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LINEAR MAPS PRESERVING THE INDEX OF OPERATORS

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Abstract. Let H be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(\mathsf{H})$ the algebra of all bounded linear operators on H. In this paper, we prove that if a surjective linear map $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ preserves the index of operators, then ϕ preserves compact operators in both directions and the induced map $\varphi : \mathcal{C}(\mathsf{H}) \longrightarrow \mathcal{C}(\mathsf{H})$, determined by $\varphi(\pi(T)) = \pi(\phi(T))$ for all $T \in \mathcal{B}(\mathsf{H})$, is a continuous automorphism multiplied by an invertible element in $\mathcal{C}(\mathsf{H})$.

1. Introduction

Throughout this paper, H will denote an infinite-dimensional separable complex Hilbert space, $\mathcal{B}(\mathsf{H})$ the algebra of all bounded linear operators on H, $\mathcal{K}(\mathsf{H})$ the closed ideal of all compact operators on H, $\mathcal{F}(\mathsf{H})$ the set of all operators on H of finite rank and $\mathcal{F}_1(\mathsf{H})$ the set of all operators on H of rank 1 . We denote by $\mathcal{C}(\mathsf{H})$ the Calkin algebra $\mathcal{B}(\mathsf{H})/\mathcal{K}(\mathsf{H})$ and by $\pi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{C}(\mathsf{H})$ the canonical quotient map.

For $T \in \mathcal{B}(\mathsf{H})$, we will denote by T^*, $N(T)$, $R(T)$, $\alpha(T)$ and $\beta(T)$, the adjoint, the kernel, the range, the nullity and the defect of T, respectively.

We recall that an operator $T \in \mathcal{B}(\mathsf{H})$ is called upper semi-Fredholm if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in \mathcal{B}(\mathsf{H})$ is called lower semi-Fredholm if $\beta(T) < \infty$. Let $\Phi_+(\mathsf{H})$ and $\Phi_-(\mathsf{H})$ denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators on H, respectively. The class of all semi-Fredholm operators is defined by $\Phi_\pm(\mathsf{H}) = \Phi_+(\mathsf{H}) \cup \Phi_-(\mathsf{H})$, while the class of all Fredholm operators is defined by $\Phi(\mathsf{H}) = \Phi_+(\mathsf{H}) \cap \Phi_-(\mathsf{H})$. It is well known (Atkinson's theorem) that :

$$T \in \Phi(\mathsf{H}) \Longleftrightarrow \pi(T) \text{ is invertible in } \mathcal{C}(\mathsf{H}).$$

An interested reader can find some basic information on Fredholm theory in [8].

Recall that for $T \in \Phi_\pm(\mathsf{H})$ the index of T is defined by $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. Note that every $T \in \Phi_\pm(\mathsf{H})$ has closed range and therefore $\beta(T) = \alpha(T^*)$. Furthermore, it is well known that in the case of separable Hilbert space the concept of index can be generalized to any operator by setting $\mathrm{ind}(T) = \alpha(T) - \alpha(T^*)$, for all $T \in \mathcal{B}(\mathsf{H})$, with the convention $\infty - \infty = 0$.

For $T \in \mathcal{B}(\mathsf{H})$, the reduced minimum modulus of T (also called the conorm of T) is defined by

$$\gamma(T) = \begin{cases} \inf \{\|Tx\| : x \in N(T)^\perp \,,\ \|x\| = 1\} & \text{if } T \neq 0, \\ +\infty & \text{if } T = 0. \end{cases}$$

The following basic properties of $\gamma(T)$ were proved in [8] :

$$\gamma(T) > 0 \Longleftrightarrow R(T) \text{ is closed},$$
$$\gamma(T) = \gamma(T^*).$$

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Over the last decade there has been a considerable interest in the so called linear preserver problems (see the survey articles [6], [7], [9]). The objective is to study additive or linear maps between two Banach algebras preserving a given class of elements of algebras. The most famous problem is Kaplansky's problem [4] asking whether bijective unital linear maps between semi-simple Banach algebras preserving invertibility in both directions are Jordan isomorphisms. Many other linear preserver problems, like the problem of characterizing linear maps preserving idempotent, algebraic operators, Fredholm operators, etc, have attracted many researchers on operators theory (see [1], [2], [3]). In this paper we study linear maps preserving the index of operators. Indeed, the following theorem is our main result :

Theorem 1.1. *Let* H *be an infinite-dimensional separable complex Hilbert space and let* $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ *be a surjective linear map. If* ϕ *preserves the index of operators, then* ϕ *preserves compact operators in both directions and the induced map* $\varphi : \mathcal{C}(\mathsf{H}) \longrightarrow \mathcal{C}(\mathsf{H})$ *determined by* $\varphi(\pi(T)) = \pi(\phi(T))$ *for all* $T \in \mathcal{B}(\mathsf{H})$ *is a continuous automorphism multiplied by an invertible element in* $\mathcal{C}(\mathsf{H})$.

The proof is done in several steps. Some auxiliary results have been proved in sections 2 and 3. The most important one is given in Section 2, where we show that a surjective linear map on $\mathcal{B}(\mathsf{H})$ preserving the index of operators preserves necessarily left(i.e upper) and right (i.e lower) semi-Fredholm operators in both directions. Finally, to give the desired result we use the following theorem established by Hou and Cui in [2].

