Methods of Functional Analysis and Topology
Vol. 23 (2017), no. 3, pp. 285–300

ON THE GRAPH $K_{1,n}$ RELATED CONFIGURATIONS OF SUBSPACES OF A HILBERT SPACE

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Abstract. We study systems of subspaces $H_1, \dots, H_N$ of a complex Hilbert space H that satisfy the following conditions: for every index $k > 1$, the set $\{\theta_{k,1}, \dots, \theta_{k,m_k}\}$ of angles $\theta_{k,i} \in (0, \pi/2)$ between H_1 and H_k is fixed; all other pairs H_k, H_j are orthogonal. The main tool in the study is a construction of a system of subspaces of a Hilbert space on the basis of its Gram operator (the G-construction).

1. Introduction

1.1. **Systems of subspaces.** A study of systems $L = (V; V_1, \dots, V_n)$ of n subspaces $V_1, \dots, V_n$ of a linear space V, $n \in \mathbb{N}$, in particular, a description of indecomposable quadruples of subspaces in V up to equivalence [1], a description of indecomposable representations in the space V of finite posets (see, for example, [6]), etc., are classical problems of algebra (see bibliography in [8]).

Let H be a complex Hilbert space and H_k, $1 \leqslant k \leqslant n$, a collection of its subspaces. An important problem of functional analysis is to study systems of subspaces

$$S = (H; H_1, \dots, H_n),$$

of the Hilbert space H or, which is the same thing, collections of the corresponding orthogonal projections $P_1, \dots, P_n$. This problem was studied in numerous publications, see, for example [8], [9] and the bibliography therein.

A description of all irreducible n-tuples of subspaces $S = (H; H_1, \dots, H_n)$ is well-known for $n \leqslant 2$. If $n = 1$, then any irreducible system S is unitarily equivalent to one of the systems $S_0 = (\mathbb{C}; 0)$ and $S_1 = (\mathbb{C}; \mathbb{C})$; if $n = 2$, then a list of pairs of irreducible systems of subspaces, up to unitary equivalence, is the following:

(1) $H = \mathbb{C}^1$,

$$S_{00} = (\mathbb{C}; 0, 0), \quad S_{01} = (\mathbb{C}; 0, \mathbb{C}), \quad S_{10} = (\mathbb{C}; \mathbb{C}, 0), \quad S_{11} = (\mathbb{C}; \mathbb{C}, \mathbb{C});$$

(2) $H = \mathbb{C}^2$,

$$S_\varphi = (\mathbb{C}^2; \langle (1, 0) \rangle, \langle \cos\varphi, \sin\varphi \rangle), \quad \varphi \in (0, \frac{\pi}{2}).$$

For $n \geqslant 3$ the problem of description of irreducible n-tuples of subspaces up to unitary equivalence is a $*$-wild problem (see. [5, 7, 4]). Moreover, the problem of description of triples of subspaces $S = (H; H_1, H_2, H_3)$ such that $H_2 \perp H_3$, is $*$-wild (see [5, 7]).

Therefore, it is natural to study a class of systems of subspaces S satisfying some condition, and, if it is possible, to describe all irreducible systems S, up to unitary equivalence, of this class.

2000 *Mathematics Subject Classification.* Primary 47A67; Secondary 46C05.

Key words and phrases. System of subspaces, Hilbert space, orthogonal projections, Gram operator.

1.2. **Some classes of systems of subspaces.** The algebras

$$\begin{aligned}\mathbb{C}\langle p_1, p_2, \ldots, p_n \mid p_j^2 = p_j, \quad & j = 1, 2, \ldots, n;\\ p_i p_j p_i = \nu p_i, \quad & |i-j| = 1;\\ p_i p_j = p_j p_i, \quad & |i-j| \geqslant 2\rangle,\end{aligned}$$

$\nu \in \mathbb{C}$, were introduced in [10] by physicists H. N. V. Temperley and E. H. Lieb in connection with studying models of statistical physics. Such algebras can be considered as $*$-algebras if $\nu = \tau_0^2 \in (0,1)$ and the involution is defined by $p_j^* = p_j$, $1 \leqslant j \leqslant n$. Let π be some $*$-representation of such a $*$-algebra in a Hilbert space H, and subspaces H_i are images of the orthogonal projections $P_i = \pi(p_i)$. Then we have obtained a system of subspaces $S = (H; H_1, \ldots, H_n)$ for which the following conditions hold:

(1) "nearby" pairs of subspaces *lie at an angle* θ_0 *to each other*, $\tau_0 = \cos\theta_0$, i.e. $P_i P_{i+1} P_i = \tau_0^2 P_i$, $P_{i+1} P_i P_{i+1} = \tau_0^2 P_{i+1}$, $i = 1, \ldots, n-1$;

(2) the rest of pairs of subspaces *"commutes"*, i.e., the identities $P_i P_j = P_j P_i$ hold.

Let A_n be a graph with a set of vertices $\{1, 2, \ldots, n\}$ and the edges $\{i, i+1\}$, $1 \leqslant i \leqslant n-1$. Then conditions on the angle between subspaces corresponding to pairs of vertices i, j connected with an edge of the graph A_n are defined by the commutation relations in (1) above, whereas, for spaces corresponding to pairs of vertices not connected with an edge, we have conditions as in (2).

Let us consider a class of systems of subspaces described by a labeled graph $\mathbb{G}$ and a mapping Θ which maps each edge of the graph into some finite subset of numbers from $[0, \pi/2)$. More precisely, let $\mathbb{G}$ be a graph without multiple edges and $\{1, 2, \ldots, n\}$ be a set of vertices of the graph. Let R be a finite subset of $\mathbb{N}$, and $E = \bigsqcup_{r\in R} E_{2r+1}$ where E_s, $s = 2r+1$, is a set of s-labeled edges. A set of pairs of vertices $\{i, j\}$ that are not connected with any edge in the graph $\mathbb{G}$ will be denoted by $\overline{E}$. Let a mapping Θ defined on edges of the labeled graph $\mathbb{G}$ be such that $\Theta : E_s \ni \{i, j\} \mapsto \Theta_{i,j}$, where $\Theta_{i,j} = \Theta_{j,i}$ is a set of r numbers $\theta_{i,j;k}$ such that $0 \leqslant \theta_{i,j;1} < \cdots < \theta_{i,j;r} < \pi/2$. Define

$$f_{i,j}(x) = \prod_{k=1}^{r} (x - \tau_{i,j;k}^2), \quad \tau_{i,j;k} = \cos\theta_{i,j;k}\,.$$

By $Sys(\mathbb{G}, \Theta)$ we will denote a class of systems of subspaces S such that

(1) if $\{i, j\} \in E_s$ then the subspaces H_i, H_j lie at the angles $\theta_{i,j;k}$ to each other, i.e.,
$$f_{i,j}(P_i P_j) P_i = 0 \quad \text{and} \quad f_{i,j}(P_j P_i) P_j = 0;$$

(2) if $\{i, j\} \in \overline{E}$ then the subspaces H_i, H_j "commute", i.e., $P_i P_j = P_j P_i$.

Systems $S \in Sys(\mathbb{G}, \Theta)$ can be considered as $*$-representations of a corresponding $*$-algebra,

$$\begin{aligned}\mathcal{TL}_{\mathbb{G},\Theta} = \mathbb{C}\langle p_1, p_2, \ldots, p_n \mid p_j^2 = p_j^* = p_j,\ & j = 1, 2, \ldots, n;\\ f_{i,j}(p_i p_j) p_i = 0,\ & \{i, j\} \in E;\\ p_i p_j = p_j p_i,\ & \{i, j\} \in \overline{E}\rangle.\end{aligned}$$

Note that this $*$-algebra can be considered as an algebra without involution. In this case, the conditions $p_j^2 = p_j^* = p_j$ in the definition above should be replaced with the conditions $p_j^2 = p_j$. Then such an algebra is a *projective algebra* (see [2], section 6).

