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OVERDAMPED MODES AND OPTIMIZATION OF RESONANCES IN LAYERED CAVITIES

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Dedicated to Eduard Tsekanovskii on the occasion of his 80th birthday.

Abstract. We study the problem of optimizing the imaginary parts $\operatorname{Im}\omega$ of quasi-normal-eigenvalues ω associated with the equation $y'' = -\omega^2 By$. It is assumed that the coefficient $B(x)$, which describes the structure of an optical or mechanical resonator, is constrained by the inequalities $0 \le b_1 \le B(x) \le b_2$. Extremal quasi-normal-eigenvalues belonging to the imaginary line $\mathrm{i}\mathbb{R}$ are studied in detail. As an application, we provide examples of ω with locally minimal $|\operatorname{Im}\omega|$ (without additional restrictions on $\operatorname{Re}\omega$) and show that a structure generating an optimal quasi-normal-eigenvalue on $\mathrm{i}\mathbb{R}$ is not necessarily unique.

1. Introduction

Under the assumption of normally passing electromagnetic waves, the Maxwell system in a multilayer medium can be reduced to the wave equation of a non-homogeneous string, $B(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t)$, where the coefficient $B(x) \ge 0$ represents spatially varying dielectric permittivity in the case of 1-D optical cavity, or the linear density of the string in Mechanics settings. When the medium is homogeneous for x outside a finite interval (a_1, a_2), i.e., when $B(x) = \nu^2$ for $x \notin [a_1, a_2]$ with a certain constant $\nu > 0$, resonances (or quasi-normal-eigenvalues) ω associated with the string equation can be defined via the eigenproblem

$$y''(x) = -\omega^2 B(x) y(x) \quad \text{a.e. for} \quad x \in (a_1, a_2), \tag{1.1}$$

$$\frac{y'(a_1)}{-i\omega\nu} = y(a_1), \quad \frac{y'(a_2)}{i\omega\nu} = y(a_2). \tag{1.2}$$

In these settings, $B(x)$ is an integrable on the interval (a_1, a_2) function that describes the inner structure of the resonator.

Because of the leakage of waves to the outer medium $\mathbb{R} \setminus [a_1, a_2]$, resonant eigenoscillations $e^{-\mathrm{i}\omega t} y(x)$ decay exponentially when $t \to +\infty$. The minus imaginary part $\beta = -\operatorname{Im}\omega$ of a resonance is positive and is called *the decay rate*, the real part $\alpha = \operatorname{Re}\omega$ plays the role of *the frequency of oscillations*.

Pure imaginary resonances $\omega \in \mathrm{i}\mathbb{R}$ in Mechanics settings correspond to the over-damping and critical damping effects, when the solution $e^{-\mathrm{i}\omega t} y(x)$ of the wave equation decays as $t \to +\infty$ without oscillations. This case has very special spectral and optimization properties, which are the subject of the present note.

Explicitly computable examples of resonances are provided by constant structures $B(x) = b$ a.e. on (a_1, a_2), where $b \in \mathbb{R}_+ \setminus \{\nu^2\}$. For such structures, all the resonances

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$\omega^{[n]} = \omega^{[n]}(b)$, $n \in \mathbb{Z}$, are given by [22, 5]

$$\omega^{[n]}(b) = -\frac{\mathrm{i}}{b^{1/2}(a_2 - a_1)} \ln \left| \frac{b^{1/2} + \nu}{b^{1/2} - \nu} \right| + \frac{\pi n}{b^{1/2}(a_2 - a_1)}, \quad n \in \mathbb{Z}. \tag{1.3}$$

Since optical cavities with specific resonant properties are needed in a number of applications including Cavity Quantum Electrodynamics, Optical Engineering, and Microscopy (see e.g. [16, 18, 20, 23]), optimization problems for resonances attracted considerable interest of specialists in numerical methods [2, 8, 9, 15, 17].

The most used numerical approach in mathematical studies of the optimization of resonances involves steepest ascent method searching for resonances with locally minimal decay rate [9, 8, 17] over various discretizations of the family of *admissible structures*

$$\mathbb{A}_0 := \{B(x) \in L^\infty_{\mathbb{R}}(a_1, a_2) \ : \ b_1 \le B(x) \le b_2 \text{ a.e. on } (a_1, a_2)\}, \tag{1.4}$$

where $b_1, b_2 \in \mathbb{R}_+$ are constraints on dielectric permittivities of materials used in fabrication. It is assumed often that the dielectric permittivity of the outer medium is also in the above range, i.e.,

$$b_1 \le \nu^2 \le b_2, \tag{1.5}$$

and is fixed (in the sense that it does not participate in the optimization).