Theorem 1.2 (Theorem 2.7 [2])**.** *Let* H *be an infinite-dimensional separable complex Hilbert space and let* $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ *be a surjective linear map, with* $\phi(I)$ *a Fredholm operator. Then the following are equivalent:*

(1) ϕ *preserves left Fredholm operators in both directions.*
(2) ϕ *preserves right Fredholm operators in both directions.*
(3) $\phi(\mathcal{K}(\mathsf{H}))=\mathcal{K}(\mathsf{H})$ *and the induced map* $\varphi : \mathcal{C}(\mathsf{H}) \longrightarrow \mathcal{C}(\mathsf{H})$ *determined by* $\varphi(\pi(T)) = \pi(\phi(T))$ *for all* $T \in \mathcal{B}(\mathsf{H})$ *is a continuous automorphism multiplied by an invertible element in* $\mathcal{C}(\mathsf{H})$.

2. Some auxiliary results

In this section we give some results needed for the proof of the main theorem. In the following, we denote by $< \cdot , \cdot >$ the inner product of H and by $\mathsf{Vect}(x_1, \dots, x_n)$ the linear space spanned by $x_1, \dots, x_n \in \mathsf{H}$.

Lemma 2.1. *Let* $F \in \mathcal{F}(\mathsf{H})$ *and* $T \in \mathcal{B}(\mathsf{H})$. *Then*

$$\alpha(T + F) < \infty \Longleftrightarrow \alpha(T) < \infty.$$

Proof. We have $N(T + F) = \{x \in \mathsf{H} \, : \, Tx = -Fx\}$, then the map

$$\begin{array}{cccc} \widetilde{T}: & N(T+F) & \longrightarrow & R(F) \\ & x & \longmapsto & \widetilde{T}(x) := T(x) \end{array}$$

is well defined, having finite rank and its kernel $N(\widetilde{T}) = N(T) \cap N(T + F)$. Hence if $\alpha(T) < \infty$ then, by the rank theorem, we get $\alpha(T+F) < \infty$. Conversely, if $\alpha(T+F) < \infty$ then $\alpha(T) = \alpha(T + F - F) < \infty$. □

Lemma 2.2. *Let* $T \in \mathcal{B}(\mathsf{H})$ *and* $F \in \mathcal{F}_1(\mathsf{H})$. *Then*

$$\alpha(T+F) = \begin{cases} \alpha(T)+1 \\ \quad or \\ \alpha(T) \\ \quad or \\ \alpha(T)-1 \end{cases}$$

with the convention $\infty + n = \infty$, *for all* $n \in \mathbb{Z}$.

Proof. By Lemma 2.1, the result is immediate if $\alpha(T) = +\infty$. We will suppose from now on that $\alpha(T) < \infty$. Let a and b be non zero vectors such that $F(x) =< x, a > b$, for all $x \in \mathsf{H}$, and let $a_0 \in N(T)$, $a_1 \in N(T)^\perp$ such that $a = a_0 + a_1$. Let G be the subspace of $N(T)$ such that $N(T) = G \oplus \mathsf{Vect}(a_0)$. Then for all $x \in G$, $Tx = 0$ and $< a, x >= 0$, so $G \subset N(T+F)$ and hence $\alpha(T) - 1 \leq \alpha(T+F)$. On the other hand, for every $x \in \mathsf{H}$, we have

$$x \in N(T+F) \Longleftrightarrow Tx = - < x, a > b.$$

This leads us to distinguish two cases depending on whether $b \in R(T)$ or not. Suppose first that $b \notin R(T)$, then $N(T+F) = N(T) \cap \mathsf{Vect}(a)^\perp = G$ and therefore $\alpha(T) - 1 \leq \alpha(T+F) = \dim(G) \leq \alpha(T)$. Now, suppose that $b \in R(T)$ and let $b_0 \in N(T)^\perp$ such that $Tb_0 = b$. Then

$$\begin{aligned} N(T+F) &= \{x \in \mathsf{H} : Tx = - < x, a > Tb_0\} \\ &= \{x \in \mathsf{H} : x + < x, a > b_0 \in N(T)\} \\ &= \{x \in \mathsf{H} : x = < x, a > b_0 + y, y \in N(T)\}. \end{aligned}$$

It follows that $N(T+F) \subset N(T) \oplus \mathsf{Vect}(b_0)$, and hence $\alpha(T+F) \leq \alpha(T) + 1$. This completes the proof. □

Lemma 2.3. *Let* $F \in \mathcal{B}(\mathsf{H})$ *be a non zero operator. The following assertions are equivalent :*

(1) $F \in \mathcal{F}_1(\mathsf{H})$.

