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REPRESENTATION OF ISOMETRIC ISOMORPHISMS BETWEEN MONOIDS OF LIPSCHITZ FUNCTIONS

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Abstract. We prove that each isometric isomorphism between the monoids of all nonnegative 1-Lipschitz maps defined on invariant metric groups and equipped with the inf-convolution law, is given canonically from an isometric isomorphism between their groups of units.

Introduction

Let X be a metric space with metric d which we briefly denote by (X, d). We denote by $Lip^1_+(X)$ the set of all nonnegative 1-Lipschitz maps on X equipped with the metric

$$\rho(f,g) := \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}, \quad \forall f,g \in Lip^1_+(X).$$

If X is a group and $f, g : X \longrightarrow \mathbb{R}$ are two functions, the inf-convolution of f and g is defined by the following formula

$$(f \oplus g)(x) \quad := \inf_{y,z\in X/yz=x} \{f(y)+g(z)\}, \quad \forall x \in X.$$

We recall the following definition.

Definition 1. *Let (X, d) be a metric group. We say that (X, d) is invariant if and only if,*

$$d(xy, xz) = d(yx, zx) = d(y, z), \quad \forall x, y, z \in X.$$

If moreover X is complete for the metric d, then we say that (X, d) is an invariant complete metric group.

Examples of invariant metric groups are given in [2]. In all the paper (X, d) and (Y, d') will be assumed to be invariant metric groups having respectively e and e' as identity element and $(\overline{X}, \overline{d})$ (resp. $(\overline{Y}, \overline{d'})$) denotes the group completion of (X, d) (resp. of (Y, d')). Recall that the group completion $(\overline{X}, \overline{d})$ of an invariant metric group (X, d), is constructed as follows (See [8]): Two Cauchy sequences u_n and v_n of X are said to be equivalent $u_n \sim v_n$ iff $\lim_n d(u_n, v_n) = 0$. The elements of the metric completion $(\overline{X}, \overline{d})$ are equivalence classes of Cauchy sequences, with $\overline{d}([x_n], [y_n]) = \lim_n d(x_n, y_n)$. Now, suppose that x_n, x'_n, y_n, y'_n are Cauchy sequences with $x_n \sim x'_n$ and $y_n \sim y'_n$. For each $n \in \mathbb{N}$ let $z_n = x_n y_n$ and $z'_n = x'_n y'_n$. Then, z_n and z'_n are Cauchy sequences, with $z_n \sim z'_n$. Thus we can define multiplication in $\overline{X}$ by $[x_n][y_n] = [x_n y_n]$ and $\overline{X}$ becomes a group in which X is isometrically and isomorphically embedded. The metric $\overline{d}$ is invariant on $\overline{X}$.

Recently, we established in [2] that the set $(Lip^1_+(X), \oplus)$ enjoys a monoid structure, having the map $\delta_e : x \mapsto d(x, e)$ as identity element and that the group completion $(\overline{X}, \overline{d})$

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of (X,d) is completely determined by the metric monoid structure of $(Lip^1_+(X),\oplus,\rho)$. In other words, $(Lip^1_+(X),\oplus,\rho)$ and $(Lip^1_+(Y),\oplus,\rho)$ are isometrically isomorphic as monoids if and only if, $(\overline{X},\overline{d})$ and $(\overline{Y},\overline{d'})$ are isometrically isomorphic as groups. Also, we proved that the group of all invertible elements of $Lip^1_+(X)$ coincides, up to isometric isomorphism, with the group completion $\overline{X}$ (See [2, Theorem 1] and [2, Theorem 2]). The main result of [2], gives a Banach-Stone type theorem. For information on general topological structure of semigroups and monoids we refer for instance to [9], [4], [1].

The representations of isometries between Banach spaces of Lipschitz maps defined on metric spaces and equipped with their natural norms, was considered by several authors [5], [10], [7]. In general, such isometries are given, under some conditions, canonically as a composition operators. Other Banach-Stone type theorems are also given for unital vector lattice structure [6].