A more restricted class of systems of subspaces can be obtained if for each pair of vertices i, j non-connected with any edge, the commutation condition is replaced with the *orthogonality* condition $P_i P_j = P_j P_i = 0$. A class of such systems of subspaces will be denoted by $Sys(\mathbb{G}, \Theta, \perp)$. Such systems can be considered as $*$-representations of the related $*$-algebra $\mathcal{TL}_{\mathbb{G},\Theta,\perp}$ (that is a factor algebra of the $*$-algebra $\mathcal{TL}_{\mathbb{G},\Theta}$).

1.3. **Basic definitions.** A system of subspaces $S = (H; H_1, \dots, H_n)$ is said to be *decomposable* if there exists an orthogonal decomposition $H = H' \oplus H''$ and two systems of subspaces, $S' = (H'; H'_1, \dots, H'_n)$ and $S'' = (H''; H''_1, \dots, H''_n)$, such that $H_k = H'_k \oplus H''_k$ for $k = 1, 2, \dots, n$. A system S is said to be *indecomposable* if it is not decomposable. It is well-known that a system S is indecomposable iff S is *irreducible*, that is, the following condition is satisfied: if a bounded linear operator $A : H \to H$ commutes with P_k, $1 \leqslant k \leqslant n$, then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

Two systems of subspaces $S = (H; H_1, \dots, H_n)$ and $S' = (H'; H'_1, \dots, H'_n)$ are said to be *unitarily equivalent* if there exists a unitary operator $U : H \to H'$ such that $H'_k = U(H_k)$ for all $1 \leqslant k \leqslant n$. Clearly, this condition is equivalent to $P'_k = UP_kU^*$, i.e., $UP_k = P'_kU$.

For a system of subspaces S, the vector $(\dim H; \dim H_1, \dots, \dim H_n)$, whose components are cardinal numbers, is called *a generalized dimension* of the system S. In the sequel, to simplify the notation, if S is such that all $\dim H_k$ are equal, the generalized dimension of S is defined to be $(\dim H; \dim H_1)$.

A system of subspaces S is called *zero* if $H_i = 0$ for $1 \leqslant i \leqslant n$. Otherwise the system S is said to be nonzero.

1.4. **On complexity of description of systems of subspaces.** As we have already mentioned, the problem of describing, up to unitary equivalence, irreducible triples of subspaces, a pair of which is orthogonal, is a $*$-wild problem. This means that this problem is no less complicated than the problem of a description, up to unitary equivalence, of a pair of self-adjoint operators.

Description problems considered in the paper for some classes of subspaces will be referred to as *complex* if such a problem is no less complicated than the problem of description of triples of subspaces $(H; H_1, H_2, H_3)$ such that (i) for some $\varepsilon > 0$ the following inequality holds:

$$P_{H_1} + P_{H_2} + P_{H_3} \leqslant (1 + \varepsilon)I \tag{1}$$

and (ii) $H_2 \perp H_3$.

The hypothesis is that the task of description of such triples of subspaces is $*$-wild.

Let us note that in [9, section 3], a more complicated description problem for triples of subspaces satisfying only the condition (i) was shown to be $*$-wild.

1.5. **Formulation of the problem and main results.** Let N be some natural number, $K_{1,N}$ a star graph with N edges, and Θ some function on edges of $K_{1,N}$ such that

$$\Theta : \{0, k\} \mapsto \Theta_k = \{\theta_{k,1}, \dots, \theta_{k,m_k}\},$$

$m_k \geq 1$, $k = 1, \dots, N$. In the paper we consider systems of subspaces $Sys(K_{1,N}, \Theta, \perp)$, in other words, we study systems of subspaces $S = (H; H_0, H_1, \dots, H_N)$ such that following conditions hold.

(Ang): Condition on the angles. For any $k = 1, \dots, N$ we have $Ang(H_0, H_k) = Ang(H_k, H_0) \subset \Theta_k$, that is,

$$\prod_{i=1}^{m_k} (P_0 P_k P_0 - \tau_{k,i}^2 P_0) = \prod_{i=1}^{m_k} (P_k P_0 P_k - \tau_{k,i}^2 P_k) = 0,$$

where $\tau_{k,i} = \cos \theta_{k,i}$, $i = 1, \dots, m_k$.

(Ort): Orthogonality condition. If $i, j \geq 1$ and $i \neq j$, then H_i and H_j are orthogonal, that is, $P_i P_j = P_j P_i = 0$.

The main results of our work are the following.

(1) We describe (up to unitary equivalence) all systems of subspaces S that satisfy (Ang), (Ort) (see Section 3.1).

(2) We give a description (up to unitary equivalence) of all irreducible systems of subspaces S which satisfy (Ang), (Ort) (see Theorem 1 and Section 3.2).

2. G-construction and the Gram operator of a system of subspaces

2.1. G-construction of a system of subspaces of a Hilbert space. In this section we will recall the G-construction and some related results used for studying systems of subspaces that satisfy (Ang) and (Ort). For more details and proofs for the results, we refer the reader to [9].

Let $H_{0,k}$, $1 \leqslant k \leqslant n$, be a collection of nonzero Hilbert spaces. Define the Hilbert space $\widetilde{H} = H_{0,1} \oplus \cdots \oplus H_{0,n}$, and let $\langle \cdot, \cdot \rangle_0$ be the corresponding scalar product. Define the subspace $\widetilde{H}_k$ of $\widetilde{H}$ by

$$\widetilde{H}_k = \{(0, \ldots, 0, \underbrace{x}_{k}, 0, \ldots, 0) \mid x \in H_{0,k}\}, \quad 1 \leqslant k \leqslant n.$$

Let $B : \widetilde{H} \to \widetilde{H}$ be a bounded non-negative self-adjoint operator such that its block decomposition $B = (B_{i,j} : H_{0,j} \to H_{0,i} \mid 1 \leqslant i, j \leqslant n)$ satisfies $B_{k,k} = I_{H_{0,k}}$, $1 \leqslant k \leqslant n$.

Set $\widetilde{H}_0 = \ker(B)$. Using the operator B, we define the scalar product in the linear space $\widetilde{H}/\widetilde{H}_0$ by

$$\langle x + \widetilde{H}_0, y + \widetilde{H}_0 \rangle = \langle Bx, y \rangle_0, \quad x, y \in \widetilde{H}.$$

Clearly, this definition is correct, that is, it does not depend on representatives of the equivalence classes. Let H be the Hilbert space completion of the space $\widetilde{H}/\widetilde{H}_0$ with respect to this scalar product.

Let $H_k = \{x + \widetilde{H}_0 \mid x \in \widetilde{H}_k\}$, $1 \leqslant k \leqslant n$. Since $\|x + \widetilde{H}_0\| = \sqrt{\langle Bx, x \rangle_0} = \|x\|_0$ for arbitrary $x \in \widetilde{H}_k$, we see that H_k is a subspace of H. Since $H_1 + \ldots + H_n = \{x + \widetilde{H}_0 \mid x \in \widetilde{H}\} = \widetilde{H}/\widetilde{H}_0$, we conclude that $H_1 + \ldots + H_n$ is dense in H.

We have thus obtained a collection of subspaces, $H_1, \ldots, H_n$, of the Hilbert space H. In such a case, we will write $(H; H_1, \ldots, H_n) = \mathcal{G}(H_{0,1}, \ldots, H_{0,n}; B)$. The construction above is called the G-construction.

