the RKPS with the kernel $\mathsf{K}_\omega^W(\lambda)$, defined by* (1.2), *let $\mathcal{L}_W:=\mathcal{K}(W)\cap L_2^m$, and let $\kappa_1=\operatorname{ind}_-(\mathcal{L}_W)$, $\kappa_2=\kappa-\kappa_1$. Assume that*

(A1) *$\mathfrak{h}_W\cap\Omega_0\neq\emptyset$;*
(A2) *The closure $\overline{\mathcal{L}_W}$ of $\mathcal{L}_W$ is nondegenerate in $\mathcal{K}(W)$.*

Then the mvf $W(\lambda)$ admits the factorization (3.2) *such that*

(i) *the RKPS $\mathcal{K}(W^{(1)})$ coincides with $\overline{\mathcal{L}_W}$ and is embedded isometrically in $\mathcal{K}(W)$;*

(ii) $\mathcal{L}_{W^{(1)}} = \mathcal{L}_W$ *and* $\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W)$.

Proof. Step 1. Verification that the closure $\overline{\mathcal{L}_W}$ *of* $\mathcal{L}_W$ *is a RKPS.*

Indeed, $\overline{\mathcal{L}_W}$ is a nondegenerate subspace of $\mathcal{K}(W)$ and hence $\overline{\mathcal{L}_W}$ is a Pontryagin space of negative index κ_1. Since $\mathcal{K}(W)$ is a RKPS, then the evaluation operator $E(\lambda)$ is bounded as an operator acting from $\mathcal{K}(W)$ to $\mathbb{C}^m$. The reproducing kernel for $\mathcal{K}(W)$ is given by

$$\mathsf{K}_\omega(\lambda) = E(\lambda)E(\omega)^*.$$

Let $F(\lambda)$ be a restriction of $E(\lambda)$ to $\overline{\mathcal{L}_W}$, [2]. $F(\lambda)$ is bounded as an operator from $\overline{\mathcal{L}_W}$ to $\mathbb{C}^m$. The reproducing kernel for $\overline{\mathcal{L}_W}$ has the form

$$\mathsf{K}_\omega^{(1)}(\lambda) = F(\lambda)F(\omega)^*.$$

Step 2. Verification that the RKPS $\overline{\mathcal{L}_W}$ *is a* $\mathcal{K}(W^{(1)})$ *space*, i.e. its kernel can be represented as

$$\mathsf{K}_\omega^{(1)}(\lambda) = \mathsf{K}_\omega^{W^{(1)}}(\lambda) := \frac{j_{pq} - W^{(1)}(\lambda)j_{pq}W^{(1)}(\omega)^*}{\rho_\omega(\lambda)},$$

for some $W^{(1)} \in \mathcal{U}_{\kappa_1}(j_{pq})$.

Let us check the conditions (1)–(3) of Theorem 3.17 for the RKPS $\overline{\mathcal{L}_W}$. The condition (1) holds, since $\mathcal{L}_W$ is R_α invariant for all $\alpha \in \mathfrak{h}_W$, the condition (2) is in force, since the de Branges identity holds for all $f, g \in \mathcal{K}(W)$ and $\overline{\mathcal{L}_W} \subset \mathcal{K}(W)$. The last condition follows from (A1). Therefore, the RKPS $\overline{\mathcal{L}_W}$ is a $\mathcal{K}(W^{(1)})$ space, for some $W^{(1)} \in \mathcal{U}_{\kappa_1}(j_{pq})$.