The local minimizers that were studied in [7, 21, 9, 8, 17] can rigorously be defined in the following way. For a fixed parameter $\nu > 0$ and a fixed interval $[a_1, a_2]$, the structure $B(\cdot) \in L^1_{\mathbb{R}}(a_1, a_2)$ completely determines the set $\Sigma(B)$ of all associated resonances (in short, resonances of B). Let $\mathbb{A} \subset L^1_{\mathbb{R}}(a_1, a_2)$ be a certain family of admissible structures B over that the optimization is performed (for brevity, *admissible family*). A pair (ω, B) is called admissible if $B \in \mathbb{A}$ and $\omega \in \Sigma(B)$. Let us define

the decay rate functional $\mathrm{Dr}(\omega, B) := -\operatorname{Im} \omega$ *on the set of all admissible pairs.*

Definition 1.1 (cf. [7, 21] for Schrdinger equations)**.** An admissible pair (ω_0, B_0) is said to be a *local minimizer of* Dr if there exists $\varepsilon > 0$ such that $\mathrm{Dr}(\omega_0, B_0) \le \mathrm{Dr}(\omega_1, B_1)$ for every admissible pair (ω_1, B_1) with $|\omega_1 - \omega_0| < \varepsilon$ and $\|B_1 - B_0\|_1 < \varepsilon$.

Under assumption (1.5), proofs of existence of such local minimizers over the family $\mathbb{A}$ have not been available (while an attempt to justify the existence on the base of numerical experiments has been done in [17]). One of the goals of the present paper is to rigorously obtain the following result.

Theorem 1.1. *Let b_1 and b_2 be constants so that $0 < b_1 < b_2$. Then at least one of the pairs $(\omega^{[0]}(b_j), b_j)$, $j = 1, 2$, is a local minimizer of the functional* Dr *over the admissible family $\mathbb{A}_0$.* (*Recall that $\omega^{[0]}(b)$ is defined by (1.3).*)

This is a particular case of more general Corollary 3.2 obtained below by the study of total multiplicities of resonances in complex domains near the imaginary line $\mathrm{i}\mathbb{R}$ (see Section 3).

As a by-product, we obtain a negative answer on the question of uniqueness of 1-D structures generating a resonance of minimal decay ω under an additional restriction on its frequency $\operatorname{Re} \omega$ (this question was discussed in [7, 10, 12]). Namely, let

$$\Sigma[\mathbb{A}] := \bigcup_{B \in \mathbb{A}} \Sigma(B)$$

be the set of *admissible resonances* (over $\mathbb{A}$). For a frequency $\alpha \in \operatorname{Re} \Sigma[\mathbb{A}]$, the minimal decay rate $\beta_{\min}(\alpha)$ is defined by

$$\beta_{\min}(\alpha) := \ \inf\{\beta \in \mathbb{R} \ : \ \alpha - \mathrm{i}\beta \in \Sigma[\mathbb{A}]\} \quad \text{(see [11]).} \tag{1.6}$$

If $\omega = \alpha - \mathrm{i}\beta_{\min}(\alpha)$ is a resonance of a certain structure $B \in \mathbb{A}$ (i.e., if the minimum is achieved in (1.6)), then the resonances ω and the structures B are said to be of *minimal decay for the frequency* α.

Example 1.2. Let $b_1 = 1$, $b_2 = 4$, and $\nu^2 = 3$. Then $0 \in \operatorname{Re}\Sigma[\mathbb{A}]$, and the resonance of minimal decay for the frequency 0 exists and is equal to $\omega = -\frac{\mathrm{i}}{2(a_2-a_1)} \ln\left(7 + 4\sqrt{3}\right)$. The two extreme allowed constant structures b_1 and b_2 are the structures of minimal decay for the frequency 0. These statements follow from Corollary 3.2 (iii) and formula (1.3).

A preliminary version of these results was posted in Section 7 of the e-print arXiv:1508.04706v1 [math.OC]. The other sections of the e-print arXiv:1508.04706v1 are concerned with optimization of resonances in $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$ and were published as [13]. They provide some background information for the present note.