2.2. Basic properties of the G-construction. Let K be a Hilbert space,

$$S = (K; K_1, \ldots, K_n)$$

an n-tuple of nonzero subspaces of K. Denote by Q_i the orthogonal projection onto K_i, $1 \leqslant i \leqslant n$.

Definition 1. The operator $G = G(S) : \oplus_{i=1}^n K_i \to \oplus_{i=1}^n K_i$ defined by its block decomposition by $G_{i,j} = Q_i \restriction_{K_j} : K_j \to K_i$, $1 \leqslant i, j \leqslant n$, is called the Gram operator of the system of subspaces S.

The crucial property of the G-construction is the following.

Proposition 1. *Suppose that $K_1 + \cdots + K_n$ is dense in K. Then the system of subspaces $\mathcal{G}(K_1, \ldots, K_n; G(S))$ is unitarily equivalent to S.*

Consider the following question: when two systems of subspaces obtained by the G-construction are unitarily equivalent?

Proposition 2. *Two systems of subspaces*

$$\mathcal{G}(H_{0,1}, \ldots, H_{0,n}; B) \quad \text{and} \quad \mathcal{G}(H'_{0,1}, \ldots, H'_{0,n}; B')$$

are unitarily equivalent iff there exists a collection of unitary operators $U_{0,k} : H'_{0,k} \to H_{0,k}$, $1 \leqslant k \leqslant n$, such that

$$B'_{i,j} = U^*_{0,i} B_{i,j} U_{0,j}. \tag{2}$$

for any $1 \leqslant i, j \leqslant n$.

Next, we are going to find conditions on the operator B under which the system of subspaces $\mathcal{G}(H_{0,1},\dots,H_{0,n};B)$ is irreducible.

For a sequence of indices $l=(i_1,\dots,i_k)$, where $1\leqslant i_1,\dots,i_k\leqslant n$, $k\geqslant 2$, define the operator $B_l=B_{i_1,i_2}\dots B_{i_{k-1}i_k}$. Let $\alpha\in\{1,2,\dots,n\}$. Denote by $\mathcal{L}_\alpha$ the set of sequences $l=(i_1,\dots,i_k)$ such that $i_1=i_k=\alpha$. Note that the set of operators B_l, $l\in\mathcal{L}_\alpha$ is a $*$-set, that is, if an operator A belongs to this set, then the operator A^* also belongs to this set.

Proposition 3. *Let $\alpha\in\{1,2,\dots,n\}$ be such that for any $k=1,2,\dots,n$ there exists a sequence of indices $l=(\alpha,\dots,k)$ such that B_l is invertible. Then the following conditions are equivalent:*

(1) *the system of subspaces $\mathcal{G}(H_{0,1},\dots,H_{0,n};B)$ is irreducible;*
(2) *the set of operators B_l, $l\in\mathcal{L}_\alpha$ is irreducible.*

2.3. **Connection between properties of a system of subspaces $\mathcal{G}(H_{0,1},\dots,H_{0,n};B)$ and properties of the operator B.** Let $S=(H;H_1,\dots,H_n)=\mathcal{G}(H_{0,1},\dots,H_{0,n};B)$. Denote by P_i the orthogonal projection onto H_i, $1\leqslant i\leqslant n$. Let $G=G(S)$ be the Gram operator of the system S, $G_{i,j}=P_i\restriction_{H_j}\colon H_j\to H_i$, $1\leqslant i,j\leqslant n$.

Proposition 4. *There exists a collection of unitary operators $U_{0,k}:H_k\to H_{0,k}$, $1\leqslant k\leqslant n$, such that $G_{i,j}=U_{0,i}^*B_{i,j}U_{0,j}$ for $1\leqslant i,j\leqslant n$.*

Proof. From Proposition 1 it follows that S is unitarily equivalent to the system

$$\mathcal{G}(H_1,\dots,H_n;G).$$

Using Proposition 2, we obtain the needed assertion. □

Using Proposition 4, we will connect properties of the system of subspaces S with properties of the operator B. Let $\alpha,\beta\in\{1,2,\dots,n\}$, $\alpha\neq\beta$.

Example 1. *Orthogonality condition.* The subspaces H_α and H_β are orthogonal iff $P_\alpha P_\beta=0$, that is, $G_{\alpha,\beta}=0$. Clearly, this condition is equivalent to $B_{\alpha,\beta}=0$.

Example 2. *Condition on the set of angles between H_α and H_β.* Let $A\subset[0,\pi/2]$. Define $\cos^2(A)=\{\cos^2\varphi\mid\varphi\in A\}$. The condition $Ang(H_\alpha,H_\beta)\subset A$ is equivalent to $\sigma(P_\alpha P_\beta P_\alpha\restriction_{H_\alpha})\subset\cos^2(A)$. Since

$$P_\alpha P_\beta P_\alpha\restriction_{H_\alpha}=G_{\alpha,\beta}G_{\beta,\alpha}=U_{0,\alpha}^*B_{\alpha,\beta}B_{\beta,\alpha}U_{0,\alpha},$$

we conclude that $Ang(H_\alpha,H_\beta)\subset A$ iff $\sigma(B_{\alpha,\beta}B_{\beta,\alpha})\subset\cos^2(A)$.

3. Description of all systems which satisfy (Ang) and (Ort)

In this section we

(1) obtain a description of all systems satisfying (Ang), (Ort);
(2) obtain a description of all *irreducible unitarily nonequivalent* systems satisfying (Ang), (Ort);
(3) will give, as an example, a description of all irreducible unitarily nonequivalent systems satisfying (Ang), (Ort) in the case $N=2$ and $m_1=2$, $m_2=3$.

3.1. **Description of all systems of subspaces $S=(H;H_0,H_1,\dots,H_N)$ which satisfy (Ang) and (Ort). 1.** A zero system $S=(H;0,\dots,0)$ satisfies all conditions. So we will consider only non-zero systems. Note that all operators B_{0k} are invertible so if a system $S=(H;H_0,\dots,H_N)$ satisfies conditions (Ang) and for some index k we have $H_k=0$, then $H_0=\dots=H_N=0$. Thus if a system of subspaces is non-zero then $H_k\neq 0$, $0\leqslant k\leqslant N$.

2. Let $S = (H; H_0, \dots, H_N)$ be a non-zero system of subspaces which satisfies conditions (Ang) and (Ort). Suppose the sum $H_0 + \dots + H_N$ is not dense in the space H. Define the systems

$$S' = (H'; H_0, \dots, H_N) \quad \text{and} \quad S'' = (H \ominus H'; 0, \dots, 0),$$

where $H' = \overline{H_0 + \dots + H_N}$. Then $S = S' \oplus S''$, S' satisfies conditions (Ang), (Ort) and S'' is a zero system of subspaces. So to describe all systems of subspaces that satisfy conditions (Ang) and (Ort) it is sufficient to describe systems that satisfy the condition that the sum of subspaces $H_0 + \dots + H_N$ is dense in the space H.