Step 3. Construction of a mvf $W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq})$ *such that* (3.2) *holds.*

Let P be the orthogonal projection in $\mathcal{K}(W)$ onto

$$\mathcal{K}(W^{(1)}) := \overline{\mathcal{L}_W}. \tag{3.41}$$

Then

$$PE(\cdot)E(\omega)^*|_{\overline{\mathcal{L}_W}} = F(\cdot)F(\omega)^* \quad (\omega \in \mathfrak{h}_W).$$

Indeed, for all $f \in \mathcal{K}(W^{(1)})$ and $u \in \mathcal{K}^m$ one obtains

$$\begin{aligned} \langle f, P(E(\cdot)E(\omega)^*u\rangle_{\mathcal{K}(W^{(1)})} &= \langle f, E(\cdot)E(\omega)^*u\rangle_{\mathcal{K}(W)} \\ &= u^*f(\omega) = \langle f, F(\cdot)F(\omega)^*u\rangle_{\mathcal{K}(W^{(1)})}. \end{aligned} \tag{3.42}$$

Let the kernel $\mathsf{K}_\omega^{(2)}(\lambda)$ be defined by

$$\mathsf{K}_\omega^{(2)}(\lambda) = \mathsf{K}_\omega(\lambda) - \mathsf{K}_\omega^{(1)}(\lambda) \quad (\omega, \lambda \in \mathfrak{h}_W).$$

The kernel $\mathsf{K}_\omega^{(2)}(\lambda)$ has $\kappa_2 = \kappa - \kappa_1$ negative squares. Indeed, for every $u, v \in \mathcal{K}^m$

$$\begin{aligned} \langle \mathsf{K}_\omega^{(2)}(\lambda)u, v\rangle &= \langle E(\omega)^*u, E(\omega)^*v\rangle_{\mathcal{K}(W)} - \langle F(\omega)^*u, F(\omega)^*v\rangle_{\mathcal{K}(W)} \\ &= \langle (1-P)E(\omega)^*u, (1-P)E(\omega)^*v\rangle_{\mathcal{K}(W)}. \end{aligned}$$

Hence one obtains the equality

$$\sum_{j,k=1}^n \langle \mathsf{K}_{\omega_j}^{(2)}(\omega_k)u_j, u_k\rangle \xi_j\overline{\xi_k} = \sum_{j,k=1}^n \langle (I-P)E(\omega_j)^*u_j, (I-P)E(\omega_k)^*u_k\rangle_{\mathcal{K}(W)}\xi_j\overline{\xi_k},$$

which shows that $\mathsf{K}_\omega^{(2)}(\lambda)$ has κ_2 negative squares.

By Theorem 3.16 there is $W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq})$ such that $W(\lambda) = W^{(1)}(\lambda)W^{(2)}(\lambda)$. Moreover, $W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq})$, since both W and $W^{(1)}$ have j_{pq}-unitary nontangential limits a.e. on Ω_0.

Step 4. Verification that $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$, $\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W)$.

The inclusion $W^{(1)} \in \mathcal{U}_\kappa^r(j_{pq})$ is implied by Lemma 3.7. Now it follows from [4, Theorem 6.14] that

$$\mathcal{K}(W) = \mathcal{K}(W^{(1)})[+]W^{(1)}\mathcal{K}(W^{(2)}). \tag{3.43}$$

Equality (3.43) implies the statement (ii). Moreover, it follows from (3.43) that

$$\mathcal{L}_{W^{(1)}} = \mathcal{K}(W^{(1)}) \cap L_2^m \subset \mathcal{K}(W) \cap L_2^m = \mathcal{L}_W.$$

On the other hand, it follows from (3.41) that

$$\mathcal{L}_{W^{(1)}} = \mathcal{K}(W^{(1)}) \cap L_2^m = \overline{\mathcal{L}_W} \cap L_2^m \supset \mathcal{L}_W.$$

Therefore, $\mathcal{L}_{W^{(1)}} = \mathcal{L}_W$ and hence $\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W)$ by Theorem 2.10. This completes the proof. □

Corollary 3.18. *Let, under the assumptions of Theorem 3.17, $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq})$, and let $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$ and $W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq})$ be the mvf's determined in Theorem 3.17. Then*

$$W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}). \tag{3.44}$$

Proof. Since $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq})$ one has $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$ and $W^{(2)} \in \mathcal{U}_{\kappa_2}^\ell(j_{pq})$. Next by Theorem 3.17 the following condition holds

$$\mathrm{ap}^r(W^{(1)}) = \mathrm{ap}^r(W), \tag{3.45}$$

and hence by Lemma 3.12 $W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq})$. □