Notation. The following sets of real and complex numbers are used: the open half-lines $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$ and the open discs $\mathbb{D}_\epsilon(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \epsilon\}$. By $\partial_x f$, $\partial_z f$, etc., we denote (ordinary or partial) derivatives with respect to (w.r.t.) x, z, etc. $L^p_{\mathbb{C}(\mathbb{R})}(a_1, a_2)$ are the Lebesgue spaces of complex- (resp., real-) valued functions, and $W^{k,p}[a_1, a_2] := \{y \in L^p_{\mathbb{C}}(a_1, a_2) \ : \ \partial^j_x y \in L^p_{\mathbb{C}}(a_1, a_2),\ 1 \le j \le k\}$ are Sobolev spaces. The corresponding standard norms are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}}$. The space of continuous complex-valued functions with the uniform norm is denoted by $C[a_1, a_2]$. The Lebesgue measure is denoted by $\operatorname{meas} E$. Let S be a subset of a normed space U over $\mathbb{C}$. For $u_0 \in U$ and $z \in \mathbb{C}$, $zS + u_0 := \{zu + u_0 \ : \ u \in S\}$. Open balls are denoted by $\mathbb{B}_\epsilon(u_0) := \{u \in U \ : \ \|u - u_0\|_U < \epsilon\}$. The closure of a set S (in the norm topology) is denoted by $\overline{S}$, the boundary of S by $\operatorname{bd} S$. For a function f defined on S, $f[S]$ is the image of S. For a functional $G(z; u)$ that maps $\mathbb{C} \times U$ to $\mathbb{C}$, we denote by $\frac{\partial G(z;u_0)}{\partial u}(u_1) := \lim_{\zeta\to 0} \frac{G(z;u_0+\zeta u_1)-G(z;u_0)}{\zeta}$ *the directional derivative* of $G(z; \cdot)$ along the vector $u_1 \in U$ at the point $u_0 \in U$. We say that a map $G : U_1 \to U_2$ between normed spaces $U_{1,2}$ is *bounded-to-bounded* if the set $G[S]$ in U_2 is bounded for any bounded S in U_1.

2. Optimization of resonances on $\mathrm{i}\mathbb{R}$

2.1. **Basic facts about resonances in 1- and 2- side open cavities.** To consider resonances in more general settings that include 1-side open cavities and the massless string approximation, we assume that $B(\cdot) \in L^1_{\mathbb{C}}(a_1, a_2)$ and that equation (1.1) is equipped with the more general boundary conditions

$$\frac{y'(a_1)}{-i\omega} = \nu_1 y(a_1), \quad \frac{y'(a_2)}{i\omega\nu_2} = y(a_2) \tag{2.1}$$

involving the (extended) constants $\nu_{1,2}$, which throughout the paper are assumed to be fixed and satisfy

$$\nu_1 \in [0, +\infty), \quad \nu_2 \in (0, +\infty], \quad \nu_1 + 1/\nu_2 \neq 0, \quad \text{and} \quad \nu_1 \le \nu_2. \tag{2.2}$$

When $\nu_2 = +\infty$, we suppose that $1/\nu_2 = 0$, and that the second boundary condition in (2.1) turns into $y(a_2) = 0$.

Resonances ω associated with $B(\cdot)$ are defined as numbers in the set $\mathbb{C} \setminus \{0\}$ such that eigenproblem (1.1), (2.1) has a nontrivial solution $y \in W^{2,1}[a_1, a_2]$ (i.e., a solution that is not identically zero). This solution y is called a (resonant) *mode*.

Denote by $\theta(x) = \theta(x, z; B)$ the solutions to $\partial^2_x y(x) = -z^2 B(x) y(x)$ satisfying

$$\theta(a_1, z; B) = 1, \quad \partial_x \theta(a_1, z; B) = -\mathrm{i}z\nu_1, \tag{2.3}$$

and by $F(z) = F(z;B)$ the functional

$$F(z;B) := \theta(a_2, z; B) + \frac{\nu_1}{\nu_2} - \frac{iz}{\nu_2}\int_{a_1}^{a_2} \theta(s, z; B)\ B(s)\,\mathrm{d}s. \tag{2.4}$$

It is easy to see that for $z \neq 0$, $F(z;B) = \theta(a_2, z; B) + \frac{\mathrm{i}\partial_x \theta(a_2,z;B)}{z\nu_2}$, and that the set of resonances $\Sigma(B)$ is the set of zeros of the function $F(\cdot;B)$. When $\nu_2 = \infty$, the formula for F turns into $F(z;B) = \theta(a_2, z; B)$.

The functional $F(z;B)$ is analytic on the Banach space $\mathbb{C} \times L^1_{\mathbb{C}}(a_1, a_2)$. By definition, *the multiplicity of a resonance of B is its multiplicity as a zero of the analytic function* $F(\cdot;B)$ (see [14, 19, 11] and references therein). A resonance is called *simple* if its multiplicity is 1. The set of non-simple resonances is denoted by $\Sigma_{\mathrm{mult}}(B)$ (non-simple resonances are often called *multiple*). There exist triples (ν_1, ν_2, B) that fit to our settings and generate multiple resonances [6, 19].

Since $F(0;B) = 1 + \frac{\nu_1}{\nu_2} > 0$, the set $\Sigma(B)$ is either empty, or consists of isolated resonances, which have finite multiplicities and can accumulate only at ∞. In the case $B(x) \geq 0$ a.e., it is easy to see that $\Sigma(B) \subset \mathbb{C}_-$ and $\Sigma(B)$ is symmetric w.r.t. the imaginary axis $\mathrm{i}\mathbb{R}$ taking multiplicities into account.