(2) *For all* $T \in \mathcal{B}(\mathsf{H})$, $\operatorname{ind}(T+F) = \begin{cases} \operatorname{ind}(T) + 1 \\ \quad or \\ \operatorname{ind}(T) \\ \quad or \\ \operatorname{ind}(T) - 1. \end{cases}$

Proof. "(1) $\Longrightarrow$ (2)" : Assume that $F \in \mathcal{F}_1(\mathsf{H})$. Let a and b be non zero vectors such that $F(\cdot) =< \cdot, a > b$. Then $F^*(\cdot) =< \cdot, b > a$. By Lemma 2.2, we see that for all $T \in \mathcal{B}(\mathsf{H})$ satisfying $\alpha(T+F) = \alpha(T)$ or $\alpha(T^* + F^*) = \alpha(T^*)$, we have $\operatorname{ind}(T+F) \in \{\operatorname{ind}(T) - 1\} \cup \{\operatorname{ind}(T)\} \cup \{\operatorname{ind}(T) + 1\}$. In particular, if $\alpha(T) = +\infty$ or $\alpha(T^*) = +\infty$ the result is immediate. So, we will show the result only for $T \in \mathcal{B}(\mathsf{H})$ such that both $\alpha(T+F) \neq \alpha(T)$ and $\alpha(T^* + F^*) \neq \alpha(T^*)$. Let $T \in \mathcal{B}(\mathsf{H})$ be a such operator. Let $a_0 \in N(T)$, $a_1 \in N(T)^\perp$ such that $a = a_0 + a_1$ and let G be the subspace of $N(T)$ satisfying $N(T) = G \oplus \mathsf{Vect}(a_0)$. We have

$$N(T+F) = \{x \in \mathsf{H} : Tx = - < x, a > b\}$$

and

$$N(T^* + F^*) = \{x \in \mathsf{H} : T^*x = - < x, b > a\}.$$

Two cases occur:

• Case 1 : $b \notin R(T)$. In this case, we obtain $N(T+F) = G$. But, since $\alpha(T+F) \neq \alpha(T)$, then $a_0 \neq 0$ and $\alpha(T+F) = \alpha(T) - 1$. On the other hand,

$$N(T^* + F^*) = \{x \in \mathsf{H} : T^*x + < x, b > a_1 = - < x, b > a_0\}.$$

But, $T^*x + < x, b > a_1 \in \overline{R(T^*)} = N(T)^\perp$ and $< x, b > a_0 \in N(T)$. Consequently,

$$N(T^* + F^*) = \{x \in \mathsf{H} : T^*x + < x, b > a_1 = 0 \text{ and } < x, b > a_0 = 0\}.$$

Now, since $a_0 \neq 0$, then

$$N(T^* + F^*) = \{x \in \mathsf{H} : T^*x = 0 \text{ and } < x, b >= 0\} = N(T^*) \cap \mathsf{Vect}(b)^\perp.$$

This implies that $\alpha(T^* + F^*) = \alpha(T^*)$ or $\alpha(T^* + F^*) = \alpha(T^*) - 1$. But, by assumption, $\alpha(T^* + F^*) \neq \alpha(T^*)$, thus $\alpha(T^* + F^*) = \alpha(T^*) - 1$ and so $\operatorname{ind}(T+F) = \operatorname{ind}(T)$.

• Case 2 : $b \in R(T)$. Let $b_0 \in N(T)^{\perp}$ such that $Tb_0 = b$. Since $b \neq 0$, then $b_0 \neq 0$. So

$$\begin{aligned} N(T+F) &= \{x \in \mathsf{H} \,:\, Tx = -<x,a> Tb_0\} \\ &= \{x \in \mathsf{H} \,:\, x + <x,a> b_0 \in N(T)\} \\ &= \{x \in \mathsf{H} \,:\, x = -<x,a> b_0 + y,\ y \in N(T)\} \\ &= \{x \in \mathsf{H} \,:\, x = \alpha b_0 + y,\ y \in N(T),\ \alpha = -<x,a>\} \\ &= \{x \in \mathsf{H} \,:\, x = \alpha b_0 + y,\ y \in N(T),\ \alpha = -<x,a> = <x, \frac{b_0}{\|b_0\|^2}>\}. \end{aligned}$$

It follows that

$$N(T+F) = (N(T) \oplus \mathsf{Vect}(b_0)) \cap \mathsf{Vect}(a + \frac{b_0}{\|b_0\|^2})^{\perp}.$$

Denote by $\mathsf{H}_1 = N(T) \oplus \mathsf{Vect}(b_0)$ and $z = a + \frac{b_0}{\|b_0\|^2}$. Then, $z = z_0 + z_1$, where $z_0 \in \mathsf{H}_1$ and $z_1 \in \mathsf{H}_1^{\perp}$. Let G_1 be the subspace of H_1 such that $\mathsf{H}_1 = G_1 \oplus \mathsf{Vect}(z_0)$. Clearly, $N(T+F) = \mathsf{H}_1 \cap \mathsf{Vect}(z)^{\perp} = G_1$. Thus, $\alpha(T+F) = \dim(G_1) = \dim(\mathsf{H}_1)$ or $\alpha(T+F) = \dim(G_1) = \dim(\mathsf{H}_1) - 1$. Since $\dim(\mathsf{H}_1) = \alpha(T) + 1$ and $\alpha(T+F) \neq \alpha(T)$, then $\alpha(T+F) = \alpha(T) + 1$.