Corollary 3.19. *Let, under the assumptions of Theorem 3.17, $W \in \mathcal{U}_\kappa^r(j_{pq})$, let $W^{(1)} \in \mathcal{U}_\kappa^r(j_{pq})$, $W^{(2)} \in \mathcal{U}(j_{pq})$ be the mvf's constructed in Theorem 3.17, and let* $\mathrm{ind}_-\mathcal{L}_W = \kappa$. *Then $W^{(2)} \in \mathcal{U}^{\ell,S}(j_{pq})$.*

Proof. Since $\mathrm{ind}_-\mathcal{L}_W = \kappa$ the space $\overline{\mathcal{L}_W} = \overline{(\mathcal{K}(W) \cap L_2^m)}$ is nondegenerate, i.e. the assumption (A2) holds. By Theorem 3.17 there exist mvf's $W^{(1)} \in \mathcal{U}_\kappa^r(j_{pq})$ and $W^{(2)} \in \mathcal{U}(j_{pq})$, such that $W = W^{(1)}W^{(2)}$ and (3.45) holds. By Corollary 3.13 $W^{(2)} \in \mathcal{U}^S(j_{pq})$. □

In the next lemma we find some sufficient conditions for a mvf $W(\lambda)$ to be regular. Denote by $\mathcal{R}^{m\times m}$ the set of rational $m \times m$-mvf's.

Lemma 3.20. *Let, under the assumptions of Theorem 3.17,* $\mathrm{ind}_-\mathcal{L}_W = \kappa$. *Then the following implications hold:*

(1) $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longrightarrow \overline{\mathcal{L}_W} = \mathcal{K}(W)$;
(2) $\mathcal{K}(\widetilde{W}) \subset L_2^{m\times m} \Longrightarrow W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$;
(3) $W \in \widetilde{L}_2^{m\times m} \cap \mathcal{R}^{m\times m} \Longrightarrow W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$.

Proof. By Theorem 3.17 and Corollary 3.19 $W = W^{(1)}W^{(2)}$, where $W^{(1)} \in \mathcal{U}_{\kappa_1}^r(j_{pq})$ and $W^{(2)} \in \mathcal{U}^S(j_{pq})$.

(1) Let $W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$ and assume that $\overline{\mathcal{K}(W) \cap L_2^m} \neq \mathcal{K}(W)$. Then

$$\mathcal{K}(W^{(1)}) = \overline{\mathcal{K}(W) \cap L_2^m} \neq \mathcal{K}(W), \tag{3.46}$$

and the equalities (3.43) and (3.46) yield $\mathcal{K}(W^{(2)}) \neq \{0\}$, i.e. $W^{(2)} \not\equiv \mathrm{const}$. But this contradicts the assumption $W \in \mathcal{U}_\kappa^{rR}(j_{pq})$.

(2) Let $\mathcal{K}(\widetilde{W}) \subset L_2^{m\times m}$, and assume that

$$W = W^{(3)}W^{(4)}, \quad \text{where} \quad W^{(3)} \in \mathcal{U}_{\kappa_1}^r(j_{pq}), \quad W^{(4)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}) \quad \text{and} \quad \kappa_3 + \kappa_4 = \kappa.$$

Then

$$\widetilde{W} = \widetilde{W}^{(4)}\widetilde{W}^{(3)}, \quad \text{where} \quad \widetilde{W}^{(3)} \in \mathcal{U}_{\kappa_3}(j_{pq}), \quad \widetilde{W}^{(4)} \in \mathcal{U}_{\kappa_4}^{r,S}(j_{pq}).$$

By Theorem 3.6

$$\mathcal{K}(\widetilde{W}) = \mathcal{K}(\widetilde{W}^{(4)}) + \widetilde{W}^{(4)}\mathcal{K}(\widetilde{W}^{(3)}). \tag{3.47}$$

Since $\mathcal{K}(\widetilde{W}) \subset L_2^{m\times m}$ and $\mathcal{K}(\widetilde{W}^{(4)}) \subset \mathcal{K}(\widetilde{W})$ one obtains $\mathcal{K}(\widetilde{W}^{(4)}) = \{0\}$ and hence $W^{(4)} \equiv$ const.