The aim of this note, is to prove the following result which gives complete representations of isometric isomorphisms for the monoid structure between $Lip^1_+(X)$ and $Lip^1_+(Y)$ in the invariant metric group framework. This result complements those given in [2] and answers positively Problem 2. in [3]. Recall that a set M with an internal law $\cdot$ is a monoid if it is a semigroup with an identity element i.e. if it satisfies the following two axioms: (1) (Associativity) For all a,b and c in M, the equation $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ holds. (2) (Identity element) There exists an element e in M such that for every element a in M, the equations $e\cdot a = a\cdot e = a$ hold.

Theorem 1. *Let (X,d) and (Y,d') be two invariant metric groups. Let Φ be a map from $(Lip^1_+(X),\oplus,\rho)$ into $(Lip^1_+(Y),\oplus,\rho)$. Then the following assertions are equivalent:*
(1) *Φ is an isometric isomorphism of monoids,*
(2) *there exists an isometric isomorphism of groups $T:(\overline{X},\overline{d})\longrightarrow(\overline{Y},\overline{d'})$ such that $\Phi(f) = (\overline{f}\circ T^{-1})_{|Y}$ for all $f\in Lip^1_+(X)$, where $\overline{f}$ denotes the unique 1-Lipschitz extension of f to $\overline{X}$ and $(\overline{f}\circ T^{-1})_{|Y}$ denotes the restriction of $\overline{f}\circ T^{-1}$ to Y.*

If M (resp. G) is a metric monoid (resp. a metric group), by $Is_m(M)$ (resp. $Is_g(G)$) we denote the group of all isometric automorphism of the monoid M (resp. of all isometric automorphism of the group G). The symbol "$\simeq$" means "*isomorphic as groups*". An immediate consequence of Theorem 1 is given in the following corollary.

Corollary 1. *Let (X,d) be an invariant metric group. Then*

$$Is_m(Lip^1_+(X)) \simeq Is_g(\overline{X}).$$

As application of the results of this note, we discover new semigroup law on $\mathbb{R}^n$ (different from the usual operation $+$) having some nice properties. We treat this question in Example 1 at Section 3, where it is shown that each finite group structure $(G,\cdot)$, extends canonically to a semigroup structure on $\mathbb{R}^n$ (where n is the cardinal of G). In other words, there always exists a semigroup law $\star_G$ on $\mathbb{R}^n$ and an injective group morphism i from $(G,\cdot)$ into $(\mathbb{R}^n,\star_G)$ such that the maximal subgroup of $(\mathbb{R}^n,\star_G)$ having $e := (0,1,1,\dots,1)$ as identity element is isomorphic to the group $G\times\mathbb{R}$. The idea is simply based on the use of the results of this paper and the identification between $(\mathbb{R}^n,\star_G)$ and $(Lip(G),\oplus)$ where G is equipped with the discrete metric, and $Lip(G)$ denotes the space of all Lipschitz map on G.

This note is organized as follows. Section 1 is concerned the proof of Theorem 1 and is divided on two subsections: in Subsection 1.1 we prove some useful lemmas and in Subsection 1.2, we give the proof of the main result Theorem 1. In Section 2, we give some properties of the group of invertible elements for the inf-convolution law. In section 3, we review the results of this paper in the algebraic case.

1. Proof of the main result

1.1. **Preliminary results.** We follow the notation of [2]. For each fixed point $x \in X$, the map δ_x is defined from X into $\mathbb{R}$ as follows

$$\begin{array}{rcl} \delta_x : X & \to & \mathbb{R}, \\ z & \mapsto & d(z,x) = d(zx^{-1}, e). \end{array}$$

We define the subset $\mathcal{G}(X)$ of $Lip^1_+(X)$ as follows

$$\mathcal{G}(X) := \{\delta_x : x \in X\} \subset Lip^1_+(X).$$

We consider the operator γ_X defined as follows

$$\begin{array}{rcl} \gamma_X : X & \to & \mathcal{G}(X), \\ x & \mapsto & \delta_x. \end{array}$$

We are going to prove some lemmas.

