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ON BARCILON'S FORMULA FOR KREIN'S STRING

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Abstract. We find conditions on two sequences of positive numbers that are sufficient for the sequences to be the Neumann and the Dirichlet spectra of a Krein string such that Barcilon's formula holds true.

1. Introduction

There exists a series of papers [1], [2], [3], [4], [8] on the so-called Barcilon formula. In [1], V. Barcilon considered the differential equation

$$y'' + \lambda p(x)y = 0, \quad 0 \le x \le b < +\infty, \tag{1.1}$$

where λ is a spectral parameter, the function $p(x)$ is continuous and $p(x) > 0$ on $[0, b]$. This equation describes small transverse vibrations of a string of linear density $p(x)$ stretched by a unit force, and b is the length of the string. Denote by $\{\mu_n\}_{n=1}^{\infty}$ the spectrum of the boundary value problem generated by (1.1) and the boundary conditions

$$y'(0) = y(b) = 0$$

and by $\{\lambda_n\}_{n=1}^{\infty}$ the spectrum of the boundary value problem generated by (1.1) and the boundary conditions

$$y(0) = y(b) = 0.$$

In [1] (Theorem 2) the following formula was stated (following C.-L. Shen [2] we call it Barcilon's formula) expressing $p(0)$ via b, $\{\mu_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$:

$$p(0) = \frac{1}{b^2\mu_1}\prod_{n=1}^{\infty}\frac{\lambda_n^2}{\mu_n\mu_{n+1}}. \tag{1.2}$$

A rigorous proof of this formula was given by C.-L. Shen [2] under rather restrictive conditions of piecewise differentiability of $p(x)$ with $p'(x)$ having a finite number of discontinuities.

In [8] using the results of [7] we show existence of a wide class of strings for which formula (1.2) is true with a nonzero finite limit $\lim\limits_{x\to+0}\frac{M(x)}{x}$ instead of $p(0)$ ($M(x)$ is a nondecreasing nonnegative mass distribution function normalized by $M(0) = 0$).

An inverse problem approach is to find information on p from known spectra. In this context it is more natural to describe validity of (1.2) in terms of spectra and avoid using conditions of piecewise differentiability. Moreover, the validity of (1.2) depends directly on asymptotics of $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$. A necessary condition for validity of (1.2) is convergence of the series on the right-hand side of it. However, this condition seems to be not sufficient. In [8] it was proved that (1.2) is true for a wide class of strings including those for which $M(x)$ is a singular function, i.e. $M'(x) \overset{a.e.}{=} 0$.

In Section 4 of the present paper we propose a sufficient condition for validity of (1.2).

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For this purpose in Sections 2 and 3 we need first to describe some results on the spectral theory of the string with a regular left end (M. G. Krein's string).

2. Certain facts from spectral theory of strings

2.1. **The string and classification of its ends.** Let I be an interval of one of the kinds $(0,b)$, $(0,b]$, $[0,b)$ or $[0,b]$ where $0 < b < \infty$. Let $M(x)$ $(M(x) < \infty$ for all $x \in I)$ be a nondecreasing function on I which can have jumps, intervals of constant value, absolutely continuous, continuous singular parts. We set $a_0 =: \inf \mathcal{F}_M$, $b_0 =: \sup \mathcal{F}_M$, where $\mathcal{F}_M$ is the set of points of growth of $M(x)$. We assume that $\mathcal{F}_M$ is an infinite set of points and that $a_0 = 0$. Let us associate with I and M a string $S(I,M)$ with the mass distribution described by $M(x)$ in the sense that $M(x_2+0) - M(x_1-0)$ is the mass of the part of the string located on $[x_1,x_2]$ for each $x_1, x_2 \in I$ and $x_1 \le x_2$ (here we set $M(-0) = M(0)$ if $0 \in I$ and $M(b+0) = M(b)$ if $b \in I$). The left (right) end of the string $S(I,M)$ is said to be *regular* if $M(+0) > -\infty$ $(M(b-0) < \infty)$. In the opposite case the end is said to be *singular*. A regular left (right) end is said to be *completely regular* if $0 \in I$ $(b \in I)$. A string with both ends regular is said to be a *regular string*. In the opposite case it is said to be a *singular string*. If the string is regular and $0 \notin I$ $(b \notin I)$ we set

$$M(0) = \lim_{x \to +0} M(x) \quad (M(b) = \lim_{x \to b-0} M(x)).$$

2.2. **The differential equation of a string.** Below we use a definition of the differential expression $l_{I,M}$ for the string $S(I,M)$ which fits to this situation.