3. Let us suppose for now that $S = (H; H_0, \dots, H_N)$ is a system of subspaces that satisfies conditions (Ang) and (Ort), is such that $H_k \neq 0$, $0 \leqslant k \leqslant N$, and the sum $H_0 + \dots + H_N$ is dense in the space H. Let $G = (G_{i,j},\ 1 \leqslant i, j \leqslant N+1)$ be the Gram operator of the system S. Then S is unitarily equivalent to the system $\mathcal{G}(H_0, \dots, H_N; G)$. If the identities

$$\prod_{i=1}^{m_k} (G_{k,0} G_{k,0}^* - \tau_{k,i}^2 I_k) = 0, \quad \prod_{i=1}^{m_k} (G_{k,0}^* G_{k,0} - \tau_{k,i}^2 I_0) = 0$$

hold for any $1 \leqslant k \leqslant n$, then the operators

$$U_k = G_{k,0} (G_{0,k} G_{k,0})^{-1/2} : H_0 \to H_k, \quad 0 \leqslant k \leqslant n$$

are properly defined unitary operators. Define

$$B_{j,k} = U_j^* G_{j,k} U_k : H_0 \to H_0, \quad 0 \leqslant j, k \leqslant n,$$

then

$$B_{0,k} = G_{0,k} G_{k,0} (G_{0,k} G_{k,0})^{-1/2} = (G_{0,k} G_{k,0})^{1/2}$$

is a self-adjoint invertible operator with spectrum $\sigma(B_{0,k}) = \sigma_k = \{\tau_{k,1}, \dots, \tau_{k,m_k}\}$, so there exists a decomposition of the identity,

$$I_0 = Q_{k,1} \oplus \dots \oplus Q_{k,r_k} \quad \text{such that} \quad B_{0,k} = \sum_{i=1}^{r_k} \tau_{k,i} Q_{k,i}.$$

Define the operator $B : \oplus_{k=1}^{N+1} H_1 \to \oplus_{k=1}^{N+1} H_1$ by its block decomposition $B = (B_{i,j})$. The proposition 2 claims that the system of subspaces $\mathcal{G}(H_0, \dots, H_n; G)$ is unitarily equivalent to $\mathcal{G}(H_0, \dots, H_0; B)$ so S is unitarily equivalent to $\mathcal{G}(H_0, \dots, H_0; B)$ as well.

Let now $S = (H; H_0, \dots, H_N) = \mathcal{G}(H_0, \dots, H_0; B)$ for some Hilbert space H_0 and some operator $B : \oplus_{k=1}^{N+1} H_0 \to \oplus_{k=1}^{N+1} H_0$ such that $B_{0,k}$ is equal to

$$B_k = \sum_{i=1}^{r_k} \tau_{k,i} Q_{k,i}, \quad I_0 = \bigoplus_{i=1}^{r_k} Q_{k,i}, \quad 1 \leqslant k \leqslant N. \tag{3}$$

Let us find conditions on B that are required for system of subspaces to satisfy conditions (Ang) and (Ort).

Conditions (Ang) are equivalent to the conditions

$$\prod_{i=1}^{r_k} (B_k^2 - \tau_{k,i}^2 I_0) = 0,$$

that hold as far as

$$\prod_{i=1}^{r_k} (B_k^2 - \tau_{k,i}^2 I_0) = \prod_{i=1}^{r_k} \Big(\sum_{j=1}^{r_k} (\tau_{k,j}^2 - \tau_{k,i}^2) Q_{k,i} \Big) = \sum_{j=1}^{r_k} \Big(\prod_{i=1}^{r_k} (\tau_{k,j}^2 - \tau_{k,i}^2) \Big) Q_{k,i} = 0.$$

Conditions (Ort) just mean that $B_{i,j} = 0$ for any $i \neq j$, $1 \leqslant i, j \leqslant N$. Thus the operator B has the form

$$B = \begin{pmatrix} I & B_1 & B_2 & \dots & B_N \\ B_1 & I & 0 & \dots & 0 \\ B_2 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_N & 0 & 0 & \dots & I \end{pmatrix}, \tag{4}$$

where $B_k = \sum_{i=1}^{r_k} \tau_{k,i} Q_{k,i}$, for some set of projections $Q_{k,j}$, $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant r_k$, such that $Q_{k,j_1} Q_{k,j_2} = 0$ for any k and $j_1 \neq j_2$. In the following this operator will be denoted by $B(\{Q_{k,j}\})$.

To apply the G-construction, the operator $B(\{Q_{k,j}\})$ is required to be non-negative. The following lemmas will be useful for studying the question when such an operator is non-negative.

Lemma 1. *Let K be a Hilbert space, $A_1, \dots, A_n$ non-negative invertible operators on K. Let $y \in K$ and $\mu_k > 0$, $1 \leqslant k \leqslant n$. If $u_k \in K$, $1 \leqslant k \leqslant n$, and $\sum_{k=1}^{n} \mu_k u_k = y$, then*

$$\sum_{k=1}^{n} \langle A_k u_k, u_k \rangle \geqslant \left\langle \left(\sum_{j=1}^{n} \mu_j^2 A_j^{-1} \right)^{-1} y, y \right\rangle,$$

with the equality taking place iff

$$u_k = \mu_k A_k^{-1} (\sum_{j=1}^{n} \mu_j^2 A_j^{-1})^{-1} y, \quad 1 \leqslant k \leqslant n.$$

See a proof of this lemma in [9].

Proposition 5. *Let the operator B be as in (4) for some self-adjoint positive operators B_k, $1 \leqslant k \leqslant N$. Then B is non-negative if and only if*

$$\sum_{k=1}^{N} B_k^2 \leqslant I_0. \tag{5}$$

Proof. The operator B is non-negative by definition iff

$$\langle Bx, x \rangle \geqslant 0, \quad x = (z, x_1, \dots, x_N) \in \oplus_{k=0}^{N} H_0.$$

Denote

$$z_0 = z_0(x_1, \dots, x_m) = \sum_{k=1}^{N} B_k x_k, \quad B_0 = B_0(x_1, \dots, x_m) = \sum_{k=1}^{N} \|x_k\|^2$$

then we will get the condition

$$\langle Bx, x \rangle = \|z\|^2 + 2\mathrm{Re} \langle z, z_0 \rangle + B_0 \geqslant 0.$$

This condition is equivalent to the condition

$$\|z + z_0\|^2 - \|z_0\|^2 + B_0 \geqslant 0.$$

The left-hand side of this inequality reaches a minimal value as z varies if $z = -z_0$. Thus the operator B is non-negative if and only if

$$B_0 \geqslant \|z_0\|^2.$$

Denoting $y_k = B_k x_k$ we can rewrite this inequality in following form:

$$\sum_{k=1}^{N} \langle B_k^{-2} y_k, y_k \rangle \geqslant \| \sum_{k=1}^{N} y_k \|^2.$$

Let $y = \sum_{k=1}^N y_k$. Then by lemma 1 the minimum of the left-hand side is equal to

$$\Big\langle \big(\sum_{k=1}^N B_k^2\big)^{-1} y, y \Big\rangle,$$

so the operator B is non-negative if and only if

$$\big(\sum_{k=1}^N B_k^2\big)^{-1} \geqslant I_0.$$

The last inequality is equivalent to inequality (5). □

The following proposition provides a description of $\operatorname{Ker} B$. It can be obtained from proofs of previous lemmas.

Proposition 6. *Let inequality* (5) *hold. A vector*

$$x = (z, x_1, \dots, x_N)$$

belongs to $\operatorname{Ker} B$ *if and only if*

(1) $z = -\sum_{k=1}^N B_k x_k$;
(2) $x_k = B_k y$, $1 \leqslant k \leqslant N$, *where* $y \in \operatorname{Ker}(I - \sum_{k=1}^N B_k^2)$.

Corollary 1. *Let inequality* (5) *hold and* H_0 *be finite dimensional. Then*

$$\dim \operatorname{Ker} B = \dim \operatorname{Ker}(I - \sum_{k=1}^N B_k^2). \tag{6}$$

A criterion for unitary equivalence of systems of subspaces (Proposition 2) in the considered case can be formulated in terms operators B_k.