(3) Assume that $W \in \widetilde{L}_2^{m\times m} \cap \mathcal{R}^{m\times m}$. Then $\mathsf{K}_\omega u \in L_2^m$ for all $\omega \in \mathfrak{h}_W$ and $u \in \mathcal{K}^m$ and hence the set $\mathcal{L}_W = \mathcal{K}(W) \cap L_2^m$ is dense in $\mathcal{K}(W)$. In fact, $\mathcal{K}(W)$ is a finite-dimensional space since W is rational, and hence $\mathcal{K}(W) = \mathcal{L}_W \subset L_2^{m\times m}$.

The assumption $W \in \widetilde{L}_2^{m\times m} \cap \mathcal{R}^{m\times m}$ implies also $\widetilde{W} \in \widetilde{L}_2^{m\times m} \cap \mathcal{R}^{m\times m}$ and hence as above one obtains $\mathcal{K}(\widetilde{W}) \subset L_2^{m\times m}$. Now the statement is implied by (2) □

Remark 3.21. In contrast with the definite case the result of Lemma 3.20 is much weaker. If $\kappa = 0$ then the statements (1) and (3) take the form (see [10, Theorems 5.86, 5.90]):

(1′) $W \in \mathcal{U}^{r,R}(j_{pq}) \Longleftrightarrow \overline{\mathcal{L}_W} = \mathcal{K}(W)$;

(3′) $W \in \widetilde{L}_2^{m\times m} \cap \mathcal{U}^r(j_{pq}) \Longrightarrow W \in \mathcal{U}^{r,R}(j_{pq})$.

In the following theorem a criterion for a rational mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ to be A-regular is proved.

Theorem 3.22. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ be a rational mvf. Then*

$$W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Longleftrightarrow \mathcal{L}_W = \mathcal{K}(W).$$

Proof. **1.** *Verification of the implication $\mathcal{L}_W = \mathcal{K}(W) \Rightarrow W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$.*
It follows from the assumption $\mathcal{L}_W = \mathcal{K}(W)$ that $W \in \widetilde{L}_2^{m\times m}$. Hence by Theorem 3.20 $W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$.

2. *Verification of the implication $W \in \mathcal{U}_\kappa^{r,R}(j_{pq}) \Rightarrow \mathcal{L}_W = \mathcal{K}(W)$.*
Assume that $\mathcal{L}_W \neq \mathcal{K}(\widetilde{W})$. Then W has a pole ω_0 on Ω_0 and hence the space $\mathcal{K}(W)$ contains a vvf $f(\lambda) = \frac{v}{\lambda - \overline{\omega_0}}$, see [4, Theorem 5.2]. A vvf $f(\lambda)$ is an eigenfunction for the backward shift operator R_α corresponding to the eigenvalue $\frac{1}{\overline{\omega}_0 - \alpha}$, $\alpha \in \Omega_+$. Since $\mathcal{K} = \mathcal{K}(\widetilde{W})$ is a RKPS with the kernel $\mathsf{K}_\omega^{\widetilde{W}}(\lambda)$ by [4, Theorem 6.9], then for every choice of $f, g \in \mathcal{K}(\widetilde{W})$ and every $\alpha, \beta \in \Omega_+$ the identity (3.39) holds if $\Omega_+ = \mathbb{D}$, or the identity (3.40) holds if $\Omega_+ = \mathbb{C}_+$. Substituting $\beta = \alpha$ and $g = f = \frac{v}{\lambda - \overline{\omega}_0}$ in (3.39) if $\Omega_+ = \mathbb{D}$ (or in (3.40), if $\Omega_+ = \mathbb{C}_+$), one obtains from (3.39) ((3.40), resp.)