Definition 2.1. Let $\mathcal{D}_{M,I}$ be the set of all functions $f(x)$ defined on I such that
1) f is locally absolutely continuous on I with respect to Lebesgue measure,
2) there exist finite left $f'_-(x)$ and right $f'_+(x)$ derivatives at each interior point x of I,
3) there exists an M-measurable function g such that for any two points x_1, x_2 of I $(x_1 \le x_2)$

$$f'_\pm(x_2) - f'_\pm(x_1) = -\int_{x_1 \pm 0}^{x_2 \pm 0} g(x)\, dM(x) \tag{2.1}$$

for each of the four combinations of signs which are the same on both sides of (2.1). Here we mean that $f'_-(0) = f'_+(0)$ and $f'_+(b) = f'_-(b)$. For a function $f \in \mathcal{D}_{I,M}$ we set $l_{M,I}[f](x) = g(x)$ where g is the function involved in (2.1).

Remark. We have defined $l_{M,I}[f](x)$ up to equivalence with respect to the M- measure. It is clear that for $f \in \mathcal{D}_{I,M}$

$$l_{I,M}[f](x) = -\frac{d}{(d)M(x)} f'_+(x) = -\frac{d}{(d)M(x)} f'_-(x)$$

at M-almost all x. Here $\frac{d}{(d)M(x)}$ is the symbol of the symmetric derivative with respect to M.

The expression

$$l_{I,M}[y] - \lambda y = 0 \quad (x \in I), \tag{2.2}$$

where λ is the spectral parameter, we call *the differential equation of the string* $S(I,M)$. A function $u \in \mathcal{D}_{I,M}$ is s said to be a solution of (2.2) if $l_{I,M}[u](x) - \lambda u(x) = 0$ for M-almost all $x \in I$. Note that for any $\lambda > 0$ each solution to (2.2) is the amplitude function of vibrations of our string with the frequency $\sqrt{\lambda}$.

2.3. **M. G. Krein's strings $S_1(I,M)$ and $S_0(I,M)$.** We deal with strings $S(I,M)$ the left ends of which are completely regular (for convenience we place them at $x=0$) while the right ends $x=b$ are either completely regular, or singular and then $b\notin I$. The ends do not bear point masses.

A string $S(I,M)$ is written as $S_1(I,M)$ if its left end is free to move without friction in the direction orthogonal to the x-axis, i.e. to the equilibrium position of the string. By $S_0(I,M)$ we denote a string $S(I,M)$ with the left end fixed. We assume that if the right end of a string $S_1(I,M)$ or $S_0(I,M)$ is completely regular then it is fixed.

We define *fundamental* functions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ of the strings $S_1(I,M)$ and $S_0(I,M)$, respectively, as the solutions of equation (2.2) which satisfy the initial conditions $\phi(0,\lambda)=1$, $\phi'_+(0,\lambda)=0$ and $\psi(0,\lambda)=0$, $\psi'_+(0,\lambda)=1$, respectively.

It is known (see [6], Sec. 2) that for any fixed $x\in I$ the functions $\phi(x,\lambda)$, $\psi(x,\lambda)$, $\phi'_-(x,\lambda)$, $\psi'_-(x,\lambda)$, $\phi'_+(x.\lambda)$ and $\psi'_+(x,\lambda)$ are real entire functions of λ of order not more than 1/2.

Remark. A meromorphic function in C or an entire function is said to be real if it attains real values for real values of the variable.

Since $\phi(x,0)=1$ and $\psi(x,0)=x$, for each fixed $x\in I$ the following representations are valid:

$$\phi(x,\lambda)=\prod_j\left(1-\frac{\lambda}{\mu_j(x)}\right),\quad \psi(x,\lambda)=x\prod_j\left(1-\frac{\lambda}{\lambda_j(x)}\right),$$

where $\mu_j(x)$, $\lambda_j(x)$, $j=1,2,\dots$ are eigenvalues of $S_1([0,x],M)$ and $S_0([0,x],M)$ with the right ends fixed.