Lemma 1. *Let (X,d) and (Y,d') be two invariant complete metric groups having respectively e and e' as identity elements. Let Φ be a map from $(Lip^1_+(X), \oplus, \rho)$ onto $(Lip^1_+(Y), \oplus, \rho)$ which is an isometric isomorphism of monoids. Then, the following assertions hold:*

(1) *for all $f \in Lip^1_+(X)$, $\inf_Y \Phi(f) = \inf_X f$ and for all $r \in \mathbb{R}^+$, $\Phi(r) = r$,*

(2) *there exists an isometric isomorphism of groups $T : (X,d) \longrightarrow (Y,d')$ such that $\Phi(r + \delta_x) = r + \delta_{T(x)} = r + \delta_x \circ T^{-1}$, for all $r \in \mathbb{R}^+$ and for all $x \in X$,*

(3) *$\Phi(f + r) = \Phi(f) + r$, for all $f \in Lip^1_+(X)$ and for all $r \in \mathbb{R}^+$.*

Proof. Since an isomorphism of monoids, sends the group of units onto the group of units, then using [2, Theorem 1], the restriction $T_1 := \Phi_{|\mathcal{G}(X)}$ is an isometric group isomorphism from $\mathcal{G}(X)$ onto $\mathcal{G}(Y)$. On the other hand, the map $\gamma_X : X \longrightarrow \mathcal{G}(X)$ gives an isometric group isomorphism by [2, Lemma 2]. Thus, the map $T := \gamma_Y^{-1} \circ T_1 \circ \gamma_X$, gives an isometric group isomorphism from X onto Y and we have that for all $x \in X$, $\Phi(\delta_x) := T_1(\delta_x) = T_1 \circ \gamma_X(x) = \gamma_Y \circ T(x) = \delta_{T(x)} = \delta_x \circ T^{-1}$.

We prove part (1). Note that $f \oplus 0 = 0 \oplus f = \inf_{x\in X} f$ for all $f \in Lip^1_+(X)$. First, we prove that $\Phi(0) = 0$. Indeed, for all $x \in X$, we have that $0 \oplus \delta_x = 0$. Thus, $\Phi(0) = \Phi(0) \oplus \Phi(\delta_x) = \Phi(0) \oplus \delta_{Tx}$. Using the surjectivity of T, we obtain that for all $y \in Y$, we have that $\Phi(0) = \Phi(0) \oplus \delta_y$. So, using the definition of the inf-convolution, we get $\Phi(0)(z) = \inf_{ts=z}\{\Phi(0)(t) + \delta_y(s)\} \leq \Phi(0)(zy^{-1})$ for all $y, z \in Y$. By taking the infimum over $y \in Y$, we obtain that $\Phi(0)(z) \leq \inf_Y \Phi(0)$, for all $z \in Y$. It follows that $\Phi(0) = \inf_Y \Phi(0)$ is a constant function. Now, since $\Phi(0)$ is a constant function, we have $2\Phi(0) = \Phi(0) \oplus \Phi(0) = \Phi(0 \oplus 0) = \Phi(0)$, it follows that $\Phi(0) = 0$. Finally, we prove that $\Phi(r) = r$ for all $r \in \mathbb{R}^+$. Indeed, since $r \oplus 0 = r$ and $\Phi(0) = 0$, it follows that $\Phi(r) = \Phi(r) \oplus 0 = \inf_Y \Phi(r)$, which implies that $\Phi(r)$ is a constant function. Using the fact that Φ is an isometry, we get that $\rho(\Phi(r), 0) = \rho(\Phi(r), \Phi(0)) = \rho(r, 0)$. In other words, $\frac{\Phi(r)}{1+\Phi(r)} = \frac{r}{1+r}$, which implies that $\Phi(r) = r$. Now, we have $\inf_{y\in Y} \Phi(f) = \Phi(f) \oplus 0 = \Phi(f) \oplus \Phi(0) = \Phi(f \oplus 0) = \Phi(\inf_{x\in X} f) = \inf_{x\in X} f$.