$$v^* j_{pq} v = 0. \tag{3.48}$$

Consider the mvf's

$$V_\varepsilon(\lambda) := I_m - \frac{\varepsilon}{2} c_{\omega_0}(\lambda) v v^* j_{pq}, \quad W_\varepsilon(\lambda) := V_\varepsilon(\lambda)^{-1} \widetilde{W}(\lambda), \quad \varepsilon > 0.$$

Then $V_\varepsilon \in \mathcal{U}(j_{pq})$ and $\mathcal{K}(V_\varepsilon) = \operatorname{span} f$ (see Example 1), $W_\varepsilon \in \mathcal{U}_{\kappa'}(j_{pq})$ for some $\kappa' \geq \kappa$,

$$\widetilde{W}(\lambda) = V_\varepsilon(\lambda) W_\varepsilon(\lambda) \tag{3.49}$$

and

$$\mathcal{K}(\widetilde{W}) \subseteq \mathcal{K}(V_\varepsilon) + V_\varepsilon(\mathcal{K}(W_\varepsilon)). \tag{3.50}$$

If $[f, f]_\mathcal{K} \leq 0$ then the following inequality holds

$$[f, f]_\mathcal{K} \leq 0 \leq [f, f]_{\mathcal{K}(V_\varepsilon)} \tag{3.51}$$

and hence the space $\mathcal{K}(V_\varepsilon)$ is contractively contained in $\mathcal{K}(\widetilde{W})$.

If $[f, f]_\mathcal{K} > 0$, then the inequality (3.51) will be satisfied for ε small enough, cf. [4, Theorem 5.4], and hence again the inclusion $\mathcal{K}(V_\varepsilon) \subset \mathcal{K}(\widetilde{W})$ will be contractive. By Theorem 3.6 one obtains $\kappa' = \kappa$ and hence $W_\varepsilon \in \mathcal{U}_\kappa(j_{pq})$. Applying the transform (2.26) one obtains the factorization

$$W(\lambda) = \widetilde{W}_\varepsilon(\lambda) \widetilde{V}_\varepsilon(\lambda),$$

where $W_\varepsilon \in \mathcal{U}_\kappa^r(j_{pq})$, $V_\varepsilon \in \mathcal{U}^S(j_{pq})$ and $V_\varepsilon \not\equiv$ const. This contradicts the assumption $W \in \mathcal{U}_\kappa^{r,R}(j_{pq})$. □

In the case $\kappa = 0$ an examples of A-regular j_{pq}-inner mvf's are provided by BP-factors of the 1-st and the 2-nd kind. In the indefinite case ($\kappa > 0$) these examples can be slightly modified.

Example 5. By Theorem 3.22 every rational mvf from $\mathcal{U}_1^r(j_{pq})$, which has no poles on Ω_0, is right A-regular, in particular, the mvf's $U_\omega(\lambda)$ in (2.23) and (2.24) belong to the class $\mathcal{U}_1^{r,R}(j_{pq})$, if $v_2 v_1^* \neq 0$.

In the following example we introduce a rational generalized j_{pq}-inner mvf with poles on the boundary Ω_0, which is not A-regular and does not admit A-regular–A-singular factorization.

Example 6. Let $\Omega_+ = \mathbb{D}$ and let the mvf $W(\lambda)$ be defined by (see [4, (7.5)])

$$W(\lambda) = (I_2 + \{b_{\beta,\alpha}(\lambda) - 1\}W_{1,2})(I_2 + \{b_{\alpha,\beta}(\lambda) - 1\}j_{pq}W_{1,2}^* j_{pq}),$$

where

$$W_{1,2} = u_1(u_2^* j_{pq} u_1)^{-1} u_2^* j_{pq}, \quad b_{\alpha,\beta}(\lambda) = \frac{\lambda - \alpha}{1 - \lambda\beta^*},$$

and u_1, u_2 are vectors in $\mathbb{C}^2$, such that $u_2^* j_{pq} u_1 \neq 0$. Then for $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\alpha = 0 \in \Omega_+$, $\beta = 1$, (notice that $\beta \notin \Omega_+$) one obtains

$$W(\lambda) = \frac{1}{2\lambda - 2} \begin{bmatrix} \lambda^2 - 3\lambda + 1 & \lambda^2 - \lambda + 1 \\ \lambda^2 - \lambda + 1 & \lambda^2 - 3\lambda + 1 \end{bmatrix}.$$

The mvf $W(\lambda)$ has the following properties:

(1) $W \in \mathcal{U}_1^r(j_{pq})$;
(2) $W(\cdot)$ is neither A-singular, nor A-regular;
(3) $W(\cdot)$ does not admit A-regular–A-singular factorization.