The set of squares of frequencies of free vibrations of a regular string is called its *spectrum*. The spectrum depends on the mass distribution and on whether the ends are fixed or free. Therefore, we mean by spectra of strings $S_1(I,M)$ and $S_0(I,M)$ the sets of eigenvalues of the boundary value problems

$$l_{I,M}[y]-\lambda y=0,\quad y'_+(0)=y(b)=0,$$

$$l_{I,M}[y]-\lambda y=0,\quad y(0)=y(b)=0,$$

respectively. It is easy to see that the spectrum $\{\mu_j\}_{j=1}^\infty:=\{\mu_j(b)\}_{j=1}^\infty$ of a completely regular string $S_1(I,M)$ is the set of zeros of the entire function $\phi(b,\lambda)$ and the spectrum $\{\lambda_j\}_{j=1}^\infty:=\{\lambda_j(b)\}_{j=1}^\infty$ of a completely regular string $S_0(I,M)$ is the set of zeros of $\psi(b,\lambda)$ with the same I and M, respectively. This is in accordance with a general definition of the spectra of the strings $S_0(I,M)$ and $S_1(I,M)$ given below.

Definition 2.2. ([5]). A function $f(z)$ of a complex variable z is said to be an *R-function* or to belong to the class (R) if

1) it is defined and holomorphic in each of the half-planes $\mathrm{Im}z>0$ and $\mathrm{Im}z<0$,
2) $f(\overline{z})=\overline{f(z)}$ $(\mathrm{Im}z\neq 0)$,
3) $\mathrm{Im}z\ \mathrm{Im}f(z)\geq 0$ $(\mathrm{Im}z\neq 0)$.

Such a function is also often called a *Nevanlinna* function.

Definition 2.3. A function f is said to be an *S-function* or to belong to the class (S) if

1) $f\in(R)$,
2) f is analytic in $\mathrm{Ext}[0,\infty)$ $(=C\backslash[0,\infty))$,
3) $f(z)\geq 0$ for all $z\in(-\infty,0)$.

We denote by $\widehat{\mathcal{L}}^2_M(I)$ the set of functions $f\in\mathcal{L}^2_M(I)$ which are identically zero in some left neighborhood of $x=b$ if the right end is singular. If the right end is regular then $\widehat{\mathcal{L}}^2_M(I)=\mathcal{L}^2_M(I)$.

Definition 2.4. A nondecreasing function $\tau(\lambda)$ on $(-\infty,\infty)$ normalized by the condition

$$\tau(\lambda)=\frac{1}{2}(\tau(\lambda+0)+\tau(\lambda-0))\ \forall\lambda\in(-\infty,\infty),\quad \tau(0)=0,$$

is said to be a spectral function of the string $S_1(I,M)$ $(S_0(I,M))$ if the mapping $U: f\mapsto\mathcal{F}$ where $f\in\widehat{\mathcal{L}}^2_M(I)$ and

$$\mathcal{F}(\lambda)=\int_0^L f(x)\phi(x,\lambda)\,dM(x)\quad \left(\mathcal{F}(\lambda)=\int_0^L f(x)\psi(x,\lambda)\,dM(x)\right)$$

maps $\widehat{\mathcal{L}}^2_M(I)$ isometrically into $\mathcal{L}^2_\tau(-\infty,\infty)$, i.e. if for each function $f\in\widehat{\mathcal{L}}^2_M(I)$ the 'Parseval identity' is true:

$$\int_{-\infty}^{\infty}|\mathcal{F}(\lambda)|^2d\tau(\lambda)=\int_I|f(x)|^2dM(x),$$

where $\mathcal{F}=Uf$. A spectral function is said to be orthogonal if U maps $\widehat{\mathcal{L}}^2_M(I)$ onto a dense part of $\mathcal{L}^2_\tau(-\infty,\infty)$. The set of points of growth of a spectral function is said to be its spectrum.

