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ON A FUNCTION SYSTEM MAKING A BASIS IN A WEIGHT SPACE

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This paper is dedicated to the 80th birthday of E. R. Tsekanovskii

Abstract. We find necessary and sufficient conditions for systems of functions generated by a second order differential equation to form a basis. The results are applied to show that Mathieu functions make a basis.

Introduction

Basis issues of function systems are among the most important problems of functional analysis. The line of investigation in this realm developed due to the work by B. S. Pavlov [9] is one of the most effective. Besides, connection of these problems of analysis with spectral theory of non-selfadjoint operators plays the main role [4, 5, 7].

This paper is a continuation of investigations started in work [6]. Let φ be a real function of the class $C^1(\mathbb{R})$, besides,

1) $\varphi(x) \geq 0$ $(\forall x \in \mathbb{R}_+)$;
2) $\varphi(-x) = (-1)^\nu \varphi(x)$ $(\nu \in \mathbb{R},\ x \in \mathbb{R}_+)$;
3) $\int\limits_0^b \varphi(x)\,dx < \infty$; $\int\limits_0^b \frac{dx}{\varphi(x)} < \infty$ $(\forall b,\ 0 < b < \infty)$.

Denote by $L^2_\varphi(-a,a)$ $(0 < a \leq \infty)$ the Hilbert space of functions with the scalar product

$$\langle f, g\rangle = \int\limits_{-a}^{a} f(x)\overline{g(x)}|\varphi(x)|\,dx.$$

Consider the function

$$e(x,\lambda) = f(x,\lambda) - \frac{i}{\lambda} f'(x,\lambda),$$

where $f(x,\lambda)$ is a solution of the integral equation

$$f(x,\lambda) - \lambda^2 \int\limits_0^x \frac{dt}{\varphi(t)} \int\limits_0^t f(s,\lambda)\varphi(s)\,ds = 1$$

(when $\varphi(x) \equiv \text{const}$, $e(x,\lambda) = e^{i\lambda x}$). In this paper the absolute bases

$$\{e\,(x,\lambda_k) : \lambda_k \in \Lambda\}$$

are studied in the space $L^2_\varphi(-a,a)$, where the set $\Lambda = \{\lambda_k \in \mathbb{C}, k \in \mathbb{Z}\}$ has no limit points and is situated at a positive distance from $\mathbb{R}$. The function $e(x,\lambda)$ has the representation

$$e(x,\lambda) = (I - \lambda B)^{-1}\mathbb{1},$$

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where B is a compact non-dissipative operator with its spectrum at zero. It is important that $e(x,\lambda_k)$ are eigenfunctions of the operator K ($\lambda_k K e(x,\lambda_k) = e(x,\lambda_k)$ where K is a one-dimensional perturbation of the operator B).

1. Preliminary information

For the Hilbert space

$$L^2_\varphi(-a,a) \stackrel{\text{def}}{=} \left\{ f(x) : \int_{-a}^{a} |f(x)|^2 |\varphi(x)|\, dx < \infty \right\}, \tag{1.1}$$

where $0 < a \leqslant \infty$, the decomposition

$$L^2_\varphi(-a,a) = L_+ \oplus L_- \tag{1.2}$$

takes place, where

$$L_\pm \stackrel{\text{def}}{=} \left\{ f_\pm(x) = \frac{1}{2}\left(f(x) \pm f(-x)\right) : f(x) \in L^2_\varphi(-a,a) \right\}. \tag{1.3}$$

We specify in $L^2_\varphi(-a,a)$ the linear operator

$$(Bf)(x) \stackrel{\text{def}}{=} i\int_0^x f_-(t)\, dt + \frac{i}{\varphi(x)} \int_0^x f_+(t)\varphi(t)\, dt. \tag{1.4}$$