2.2. **Local extrema on $\mathrm{i}\mathbb{R}$ with functions as constraints.** Optimization of resonances will be considered over the admissible family

$$\mathbb{A} := \{B(x) \in L^1_{\mathbb{R}}(a_1, a_2)\ :\ b_1(x) \leq B(x) \leq b_2(x) \text{ a.e. on } (a_1, a_2)\}, \tag{2.5}$$

where $b_1(\cdot)$ and $b_2(\cdot)$ are certain Lebesgue integrable functions defined on (a_1, a_2) such that

$$0 \leq b_1(x) \leq b_2(x) \quad \text{on} \quad (a_1, a_2) \quad \text{and} \tag{2.6}$$

$$\operatorname{meas} E > 0, \quad \text{where} \quad E = \{x \in (a_1, a_2)\ :\ b_1(x) < b_2(x)\}. \tag{2.7}$$

This more general definition of $\mathbb{A}$ allows one to consider uncertain resonances [13] and an important for applications situation when only some parts of the device are suitable for modifications to achieve better resonant properties (see references in [13]).

Besides Definition 1.1, we will use several other definitions of local minima and maxima. The next one includes an additional restriction on $\operatorname{Re}\omega$. We say that an admissible pair (ω_0, B_0) is *a local maximizer of* $\operatorname{Im}\omega$ *for the frequency* $\operatorname{Re}\omega_0$ if there exist $\varepsilon > 0$ such that $\operatorname{Im}\omega_1 \leq \operatorname{Im}\omega_0$ for any admissible pair (ω_1, B_1) satisfying the following three conditions: $\operatorname{Re}\omega_1 = \operatorname{Re}\omega_0$, $|\omega_1 - \omega_0| < \varepsilon$, and $\|B_1 - B_0\|_1 < \varepsilon$. *Local minimizers of* $\operatorname{Im}\omega$ *for a particular frequency* α are defined in a similar way. A pair (ω, B) is a *local extremizer of* $\operatorname{Im}\omega$ *for a frequency* α if it is a local minimizer of $\operatorname{Im}\omega$ for α or a local maximizer of $\operatorname{Im}\omega$ for α.

The following result, which can be seen as a generalization of [11, Theorem 2.5], is one of our main tools.

Proposition 2.1. *If (ω_0, B_0) is a local extremizer of $\operatorname{Im}\omega$ for the frequency 0, then either $B_0(x) = b_1(x)$ a.e., or $B_0(x) = b_2(x)$ a.e..*

Since the steps of the proof are somewhat different from that of [11], the rest of this section gives a sketch. It is based on the resonances' perturbation theory [11, 13].

Let ω be a resonance of $B \in L^1_{\mathbb{C}}$ of multiplicity $m \in \mathbb{N}$. Let $V \in L^1_{\mathbb{C}}$. Then, for $\zeta \in \mathbb{D}_\delta(0) \setminus \mathrm{i}(-\delta, 0)$ with $\delta > 0$ small enough, all resonances (taking multiplicities into account) of $B + \zeta V$ lying in a vicinity of ω are given by $\Omega_j(\zeta)$, $j = 1, \dots, m$, where Ω_j can be enumerated such that they are continuous functions in $\zeta \in \mathbb{D}_\delta(0) \setminus \mathrm{i}(-\delta, 0)$ and have the following asymptotics as $\zeta \to 0$:

$$\Omega_j(\zeta) = \omega + [K(\omega, B; V)\zeta]^{1/m} + o(|\zeta|^{1/m}), \quad \text{where } K(\omega, B; V) := -\frac{m!\,\frac{\partial F(\omega,B)}{\partial B}(V)}{\partial_z^m F(\omega; B)}. \tag{2.8}$$

When $\omega \in \Sigma(B)$, the directional derivative of F in the direction $V \in L^1_{\mathbb{C}}$ equals

$$\frac{\partial F(\omega, B)}{\partial B}(V) = \frac{\omega^2}{\partial_x \theta(a_2, \omega, B)} \int_{a_1}^{a_2} \theta^2(s, \omega; B)\, V(s)\, \mathrm{d}s. \tag{2.9}$$

In the case $\omega \in \mathrm{i}\mathbb{R}$, the mode $\theta(x) = \theta(x, \omega; B)$ associated with ω has specific properties different from that of the case $\omega \notin \mathrm{i}\mathbb{R}$. These properties are described by the following lemma, which is easy to obtain from (1.1), (2.1), and (2.3).