The function

$$T(z):=\lim_{x\to b-0}\frac{\psi(x,z)}{\phi(x,z)},\quad z\in(C\backslash[0,+\infty)) \tag{2.3}$$

is said to be the *coefficient of dynamic compliance* of the string $S_1(I,M)$. If the right end $x=b$ is completely regular then $T(z)=\frac{\psi(b,z)}{\phi(b,z)}$ is a meromorphic function. In any case $T(z)$ is an S-function.

Being an S-function, $T(z)$ has the spectral function $\tau^{(1)}(\lambda)$ which is constant on $(-\infty,0)$. Since $\inf\mathcal{F}_M=0$, we have (see [5], Sec. 5, [6], Sec. 10):

$$T(z)=\int_{-0}^{+\infty}\frac{d\tau^{(1)}(\lambda)}{\lambda-z},\quad z\in(C\backslash[0,+\infty)). \tag{2.4}$$

We keep the norming

$$\tau^{(1)}(\lambda)=\tau^{(1)}(-0)\quad\text{for}\quad\lambda<0,$$

$$\tau^{(1)}(\lambda)=\frac{1}{2}(\tau^{(1)}(\lambda+0)+\tau^{(1)}(\lambda-0))\quad\forall\lambda\in R,\quad\tau^{(1)}(0)=0,$$

which is common for R-functions (recall that $(S)\subset(R)$). Being normed this way the function $\tau^{(1)}(\lambda)$ is a spectral function of a string $S_1(I,M)$ (see [6], Sec. 3, Main Theorem and Sec. 10, Theorem 10.1). This spectral function of the string $S_1(I,M)$ is called its *main spectral function*. The main spectral function is orthogonal. Notice that

$$\int_{-0}^{+\infty}\frac{d\tau^{(1)}(\lambda)}{1+\lambda}<\infty.$$

In case of a singular string, $\tau^{(1)}(\lambda)$ is its unique spectral function with nonnegative spectrum. The spectrum of the main spectral function of a string $S_1(I,M)$ is said to be the spectrum of this string.

2.4. **Kasahara's theorem.** It was shown in [7] that if a string $S_1(I,M)$ is such that there exists a nonzero finite limit $\lim\limits_{x\to+0}\frac{M(x)}{x}$ then the limits $\lim\limits_{z\to+\infty}(T(-z)z^{\frac{1}{2}})$ and $\frac{\pi}{2}\lim\limits_{\lambda\to+\infty}\tau^{(1)}(\lambda)\lambda^{-\frac{1}{2}}$ also exist, neither of them equals zero, and

$$\lim_{x\to+0}\frac{M(x)}{x}=\left(\lim_{z\to+\infty}(T(-z)z^{\frac{1}{2}})\right)^{-2}=\left(\frac{\pi}{2}\lim_{\lambda\to+\infty}\tau^{(1)}(\lambda)\lambda^{-\frac{1}{2}}\right)^{-2}. \tag{2.5}$$

Remark. Equations (2.5) remain true if $\tau^{(1)}(\lambda)$ is changed for any other spectral function of the same string (see [7], Lemma 6).

In the case of a string $S(I,M)$ with a regular right end we have $T(z)=\frac{\psi(b,z)}{\phi(b,z)}$. Therefore, the first equation in (2.5) is an analogue of the one which is also called *Barcilon formula* in [3].

Some years after [7], a paper by Kasahara [9] appeared where the results of [7] were generalized and inverted. In particular, Theorem 2 in [9] implies that if one of the limits in (2.5) exists and is not zero, also the other limits in (2.5) exist and (2.5) is valid.

It should be noticed that unlike [1], [2], [3] the results in [7] and [9] were obtained without any assumption on continuity or piecewise differentiability of the density of the string. By the way, the first limit in (2.5), i.e. the right derivative of the function $M(x)$ at $x=0$, i.e. the density of the string at $x=0$ can exist and be finite and nonzero even in the case where $M(x)$ is a pure jump function.

2.5. **The main spectral function of the string $S_1(I,M)$ and the length of the string.** If $\tau^{(1)}(\lambda)$ is the main spectral function of a string $S_1(I,M)$ (regular or singular), then

$$\int_{-0}^{+\infty}\frac{d\tau^{(1)}(\lambda)}{\lambda}=b \tag{2.6}$$

in both cases of finite and infinite b. This result was obtained by M. G. Krein [10].