It is easy to show [8] that if

$$b = \int_{\mathbb{R}} \varphi(x)\, dx < \infty, \quad \tilde{b} = \int_{\mathbb{R}} \frac{dx}{\varphi(x)} < \infty, \tag{1.5}$$

then the operator B (1.5) is bounded. The operator B (1.5) is non-dissipative and has the two-dimensional imaginary component

$$\frac{B - B^*}{i} f = \sum_{\alpha,\beta=1}^{2} \langle f_1, g_2 \rangle (J_p)_{\alpha,\beta} g_\beta \quad \left(f \in L^2_\varphi(-a,a)\right), \tag{1.6}$$

where

$$J_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad g_1 = \left(\frac{\tilde{b}}{4b}\right)^{\frac{1}{4}} \mathbb{1}, \quad g_2(x) = \left(\frac{\tilde{b}}{4b}\right)^{\frac{1}{4}} \left\{ \frac{\chi_+(x)}{\varphi(x)} - \frac{\chi_-(x)}{\varphi(-x)} \right\},$$

besides, $\mathbb{1} = \chi(x)$ and $\chi_\pm(x)$ are characteristic functions of the sets $[-a,a]$ and $\mathbb{R}_\pm \cap [-a,a]$, correspondingly.

Let us consider the integration operator

$$(\mathbb{J}f)(x) = i\int_0^x f(x)\, dt \tag{1.7}$$

in the space $L^2(-a,a)$ without weight, besides, it is obvious that $\mathbb{J} = B$, when $\varphi(x) \equiv 1$. If, for $\varphi(x)$, (1.5) takes place and $\varphi(x) \neq 0$, $\frac{1}{\varphi(x)} \neq 0$ ($\forall x \in [0,a)$), then [8] the operator B (1.4) is similar to the operator $\mathbb{J}$ (1.7).

Denote by $e(x,\lambda)$ the function

$$e(x,\lambda) = (I - \lambda B)^{-1} \mathbb{1}, \tag{1.8}$$

where B is given by (1.4). Then

$$Be(x,\lambda) = \frac{e(x,\lambda) - \mathbb{1}}{\lambda}, \tag{1.9}$$

the function $e(x,\lambda)$ is equal to

$$e(x,\lambda) = f(x,\lambda) - \frac{i}{\lambda} f'(x,\lambda), \tag{1.10}$$

where $f(x,\lambda)$ is a solution of the integral equation

$$f(x,\lambda) = 1 - \lambda^2 \int_0^x \frac{dt}{\varphi(t)} \int_0^t f(x,\lambda)\varphi(s)\,ds. \tag{1.11}$$

2. Perturbations of the operator B

Let us consider the operator K,

$$Kh \overset{\text{def}}{=} Bh + \langle h, g\rangle \mathbb{1} \quad (h \in L^2_\varphi(-a;a)) \tag{2.1}$$

in the space $L^2_\varphi(-a;a)$ where B is given by (1.4) and $\mathbb{1} = \chi_{(-a,a)}(x)$.

Lemma 1. *For the Fredgolm resolvents $K(\lambda) = K(I - \lambda K)$, $B(\lambda) = B(I - \lambda K)^{-1}$ of the operators K (2.1) and B (1.4), the formula*

$$K(\lambda)f = B(\lambda)f + \frac{\langle (I - \lambda B)^{-1} f, g\rangle}{1 - \lambda\langle e, g\rangle} e \tag{2.2}$$

is true $\forall f \in L^2_\varphi(-a,a)$ (1.1) where e is given by (1.8).

Proof. Let $(K - zI)^{-1} f = h$, then

$$f = (B - zI)h + \langle h, g\rangle \mathbb{1}$$

or

$$R_B(z)f = h + \langle h, g\rangle R_B(z)\mathbb{1}, \tag{2.3}$$

where $R_B(z) = (B - zI)^{-1}$. This implies

$$\langle R_B(z)f, g\rangle = \langle h, g\rangle (1 + \langle R_B(z)\mathbb{1}, g\rangle),$$

therefore

$$R_K(z)f = R_B(z)f - \frac{\langle R_B(z)f, g\rangle}{1 + \langle R_B(z)\mathbb{1}, g\rangle} R_B(z)\mathbb{1}$$

in view of (2.3) and $h = R_K(z)f$, where $R_K(z) = (K - zI)^{-1}$. Assuming that $z = \lambda^{-1}$, we obtain

$$(I - \lambda K)^{-1} f = (I - \lambda B)^{-1} f + \frac{\lambda\langle (I - \lambda B)^{-1} f, g\rangle}{1 - \lambda\langle (I - \lambda B)^{-1}\mathbb{1}, g\rangle} (I - \lambda B)^{-1}\mathbb{1}.$$