Indeed, the kernel

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda) j_{pq} W(\omega)^*}{1 - \lambda\overline{\omega}} = \frac{1}{2(\lambda - 1)(\overline{\omega} - 1)} \begin{bmatrix} 2 - \lambda - \overline{\omega} & \lambda - \overline{\omega} \\ -(\lambda - \overline{\omega}) & -(2 - \lambda - \overline{\omega}) \end{bmatrix} \tag{3.52}$$

has 1 negative square in $\mathfrak{h}_W^+$; $W(\lambda)$ is j_{pq}-unitary a.e. on $\mathbb{T}$, hence $W \in \mathcal{U}_1(j_{pq})$. The PG-transformation $S = PG(W)$ of W takes the form

$$S(\lambda) = \frac{1}{\lambda^2 - 3\lambda + 1} \begin{bmatrix} -2\lambda(\lambda - 1) & \lambda^2 - \lambda + 1 \\ -(\lambda^2 - \lambda + 1) & 2(\lambda - 1) \end{bmatrix}.$$

If λ_1 and λ_2 are two zeros of the polynomial $\lambda^2 - 3\lambda + 1$, such that $\lambda_1 \in \mathbb{D}$ and $\lambda_2 \notin \mathbb{D}$, then the left KL- factorization of $s_{21}(\lambda)$ takes the form

$$s_{21}(\lambda) = -\frac{\lambda^2 - \lambda + 1}{\lambda^2 - 3\lambda + 1} = b_\ell^{-1} s_\ell = s_r b_r^{-1},$$

where $b_r(\lambda) = b_\ell(\lambda) = \frac{\lambda - \lambda_1}{1 - \overline{\lambda_1}\lambda}$ and hence $s_{21} \in \mathcal{S}_1$ and $W \in \mathcal{U}_1^r(j_{pq})$.

Since the function

$$b_\ell s_{22} = \frac{\lambda - \lambda_1}{1 - \overline{\lambda_1}\lambda} \cdot \frac{2(\lambda - 1)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \frac{2(\lambda - 1)}{(1 - \overline{\lambda_1}\lambda)(\lambda - \lambda_2)}, \quad \lambda_2 \notin \mathbb{D}.$$

is outer, the factor b_2 in (2.20) is missing, that is $b_2 = 1$. The function

$$s_{11} b_r = -\frac{2\lambda(\lambda - 1)}{\lambda^2 - 3\lambda + 1} \cdot \frac{\lambda - \lambda_1}{1 - \overline{\lambda_1}\lambda} = -\frac{2\lambda(\lambda - 1)}{(\lambda - \lambda_2)(1 - \overline{\lambda_1}\lambda)}$$

has an inner factor $b_1 = \lambda$. Therefore, the associated pair $\mathrm{ap}^r(W)$ coincides with $\{\lambda, 1\}$ and by Theorem 3.1 the mvf $W(\cdot)$ is not A-singular.

The RKPS $\mathcal{K}(W)$ and the subspace $\mathcal{L}_W$ take the form

$$\mathcal{K}(W) = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\lambda-1}\begin{bmatrix}1\\-1\end{bmatrix}\right\}, \quad \mathcal{L}_W = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

By Theorem 3.22 the mvf $W(\lambda)$ is not A-regular, since $\mathcal{L}_W \neq \mathcal{K}(W)$.