We prove part (2). Let $r \in \mathbb{R}^+$ and set $g = \Phi(r + \delta_e) \in Lip^1_+(Y)$. We first prove that $g = r + \delta_{e'}$. Using part(1), we have that $r = \Phi(r) = \Phi(\inf_{x\in X}(r + \delta_e)) = \inf_{y\in Y} \Phi(r + \delta_e) \leq \Phi(r + \delta_e) = g$. Thus $g - r \geq 0$ and so $g - r \in Lip^1_+(Y)$. On the other hand, since $Lip^1_+(Y)$ is a monoid having $\delta_{e'}$ as identity element, we have that $g = (g-r) \oplus (r+\delta_{e'}) = (r + \delta_{e'}) \oplus (g - r)$. Now, since Φ^{-1} is a monoid morphism, we get that

$$\begin{array}{rcl} r + \delta_e & = & \Phi^{-1}(g) \\ & = & \Phi^{-1}(g - r) \oplus \Phi^{-1}(r + \delta_{e'}) = \Phi^{-1}(r + \delta_{e'}) \oplus \Phi^{-1}(g - r). \end{array}$$

As above we prove that $\Phi^{-1}(r+\delta_{e'})-r \geq 0$. Thus, $\Phi^{-1}(r+\delta_{e'})-r \in Lip^1_+(X)$. Since r is a constant function, the above equality is equivalent to the following one:

$$\delta_e \quad = \quad \Phi^{-1}(g-r) \oplus (\Phi^{-1}(r+\delta_{e'})-r) = (\Phi^{-1}(r+\delta_{e'})-r) \oplus \Phi^{-1}(g-r).$$

Since from [2, Theorem 1], the invertible elements in $Lip^1_+(X)$ are exactly the element of $\mathcal{G}(X)$ and since $\mathcal{G}(X)$ is a group by [2, Lemma 2], we deduce from the above equality that $\Phi^{-1}(r+\delta_{e'})-r \in \mathcal{G}(X)$ and $\Phi^{-1}(g-r) \in \mathcal{G}(X)$ and there exist $\alpha(r), \beta(r) \in X$ such that

$$\begin{cases} e = \alpha(r)\beta(r) \\ \Phi^{-1}(r+\delta_{e'})-r = \delta_{\alpha(r)} \\ \Phi^{-1}(g-r) = \delta_{\beta(r)} \end{cases}.$$

This implies that

$$\begin{cases} e = \alpha(r)\beta(r) \\ \Phi(r+\delta_{\alpha(r)}) = r+\delta_{e'} \\ g = r+\Phi(\delta_{\beta(r)}) = r+\delta_{T(\beta(r))} \end{cases}. \tag{1}$$

We need to prove that $\alpha(r)=\beta(r)=e$ for all $r \in \mathbb{R}^+$. Indeed, since Φ is an isometry, we have that

$$\rho(\Phi(r+\delta_{\alpha(r)}), \Phi(\delta_e)) = \rho(r+\delta_{\alpha(r)}, \delta_e).$$

Using the above formula, the second equations in (1) and the definition of the metric ρ with the fact that $\Phi(\delta_e)=\delta_{e'}$, we get

$$\begin{aligned} \frac{r}{1+r} &= \rho(r+\delta_{e'}, \delta_{e'}) \\ &= \rho(\Phi(r+\delta_{\alpha(r)}), \Phi(\delta_e)) \\ &= \rho(r+\delta_{\alpha(r)}, \delta_e) \\ &= \sup_{t\in X} \frac{|r+\delta_{\alpha(r)}(t)-\delta_e(t)|}{1+|r+\delta_{\alpha(r)}(t)-\delta_e(t)|} \\ &\geq \frac{r+\delta_{\alpha(r)}(e)}{1+r+\delta_{\alpha(r)}(e)}. \end{aligned}$$

A simple computation of the above inequality, gives that $\delta_{\alpha(r)}(e) \leq 0$ i.e. $d(\alpha(r), e) \leq 0$. In other words, we have that $\alpha(r)=e$ for all $r \in \mathbb{R}^+$. On the other hand, using the first equation of (1), we get that $\beta(r)=e$ for all $r \in \mathbb{R}^+$. It follows from the equation (1) that $\Phi(r+\delta_e)=r+\delta_{e'}$ for all $r \in \mathbb{R}^+$. Now, it is easy to see that for all $r \in R^+$ and all $x \in X$ we have

$$r+\delta_x = (r+\delta_e) \oplus \delta_x.$$

It follows that

$$\begin{aligned} \Phi(r+\delta_x) &= \Phi(r+\delta_e) \oplus \Phi(\delta_x) \\ &= (r+\delta_{e'}) \oplus \delta_{T(x)} \\ &= r+\delta_{T(x)}. \end{aligned}$$

Since T is isometric, we obtain that $\Phi(r+\delta_x) = r+\delta_{T(x)} = r+\delta_x \circ T^{-1}$.