Lemma 2.2. *Let $\omega \in \Sigma(B) \cap \mathrm{i}\mathbb{R}$ and $B(x) \geq 0$ a.e. on (a_1, a_2). Then*
(i) *$\theta(x) \in \mathbb{R}$ for all $x \in [a_1, a_2]$.*
(ii) *There exists a subinterval $[x_*, x^*]$ of $[a_1, a_2]$ such that*

$$\theta(x)\partial_x\theta(x) < 0 \quad \text{if} \quad x < x_*, \quad \theta(x)\partial_x\theta(x) = 0 \quad \text{if} \quad x \in [x_*, x^*],$$
$$\text{and} \quad \theta(x)\partial_x\theta(x) > 0 \quad \text{if} \quad x > x^*.$$

(iii) *If $x_* < x^*$, then $B(x) = 0$ a.e. on (x_*, x^*) and $\theta(x)$ is a nonzero constant function on $[x_*, x^*]$. In particular, if $B(x) > 0$ a.e. on (a_1, a_2), then $x_* = x^*$.*
(iv) *The function $\theta(x)$ has at most one zero in $[a_1, a_2]$. More precisely, if $\theta(x_0) = 0$, then $x_0 = x_* = x^*$.*

Assume that (ω_0, B_0) is a local extremizer of $\operatorname{Im}\omega$ for frequency 0 over $\mathbb{A}$. Assume that $B_0(\cdot) \neq b_1(\cdot)$ and $B_0(\cdot) \neq b_2(\cdot)$ (in the sense of L^1-space). Then it follows from (2.9) and Lemma 2.2 (iv), that there exist $V_1^\pm \in \mathbb{A} - B_0$ such that $\pm\frac{\partial F(\omega_0; B_0)}{\partial B}(V_1^\pm) > 0$. This, formula (2.8), the symmetry of $\Sigma(B)$ w.r.t. $\mathrm{i}\mathbb{R}$, and the arguments of [11, Section 4.4] allows one to show that there exist $V_2^\pm \in \mathbb{A} - B_0$ and $\omega_\pm(\zeta) \in \Sigma(B_0 + \zeta V_2^\pm)$ such that for small enough $\zeta > 0$,

$$\operatorname{Re}\omega_\pm(\zeta) = 0, \quad \pm[\operatorname{Im}\omega_\pm(\zeta) - \operatorname{Im}\omega_0] > 0, \quad \text{and} \quad \omega_\pm(\zeta) \to \omega_0 \quad \text{as} \quad \zeta \to 0.$$

(Note that V_2^+ and V_2^- are not necessarily different. When m is even, one of the directions $V_1^\pm$ can be taken as $V_2^+ = V_2^-$.) Thus, (ω_0, B_0) is not a local extremizer for the frequency 0. This contradiction concludes the proof of Proposition 2.1.

Remark 2.1. Problems of other types involving optimization of resonances on $\mathrm{i}\mathbb{R}$ in the context of optimal damping were studied in [3, 4] (see also [5]).

3. Local minimizers of Dr and non-uniqueness

A frequency $\alpha \in \mathbb{R}$ is called *admissible* (over $\mathbb{A}$ defined by (2.5)) if it is the frequency of some admissible resonance ω, i.e., if $\alpha = \operatorname{Re}\omega$ for certain $\omega \in \Sigma[\mathbb{A}]$.

An admissible frequency α_0 is *a local minimizer for $\beta_{\min}$* if there exists $\varepsilon > 0$ such that $\beta_{\min}(\alpha_0) \leq \beta_{\min}(\alpha)$ for all admissible α in $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. If for certain $\varepsilon > 0$ and all admissible α from a punctured neighborhood $(\alpha_0 - \varepsilon, \alpha_0) \cup (\alpha_0, \alpha_0 + \varepsilon)$ the strict inequality $\beta_{\min}(\alpha_0) < \beta_{\min}(\alpha)$ holds, α_0 is said to be *a strict local minimizer for $\beta_{\min}$*.

It is obtained in [11, 13] by weak compactness arguments that

$$\text{the set of admissible resonances} \quad \Sigma[\mathbb{A}] \quad \text{is closed.} \tag{3.1}$$

This and Proposition 2.1 imply that,

$$\text{if} \quad 0 \in \operatorname{Re}\Sigma[\mathbb{A}], \quad \text{then} \quad \mathrm{i}\mathbb{R} \cap [\Sigma(b_1) \cup \Sigma(b_2)] \quad \text{is a nonempty closed set and} \quad \beta_{\min}(0) = \min_{\omega \in \mathrm{i}\mathbb{R} \cap [\Sigma(b_1) \cup \Sigma(b_2)]} |\operatorname{Im}\omega|. \tag{3.2}$$