It should be mentioned that if for some $\epsilon>0$ the interval $[0,\epsilon)$ has zero $\tau^{(1)}$-measure then the integral in (2.6) is finite and (2.4) implies $T(0)=b<\infty$.

3. Relation between the discrete spectra of the strings $S_1(I,M)$ and $S_0(I,M)$ and the behavior of $M(x)$ at $x\to+0$

3.1. **Spectral function via two spectra.** Let us remind that in case of $M(b)=\infty$ the necessary and sufficient condition for discreteness of the spectrum is

$$\lim_{x\to b-0}M(x)(b-x)=0.$$

Let a string $S(I,M)$ have the length b, the string $S_1(I,M)$ generated by $S(I,M)$ have discrete spectrum $\{\mu_k\}_{k=1}^{\infty}$ where $0<\mu_1<\mu_2<\dots$ and let $\{\lambda_k\}_{k=1}^{\infty}$ where $\lambda_1<\lambda_2<\dots$ be the spectrum of the string $S_0(I,M)$ generated by the string $S(I,M)$. It is known that these spectra interlace:

$$0<\mu_1<\lambda_1<\mu_2<\lambda_2<\cdots$$

Kasahara's theorem mentioned above shows that information about the behavior of $M(x)$ at $x\to+0$ can be extracted from the behavior of the main spectral function $\tau^{(1)}(\lambda)$ of the string $S_1(I,M)$ at $\lambda\to+\infty$. We will use the following theorems (see [8], Theorems 4.1 and 4.2).

Theorem 3.1. *The function $T(z)$ defined by (2.3) admits representation*

$$T(z)=b\prod_{k=1}^{\infty}\frac{1-\frac{z}{\lambda_k}}{1-\frac{z}{\mu_k}},\quad z\in(C\backslash\{\mu_k\}_{k=1}^{\infty}) \tag{3.1}$$

independently of whether the string is regular or not.

Theorem 3.2. *Let two sequences* $\{\mu_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *of interlacing real numbers be given:*

$$0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \cdots \tag{3.2}$$

Then there exists a unique string $S(I, M)$ *of a given finite length* b *such that the spectrum of the string* $S_1(I, M)$ *coincides with* $\{\mu_n\}_{n=1}^{\infty}$ *and the spectrum of the string* $S_0(I, M)$ *coincides with* $\{\lambda_n\}_{n=1}^{\infty}$.

In what follows we will say that the string $S(I, M)$, the existence of which is stated in Theorem 3.2, corresponds to the data b, $\{\mu_k\}_{k=1}^{\infty}$, $\{\lambda_k\}_{k=1}^{\infty}$.

4. Main result

Now we are ready to state our result.

Theorem 4.1. *Let two sequences* $\{\mu_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *satisfy (3.2), let the product*

$$\frac{1}{b^2\mu_1}\prod_{n=1}^{\infty}\frac{\lambda_n^2}{\mu_n\mu_{n+1}} \tag{4.1}$$

converge and let

$$\sum_{k\in N\setminus\mathcal{K}}\left(\frac{\lambda_k-\mu_k}{\mu_{k+1}-\lambda_k}+\frac{\mu_{k+1}-\lambda_k}{\lambda_k-\mu_k}-2\right)<\infty, \tag{4.2}$$

where $\mathcal{K}$ *is the set of all those* k*-s for which*

$$\lambda_k \in \left(\mu_k, \frac{2\mu_k\mu_{k+1}}{\mu_k+\mu_{k+1}}\right) \cup \left[\frac{\mu_k+\mu_{k+1}}{2}, \mu_{k+1}\right).$$

Then there exists a unique string of length b *with two spectra* $\{\mu_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *and for this string*

$$\lim_{x\to+0}\frac{M(x)}{x} = \frac{1}{b^2\mu_1}\prod_{n=1}^{\infty}\frac{\lambda_n^2}{\mu_n\mu_{n+1}}.$$