On the other hand, we have

$$N(T^* + F^*) = \{x \in \mathsf{H} \,:\, T^*x = -<x,b> a\}.$$

Suppose that $a \notin R(T^*)$. Then $N(T^*+F^*) = N(T^*) \cap \mathsf{Vect}(b)^{\perp}$, but $b \in R(T) \subset \overline{R(T)} = N(T^*)^{\perp}$, hence $N(T^*+F^*) = N(T^*)$ and so $\alpha(T^*+F^*) = \alpha(T^*)$. But this contradicts the hypothesis. So, necessarily $a \in R(T^*)$, which implies that $a = T^*x_1$ for some non zero vector x_1 in $N(T^*)^{\perp}$. Just like in the beginning of this proof, we get

$$\begin{aligned} N(T^*+F^*) &= \{x \in \mathsf{H} \,:\, T^*x = -<x,b> T^*x_1\} \\ &= (N(T^*) \oplus \mathsf{Vect}(x_1)) \cap \mathsf{Vect}(b + \frac{x_1}{\|x_1\|^2})^{\perp} \end{aligned}$$

and $\alpha(T^*+F^*) = \alpha(T^*) + 1$. Thus, $\mathrm{ind}(T+F) = \mathrm{ind}(T)$.

"(2) $\Longrightarrow$ (1)" : Suppose that, for all $T \in \mathcal{B}(\mathsf{H})$, $\mathrm{ind}(T+F) \in \{\mathrm{ind}(T)-1\} \cup \{\mathrm{ind}(T)\} \cup \{\mathrm{ind}(T)+1\}$. In particular, if $T = -F$, we get $\mathrm{ind}(F) \in \{-1, 0, 1\}$. Assume, to the contrary, that F is not of rank one, then two cases can hold :

• Case 1 : $\mathrm{ind}(F) \in \{-1, 1\}$. Then $\alpha(F)$ and $\alpha(F^*)$ are finite and so that $\dim(N(F))^{\perp} = +\infty$ and $\mathrm{rank}(F) = +\infty$. Let H_3, H_4 and H_5 be three closed infinite-dimensional subspaces of $\overline{R(F)}$ such that $\overline{R(F)} = \mathsf{H}_3 \oplus \mathsf{H}_4 \oplus \mathsf{H}_5$. Denote by $\mathsf{H}_1 = F^{-1}(\mathsf{H}_3) \cap N(F)^{\perp}$, and let H_2 be the closed subspace of $N(F)^{\perp}$ such that $N(F)^{\perp} = \mathsf{H}_1 \oplus \mathsf{H}_2$. Note that H_1 must be of infinite dimensional, else if $\dim(R(F) \cap \mathsf{H}_3) < \infty$, then $\overline{R(F)} \subset (R(F) \cap \mathsf{H}_3) \oplus \mathsf{H}_4 \oplus \mathsf{H}_5$, which is a contradiction. Similarly, H_2 must be also of infinite dimensional. Let us consider the operator $T \in \mathcal{B}(\mathsf{H})$ defined as follow :

$$\begin{cases} T_{|\mathsf{H}_1} : \mathsf{H}_1 \longrightarrow \mathsf{H}_3,\ T_{|\mathsf{H}_1} = -F_{|\mathsf{H}_1}, \\ T_{|\mathsf{H}_2} : \mathsf{H}_2 \longrightarrow \mathsf{H}_4 \text{ be an invertible operator}, \\ T_{|N(F)} : N(F) \longrightarrow N(F^*) \text{ be an arbitrary bounded operator.} \end{cases}$$

Then $\overline{R(T)} = \overline{T(\mathsf{H}_1) + T(\mathsf{H}_2) + T(N(F))} \subset \mathsf{H}_3 \oplus \mathsf{H}_4 \oplus N(F^*)$. This implies that $\mathsf{H}_5 \oplus \overline{R(T)}$, hence $\alpha(T^*) = +\infty$. But, $T_{|\mathsf{H}_1 \oplus \mathsf{H}_2}$ is one-to-one, so $\alpha(T) < \infty$, and hence $\mathrm{ind}(T) = -\infty$. In the other hand, since $\mathsf{H}_1 \subset N(T+F)$, then $\alpha(T+F) = +\infty$. Therefore, $\mathrm{ind}(T+F) = 0$ or $\mathrm{ind}(T+F) = +\infty$, which is a contradiction.

• Case 2 : $\mathrm{ind}(F) = 0$. In this case three sub-cases can hold :

◦ If $\alpha(F) = \alpha(F^*) < \infty$, then, as in Case 1, we can find $T \in \mathcal{B}(\mathsf{H})$ such that $\mathrm{ind}(T) = -\infty$ and $\mathrm{ind}(T+F) = 0$ or $\mathrm{ind}(T+F) = +\infty$, which is a contradiction.