Taking into account $(I - \lambda B)^{-1} - 1 = \lambda B(I - \lambda B)^{-1}$ and definition (1.13) of the function $e(x\lambda)$, we obtain

$$K(I - \lambda K)^{-1} f = B(I - \lambda B)^{-1} f + \frac{\langle (I - \lambda B)^{-1} f, g\rangle}{1 - \lambda\langle e, g\rangle} e;$$

which gives us (2.2). □

Hereinafter, the function

$$n(\lambda) \overset{\text{def}}{=} 1 - \lambda\langle e, g\rangle \tag{2.4}$$

plays an important role. Formula (2.2) implies that the Fredholm spectrum of the completely continuous operator K (2.1) coincides with the set

$$\Lambda = \{\lambda \in \mathbb{C} : n(\lambda) = 0\}. \tag{2.5}$$

If $\lambda_n \in \Lambda$, then

$$Ke(x, \lambda_n) = \frac{1}{\lambda_n} e(x, \lambda_n) \tag{2.6}$$

and thus $e(x, \lambda_n)$ is an eigenfunction of the operator K. Indeed,

$$\begin{aligned} Ke(x, \lambda_n) &= Be + \langle e, g \rangle 1\!\!1 = \frac{e(x, \lambda_n) - 1\!\!1}{\lambda_n} + \langle e, g \rangle 1\!\!1 \\ &= \frac{1}{\lambda_n} e(x, \lambda_n) - \frac{1}{\lambda_n} n(\lambda_n) 1 = \frac{1}{\lambda_n} e(x, \lambda_n) \end{aligned}$$

since $\lambda_n \in \Lambda$ (2.5), in view of (1.9).

Dimension of the root subspace of the operator K corresponding to the proper number λ_n^{-1} is equal to the multiplicity of the root λ_n of the function $n(\lambda)$ (2.4).

The problem of description of bases of the type $\{e(x, \lambda_n)\}^\infty$ is closely connected with a study of the operator K (2.1).

Theorem 1. *Suppose that the function $\varphi(x)$ is such that (1.5) take place and let the totality $\{e(x, \lambda_n)\}$ ($\lambda_n \in \Lambda, 0 \notin \Lambda$) form an absolute basis in $L^2_\varphi(-a, a)$ (1.1). Then there exists a single function $g \in L^2_\varphi(-a, a)$ such that the equalities (2.6) are true for an operator K of type (2.1).*

Proof. The fact that the totality $\{e(x, \lambda_n)\}$ forms a basis implies that the sequence $\{\lambda_n\}$ coincides with the set of zeros of some entire function and thus λ_n^{-1} is bounded when $\lambda_n \in \Lambda$. We define the operator K in the space $L^2_\varphi(-a, a)$ which on the basis of the vector $e(x, \lambda_n)$ acts by formula (2.6), then K is bounded in view of the uniform boundedness of λ_n^{-1}. We search for K as $K = B + \Gamma$, where B is given by (1.5). Formulas (2.6) and (1.14) imply that

$$\frac{1}{\lambda_n} e(x, \lambda_n) = \frac{1}{\lambda_n} (e(x, \lambda_n) - 1\!\!1 + \Gamma e(x, \lambda_n),$$

therefore $\Gamma e(x, \lambda_n) = \frac{1}{\lambda_n} 1\!\!1$. The operator Γ is bounded since B and K have one-dimensional image. This implies that there exists a single function $g \in L^2_\varphi(-a, a)$ such that $\Gamma = \langle ., g \rangle 1\!\!1$, this proves the statement. □