Notice, that the fact that $W(\lambda)$ is not right A-regular can be also checked directly. Indeed, $W(\lambda)$ admits the factorization

$$W(\lambda) = W^{(1)}(\lambda)U^{(2)}(\lambda),$$

where $U^{(2)}(\lambda)$ is the mvf from Example 3 and

$$W^{(1)}(\lambda) = W(\lambda)(U^{(2)}(\lambda))^{-1} = \frac{1}{2(1-\lambda)}\begin{bmatrix}3\lambda-2 & -\lambda(2\lambda-1)\\ \lambda-2 & -\lambda(2\lambda-3)\end{bmatrix}.$$

The corresponding reproducing kernel $\mathsf{K}_\omega^{W^{(1)}}(\lambda)$ and the RKPS $\mathcal{K}(W^{(1)})$ take the form

$$\mathsf{K}_\omega^{W^{(1)}}(\lambda) = \frac{-1}{2(1-\lambda)(1-\overline{\omega})}\begin{bmatrix}2\lambda\overline{\omega}-\lambda-\overline{\omega} & 2\lambda\overline{\omega}-3\lambda-\overline{\omega}+2\\ 2\lambda\overline{\omega}-\lambda-3\overline{\omega}+2 & 2\lambda\overline{\omega}-3\lambda-3\overline{\omega}+4\end{bmatrix},$$

$$\mathcal{K}(W^{(1)}) = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\lambda-1}\begin{bmatrix}1\\-1\end{bmatrix}\right\}.$$

It is easily checked that $\kappa_-(\mathcal{K}(W^{(1)})) = 1$ and hence $W^{(1)} \in \mathcal{U}_1^r(j_{11})$. Since $U^{(2)} \in \mathcal{U}^S(j_{11})$ and $U^{(2)} \not\equiv \text{const}$ it shows that $W(\lambda)$ is not A-regular.

Moreover, the mvf $W(\lambda)$ does not admit right A-regular–A-singular factorization. Indeed, if

$$W(\lambda) = W^{(3)}(\lambda)W^{(4)}(\lambda), \quad W^{(3)} \in \mathcal{U}_{\kappa_3}^{r,R}(j_{11}), \quad W^{(4)} \in \mathcal{U}_{\kappa_4}^{\ell,S}(j_{11}), \tag{3.53}$$

then $W^{(3)}(\lambda)$ and $W^{(4)}(\lambda)$ are factors of degree 1, since W is neither right A-regular nor A-singular mvf. If $\kappa_3 = 0$ then the mvf $W^{(3)}$ is a BP-factor of the 1-st kind with pole at ∞,

$$W^{(3)}(\lambda) = I + (\lambda-1)vv^*j_{pq}, \quad v^*j_{pq}v = 1, \tag{3.54}$$

where $v \in \mathbb{C}^2$ is determined by $v^*j_{pq}W^{(3)}(0) = 0$.

However, the equation $v^*j_{pq}W(0) = 0$ has a unique (up to a j_{pq}-unitary factor) solution $v = \begin{bmatrix}1\\1\end{bmatrix}$ and this vector does not satisfy the condition $v^*j_{pq}v = 1$.

In the case $\kappa_3 = 1$ the mvf $W^{(3)}$ admits the representation (2.23) (see Example 1)

$$W^{(3)}(\lambda) = I - (\lambda-1)vv^*j_{pq}, \quad \text{where} \quad v^*j_{pq}v = -1$$

and again $v \in \mathbb{C}^2$ is determined by $v^*j_{pq}W^{(3)}(0) = 0$. But this implies $v^*j_{pq}W(0) = 0$ and solution $v = \begin{bmatrix}1\\1\end{bmatrix}$ of the equation $v^*j_{pq}W(0) = 0$ does not satisfies $v^*j_{pq}v = -1$.

This proves that the mvf $W(\lambda)$ does not admit the factorization (3.53).