∘ If $\alpha(F) = \alpha(F^*) = +\infty$ and $\operatorname{rank}(F) = +\infty$. Let H_3, H_4 and H_5 be three closed infinite-dimensional subspaces of $\overline{R(F)}$ such that $\overline{R(F)} = \mathsf{H}_3 \oplus \mathsf{H}_4 \oplus \mathsf{H}_5$. Denote by $\mathsf{H}_1 = F^{-1}(\mathsf{H}_3) \cap N(F)^{\perp}$, and let H_2 be the closed subspace of $N(F)^{\perp}$ such that $N(F)^{\perp} = \mathsf{H}_1 \oplus \mathsf{H}_2$. Note that, as in Case 1, H_1 and H_2 must be of infinite dimensional. Now, consider the operator $T \in \mathcal{B}(\mathsf{H})$ defined as follow :

$$\begin{cases} T_{|\mathsf{H}_1} : \mathsf{H}_1 \longrightarrow \mathsf{H}_3,\ T_{|\mathsf{H}_1} = -F_{|\mathsf{H}_1}, \\ T_{|\mathsf{H}_2} : \mathsf{H}_2 \longrightarrow \mathsf{H}_4 \text{ be an arbitrary invertible operator,} \\ T_{|N(F)} : N(F) \longrightarrow N(F^*) \text{ be an arbitrary invertible operator.} \end{cases}$$

We get $\operatorname{ind}(T) = -\infty$ and $\operatorname{ind}(T+F) = 0$ or $\operatorname{ind}(T+F) = +\infty$, which is a contradiction.

∘ If $\alpha(F) = \alpha(F^*) = +\infty$ and $\operatorname{rank}(F)$ is finite. In this sub-case, let $p = \operatorname{rank}(F)$. Then, $p \geq 2$ and $N(F)^{\perp} = \mathsf{Vect}(x_1, \ldots, x_p)$, for some linearly independent vectors $x_1, \ldots, x_p$. For $i = 1, \ldots, p$, let $y_i = Fx_i$. Let $S : N(F) \longrightarrow N(F^*)$ be an one-to-one linear map satisfying $\gamma(S) = 0$ and $\overline{R(S)} = N(F^*)$. Let z_1 and z_2 be two linearly independent vectors in $N(F^*) \backslash R(S)$ and let $T \in \mathcal{B}(\mathsf{H})$ be the one-to-one operator defined as follow :

$$\begin{cases} T_{|N(F)} = S, \\ Tx_k = y_k,\ 3 \leq k \leq p, \\ Tx_1 = z_1,\ Tx_2 = z_2. \end{cases}$$

It follows that

$$\begin{aligned} \overline{R(T)} &= \overline{T(N(F)) + \mathsf{Vect}(Tx_1) + \cdots + \mathsf{Vect}(Tx_p)} \\ &= \overline{N(F^*) + \mathsf{Vect}(y_3, \ldots, y_p)} \\ &= N(F^*) + \mathsf{Vect}(y_3, \ldots, y_p). \end{aligned}$$

Hence, $\mathsf{Vect}(y_1, y_2) \oplus \overline{R(T)} = \mathsf{H}$ and therefore $\operatorname{ind}(T) = -2$. On the other hand, it is clear that $T + F$ is one-to-one and $y_i \in R(T + F)$, for $i = 3, \ldots, p$. Furthermore, for $i = 1, 2$, we have $y_i = (T + F)x_i - z_i \in \overline{R(T + F)}$ and we have also $N(F^*) = \overline{S(N(F))} = \overline{T(N(F))} = \overline{(T + F)(N(F))} \subset \overline{R(T + F)}$. Hence, $\alpha(T^* + F^*) = 0$. It follows that $\operatorname{ind}(T + F) = 0 \notin \{\operatorname{ind}(T)\} \cup \{\operatorname{ind}(T) - 1\} \cup \{\operatorname{ind}(T) + 1\}$, which is a contradiction. So $\operatorname{rank}(F) = 1$. This completes the proof. □

3. Linear maps preserving the index of operators

An additive surjective map $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ is said to preserve the index of operators if it satisfies:

$$\operatorname{ind}(\phi(T)) = \operatorname{ind}(T), \quad \forall\, T \in \mathcal{B}(\mathsf{H}).$$

An additive surjective map $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ is said to preserve a given subset $\mathcal{A}$ of $\mathcal{B}(\mathsf{H})$ in both directions, if it satisfies the following equivalence:

$$(T \in \mathcal{A} \Longleftrightarrow \phi(T) \in \mathcal{A}), \quad \forall\, T \in \mathcal{B}(\mathsf{H}).$$

Lemma 3.1. *Let $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ be an additive surjective map preserving the index of operators. Then*

(1) *ϕ is injective.*
(2) *ϕ preserves the set of rank one operators in both directions.*

Proof. 1. Let $F \in N(\phi)$, then for all $T \in \mathcal{B}(\mathsf{H})$,

$$\operatorname{ind}(T + F) = \operatorname{ind}(\phi(T + F)) = \operatorname{ind}(\phi(T)) = \operatorname{ind}(T).$$

By Lemma 2.2, it follows that F is of rank less then one. Assume, to the contrary, that F is not zero and let $x_0 \in \mathsf{H}$ such that $N(F)^{\perp} = \mathsf{Vect}(x_0)$, then $R(F) = \mathsf{Vect}(Fx_0)$.