Let a sequence of complex numbers $\Lambda = \{\lambda_k : k \in \mathbb{Z}\}$ lie at a positive distance from $\mathbb{R}$ and have single limit point ∞. We divide Λ into two parts,

$$\Lambda_+ = \{\lambda_K \in \Lambda : \operatorname{Im} \lambda_K > 0\}, \quad \Lambda_- = \{\lambda_K \in \Lambda : \operatorname{Im} \lambda_K < 0\}, \tag{2.7}$$

a sequences of numbers in $\mathbb{C}_+$ and $\mathbb{C}_-$, correspondingly.

We recall [4] that a set $\{\lambda_k\}_{k \in \mathbb{Z}}$ satisfies the Carleson condition if

$$\inf_k \prod_{j \neq k} \left| \frac{\lambda_k - \lambda_j}{\lambda_k - \overline{\lambda_j}} \right| > 0. \tag{2.8}$$

A weight $\omega(x)$ satisfies the A_2 condition (or the Muckenhaupt condition) [3, 4] if

$$\sup_\Delta \Big(\frac{1}{|\Delta|} \int_\Delta \omega \, dx \Big) \Big(\frac{1}{\Delta} \int \frac{1}{\omega} \, dx \Big) < \infty, \tag{2.9}$$

where Δ runs over the set of intervals from $\mathbb{R}$ and $|\Delta|$ is the length of Δ.

3. Function $n(\lambda)$

I. Description of the class of functions $n(\lambda)$ (2.4) is based on properties of the functions $\langle e, g\rangle$, where $e(x,\lambda)$ is given by (1.8) and $g \in L^2_\varphi(-a,a)$.

Lemma 2. *The resolvent $(I-\lambda\mathbb{J})^{-1}$ of the operator $\mathbb{J}$ (1.12) is given by the formula*

$$\left((I-\lambda\mathbb{J})^{-1}h\right)(x) = h(x) + i\lambda\int_0^x e^{i\lambda t}h(x-t)\,dt \tag{3.1}$$

$\forall h \in L^2(-a,a)$.

Let A be a bounded (or bounded invertible) operator from $L^2_\varphi(-a,a)$ into $L^2(-a,a)$ realizing the similarity of B (1.5), $\mathbb{J}$ (1.12), $AB = \mathbb{J}A$ [6, 8]. Then

$$\langle e,h\rangle = \left\langle (I-\lambda B)^{-1}\mathbb{1}, h\right\rangle = \left\langle A(I-\lambda B)^{-1}\mathbb{1}, {A^*}^{-1}h\right\rangle = \left\langle (I-\lambda J)^{-1}f, H\right\rangle,$$

where $f = A\mathbb{1} \in L^2(-a,a)$ and $H = (A^*)^{-1}f \in L^2(-a,a)$. Using (3.1), we find

$$\begin{aligned}\langle e,h\rangle &= \int_{-a}^{a}\Big(f(x) + i\lambda\int_0^x e^{i\lambda t}f(x-t)\,dt\Big)\overline{H}(x)\,dx\\ &= \langle f,H\rangle + i\lambda\int_0^a\int_0^x e^{i\lambda t}f(x-t)\,dt\overline{H}(x)\,dx + i\lambda\int_{-a}^0\int_0^x e^{i\lambda t}f(x-t)\,dt\overline{H}(x)\,dx,\end{aligned}$$

which, after the change of order of integration, gives us

$$\langle e,h\rangle = \langle \mathbb{1},h\rangle + i\lambda\int_{-a}^{a} e^{i\lambda t}\psi(t)\,dt, \tag{3.2}$$

where

$$\psi(t) = \begin{cases} \int_t^a f(\xi-t)\overline{H}(\xi)\,d\xi, & t\in[0,a],\\ -\int_{-a}^t f(\xi-t)\overline{H}(\xi)\,d\xi, & t\in[-a,0].\end{cases} \tag{3.3}$$

Since $f, H \in L^2(-a,a)$, we have $\psi \in L^2(-a,a)$. The function $\langle e,h\rangle$ represents an analogue of the Fourier transform of the function $\overline{h}(x)$.