Now, we prove part (3). Let $f \in Lip^1_+(X)$ and $r \in \mathbb{R}^+$. It is easy to see that $f+r = f \oplus (r+\delta_e)$. So, using part(2), we obtain that $\Phi(f+r) = \Phi(f) \oplus \Phi(r+\delta_e) = \Phi(f) \oplus (r+\delta_{e'}) = \Phi(f)+r$. □

Lemma 2. *Let (X,d) be an invariant metric group. Let $f \in Lip^1_+(X)$. Then, for all $x \in X$ and all positive real number a such that $a \geq f(x)$, we have that*

$$f(x) = (\inf(\delta_e, a) \oplus f)(x).$$

Proof. Let $x \in X$ and $a \geq 0$ such that $f(x) \leq a$. We have that

$$\begin{aligned}(\inf(\delta_e, a) \oplus f)(x) &= \inf_{t \in X} \{\inf(d(xt^{-1}, e), a) + f(t)\} \\ &= \inf_{t \in X} \{f(t) + \inf(d(t,x), a)\} \\ &= \min\{\inf_{t \in X/d(t,x) \leq a} \{f(t) + \inf(d(t,x), a)\}; \\ &\qquad \inf_{t \in X/d(t,x) \geq a} \{f(t) + \inf(d(t,x), a)\}\} \\ &= \min\{\inf_{t/d(t,x) \leq a} \{f(t) + d(t,x)\}, \inf_{t/d(t,x) \geq a} \{f(t) + a\}\}.\end{aligned}$$

Since f is 1-Lipschitz we have that $f(x) = \inf_{t/d(t,x) \leq a}\{f(t) + d(t,x)\}$. It follows that

$$(\inf(\delta_e, a) \oplus f)(x) = \min\{f(x), \inf_{t/d(t,x) \geq a} \{f(t)\} + a\} = f(x).$$

□

Lemma 3. *Let (X,d) be an invariant metric group. Then, the following assertions hold.*

(1) *For each $f \in Lip^1_+(X)$ and for each bounded function $h \in Lip^1_+(X)$, the function $f \oplus h \in Lip^1_+(X)$ is bounded.*

(2) *Let $f, g \in Lip^1_+(X)$, then the following assertions are equivalent:*

(*a*) *$f \leq g$,*

(*b*) *$h \oplus f \leq h \oplus g$, for all function $h \in Lip^1_+(X)$ which is bounded.*

Proof. (1) Since $0 \leq f \oplus h(x) \leq f(e) + h(x)$ for all $x \in X$ and since h is bounded, it follows that $f \oplus h$ is bounded. On the other hand, $f \oplus h \in Lip^1_+(X)$ since $Lip^1_+(X)$ is a monoid.

(2) Part $(a) \Longrightarrow (b)$ is easy. Let us prove part $(b) \Longrightarrow (a)$. Indeed, let $x \in X$ and chose a positive real number $a \geq \max(f(x), g(x))$. Set $h := \inf(\delta_e, a)$. It is clear that $h \in Lip^1_+(X)$ and is bounded. So, from the hypothesis (b) we have that $(\inf(\delta_e, a) \oplus f) \leq (\inf(\delta_e, a) \oplus g)$. Using Lemma 2, we obtain that $f(x) \leq g(x)$. □

Lemma 4. *Let A be a nonempty set and $f, g : A \longrightarrow \mathbb{R}$ be two functions. Then, the following assertions are equivalent:*

(1) $\sup_{x \in A} |f(x) - g(x)| < +\infty$,

(2) $\sup_{x \in A} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} < 1$.