Proposition 7. *Systems of subspaces* $\mathcal{G}(H_0, \dots, H_0; B)$ *and* $\mathcal{G}(H_0', \dots, H_0'; B')$ *are unitarily equivalent if and only if the sets of operators* $\{B_k\}$ *and* $\{B_k'\}$ *are unitarily equivalent.*

In the rest of this section we will consider the case where $B = B(\{Q_{k,j}\})$. Note that the system of operators generated by the set of operators $\{Q_{k,j}\}$, $Q_{k,j_1} Q_{k,j_2} = 0$, is the same as the one generated by the set $\{B_k = \sum_{j=1}^{r_k} \tau_{k,j} Q_{k,j}\}$. To prove this, it is sufficient to see that

$$Q_{k,j} = q_{k,j}(B_k), \quad q_{k,j}(x) = \prod_{i \neq j} \frac{x - \tau_{k,i}}{\tau_{k,j} - \tau_{k,i}}.$$

Proposition 5 in the considered case can be formulated more precisely.

Proposition 8. *Let*

$$\xi(\tau) = 1 - \sum_{k=1}^n \tau_{k,1}^2$$

and

$$M = \bigcup_{k=1}^N M_k, \quad M_k = \{(k,j) \,|\, 2 \leqslant j \leqslant r_k\}, \quad k = 1 \leqslant k \leqslant N.$$

The operator $B = B(\{Q_{k,j}\})$ *is non-negative if and only if*

$$\sum_{(k,j) \in M} \delta_{k,j} Q_{k,j} \leqslant \xi(\tau) I_0, \tag{7}$$

where $\delta_{k,j} = \tau_{k,j}^2 - \tau_{k,1}^2$, $(k,j) \in M$.

Proof. Substituting $B_k = \sum_{j=1}^{r_k} \tau_{k,j} Q_{k,j}$ into inequality (5) we will get

$$\sum_{k=1}^{N} \sum_{j=1}^{r_k} \tau_{k,j}^2 Q_{k,j} \leqslant I_0.$$

Decreasing both side of this inequality by $\sum_{k=1}^{N} \tau_{k,1}^2 I_0$ we will get (7) □

So considering all the sets of operators $\{Q_{k,j}\}$ that are different up to unitarily equivalence and satisfy inequality (7), and applying the G-construction we will obtain all non-equivalent non-zero systems of subspaces S satisfying conditions (Ang) and (Ort) such that $H_0 + \cdots + H_N$ is dense in the Hilbert space H.

Note that condition (7) requires that $\xi(\tau) \geqslant 0$ if $\delta_{k,i} > 0$. So if $\xi(\tau) < 0$ then there is a no non-zero system of subspaces S that satisfy (Ang) and (Ort). In the following we will suppose that $\xi(\tau) \geqslant 0$.

3.2. **Description of all irreducible unitarily nonequivalent systems S satisfying (Ang), (Ort).** Before providing a description of all irreducible systems of subspaces let us note that

- up to unitarily equivalence there exists the unique zero irreducible system of subspaces $S = (\mathbb{C}^1; 0, \dots, 0)$;
- for any non-zero irreducible system of subspaces $S = (H; H_1, \dots, H_{N+1})$, the sum $H_1 + \cdots + H_{N+1}$ is dense in H.

Moreover a criterion for irreducibility (proposition 3) of systems of subspaces in terms of operators B_k can be formulated in the following form.

Proposition 9. *Let the operator B be as in* (4) *for some self-adjoint positive operators B_k, $1 \leqslant k \leqslant N$. Then the system of subspaces $\mathcal{G}(H_0, \dots, H_0; B)$ is irreducible if and only if the set of operators $\{B_k\}$ is irreducible.*

So up to unitary equivalence, all non-zero irreducible systems of subspaces satisfying (Ang) and (Ort) are $S = \mathcal{G}(H_0, \dots, H_0; B(\{Q_{k,j}\}))$, where H_0 is a Hilbert space and $Q_{k,j}$ ($Q_{k,j_1} Q_{k,j_2} = 0$, $j_1 \neq j_2$, $1 \leqslant k \leqslant N$, $1 \leqslant j, j_1, j_2 \leqslant r_k$) is an irreducible family of orthogonal projections in the Hilbert space H_0 such that inequality (7) holds.

Since systems of subspaces S and S' are unitarily equivalent if and only if the families of orthogonal operators $\{Q_{k,j}\}$ and $\{Q'_{k,j}\}$ are unitarily equivalent, the problem under consideration is equivalent to the problem of describing, up to unitary equivalnce, all irreducible families of orthogonal projections $\{Q_{k,j}\}$ that satisfy to inequality (7).

If $\xi(\tau) = 0$ then $Q_{k,j} = 0$, $(k, j) \in M$. So $Q_{k,1} = I_0$, $1 \leqslant k \leqslant N$ and $\sum_{k=1}^{N} B_k^2 = I_0$. Because family of operators $Q_{k,j}$ should be irreducible, $H_0 = \mathbb{C}^1$. Using formula (6) we will find that $\dim H = N$ and the generalized dimension of the system S is equal to $(N; 1)$.

Let us consider the case $\xi(\tau) > 0$.

A set of index pairs M can be split into three parts,

$$\begin{aligned} M_l &= \{(k, j) \in M \mid \delta_{k,j} < \xi(\tau)\}, \\ M_e &= \{(k, j) \in M \mid \delta_{k,j} = \xi(\tau)\}, \\ M_g &= \{(k, j) \in M \mid \delta_{k,j} > \xi(\tau)\}. \end{aligned}$$

If $(k_0, j_0) \in M_g$ then, obviously, $Q_{k_0,j_0} = 0$.

If $(k_0, j_0) \in M_e$ then, by inequality (7),

$$Q_{k_0,j_0} Q_{k,j} = 0, \quad (k, j) \in M \setminus M_{k_0}.$$

Since the family $\{Q_{k,j}\}$ is irreducible, $Q_{k_0,j_0} = 0$ or $Q_{k_0,j_0} = I$. Furthermore, if $Q_{k_0,j_0} = I$, then

$$Q_{k,j} = 0, \quad (k,j) \in M \setminus \{(k_0, j_0)\}.$$

In this case, $H_0 = \mathbb{C}^1$ and, by formula (6), we can find $\dim \operatorname{Ker} B = 1$ so $\dim H = N$ and the generalized dimension of the system S is equal to $(N; 1)$.

Thus we have found $|M_e|$ irreducible unitarily non equivalent systems of subspaces S satisfying (Ang) and (Ort) corresponding to elements $(k_0, j_0) \in M_e$. It remains to consider the case where $Q_{k_0,j_0} = 0$ for all $(k_0, j_0) \in M_e$. In this case, inequality (7) can be rewritten as

$$\sum_{(k,j)\in M_l} \delta_{k,j} Q_{k,j} \leqslant \xi(\tau) I_0. \tag{8}$$

Let us introduce index sets $M_{l,k} = M_l \cap M_k$, $1 \leqslant k \leqslant N$. Then N_l will be defined as a number of indexes k such that $M_{l,k} \neq \varnothing$ and d_l will be defined as $\max_k |M_{l,k}|$.

Without loss of generality, we assume that $|M_{l,1}| \geqslant |M_{l,2}| \geqslant \cdots \geqslant |M_{l,N}|$.

1. In the case $N_l \geqslant 3$, define a set $M_{l,0}$ to be $\{(1,2),(2,2),(3,2)\}$ and, in the case $N_l = 2$, $d_l \geqslant 2$, define it as $\{(1,2),(1,3),(2,2)\}$. Then define $\delta = \max\{\delta_{k,j} \mid (k,j) \in M_{l,0}\}$ and put $Q_{k,j} = 0$, if $(k,j) \in M_l \setminus M_{l,0}$. Thus, for $\varepsilon = \xi(\tau)/\delta - 1 > 0$ and three orthogonal projections $Q_{k,j}$, $(k,j) \in M_{l,0}$ such that

$$\sum_{(k,j)\in M_{l,0}} Q_{k,j} \leqslant (1+\varepsilon) I_0$$

the following inequalities hold:

$$\sum_{(k,j)\in M_{l,0}} \delta_{k,j} Q_{k,j} \leqslant \delta \sum_{(k,j)\in M_{l,0}} Q_{k,j} \leqslant \xi(\tau) I_0.$$

So the task of describing all irreducible representations in these cases is complex.