Lemma 3. *For the function*

$$\tilde{h}(\lambda) = \langle e,h\rangle, \tag{3.4}$$

where e is given by (1.13), the representation

$$\tilde{h}(\lambda) = \tilde{h}(0) + i\lambda\int_{-a}^{a} e^{i\lambda t}\psi(t)\,dt \tag{3.5}$$

is true, where $\psi(t)$ is given by formula (3.3) and belongs to $L^2(-a,a)$.

Proof of (3.5) follows from (3.2) since $e(x,0) = \mathbb{1}$.

Definition ([1])**.** The function $f(\lambda)$ belongs to **the Bernstein class** B_σ if $f(\lambda)$ is an entire function of the exponential type $\leqslant \sigma$ and

$$\sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

It is known that $\forall f \in B_\sigma$ representation (3.5) takes place ($\sigma = a$). Thus $\tilde{h}(\lambda)$ (3.4) belongs to the class B_a.

Observation 1. *For $n(\lambda)$ (2.4),*

$$h\left(n, \pm\frac{\pi}{2}\right) = a \tag{3.6}$$

takes place, in view of (3.4), (3.5). Moreover, for $h(\lambda) = \lambda^{-1}(n(\lambda) - n(0))$, the inclusion $\lambda^{-1}(h(\lambda) - h(0)) \in L^2(\mathbb{R})$ is true.

Observation 2. *Let $q(\lambda) \geqslant 0$ ($\forall \lambda \in \mathbb{R}$) and $q \in L^1(\mathbb{R})$. Then the relations $hq \in L^1(\mathbb{R})$, $h\sqrt{q} \in L^2(\mathbb{R})$ take place $\forall h \in B_a$. In particular, this is true as $q = |\varphi|$ ($q = |\varphi|^{-1}$), in virtue of (1.9).*

Lemma 4. *Let $q(\lambda) \geqslant 0$ ($\forall \lambda \in \mathbb{R}$) and $q \in L^1(\mathbb{R})$, and the function $f = A\mathbb{1}$ be such that $\exists f'$ and $f' \in L^2(-a, a)$. Then for $\tilde{h}$ (3.4) the estimation*

$$\int_{\mathbb{R}} \left|\tilde{h}(\lambda)\right|^2 q(\lambda)\, d\lambda \leqslant C\|h\|^2_{L^2_\varphi(-a,a)} \tag{3.7}$$

is true.

II. Consider the inverse transform to $h \to \tilde{h}$ (3.4). Let us show that $\forall p \in L^2_\varphi(-a, a)$ there is such a function $\hat{p}(\lambda)$ that

$$p(x) = \int_{\mathbb{R}} e(x, \lambda)\hat{p}(\lambda)\, d\lambda, \tag{3.8}$$

where e is given by (1.13). We apply the operator A ($L^2_\varphi(-a, a) \to L^2(-a, a)$) to both sides of the equality. Then

$$P(x) = \int_{\mathbb{R}} \hat{p}(\lambda)\Big\{f(0)e^{i\lambda x} + \int_0^x e^{i\lambda\xi} f'(x - \xi)\, d\xi\Big\} d\lambda \tag{3.9}$$

in virtue of $Ae = (I - \lambda\mathbb{J})^{-1} f$, where $P = A_p$, $f = A\mathbb{1}$. Consequently,

$$f(0)r(x) + \int_0^x r(\xi) f'(x - \xi)\, d\xi = P(x), \tag{3.10}$$

where

$$r(x) = \int_{\mathbb{R}} \hat{p}(\lambda)e^{i\lambda x} d\lambda \tag{3.11}$$

under the assumption that integral (3.11) converges. Let, as in Lemma 3.3, $f(0) \neq 0$ and $f' \in L^2(-a, a)$. Then the Volterra equation of the second kind (3.10) always has a unique solution [10] $r(x) \in L^2(-a, a)$, in virtue of $P, f' \in L^2(-a, a)$. Hence it follows that $\hat{p}(\lambda)$ is the inverse Fourier transform of the function $r(x) \in L^2(-a, a)$ and thus it is an entire function of the exponential type $\leqslant a$ such that $\hat{p} \in L^2(\mathbb{R})$.