Let $S : N(F) \longrightarrow N(F^*)$ be one-to-one satisfying $\gamma(S) = 0$ and $\overline{R(S)} = N(F^*)$. Let $z_0 \in N(F^*) \backslash R(S)$ and let $L \in \mathcal{B}(\mathsf{H})$ be the operator defined as follow : $L(x_0) = z_0$ and

$L_{|N(F)} = S$. Then, L is one-to-one, $N(F^*) = \overline{R(S)} \subset \overline{R(L)}$ and $R(L) = \mathsf{Vect}(z_0) + R(S) \subset \overline{R(S)} = N(F^*)$. So $\overline{R(L)} = N(F^*) = R(F)^\perp$, hence $\mathrm{ind}(L) = -1$.

On the other hand, $F + L$ is one-to-one and $R(S) \subset R(F + L)$, so $N(F^*) \subset \overline{R(L + F)}$. Furthermore $Fx_0 = (F + L)x_0 - z_0$, then $Fx_0 \in \overline{R(L + F)}$ and finally $\mathsf{H} \subset \overline{R(L + F)}$. Therefore, $\mathrm{ind}(L + F) = 0 \neq \mathrm{ind}(L)$ which is a contradiction. So $F = 0$ and consequently ϕ is injective.

2. Let $F \in \mathcal{F}_1(\mathsf{H})$. Since ϕ is injective, then $S = \phi(F)$ is not zero. Choose any T in $\mathcal{B}(\mathsf{H})$, as ϕ is surjective there exists $T_1 \in \mathcal{B}(\mathsf{H})$ such that $\phi(T_1) = T$. It follows that

$$\mathrm{ind}(S + T) = \mathrm{ind}(\phi(F + T_1)) = \mathrm{ind}(F + T_1) \in \{\mathrm{ind}(T_1)\} \cup \mathrm{ind}(T_1) + 1\} \cup \{\mathrm{ind}(T_1) - 1\}.$$

But, $\mathrm{ind}(T) = \mathrm{ind}(T_1)$, so $\mathrm{ind}(S+T) \in \{\mathrm{ind}(T)\} \cup \{\mathrm{ind}(T)+1\} \cup \{\mathrm{ind}(T)-1\}$. Therefore, by Lemma 2.2, F must be of rank one. Finally, as ϕ is bijective and ϕ^{-1} preserves also the index of operators, then for every $F \in \mathcal{B}(\mathsf{H})$, $\mathrm{rank}(F) = 1$ if and only if $\mathrm{rank}(\phi(F)) = 1$. This completes the proof. □

Our next purpose is to show that ϕ preserves the set of compact operators. We need several preliminary observations. Let us consider the set

$$\Omega_0(\mathsf{H}) = \big\{T \in \Phi(\mathsf{H}) \ : \ \mathrm{ind}(T) \in \mathbb{Z}\backslash\{0\}\big\}.$$

Lemma 3.2. *Let $K \in \mathcal{B}(\mathsf{H})$. The following assertions are equivalent:*

(1) $K \in \mathcal{K}(\mathsf{H})$.
(2) $T + K \in \Omega_0(\mathsf{H})$, *for all* $T \in \Omega_0(\mathsf{H})$.
(3) $\mathrm{ind}(T + K) \in \mathbb{Z}\backslash\{0\}$, *for all* $T \in \Omega_0(\mathsf{H})$.

Proof. "(1) $\Leftrightarrow$ (2)": Let $P(\Omega_0(\mathsf{H})) = \{A \in \mathcal{B}(\mathsf{H}) \ : \ T + A \in \Omega_0(\mathsf{H}), \forall T \in \Omega_0(\mathsf{H})\}$. By [8, Theorem 17, p. 161], $\Omega_0(\mathsf{H})$ is an open subset of $\mathcal{B}(\mathsf{H})$ and for all $T \in G(\mathsf{H})$: the set of all invertible operators on H, we have $TL \in \Omega_0(\mathsf{H})$ and $LT \in \Omega_0(\mathsf{H})$, for all $L \in \Omega_0(\mathsf{H})$. It follows, by [5, Theorem 2.4], that $\Omega_0(\mathsf{H})$ is a closed two sided ideal of $\mathcal{B}(\mathsf{H})$. Finally, since H is a separable Hilbert space and $P(\Omega_0(\mathsf{H}))$ contains $\mathcal{K}(\mathsf{H})$(see [8, Theorem 16, p. 161]), then $P(\Omega_0(\mathsf{H})) = \mathcal{K}(\mathsf{H})$.

"(1) $\Leftrightarrow$ (3)": If $K \in \mathcal{K}(\mathsf{H})$. Then, for all T in $\Omega_0(\mathsf{H}), T + K \in \Omega_0(\mathsf{H})$ and in particular $\mathrm{ind}(T + K) \in \mathbb{Z}\backslash\{0\}$.