Proof. Suppose that (1) holds. Using [2, Lemma 1], we have that $\sup_{x \in A} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = \frac{\sup_{x \in A}|f(x)-g(x)|}{1+\sup_{x \in A}|f(x)-g(x)|} < 1$. Now, suppose that (2) holds. Set $\alpha = \sup_{x \in A} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} < 1$. Then, we obtain that $|f(x)-g(x)| \leq \frac{\alpha}{1-\alpha}$, for all $x \in A$. This implies that $\sup_{x \in A} |f(x) - g(x)| < +\infty$. □

Lemma 5. *Let (X,d) and (Y,d') be two invariant complete metric groups. Let*

$$\Phi : (Lip^1_+(X), \rho) \longrightarrow (Lip^1_+(Y), \rho)$$

be an isometric isomorphism of monoids. Then, for all $f, g \in Lip^1_+(X)$, we have

$$f \leq g \Longleftrightarrow \Phi(f) \leq \Phi(g).$$

Proof. The proof is divided in two cases.

Case 1: (The case where f and g are bounded.) Let $f, g \in Lip^1_+(X)$ be bounded functions. In this case we have $\sup_{x\in X} |f(x) - g(x)| < +\infty$, so using Lemma 4 and the fact that Φ is isometric, we get also that $\sup_{y\in Y} |\Phi(f)(y) - \Phi(g)(y)| < +\infty$. Using [2, Lemma 1] and the fact that Φ is isometric, we obtain that

$$\frac{\sup_{y\in Y} |\Phi(f)(y) - \Phi(g)(y)|}{1 + \sup_{y\in Y} |\Phi(f)(y) - \Phi(g)(y)|} = \frac{\sup_{x\in X} |f(x) - g(x)|}{1 + \sup_{x\in X} |f(x) - g(x)|}.$$

This implies that

$$\sup_{y\in Y} |\Phi(f)(y) - \Phi(g)(y)| = \sup_{x\in X} |f(x) - g(x)|.$$

Set $r := \sup_{y\in Y} |\Phi(f)(y) - \Phi(g)(y)| = \sup_{x\in X} |f(x) - g(x)| < +\infty$. By applying the above arguments to $f + r$ and g which are bounded, we also get that

$$\sup_{y\in Y} |\Phi(f + r)(y) - \Phi(g)(y)| = \sup_{x\in X} |(f + r)(x) - g(x)|.$$

Using the fact that $\Phi(f + r) = \Phi(f) + r$ (by Lemma 1) and the choice of the number r, we get that

$$\sup_{x\in X}\{\Phi(f)(x) - \Phi(g)(x) + r\} = \sup_{x\in X}\{f(x) - g(x) + r\},$$

which implies that

$$\sup_{y\in Y}\{\Phi(f)(y) - \Phi(g)(y)\} = \sup_{x\in X}\{f(x) - g(x)\}.$$

It follows that $f \leq g \Longleftrightarrow \Phi(f) \leq \Phi(g)$. Replacing Φ by Φ^{-1} we also have $k \leq l \Longleftrightarrow \Phi^{-1}(k) \leq \Phi^{-1}(l)$, for all bounded functions $k, l \in Lip^1_+(Y)$.

Case 2: (The general case.) First, note that for each bounded function $k \in Lip^1_+(Y)$, we have that $\Phi^{-1}(k) \in Lip^1_+(X)$ is bounded. Indeed, there exists $r \in \mathbb{R}^+$ such that $0 \leq k \leq r$. Using the above case, we get that $\Phi^{-1}(0) \leq \Phi^{-1}(k) \leq \Phi^{-1}(r)$. This shows that $\Phi^{-1}(k)$ is bounded, since $\Phi^{-1}(0) = 0$ and $\Phi^{-1}(r) = r$ by Lemma 1.