More generally, (3.1) implies that $\omega = \alpha - \mathrm{i}\beta_{\min}(\alpha)$ is a resonance of a certain admissible structure $B \in \mathbb{A}$ whenever $\alpha \in \operatorname{Re}\Sigma[\mathbb{A}]$, i.e., there exists the resonance of minimal decay and at least one structure of minimal decay for every admissible frequency. Taking $\Sigma[\mathbb{A}] \subset \mathbb{C}_-$ into account, we see that $\beta_{\min}(\alpha) > 0$ for each $\alpha \in \operatorname{Re}\Sigma[\mathbb{A}]$. The reflected w.r.t. $\mathbb{R}$ graph $\{\alpha - \mathrm{i}\beta_{\min}(\alpha) \ : \ \alpha \in \operatorname{Re}\Sigma[\mathbb{A}]\}$ of the function $\beta_{\min}(\cdot)$ plays the role of the Pareto optimal frontier for minimization of $|\operatorname{Im}\omega|$ [10, 12].

The following result provides information about the shape of the Pareto optimal frontier for low frequencies.

Theorem 3.1. *Suppose that $\mathbb{A}$ is defined by (2.5) and $0 \in \operatorname{Re}\Sigma[\mathbb{A}]$. Then $\alpha = 0$ is a strict local minimizer of $\beta_{\min}$ whenever $\omega = -\mathrm{i}\beta_{\min}(0)$ does not belong to $\Sigma_{\mathrm{mult}}(b_1) \cup \Sigma_{\mathrm{mult}}(b_2)$ (i.e., whenever ω is not a multiple resonance for each of the two extreme allowed structures $b_1(\cdot)$ and $b_2(\cdot)$).*

The proof is given in Subsection 3.1. To show that Theorem 3.1 and Proposition 2.1 imply Theorem 1.1, let us consider in more details the case when b_1 and b_2 are constants. Note also that each local minimizer α of $\beta_{\min}$ and any associated B of minimal decay compose a local minimizer $(\alpha - \mathrm{i}\beta_{\min}(\alpha), B)$ of the decay rate functional Dr.

When $B(x)$ equals a.e. a constant $b \in [0, +\infty)$, resonances can be calculated explicitly [13]. Namely,

$$\Sigma(b) = \varnothing \quad \text{if} \quad b^{1/2} = \nu_1 \quad \text{or} \quad b^{1/2} = \nu_2; \tag{3.3}$$

$$\Sigma(b) = \{\omega^{[0]}(0)\} \quad \text{if} \quad b = 0 \neq \nu_1, \quad \text{where} \quad \omega^{[0]}(0) := -\mathrm{i}\frac{1/\nu_1 + 1/\nu_2}{a_2 - a_1}. \tag{3.4}$$

When $b \in \mathbb{R}_+ \setminus \{\nu_1^2, \nu_2^2\}$, one has $\Sigma(b) = \{\omega^{[n]}(b)\}_{n \in \mathbb{Z}}$ with

$$\begin{aligned} \omega^{[n]}(b) = & -\frac{\mathrm{i}}{2b^{1/2}(a_2 - a_1)} \ln \left| \frac{1+K_1}{1-K_1} \right| \\ & + \frac{\pi}{b^{1/2}(a_2 - a_1)} \times \begin{cases} n, & \text{if } b^{1/2} \notin [\nu_1, \nu_2] \\ (n + 1/2), & \text{if } b^{1/2} \in (\nu_1, \nu_2) \end{cases}, \end{aligned} \tag{3.5}$$

where $K_1 = K_1(b) := \frac{1+\nu_1/\nu_2}{\frac{\nu_1}{b^{1/2}} + \frac{b^{1/2}}{\nu_2}} > 0$; moreover,

$$\text{in the cases (3.5) and (3.4) all resonances are simple.} \tag{3.6}$$

Note that

$$1 > K_1(b) \ \Leftrightarrow \ b^{1/2} \notin [\nu_1, \nu_2], \quad 1 < K_1(b) \ \Leftrightarrow \ b^{1/2} \in (\nu_1, \nu_2), \quad \text{and} \quad K_1 = 1 \ \Leftrightarrow \ b^{1/2} \in \{\nu_1, \nu_2\}, \tag{3.7}$$

and that, in the case $\nu_1 = \nu_2 = \nu$, (3.5) turns into (1.3).