Conversely, suppose that for every T in $\Omega_0(\mathsf{H})$, we have $\mathrm{ind}(T+K) \in \mathbb{Z}\backslash\{0\}$. Assume, to the contrary, that $K \notin \mathcal{K}(\mathsf{H})$. Since, the assumptions (1) and (2) are equivalent, it follows that there exists $T \in \Omega_0(\mathsf{H})$ such that $T + K \notin \Omega_0(\mathsf{H})$. But, by hypothesis, $\mathrm{ind}(T + K) \in \mathbb{Z}\backslash\{0\}$, so $T + K$ is not Fredholm. Hence, $T + K$ is not upper semi-Fredholm or $T + K$ is not lower semi-Fredholm. Therefore, by [8, Theorem 19, p. 162], there exists $K_1 \in \mathcal{K}(\mathsf{H})$ such that $\alpha(T + K + K_1) = +\infty$ or $\alpha(T^* + K^* + K_1^*) = +\infty$. Hence, $\mathrm{ind}(T + K + K_1) \in \{+\infty, 0, -\infty\}$. But, $T_1 = T + K_1$ is Fredholm and $\mathrm{ind}(T_1) = \mathrm{ind}(T) \neq 0$. So, $T_1 \in \Omega_0(\mathsf{H})$ and $\mathrm{ind}(T_1+K) \in \mathbb{Z}\backslash\{0\}$, which is a contradiction. Therefore, $K \in \mathcal{K}(\mathsf{H})$. The proof is completed. □

Proposition 3.1. *Let $\phi : \mathcal{B}(\mathsf{H}) \longrightarrow \mathcal{B}(\mathsf{H})$ be an additive surjective map preserving the index of operators.Then*

(1) *ϕ preserves $\Omega_0(\mathsf{H})$ in both directions.*
(2) *ϕ preserves $\mathcal{K}(\mathsf{H})$ in both directions.*
(3) *ϕ preserves Fredholm operators in both directions.*
(4) *ϕ preserves upper semi-Fredholm operators in both directions.*
(5) *ϕ preserves lower semi-Fredholm operators in both directions.*

Proof. Since ϕ is bijective and ϕ^{-1} preserves also the index of operators, it is then sufficient to prove that ϕ preserves the sets cited in the preceding theorem in one direction.

1. Let $T \in \Omega_0(\mathsf{H})$. Then T is Fredholm and $\operatorname{ind}(T) \neq 0$. Let $S = \phi(T)$, then $\operatorname{ind}(S) = \operatorname{ind}(T) \in \mathbb{Z}\backslash\{0\}$, so $\alpha(S) \neq \alpha(S^*)$ and both $\alpha(S)$ and $\alpha(S^*)$ are finite. Assume, to the contrary, that S is not Fredholm, then $\gamma(S) = \gamma(S^*) = 0$. Here we discus two cases :

Suppose first that $\alpha(S^*) \neq 0$. Then $\overline{R(S)} \neq \mathsf{H}$. Let $b \in \mathsf{H}\backslash\overline{R(S)}$, then $b = b_0 + b_1$, where $b_0 \in N(S^*)$ and $b_1 \in N(S^*)^\perp = \overline{R(S)}$. Note that b_0 must not be zero. Choose $a \in N(S)^\perp\backslash R(S^*) = \overline{R(S^*)}\backslash R(S^*)$ and consider the rank one operator $F =< \cdot, a > b$. Then, $N(S+F) = N(S) \cap \mathsf{Vect}(a)^\perp = N(S)$ and $N(S^* + F^*) = N(S^*) \cap \mathsf{Vect}(b)^\perp = G$, where G is the subspace of $N(S^*)$ satisfying $N(S^*) = G \oplus \mathsf{Vect}(b_0)$. It follows that $\alpha(S+F) = \alpha(S)$ and $\alpha(S^*+F^*) = \alpha(S^*)-1$. Hence, $\operatorname{ind}(S+F) = \operatorname{ind}(S)+1$. But, since ϕ^{-1} preserves the rank one operators, $\operatorname{ind}(S+F) = \operatorname{ind}(T+\phi^{-1}(F)) = \operatorname{ind}(T) = \operatorname{ind}(S)$, and thus we will end up with a contradiction.

Now, suppose that $\alpha(S) \neq 0$. Similarly, we can find $F \in \mathcal{F}_1(\mathsf{H})$ such that $\operatorname{ind}(S+F) \neq \operatorname{ind}(S)$ which is contradiction. Thus $S \in \Omega_0(\mathsf{H})$.

2. Let $K \in \mathcal{K}(\mathsf{H})$ and let $T \in \Omega_0(\mathsf{H})$, as ϕ is surjective and preserves $\Omega_0(\mathsf{H})$ in both directions, then there exists $T_0 \in \Omega_0(\mathsf{H})$ such that $\phi(T_0) = T$. It follows that $\operatorname{ind}(\phi(K)+T) = \operatorname{ind}(K+T_0) = \operatorname{ind}(T_0) \in \mathbb{Z}\backslash\{0\}$. By Lemma 3.2, we get $\phi(K) \in \mathcal{K}(\mathsf{H})$.

3. In Lemma 3.2, We have shown that ϕ preserves $\Omega_0(\mathsf{H})$ i.e ϕ preserves Fredholm operators of non zero index. Now, let $T \in \mathcal{B}(\mathsf{H})$ be a Fredholm operator of index zero and let $S = \phi(T)$. Then $\operatorname{ind}(S) = \operatorname{ind}(T) = 0$ and so $\alpha(S) = \alpha(S^*)$. Assume, to the contrary, that S is not Fredholm. Then, three cases can hold :

• Case 1 : $\alpha(S) = \alpha(S^*) = 0$ and $\gamma(S) = 0$. Choose a vector z in $\overline{R(S)}\backslash R(S) = \mathsf{H}\backslash R(S)$ and let $F =< \cdot, a > z$, where $a = -\dfrac{S^*z}{\|z\|^2}$. Then

$$N(S+F) = N(S) \cap \mathsf{Vect}(a)^\perp \subset N(S) = \{0\},$$

thus $\alpha(S+F) = 0$.