Now, let $f, g \in Lip^1_+(X)$ be fixed functions such that $f \leq g$. Let $k \in Lip^1_+(Y)$ be any bounded function. It follows that $\Phi^{-1}(k) \oplus f \leq \Phi^{-1}(k) \oplus g$. From part(1) of Lemma 3, we have that $\Phi^{-1}(k) \oplus f, \Phi^{-1}(k) \oplus g \in Lip^1_+(X)$ are bounded. Using *Case 1*, we get that $\Phi(\Phi^{-1}(k)\oplus f) \leq \Phi(\Phi^{-1}(k)\oplus g)$. Since Φ is a morphism, we have that $k\oplus\Phi(f) \leq k\oplus\Phi(g)$, which implies that $\Phi(f) \leq \Phi(g)$ by using part(2) of Lemma 3. The converse is true by changing Φ by Φ^{-1}. □

Lemma 6. *Let (X, d) and (Y, d') be two invariant metric groups and let Φ be a bijective map $\Phi : (Lip^1_+(X), \oplus, \rho) \longrightarrow (Lip^1_+(Y), \oplus, \rho)$. Then, the following assertions are equivalent:*

(1) *for all $f, g \in Lip^1_+(X)$, we have that $(f \leq g \Longleftrightarrow \Phi(f) \leq \Phi(g))$,*

(2) *for all $f', g' \in Lip^1_+(Y)$, we have that $(f' \leq g' \Longleftrightarrow \Phi^{-1}(f') \leq \Phi^{-1}(g'))$,*

(3) *for all family $(f_i)_{i\in I} \subset Lip^1_+(X)$, where I is any nonempty set, we have $\Phi(\inf_{i\in I} f_i) = \inf_{i\in I} \Phi(f_i)$.*

Proof. Part (1) $\Longleftrightarrow$ (2) is clear. Let us prove (1) $\Longrightarrow$ (3). Let $(f_i)_{i\in I} \subset Lip^1_+(X)$, where I is any nonempty set. First, it is easy to see that the infimum of a nonempty family of nonnegative and 1-Lipschitz functions is also nonnegative and 1-Lipschitz function. So, $\inf_{i\in I} f_i \in Lip^1_+(X)$. For all $i \in I$, we have that $\inf_{i\in I} f_i \leq f_i$, which implies by hypothesis that $\Phi(\inf_{i\in I} f_i) \leq \Phi(f_i)$ for all $i \in I$. Consequently we have that $\Phi(\inf_{i\in I} f_i) \leq \inf_{i\in I} \Phi(f_i)$. On the other hand, since $\inf_{i\in I} \Phi(f_i) \leq \Phi(f_i)$ for all $i \in I$, using (2), we have that $\Phi^{-1}(\inf_{i\in I} \Phi(f_i)) \leq f_i$, for all $i \in I$. It follows that, $\Phi^{-1}(\inf_{i\in I} \Phi(f_i)) \leq \inf_{i\in I} f_i$. Using (1), we obtain that $\inf_{i\in I} \Phi(f_i) \leq \Phi(\inf_{i\in I} f_i)$. Hence, $\inf_{i\in I} \Phi(f_i) = \Phi(\inf_{i\in I} f_i)$. Now, let us prove that (3) $\Longrightarrow$ (1). First, let us show that from (3) we also have that

$\Phi^{-1}(\inf_{i\in I} g_i) = \inf_{i\in I}\Phi^{-1}(g_i)$, where I is a nonempty set and $g_i \in Lip^1_+(Y)$ for all $i \in I$. Indeed, since Φ is bijective, there exists $(f_i)_{i\in I} \subset Lip^1_+(X)$ such that $g_i = \Phi(f_i)$ for all $i \in I$. Thus, $\inf_{i\in I} g_i = \inf_{i\in I}\Phi(f_i) = \Phi(\inf_{i\in I} f_i) = \Phi(\inf_{i\in I}\Phi^{-1}(g_i))$, which implies that $\Phi^{-1}(\inf_{i\in I} g_i) = \inf_{i\in I}\Phi^{-1}(g_i)$. Now, let $f, g \in Lip^1_+(X)$. We have that $f \leq g \iff f = \inf(f, g)$, so if $f \leq g$ then $\Phi(f) = \Phi(\inf(f, g)) = \inf(\Phi(f), \Phi(g))$. This implies that $\Phi(f) \leq \Phi(g)$. Conversely, if $\Phi(f) \leq \Phi(g)$ then $\Phi(f) = \inf(\Phi(f), \Phi(g))$ and so $f = \Phi^{-1}(\Phi(f)) = \Phi^{-1}(\inf(\Phi(f), \Phi(g))) = \inf(\Phi^{-1}(\Phi(f)), \Phi^{-1}(\Phi(g))) = \inf(f, g)$. This implies that $f \leq g$. □

1.2. **Proof of the main result.** Now, we give the proof of the main result.