2. Let $N_l = 0$. Then $Q_{k,j} = 0$, $(k,j) \in M_l$. Because the family of the orthogonal projections $\{Q_{k,j}\}$ is irreducible, $H_0 = \mathbb{C}^1$. Using equality (6), we will get $\dim \operatorname{Ker} B = 0$. Then $\dim H = N+1$; $\dim H_k = 1$ for all $1 \leqslant k \leqslant N+1$. Thus, the generalized dimension of the system S is equal to $(N+1; 1)$.

3. Let $N_l = 1$. Then equality (7) will become

$$\sum_{j=2}^{d_l+1} \delta_{1,j} Q_{1,j} \leqslant \xi(\tau) I_0.$$

Because all $\delta_{1,j} < \xi(\tau)$ and $Q_{1,j_1} \perp Q_{1,j_2}$, $j_1 \neq j_2$, this inequality holds for any such a family of orthogonal projections. Furthermore, the family $Q_{1,j}$, $1 \leqslant j \leqslant d_l + 1$, is irreducible and $\sum_{j=1}^{d_l+1} Q_{1,j} = I_0$ so $H_0 = \mathbb{C}^1$ and there exist $d_l + 1$ possible cases

$$\pi_j : \quad Q_{1,j} = I_0, \quad Q_{1,i} = 0, \quad i \neq j, \quad 1 \leqslant j \leqslant d_l + 1.$$

In all these cases $\dim \operatorname{Ker} B = 0$ so the generalized dimension of the related systems is equal to $(N+1, 1)$.

4. Let $N_l = 2$ and $d_l = 1$. Then inequality (8) can be rewritten as

$$\delta_{1,2} Q_{1,2} + \delta_{2,2} Q_{2,2} \leqslant \xi(\tau) I_0. \tag{9}$$

The following description of all irreducible pairs of orthogonal projections R_1 and R_2 in Hilbert space, up to unitary equivalence (not only those that satisfy inequality (9)) is well known (see, for example [3]). All such pairs can be split into

(1) four irreducible pairs of orthogonal projections in $H_0 = \mathbb{C}^1$:

$$\pi_{00} : R_1 = 0, R_2 = 0, \quad \pi_{01} : R_1 = 0, R_2 = I,$$
$$\pi_{10} : R_1 = I, R_2 = 0, \quad \pi_{11} : R_1 = I, R_2 = I;$$

(2) a family of pairs π_φ, $\varphi \in (0, \pi/2)$, in $H_0 = \mathbb{C}^2$:

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix}. \tag{10}$$

Let us find those of them that satisfy inequality (9). Consider the following cases.

4.1. In cases $H_0 = \mathbb{C}^1$, (i) $Q_{1,2} = Q_{2,2} = 0$, (ii) $Q_{1,2} = I$ and $Q_{2,2} = 0$, or (iii) $Q_{1,2} = 0$ and $Q_{2,2} = I$, inequality (9) holds and the generalized dimension of the system S is equal to $(N + 1; 1)$ if $\dim \operatorname{Ker} B = 0$.

4.2. Let $H_0 = \mathbb{C}^1$, $Q_{1,2} = Q_{2,2} = I$. Inequality (9) holds if and only if $\xi(\tau) \geqslant \delta_{1,2} + \delta_{2,2}$. In this case, the generalized dimension of the system S is equal to

(1) $(N; 1)$ if $\xi(\tau) = \delta_{1,2} + \delta_{2,2}$ for $\dim \operatorname{Ker} B = 1$;
(2) $(N + 1; 1)$ if $\xi(\tau) > \delta_{1,2} + \delta_{2,2}$ for $\dim \operatorname{Ker} B = 0$.

4.3. Let us consider the case where $H_0 = \mathbb{C}^2$ and the operators $Q_{1,2}$, $Q_{2,2}$ are represented by formulas (10), $\varphi \in (0, \pi/2)$. Inequality (9) holds if and only if the (2×2)-matrix (operator)

$$M = \xi(\tau)I - \delta_{1,2}Q_{1,2} - \delta_{2,2}Q_{2,2} = \begin{pmatrix} \xi(\tau) - \delta_{1,2} - \delta_{2,2}\cos^2\varphi & -\delta_{2,2}\cos\varphi\sin\varphi \\ -\delta_{2,2}\cos\varphi\sin\varphi & \xi(\tau) - \delta_{2,2}\sin^2\varphi \end{pmatrix}$$

is non-negative. This means that the diagonal elements and the determinant of the matrix M are non-negative. An element $(M)_{1,1} \geqslant 0$ if and only if

$$\cos^2\varphi \leqslant \frac{\xi(\tau) - \delta_{1,2}}{\delta_{2,2}}.$$

We also have that $(M)_{2,2} > 0$ for any φ. It is easy to check that

$$\det M = (\xi(\tau) - \delta_{1,2})(\xi(\tau) - \delta_{2,2}) - \delta_{1,2}\delta_{2,2}\cos^2\varphi.$$

So the condition $\det M \geqslant 0$ can be rewritten as follows:

$$\cos^2\varphi \leqslant \frac{\xi(\tau) - \delta_{1,2}}{\delta_{2,2}} \frac{\xi(\tau) - \delta_{2,2}}{\delta_{1,2}} = \eta(\tau).$$

The following cases are possible.

4.3.1. Let $\xi(\tau) \geqslant \delta_{1,2} + \delta_{2,2}$. Then for any $\varphi \in (0, \pi/2)$ the matrix M is non-negative. Using formula (6) we will get $\dim \operatorname{Ker} B = 0$. So $\dim H = 2N + 2$, $\dim H_k = \dim H_0 = 2$ for any $1 \leqslant k \leqslant N+1$. The generalized dimension of the system S is equal to $(2N+2; 2)$.

4.4.2. Suppose that $\xi(\tau) < \delta_{1,2} + \delta_{2,2}$. Then define an angle

$$\varphi(\tau) = \arccos\sqrt{\eta(\tau)} \in (0, \pi/2).$$

The matrix M is non-negative if and only if $\varphi \in [\varphi(\tau), \pi/2)$.

If $\varphi \in (\varphi(\tau), \pi/2)$ then M is positive and the generalized dimension of the system S is equal to $(2N + 2; 2)$.

If $\varphi = \varphi(\tau)$ then $\operatorname{Ker} M$ is one-dimensional, so $\dim \operatorname{Ker} B = 1$ and the generalized dimensional of the system S is equal to $(2N + 1; 2)$.

3.3. **Classification theorem.** Let us formulate the results obtained in subsection 3.2 as a theorem.

Theorem 1. *If $\xi(\tau) < 0$ then there does not exist a non-zero system of subspaces S that satisfies conditions (Ang) and (Ort).*

In the case where $\xi(\tau) = 0$, up to unitary equivalence, there exists a unique non-zero irreducible system of subspaces S that satisfies conditions (Ang)and (Ort). Its generalized dimension is $(N; 1)$.

In the case where $\xi(\tau) > 0$, up to unitary equivalence, all non-zero irreducible systems of subspaces S that satisfy conditions (Ang) and (Ort) can be described as follows.