On the other hand, since S^* is one-to-one, it follows that

$$N(S^*+F^*) = \{x \in \mathsf{H} \,:\, S^*x = -< x, z > a\} = \{x \in \mathsf{H} \,:\, x =< x, z > \frac{z}{\|z\|^2}\} = \mathsf{Vect}(z).$$

Hence, $\alpha(S^*+F^*) = 1$ and $\operatorname{ind}(S+F) = -1 \neq \operatorname{ind}(S)$. But, as T is Fredholm operator of index zero and ϕ^{-1} preserves rank one operators, then

$$\operatorname{ind}(S+F) = \operatorname{ind}(T+\phi^{-1}(F)) = \operatorname{ind}(T) = \operatorname{ind}(S),$$

this is a contradiction.

• Case 2 : $\alpha(S) = \alpha(S^*) = +\infty$.

Let H_1 and H_2 be two closed infinite-dimensional subspaces of $N(S)$ such that $N(S) = \mathsf{H}_1 \oplus \mathsf{H}_2$, and let $K_1 : \mathsf{H}_1 \longrightarrow N(S^*)$ be an one-to-one compact operator such that $\overline{R(K_1)} = N(S^*)$. Consider the operator $K \in \mathcal{K}(\mathsf{H})$ defined as follow :

$$\begin{cases} K_{|\mathsf{H}_1} = K_1, \\ K_{|\mathsf{H}_2} = 0, \\ K_{|N(S)^\perp} = 0. \end{cases}$$

Obviously $(K+S)(\mathsf{H}_2) = 0$, then $\alpha(K+S) = +\infty$. On the other hand

$$\overline{(K+S)(\mathsf{H}_1)} = \overline{\mathcal{K}(\mathsf{H})_1)} = \overline{K_1(\mathsf{H}_1)} = N(S^*)$$

and

$$\overline{(K+S)N(S)^\perp} = \overline{S(N(S)^\perp)} = \overline{R(S)}.$$

Therefore $\alpha(K^*+S^*) = 0$, so $\operatorname{ind}(S+K) = +\infty$. But, as T is Fredholm operator of index zero and ϕ^{-1} preserves compact operators, then $\operatorname{ind}(S+K) = \operatorname{ind}(T+\phi^{-1}(K)) = 0$, this is a contradiction.

• Case 3 : $\alpha(S) = \alpha(S^*) = p \geq 1$ and $\gamma(S) = 0$. Then $\gamma(S^*) = 0$ and, just like in the proof of the assertion 1. we can find $F \in \mathcal{F}_1(\mathsf{H})$ such that $\text{ind}(S + F) \neq \text{ind}(S)$. But, $\text{ind}(S + F) = \text{ind}(T + \phi^{-1}(F)) = \text{ind}(T) = \text{ind}(S)$, and thus we will end up with a contradiction. Then S is Fredholm and so, ϕ preserves Fredholm operators.

4. Let T be an upper semi-Fredholm operator and let $S = \phi(T)$. Then, if T is Fredholm then S is Fredholm and so upper semi-Fredholm. Else, suppose that T is not lower semi-Fredholm, then $\alpha(T)$ is finite, $\gamma(T) > 0$ and $\alpha(T^*) = +\infty$, so $\text{ind}(T) = -\infty$. Assume, to the contrary, that S is not upper semi-Fredholm, by [8, Theorem 18, p. 161]), there exists $K \in \mathcal{K}(\mathsf{H})$ such that $\alpha(S+K) = +\infty$. This implies $\text{ind}(S+K) = +\infty$ or $\text{ind}(S + K) = 0$. But, since ϕ preserves compact operators in both directions and the index of semi-Fredholm operators is invariant under compact perturbations (see [8, Theorem 16, p. 161]), then $\text{ind}(S + K) = \text{ind}(T + \phi^{-1}(K)) = \text{ind}(T) = -\infty$, this is a contradiction. Hence, ϕ preserves upper semi-Fredholm operators.

5. Similarly to the proof of the assertion 4. we can show that ϕ preserves lower semi-Fredholm operators. □

Proof of Theorem 1.1. In Proposition 3.1, we have proved that ϕ preserves Fredholm operators in both directions. In particular, $\phi(I)$ is a Fredholm operator. We have also shown that ϕ preserves upper semi-Fredholm operators in both directions. Hence ϕ satisfies the hypothesis of Theorem 1.2 which gives the desired result. □

Remark. In this work, we remark that the generalized notion of index on $\mathcal{B}(\mathsf{H})$ is not invariant under compact perturbations. This pushed us to ask whether a linear map on $\mathcal{B}(\mathsf{H})$ preserving index of operators is an automorphism or not. But, the method that we have used failed to give the desired result.

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