Proof. Using (2.5) and (3.1) we obtain

$$\lim_{x\to+0}\frac{M(x)}{x} = \frac{1}{\mu_1 b^2}\prod_{k=1}^{\infty}\frac{\lambda_k^2}{\mu_k\mu_{k+1}}\lim_{z\to+\infty}\prod_{k=1}^{\infty}\frac{(\mu_k+z)(\mu_{k+1}+z)}{(\lambda_k+z)^2}.$$

Let us consider the function

$$\left|1-\frac{(\mu_k+z)(\mu_{k+1}+z)}{(\lambda_k+z)^2}\right|, \quad z\in[0,\infty).$$

It attains its maximum value

$$\frac{\lambda_k^2-\mu_k\mu_{k+1}}{\lambda_k^2}$$

at $z = 0$ if

$$z_k =: \frac{\frac{\mu_k+\mu_{k+1}}{2}\lambda_k-\mu_k\mu_{k+1}}{\frac{\mu_k+\mu_{k+1}}{2}-\lambda_k}\le 0,$$

i.e. if $k\in\mathcal{K}$. If $k\in N\setminus\mathcal{K}$ then

$$\max_{z\in[0,\infty)}\left|1-\frac{(\mu_k+z)(\mu_{k+1}+z)}{(\lambda_k+z)^2}\right| = \frac{\left(\frac{\mu_k+\mu_{k+1}}{2}-\lambda_k\right)^2}{(\mu_{k+1}-\lambda_k)(\lambda_k-\mu_k)}.$$

It is easy to check that under the condition (4.2)

$$\sum_{k\in N\setminus\mathcal{K}} \frac{\left(\frac{\mu_k+\mu_{k+1}}{2}-\lambda_k\right)^2}{(\mu_{k+1}-\lambda_k)(\lambda_k-\mu_k)} < \infty.$$

Since the product in (4.1) converge we have

$$\sum_{k=1}^{\infty} \frac{|\lambda_k^2-\mu_k\mu_{k+1}|}{\lambda_k^2} < \infty.$$

Thus,

$$\sum_{k=1}^{\infty} \left|1-\frac{(\mu_k+z)(\mu_{k+1}+z)}{(\lambda_k+z)^2}\right|$$

can be majorized by a convergent series independent of z. Therefore,

$$\lim_{z\to+\infty} \prod_{k=1}^{\infty} \frac{(\mu_k+z)(\mu_{k+1}+z)}{(\lambda_k+z)^2} = 1.$$

Theorem is proved. □

References

1. V. Barcilon, *Explicit solution of the inverse problem for a vibrating string*, J. Math. Anal. Appl. **93** (1983), 222–234.
2. C.-L. Shen, *The Liouville transformation, the density function, and the complete transformability problem of the potential equation*, Inverse Problems **21** (2005), 615–634.
3. C.-L. Shen, *On the Barcilon formula for the string equation with a piecewise continuous density function*, Inverse Problems **21** (2005), 635–655.
4. M. Dehghan, A. Jodayree, *A reconstructing the potential function of the indefinite Sturm-Liouville problem using infinite product forms*, Electr. J. Differ. Equations **2013** (2013), no. 182, 116.
5. I. S. Kac, M. G. Krein, *R-functions – analytic functions mapping the upper half plane into itsel*, Amer. Math. Soc. Transl. Ser. 2 **103** (1974), 1–18.
6. I. S. Kac, M. G. Krein, *On spectral functions of the string*, Amer. Math. Soc. Transl. Ser. 2 **103** (1974), 19–102.
7. I. S. Kac, *Generalization of an asymptotic formula of V.A. Marchenko for spectral functions of a second-order boundary value problem*, Izv. Akad. Nauk SSSR, Ser. Mat. **37** (1973), no. 2, 422–436. (Russian). English transl. Math. USSR Izvestija **7** (1973), no. 2, 424–438.
8. I. S. Kac, V. Pivovarchik, *On the density of the mass distribution of a string at the origin*, Int. Eq. Oper. Theory **81** (2015), 581–599.
9. Y. Kasahara, *Spectral theory of generalized second order differential operators and its applications to Markov processes*, Japan J. Math. **1** (1975), 67–84.
10. M. G. Krein, *On some cases of effective determination of the density of an inhomogeneous cord from its spectral function*, Doklady Akad. Nauk SSSR **93** (1953), 617–620. (Russian)

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