Theorem 2. *Let a function $f = A\mathbb{1}$ from $L^2(-a, a)$ be such that $f(0) \neq 0$, f' exists and belongs to $L^2(-a, a)$. Then for any function $p \in L^2_\varphi(-a, a)$ there exist a unique $\hat{p}(\lambda)$, an entire function of exponential type $\leqslant a$, belonging to $L^2(\mathbb{R})$ if $\lambda \in \mathbb{R}$, such that representation (3.9) holds true. Besides, the estimation*

$$\|p(x)\|_{L^2_\varphi(-a,a)} \leqslant C\, \|\hat{p}(\lambda)\|_{L^2(\mathbb{R})} \tag{3.12}$$

takes place.

Proof. It is left to prove estimation (3.12). For $P = Ap$, in view of invertibility of A, $\|p\| < N\|P\|$ takes place. Therefore we should estimate the norm of P in $L^2(-a, a)$. Formula (3.10) implies that

$$\|P\|^2_{L^2(-a,a)} \leqslant |f(0)|^2\|r\|^2 + 2|f_0|\|r\| \cdot \Big\| \int_0^x r(\xi)f'(x-\xi)\,d\xi \Big\| + \Big\| \int_0^x r(\xi)f'(x-\xi)\,d\xi \Big\|^2.$$

We note that $\|r\| = \|\hat{p}\|$ in view of unitarity of the Fourier transform and (3.11), and since

$$\Big| \int_0^x r(\xi)f'(x-\xi)\,d\xi \Big|^2 \leqslant \int_0^x |r(\xi)|^2 d\xi \int_0^x |f'(x-\xi)|^2\,d\xi$$

we conclude from $f' \in L^2(-a, a)$ that

$$\Big| \int_0^x r(\xi)f'(x-\xi)\,d\xi \Big|^2 \leqslant \|r\|^2 K^2.$$

Therefore,

$$\|p\|^2 \leqslant \|\hat{p}\|^2 \left(|f(0)|^2 + 2|f(0)|K + K^2\right),$$

which proves inequality (3.12). □

Theorem 3. *Let the function φ (1.1) have properties (1.9) and $n(\lambda)$ be given by*

$$n(\lambda) = 1 - \lambda \langle e, g\rangle,$$

where $g \in L^2_\varphi(-a, a)$ and e is given by formula (1.13), besides, the operator B in $L^2_\varphi(-a, a)$ is given by (1.15). If the roots of $n(\lambda)$ do not lie on $\mathbb{R}$, then the following conditions are equivalent:

1) *$\forall h \in L^2_\varphi(-a, a)$ the estimation*

$$\int_{\mathbb{R}} |\varphi(z)|^{-1}|n(z)|^2 \left|\left\langle (I - zB)^{-1}h, g\right\rangle_{L^2_\varphi(-a,a)}\right|^2 dz \leqslant M\|h\|^2_{L^2_\varphi(-a,a)} \tag{3.13}$$

takes place.

2) *The weight $\omega^2(\lambda) = |\varphi(\lambda)||n(\lambda)|^2$ satisfies the A_2 condition on $\mathbb{R}$.*

4. On the family $\{e(x, \lambda_k)\}$ making a basis

The main result is as follows.