Let us define

$$\begin{aligned} K_2(b) \ &:= \ \operatorname{sgn}[1 - K_1(b)] \left| \frac{1 - K_1(b)}{1 + K_1(b)} \right|^{1/\sqrt{b}} \qquad \text{for } b > 0, \\ \text{and} \quad K_2(0) \ &:= \ \exp(-2/\nu_1 - 2/\nu_2) \quad (\text{when } \nu_1 = 0, \text{ we suppose } K_2(0) := 0). \end{aligned}$$

Corollary 3.2. *Let b_1 and b_2 be constants such that $b_1^{1/2} < \nu_1$ or $b_2^{1/2} > \nu_2$. Then*
(i) *$\alpha = 0$ is a strict local minimizer for $\beta_{\min}$ and $\beta_{\min}(0) = -\frac{1}{2(a_2 - a_1)} \times \ln \max\{K_2(b_1), K_2(b_2)\}$.*
(ii) *If $K_2(b_1) > K_2(b_2)$ ($K_2(b_1) < K_2(b_2)$), then b_1 (resp., b_2) is the unique structure of minimal decay for $\alpha = 0$.*
(iii) *If $K_2(b_1) = K_2(b_2)$, then b_1 and b_2 are structures of minimal decay for $\alpha = 0$.*

Proof. It follows from (3.5) that 0 is an admissible frequency. It follows from (3.4), (3.5), and (3.7) that $K_2(b) \leq 0$ exactly when $\Sigma(b) \cap \mathrm{i}\mathbb{R} = \varnothing$. On the other side, $K_2(b) > 0$ exactly when $\Sigma(b) \cap \mathrm{i}\mathbb{R} = \{\omega^{[0]}(b)\}$, where $\omega^{[0]}(b)$ is defined by (3.4), (3.5). In this case, $K_2(b) = \exp\left(2(a_2 - a_1) \operatorname{Im}\omega^{[0]}(b)\right)$. This, (3.2), (3.6), and Theorem 3.1 complete the proof. □

Note that for Example 1.2, the case (iii) of Corollary 3.2 takes place.

Remark 3.1. The following result of independent interest can be obtained immediately from (3.2): if the constraint functions $b_1(\cdot)$ and $b_2(\cdot)$ are such that the two corresponding sets of resonances $\Sigma(b_1)$ and $\Sigma(b_2)$ have no points in $\mathrm{i}\mathbb{R}$, then $\Sigma(B)$ have no points in $\mathrm{i}\mathbb{R}$ for any $B(\cdot)$ satisfying $b_1(x) \le B(x) \le b_2(x)$ a.e.. This case takes place, for example, when b_1 and b_2 are constants such that $\nu_1 \le b_1^{1/2} < b_2^{1/2} \le \nu_2$. This result can be seen as a generalization of [5, Theorem 4.2 (i)].

3.1. **Proof of Theorem 3.1.** If 0 is an isolated point of $\operatorname{Re}\Sigma[\mathbb{A}]$, the theorem is obvious from the definition of a strict local minimizer of $\beta_{\min}$.

Assume that 0 is not an isolated point of $\operatorname{Re}\Sigma[\mathbb{A}]$. Since $\Sigma[\mathbb{A}]$ and $\Sigma(B)$ for each $B \in \mathbb{A}$ are symmetric w.r.t. $\mathrm{i}\mathbb{R}$, there are sequences of admissible frequencies that converge to 0 from the left and from the right. Consider

$$\beta_* := \liminf_{\substack{\alpha\to 0- \\ \alpha\in\operatorname{Re}\Sigma[\mathbb{A}]}} \beta_{\min}(\alpha) \qquad \Big(\text{then also } \beta_* = \liminf_{\substack{\alpha\to 0+ \\ \alpha\in\operatorname{Re}\Sigma[\mathbb{A}]}} \beta_{\min}(\alpha)\ \Big).$$

Note that $\beta_* \in \mathbb{R}_+ \cup \{+\infty\}$ since $\Sigma[\mathbb{A}] = \overline{\Sigma[\mathbb{A}]} \subset \mathbb{C}_-$. If $\beta_* = +\infty$, the theorem is also obvious.

Consider the case $\beta_* < +\infty$.

We need the following result of [13] on local weak continuity of resonances. Let us fix a countable family $\{f_n\}_{n=1}^\infty$ of continuous functions that is dense in $C[a_1, a_2]$. This family generates the metric $\rho_{\mathbb{M}}(\mathrm{d}M_1, \mathrm{d}M_2) := \sum_{n=1}^{\infty} \frac{|\int_{a_1}^{a_2} f_n(\mathrm{d}M_1 - \mathrm{d}M_2)|}{2^n(1 + |\int_{a_1}^{a_2} f_n(\mathrm{d}M_1 - \mathrm{d}M_2)|)}$ on the space $\mathbb{M}$ of complex Borel measures on $[a_1, a_2]$. The weak* topology on any closed ball in $\mathbb{M}$ coincides with the topology generated by the metric $\rho_{\mathbb{M}}$. We will use the metric $\rho(B_1, B_2) := \rho_{\mathbb{M}}(B_1 \mathrm{d}x, B_2 \mathrm{d}x)$ on $L^1_{\mathbb{C}}(a_1, a_2)$, where $B_j\,\mathrm{d}x$ are absolutely continuous measures corresponding to the functions $B_j \in L^1_{\mathbb{C}}$.