(1) $N_l = 0$:
   (a) *There are $|M_e|$ systems of subspaces with generalized dimension $(N; 1)$;*
   (b) *there is one system of subspaces with generalized dimension $(N + 1; 1)$.*
(2) $N_l = 1$:
   (a) *There are $|M_e|$ systems of subspaces with generalized dimension $(N; 1)$;*
   (b) *there are $|M_l|+1$ systems of subspaces with generalized dimension $(N+1; 1)$.*
(3) $N_l = 2$, $d_l = 1$, $\sum\limits_{(k,j)\in M_l} \delta_{k,j} > \xi(\tau)$:
   (a) *There are $|M_e|$ systems of subspaces with generalized dimension $(N; 1)$;*
   (b) *there are three systems of subspaces with generalized dimension $(N + 1; 1)$;*
   (c) *there as an infinite family of systems of subspaces with generalized dimension $(2N + 2; 2)$ parameterized with the angle $\varphi \in (\varphi(\tau), \pi/2)$ where $\varphi(\tau) \in (0, \pi/2)$;*
   (d) *there is one system of subspaces with generalized dimension $(2N + 1; 2)$; related to the angle $\varphi = \varphi(\tau)$.*
(4) $N_l = 2$, $d_l = 1$ *and* $\sum\limits_{(k,j)\in M_l} \delta_{k,2} = \xi(\tau)$:
   (a) *There are $|M_e|$ systems of subspaces with generalized dimension $(N; 1)$;*
   (b) *there are three systems of subspaces with generalized dimension $(N + 1; 1)$;*
   (c) *there is one system of system with generalized dimension $(N; 1)$;*
   (d) *there is an infinite family of systems of subspaces with generalized dimension $(2N + 2; 2)$ parameterized by the angle $\varphi \in (0, \pi/2)$.*
(5) $N_l = 2$, $d_l = 1$ *and* $\sum\limits_{(k,j)\in M_l} \delta_{k,j} < \xi(\tau)$:
   (a) *There are $|M_e|$ systems of subspaces with generalized dimension $(N; 1)$;*
   (b) *there are four systems of subspaces with generalized dimension $(N + 1; 1)$;*
   (c) *there is an infinite family of systems of subspaces with generalized dimension $(2N + 2; 2)$ parameterized by the angle $\varphi \in (0, \pi/2)$.*
(6) $N_l \geqslant 3$ *or* $N_l = 2$, $d_l \geqslant 2$: *The task of description of all irreducible unitary non-equivalent systems S that satisfy conditions (Ang) and (Ort) is complex.*

3.4. **Examples.** Let us consider the results provided by the theorem in some simple cases.

Example 3. Let the relations for a pairs P_0, P_k be the same,

$$(P_0P_kP_0 - \tau_1^2P_0)(P_0P_kP_0 - \tau_2^2P_0) = 0 \quad \text{and} \quad (P_kP_0P_k - \tau_1^2P_k)(P_kP_0P_k - \tau_2^2P_k) = 0,$$

where $0 < \tau_1 < \tau_2 < 1$.

In this case

a) $\xi(\tau) = 1 - N \cdot \tau_1^2$, so $\xi(\tau) \geqslant 0$ if and only if $\tau_1^2 \leqslant N^{-1}$.
b) $\delta_{k,2} = \tau_2^2 - \tau_1^2$, so $\delta_{k,2} < \xi(\tau)$ if and only if $\tau_2^2 < 1 - (N - 1) \cdot \tau_1^2$, therefore,
   - if $\tau_2^2 > 1 - (N - 1)\tau_1^2$ then $M = M_g$;
   - if $\tau_2^2 = 1 - (N - 1)\tau_1^2$ then $M = M_e$;

- if $\tau_2^2 < 1-(N-1)\tau_1^2$ then $M = M_l$.

c) In the case $N = 2$ we will get $\xi(\tau) = 1-2\tau_1^2$ and $\delta_{1,2}+\delta_{2,2} = 2(\tau_2^2-\tau_1^2)$, so $\delta_{1,2}+\delta_{2,2} \leqslant \xi(\tau)$ if and only if $\tau_2^2 \leqslant \frac{1}{2}$.

Thus theorem 1 for these cases can be reformulated as the following propositions.

Proposition 10. *Let $N = 1$. Then there exist two systems of subspaces with generalized dimension* $(2;1)$.

Proposition 11. *Let $N = 2$.*

If $\frac{1}{2} < \tau_1^2 < 1$ then there is no a non-zero system of subspaces S that satisfies conditions (Ang) and (Ort).

In the case where $\tau_1^2 = \frac{1}{2}$, up to unitary equivalence, there exists a unique non-zero irreducible system of subspaces S that satisfies conditions (Ang) and (Ort). Its generalized dimension is $(2;1)$.

In the case where $0 < \tau_1^2 < \frac{1}{2}$, up to unitary equivalence, all non-zero irreducible systems of subspaces S that satisfy conditions (Ang) and (Ort) can be described as follows.

(1) $1-\tau_1^2 < \tau_2^2 < 1$:
  (a) *There is one system of subspaces with generalized dimension* $(3;1)$.
(2) $\tau_2^2 = 1-\tau_1^2$:
  (a) *There are two systems of subspaces with generalized dimension* $(2;1)$*;*
  (b) *one system of subspaces with generalized dimension* $(3;1)$.
(3) $\frac{1}{2} < \tau_2^2 < 1-\tau_1^2$:
  (a) *There are three systems of subspaces with generalized dimension* $(3;1)$*;*
  (b) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle $\varphi \in (\varphi(\tau),\pi/2)$, where $\varphi(\tau) \in (0,\pi/2)$;*
  (c) *there is one system of subspaces with generalized dimension* $(5;2)$*; related to angle $\varphi = \varphi(\tau)$.*
(4) $\tau_2^2 = \frac{1}{2}$*:*
  (a) *There are three systems of subspaces with generalized dimension* $(3;1)$*;*
  (b) *there is one system of subspaces with generalized dimension* $(2;1)$*;*
  (c) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle $\varphi \in (0,\pi/2)$.*
(5) $\tau_2^2 < \frac{1}{2}$*:*
  (a) *There are four systems of subspaces with generalized dimension* $(3;1)$*;*
  (b) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle $\varphi \in (0,\pi/2)$.*

Proposition 12. *Let $N \geqslant 3$.*

If $N^{-1} < \tau_1^2 < 1$ then there is no a non-zero system of subspaces S that satisfies conditions (Ang) and (Ort).

In the case where $\tau_1^2 = N^{-1}$, up to unitary equivalence, there exists a unique non-zero irreducible system of subspaces S that satisfies conditions (Ang) and (Ort). Its generalized dimension is $(N;1)$.

In the case where $0 < \tau_1^2 < N^{-1}$, up to unitary equivalence, all non-zero irreducible systems of subspaces S that satisfy conditions (Ang) and (Ort) can be described as follows.

(1) $1-(N-1)\cdot\tau_1^2 < \tau_2^2 < 1$:
  (a) There is one system of subspaces with generalized dimension $(N+1;1)$.
(2) $\tau_2^2 = 1-(N-1)\cdot\tau_1^2$:
  (a) There are N systems of subspaces with generalized dimension $(N;1)$;
  (b) there is one system of subspaces with generalized dimension $(N+1;1)$.
(3) $0 < \tau_2^2 < 1-(N-1)\cdot\tau_1^2$: The task of describing all irreducible unitary non-equivalent systems of subspaces S that satisfy conditions (Ang) and (Ort) is complex.

Example 4. Let $N = 2$ and the relations for a pairs P_0, P_k, $k = 1, 2$ be as follows:

$$\prod_{j=1}^{4-k}(P_0P_kP_0 - j\tau_k^2P_0) = 0, \quad \prod_{j=1}^{4-k}(P_kP_0P_k - j\tau_k^2P_k) = 0,$$

where $0 < \tau_1 < 1$, $0 < \tau_2 < 1$.