Theorem 4. *Suppose that the function $\varphi(x)$ has property (1.5) and the set $\Lambda = \{\lambda_k \in \mathbb{C} : k \in \mathbb{Z}\}$ lies at a positive distance from the axis $\mathbb{R}$ and let the function $f = A\mathbb{1}$ (A is the operator from $L^2_\varphi(-a, a)$ into $L^2(-a, a)$ realizing similarity of the operators B (1.4) and $\mathbb{J}$ (1.7)) be such that $f(0) \neq 0$ and $f'(x)$ exists almost everywhere, besides, $f' \in L^2(-a, a)$. In order that the family*

$$\{e(x, \lambda_k), \lambda_k \in \Lambda\} \quad (0 \notin \Lambda) \tag{4.1}$$

be an unconditional basis in $L^2_\varphi(-a, a)$ (1.1), it is necessary and sufficient that Λ form a set of roots of an entire function of the exponential type n such that

1) *$\lambda^{-1}(n(\lambda) - n(0)) \in L^2_\varphi(\mathbb{R})$;*
2) *$h\left(n, \pm\frac{\pi}{2}\right) = a$;*

3) *the weight* $\omega^2(\lambda) = |\varphi(\lambda)||n(\lambda)|^2$ *satisfies the* A_2 *condition (2.9);*
4) *the roots* $n(\lambda)$ *are simple and the sequences* $\Lambda_\pm$ *(2.7) satisfy the Carleson condition (2.8).*

The proof of the theorem repeats the reasoning of work [6].

In conclusion of the paper, we consider one important special case. Note that the integral equation (1.11) for $f(x, \lambda)$ is equivalent to the Cauchy problem

$$\left\{ \begin{array}{l} f''(x,\lambda) + \dfrac{\varphi'(x)}{\varphi(x)} f'(x,\lambda) + \lambda^2 f(x,\lambda) = 0 \\ f(0,\lambda) = 1, \quad \varphi(x) f'(x,\lambda)|_{x=0} = 0 \end{array} \right. . \tag{4.2}$$

We rewrite equation (4.2) as

$$(\varphi(x) f'(x,\lambda))' + \lambda^2 \varphi(x) f(x,\lambda) = 0. \tag{4.3}$$

Let

$$f(x;\lambda) = u(\xi;\lambda), \quad \xi(x) \stackrel{\text{def}}{=} \int\limits_0^x \frac{dt}{\varphi(t)} + C \quad (C \in \mathbb{R}), \tag{4.4}$$

then equation (4.3) becomes

$$u''(\xi,\lambda) + \lambda^2 q(\xi) u(\xi,\lambda) = 0, \tag{4.5}$$

where $q(\xi) = \varphi^2(x(\xi))$ and $x(\xi)$ is a function inverse to $\xi(x)$ (4.4), which always exists since $\varphi(x) > 0$ ($x \in \mathbb{R}_+$) and in virtue of (1.5).

Suppose that

$$\varphi(x) = \sqrt{b^2 - x^2} \quad (b \in \mathbb{R}_+, x \in [0, b]). \tag{4.6}$$

Then the function $\xi(x)$ (4.4) equals

$$\xi(x) = \arcsin \frac{x}{b}, \quad x = b \sin \xi,$$

therefore $q(\xi)$ in equation (4.5) is given by

$$q(\xi) = b^2 - b^2 \sin^2 \xi = \frac{b^2}{2}(1 + \cos 2\xi). \tag{4.7}$$

As a result, we obtain the well-known Mathieu equation

$$u''(\xi,\lambda) + \lambda^2 \frac{b^2}{2}(1 + \cos 2\xi) u(\xi,\lambda) = 0. \tag{4.8}$$

The Mathieu functions of the first kind $S(\xi,\lambda)$, $C(\xi,\lambda)$ are solutions of such an equation, besides, $S(\xi,\lambda)$ ($C(\xi,\lambda)$) is an odd (even) function of ξ. Using (1.10), we obtain the functions $e(x,\lambda)$ in this case,

$$e(x,\lambda) = C\left(\operatorname{arctg} \frac{x}{b}, \lambda\right) - \frac{i}{\lambda\sqrt{b^2 - x^2}} S\left(\operatorname{arctg} \frac{x}{b}, \lambda\right), \tag{4.9}$$

where $C(\xi,\lambda)$ and $S(\xi,\lambda)$ are even and odd Mathieu functions of the first kind. Using Theorem 6, we obtain that the functions (4.9) form a basis.

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