The *total multiplicity of resonances of B in a set* $\mathcal{D} \subset \mathbb{C}$ is the sum of multiplicities of all $\omega \in \Sigma(B) \cap \mathcal{D}$. Let $R > 0$ and let $\overline{\mathbb{B}_R(0)} = \{\|f\|_1 \le R\}$ be the closed ball in $L^1_{\mathbb{C}}(a_1, a_2)$.

Proposition 3.3. ([13]) . *Let $B_0 \in L^1_{\mathbb{C}}(a_1, a_2)$ and $R \ge \|B_0\|_1$. Let $\mathcal{D}$ be an open bounded subset of $\mathbb{C}$ such that its boundary* $\operatorname{bd}\mathcal{D}$ *does not contain resonances of B_0. Then there exists a neighborhood $W \subset \overline{\mathbb{B}_R(0)}$ of B_0 in the topology of the metric space $(\overline{\mathbb{B}_R(0)}, \rho)$ such that, for every $B \in W$, there are no resonances of B on* $\operatorname{bd}\mathcal{D}$*, and the total multiplicity of resonances of B in $\mathcal{D}$ coincides with that of B_0.*

Lemma 3.4. *Assume that $0 \in \operatorname{Re}\Sigma[\mathbb{A}]$ is not an isolated point of $\operatorname{Re}\Sigma[\mathbb{A}]$ and that $\beta_* < +\infty$. Then there exist sequences $B_n \in \mathbb{A}$ and $\omega_{\pm n} \in \Sigma(B_n)$, $n \in \mathbb{N}$, such that $\operatorname{Im}\omega_{\pm n} \to -\beta_*$, $\operatorname{Re}\omega_{\pm n} \to 0\pm$, and the sequence B_n converges to a certain $B_* \in \mathbb{A}$ w.r.t. the metric ρ.*

Proof. The existence of $\{B_n\}_{n\in\mathbb{N}} \subset \mathbb{A}$ generating resonances $\omega_{\pm n} \in \Sigma(B_n)$ that converge to $\omega_* = -\mathrm{i}\beta_*$ follows from the assumption that 0 is not an isolated point of $\operatorname{Re}\Sigma[\mathbb{A}]$. So it is enough to show that $\{B_n\}_{n\in\mathbb{N}}$ contains a ρ-convergent subsequence.

For each $B \in \mathbb{A}$ there exists unique $H(x) = H(x; B)$ such that $B(x) = (b_2(x) - b_1(x))H(x) + b_1(x)$ a.e. on (a_1, a_2) and $H(x)$ belongs to the set $\mathbb{A}_*$ of $L^\infty_{\mathbb{R}}$-functions satisfying $0 \le H(x) \le 1$ a.e. on (a_1, a_2) and $H(x) = 0$ a.e. on $(a_1, a_2) \setminus E$ (for E, see (2.7)). Let us take $R > \|b_2\|_1$. One can see that

(3.8) the map $H(\cdot; B) \mapsto B(\cdot)$ is a bijection of $\mathbb{A}_*$ to $\mathbb{A}$ which is continuous from the weak* topology of $L^\infty_{\mathbb{R}}$ to the metric topology of $(\overline{\mathbb{B}_R(0)}, \rho)$

(the continuity is sequential w.r.t. the corresponding induced topologies). From this and the sequential Banach-Alaoglu theorem applied to $\mathbb{A}_*$, we see that $\{B_n\}_{n\in\mathbb{N}}$ contains a ρ-convergent subsequence. This completes the proof. □

Now, we are ready to show that

$$\omega_* = -\mathrm{i}\beta_* \quad \text{is a multiple resonance of} \quad B_* \tag{3.9}$$

(for any B_* satisfying the statement of Lemma 3.4). Indeed, $\omega_* \in \Sigma(B_*)$ due to Proposition 3.3 and Lemma 3.4. However, Lemma 3.4 implies that for every $\varepsilon > 0$, the disc $\mathbb{D}_\varepsilon(\omega_*)$ contains at least two distinct resonances $\omega_{\pm n}$ of B_n if n is large enough. Hence, it follows from Proposition 3.3 that the multiplicity of ω_* as a resonance of B_* is at least 2.

By the assumptions of the theorem, $\omega = -\mathrm{i}\beta_{\min}(0)$ is a simple resonance of any structure B of minimal decay for the frequency 0 (note that Proposition 2.1 implies that B is either b_1, or b_2). This yields $\omega_* \neq \omega$ (since they have different multiplicities). Thus, $\beta_* > \beta_{\min}(0)$ and so 0 is a strict local minimizer of $\beta_{\min}$.

Remark 3.2. When this note was in preparation, examples showing non-uniqueness of structures of minimal decay for a 3-D Schrödinger equation with point interactions were constructed in [1]. These examples are of very different nature.

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