3.4. **Existence of A-regular–A-singular factorizations.**

Theorem 3.23. *Let $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{R}^{m\times m}$. Then the following statements are equivalent:*

(1) *W admits the factorization*

$$W = W^{(1)}W^{(2)}, \quad \text{where} \quad W^{(1)} \in \mathcal{U}_{\kappa_1}^{r,R}(j_{pq}) \quad \text{and} \quad W^{(2)} \in \mathcal{U}_{\kappa_2}^{\ell,S}(j_{pq}) \tag{3.55}$$

with $\kappa = \kappa_1 + \kappa_2$;

(2) *$\mathcal{L}_W$ is a nondegenerate subspace of $\mathcal{K}(W)$.*

Moreover, if (2) is the case then the factors $W^{(1)}$ and $W^{(2)}$ in (3.55) *are uniquely determined up to j_{pq}-unitary factors.*

Proof. **1.** *Verification of implication* (2) $\Longrightarrow$ (1). Consider the factorization $W = W^{(1)}W^{(2)}$, constructed in Theorem 3.17, in which $W^{(1)} \in \mathcal{U}^{r}_{\kappa_1}(j_{pq})$ and $W^{(2)} \in \mathcal{U}_{\kappa_2}(j_{pq})$. By Lemma 3.8 $W^{(2)} \in \mathcal{U}^{\ell}_{\kappa_2}(j_{pq})$ and by Corollary 3.18 $W^{(2)} \in \mathcal{U}^{\ell,S}_{\kappa_2}(j_{pq})$. Since

$$\mathcal{K}(W^{(1)}) = \overline{\mathcal{L}_W} = \mathcal{L}_W \subset L_2^m,$$

and $W^{(1)} \in \mathcal{R}^{m\times m}$ then also $\widetilde{W}^{(1)} \in \widetilde{L}_2^{m\times m}$ and in view of Lemma 3.20 $W^{(1)} \in \mathcal{U}^{r,R}_{\kappa_1}(j_{pq})$.
2. *Verification of implication* (1) $\Longrightarrow$ (2). Let W admits the factorization (3.55) with $\kappa = \kappa_1 + \kappa_2$. By Theorem 3.6 the following equality holds

$$\mathcal{K}(W) = \mathcal{K}(W^{(1)}) + W^{(1)}\mathcal{K}(W^{(2)}). \tag{3.56}$$

Since $W^{(1)} \in \mathcal{U}^{r,R}_{\kappa_1}(j_{pq})$ it has no zeros on Ω_0 and hence $W^{(1)}\mathcal{K}(W^{(2)}) \cap L_2^m = \{0\}$. This implies $W^{(1)}\mathcal{K}(W^{(2)}) \cap \mathcal{K}(W^{(1)}) = \{0\}$ and hence by Theorem 3.6 the sum in (3.56) is orthogonal. Therefore, the subspace $\mathcal{L}_W = \mathcal{K}(W) \cap L_2^m = \mathcal{K}(W^{(1)})$ is nondegenerate in $\mathcal{K}(W)$.
3. *Verification of uniqueness of* (3.55). Assume now that $W = W^{(3)}W^{(4)}$ is another factorization of W, such that $W^{(3)} \in \mathcal{U}^{r,R}_{\kappa_3}(j_{pq})$ and $W^{(4)} \in \mathcal{U}^{S}_{\kappa_4}(j_{pq})$.

Then by Theorem 3.22 $\mathcal{L}_{W^{(3)}} = \mathcal{K}(W^{(3)})$. Therefore, $\mathcal{K}(W^{(3)}) \subset L_2^m$ and hence $W^{(3)} \subset \widetilde{L}_2^{m\times m}$. Applying Lemma 3.11, one obtains the equality

$$\mathrm{ap}^r(W^{(3)}) = \mathrm{ap}^r(W).$$

which implies $(\mathcal{K}(W^{(3)}) =)\mathcal{L}_{W^{(3)}} = \mathcal{L}_W$. Besides, in view of Theorem 3.20

$$\mathcal{K}(W^{(1)}) = \mathcal{L}_{W^{(1)}} = \mathcal{L}_W.$$

Thus, by [18, Theorem 4.19] $W^{(3)} = W^{(1)}V$ and, hence, $W^{(4)} = V^{-1}W^{(2)}$, where V is a constant j_{pq}-unitary matrix. □

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