In this case

a) $\xi(\tau) = 1 - \tau_1^2 - \tau_2^2$, so $\xi(\tau) \geqslant 0$ if and only if $\tau_1^2 + \tau_2^2 \leqslant 1$.

b) $\delta_{k,2} = \tau_k^2$, $k = 1, 2$, $\delta_{1,3} = 2\tau_1^2$. So
- $\delta_{1,2} \leqslant \xi(\tau)$ if and only if $2\tau_1^2 + \tau_2^2 \leqslant 1$;
- $\delta_{2,2} \leqslant \xi(\tau)$ if and only if $\tau_1^2 + 2\tau_2^2 \leqslant 1$;
- $\delta_{2,3} \leqslant \xi(\tau)$ if and only if $\tau_1^2 + 3\tau_2^2 \leqslant 1$.

c) Moreover, $\delta_{1,2} + \delta_{2,2} = 2(\tau_1^2 + \tau_2^2)$, so $\delta_{1,2} + \delta_{2,2} \leqslant \xi(\tau)$ if and only if $\tau_1^2 + \tau_2^2 \leqslant \frac{1}{2}$.

Thus theorem 1 in this case can be reformulated as the following proposition.

Proposition 13. *If $\tau_1^2+\tau_2^2 > 1$ then there does not exist a non-zero system of subspaces S that satisfies conditions (Ang) and (Ort).*

In the case $\tau_1^2 + \tau_2^2 = 1$, up to unitary equivalence, there exists a unique non-zero irreducible system of subspaces S that satisfies conditions (Ang)and (Ort). Its generalized dimension is $(2; 1)$.

In the case $\tau_1^2 + \tau_2^2 < 1$, up to unitary equivalence, all non-zero irreducible systems of subspaces S that satisfy conditions (Ang) and (Ort) can be described as follows.

(1) $\tau_1^2 + 2\tau_2^2 > 1$ *and* $2\tau_1^2 + \tau_2^2 > 1$:
  (a) *There is one system of subspaces with generalized dimension* $(3; 1)$.

(2) $\tau_1^2 + 2\tau_2^2 = 1$, $\tau_2^2 < \frac{1}{3}$ *or* $2\tau_1^2 + \tau_2^2 = 1$, $\tau_2^2 > \frac{1}{3}$:
  (a) *There is one system of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there is one system of subspaces with generalized dimension* $(3; 1)$.

(3) $\tau_1^2 = \tau_2^2 = \frac{1}{3}$:
  (a) *There are two systems of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there is one system of subspaces with generalized dimension* $(3; 1)$.

(4) $2\tau_1^2 + \tau_2^2 < 1$ *and* $\tau_1^2 + 2\tau_2^2 > 1$ *or* $\tau_1^2 + 2\tau_2^2 < 1$, $\tau_1^2 + 3\tau_2^2 > 1$ *and* $2\tau_1^2 + \tau_2^2 > 1$:
  (a) *There are two systems of subspaces with generalized dimension* $(3; 1)$.

(5) $\tau_1^2 + 3\tau_2^2 = 1$ *and* $2\tau_1^2 + \tau_2^2 > 1$:
  (a) *There is one system of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there are two systems of subspaces with generalized dimension* $(3; 1)$.

(6) $\tau_1^2 + 3\tau_2^2 < 1$ *and* $2\tau_1^2 + \tau_2^2 > 1$:
  (a) *There are three systems of subspaces with generalized dimension* $(3; 1)$.

(7) $2\tau_1^2 + \tau_2^2 < 1$ *and* $\tau_1^2 + 2\tau_2^2 = 1$ *or* $\tau_1^2 + 2\tau_2^2 < 1$, $\tau_1^2 + 3\tau_2^2 > 1$ *and* $2\tau_1^2 + \tau_2^2 = 1$:
  (a) *There is one system of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there are two systems of subspaces with generalized dimension* $(3; 1)$.

(8) $\tau_2^2 = \frac{1}{5}$ *and* $\tau_1^2 = \frac{2}{5}$:
  (a) *There are two systems of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there are two systems of subspaces with generalized dimension* $(3; 1)$.

(9) $\tau_1^2 + 3\tau_2^2 < 1$ *and* $2\tau_1^2 + \tau_2^2 = 1$:
  (a) *There is one system of subspaces with generalized dimension* $(2; 1)$*;*
  (b) *there are three systems of subspaces with generalized dimension* $(3; 1)$.

(10) $\tau_1^2 + 2\tau_2^2 < 1$, $\tau_1^2 + 3\tau_2^2 > 1$, $2\tau_1^2 + \tau_2^2 < 1$ *and* $\tau_1^2 + \tau_2^2 > \frac{1}{2}$:
  (a) *There are three systems of subspaces with generalized dimension* $(3; 1)$*;*
  (b) *there is an infinite family of systems of subspaces with generalized dimension* $(6; 2)$ *parameterized by the angle* $\varphi \in (\varphi(\tau), \pi/2)$ *where* $\varphi(\tau) \in (0, \pi/2)$*;*
  (c) *there is one system of subspaces with generalized dimension* $(5; 2)$*; related to the angle* $\varphi = \varphi(\tau)$.

(11) $\tau_1^2 + 3\tau_2^2 = 1$, $2\tau_1^2 + \tau_2^2 < 1$ *and* $\tau_1^2 + \tau_2^2 > \frac{1}{2}$:
   (a) *There is one system of subspaces with generalized dimension* $(2;1)$*;*
   (b) *there are three systems of subspaces with generalized dimension* $(3;1)$*;*
   (c) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle* $\varphi \in (\varphi(\tau), \pi/2)$*, where* $\varphi(\tau) \in (0, \pi/2)$*;*
   (d) *there is one system of subspaces with generalized dimension* $(5;2)$*; related to the angle* $\varphi = \varphi(\tau)$.

(12) $\tau_1^2 + 2\tau_2^2 < 1$, $\tau_1^2 + 3\tau_2^2 > 1$ *and* $\tau_1^2 + \tau_2^2 = \frac{1}{2}$:
   (a) *There is one system of subspaces with generalized dimension* $(2;1)$*;*
   (b) *there are three systems of subspaces with generalized dimension* $(3;1)$*;*
   (c) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle* $\varphi \in (0, \pi/2)$.

(13) $\tau_1^2 = \tau_2^2 = \frac{1}{4}$:
   (a) *There are two systems of subspaces with generalized dimension* $(2;1)$*;*
   (b) *there are three systems of subspaces with generalized dimension* $(3;1)$*;*
   (c) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle* $\varphi \in (0, \pi/2)$.

(14) $\tau_1^2 + \tau_2^2 < \frac{1}{2}$ *and* $\tau_1^2 + 3\tau_2^2 > 1$:
   (a) *There are four systems of subspaces with generalized dimension* $(3;1)$*;*
   (b) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle* $\varphi \in (0, \pi/2)$.

(15) $\tau_1^2 + \tau_2^2 < \frac{1}{2}$ *and* $\tau_1^2 + 3\tau_2^2 = 1$:
   (a) *There is one system of subspaces with generalized dimension* $(2;1)$*;*
   (b) *there are four systems of subspaces with generalized dimension* $(3;1)$*;*
   (c) *there is an infinite family of systems of subspaces with generalized dimension* $(6;2)$ *parameterized by the angle* $\varphi \in (0, \pi/2)$.

(16) *if* $2\tau_1^2 + \tau_2^2 < 1$ *and* $\tau_1^2 + 3\tau_2^2 < 1$ *then the task of describing all irreducible unitary non-equivalent systems of subspaces* S *that satisfy conditions (Ang) and (Ort) is* $*$*-wild.*

Acknowledgements. The author is sincerely grateful to Yu. S. Samoilenko and I. S. Feshchenko for a discussion of the results exposed in the paper.

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Received 23/03/2017