Proof of Theorem 1. We know from [2, Lemma 3] that the map

$$\begin{array}{rcl}\chi_X : (Lip^1_+(X), \oplus, \rho) & \to & (Lip^1_+(\overline{X}), \oplus, \rho),\\ f & \mapsto & \overline{f}\end{array}$$

is an isometric isomorphism of monoids, where $\overline{f}$ denotes the unique 1-Lipschitz extension of f to $\overline{X}$. Given a map $\Phi : (Lip^1_+(X), \oplus, \rho) \longrightarrow (Lip^1_+(Y), \oplus, \rho)$, we define the map $\overline{\Phi} : (Lip^1_+(\overline{X}), \oplus, \rho) \longrightarrow (Lip^1_+(\overline{Y}), \oplus, \rho)$ by $\overline{\Phi} := \chi_Y \circ \Phi \circ \chi_X^{-1}$. Then, Φ is an isometric isomorphism of monoids if and only if $\overline{\Phi}$ is an isometric isomorphism of monoids.

(1) $\Longrightarrow$ (2). Since $Lip^1_+(\overline{X})$ is a monoid having $\delta_e : \overline{X} \ni x \mapsto \overline{d}(x, e)$ as identity element, we have that $\overline{f} = \delta_e \oplus \overline{f}$ for all $\overline{f} \in Lip^1_+(\overline{X})$. Thus, $\overline{f} = \inf_{t\in\overline{X}}\{\overline{f}(t) + \delta_t\}$ for all $\overline{f} \in Lip^1_+(\overline{X})$. Using Lemma 6 together with Lemma 5, we have that for all $\overline{f} \in Lip^1_+(\overline{X})$, $\overline{\Phi}(\overline{f}) = \overline{\Phi}(\inf_{t\in\overline{X}}\{\overline{f}(t) + \delta_t\}) = \inf_{t\in\overline{X}}\overline{\Phi}(\overline{f}(t) + \delta_t)$. Using Lemma 1, there exists an isometric isomorphism of groups $T : (\overline{X}, \overline{d}) \longrightarrow (\overline{Y}, \overline{d'})$ such that $\overline{\Phi}(\overline{f}(t) + \delta_t) = \overline{f}(t) + \delta_{T(t)}$, for all $t \in \overline{X}$. Thus, we get that $\overline{\Phi}(\overline{f}) = \inf_{t\in\overline{X}}\{\overline{f}(t) + \delta_{T(t)}\}$. Equivalently, for all $y \in \overline{Y}$, we have

$$\begin{array}{rcl}\overline{\Phi}(\overline{f})(y) & = & \inf_{t\in\overline{X}}\{\overline{f}(t) + \delta_{T(t)}(y)\}\\ & = & \inf_{t\in\overline{X}}\{\overline{f}(t) + \overline{d'}(y, T(t))\}\\ & = & \inf_{t\in\overline{X}}\{\overline{f}(t) + \overline{d}(T^{-1}(y), t)\}\\ & = & (\delta_e \oplus \overline{f})(T^{-1}(y))\\ & = & \overline{f}(T^{-1}(y))\\ & = & \overline{f} \circ T^{-1}(y).\end{array}$$

From the formulas $\Phi = \chi_Y^{-1} \circ \overline{\Phi} \circ \chi_X$, we get that $\Phi(f) = (\overline{f} \circ T^{-1})_{|Y}$ for all $f \in Lip^1_+(X)$.

(2) $\Longrightarrow$ (1). If $T : (\overline{X}, \overline{d}) \longrightarrow (\overline{Y}, \overline{d'})$ is an isometric isomorphism of groups, then clearly the map $\overline{\Phi}$ defined by $\overline{\Phi}(\overline{f}) := \overline{f} \circ T^{-1}$ for all $\overline{f} \in Lip^1_+(\overline{X})$, gives an isometric isomorphism from $(Lip^1_+(\overline{X}), \oplus, \rho)$ onto $(Lip^1_+(\overline{Y}), \oplus, \rho)$. Thus, the map $\Phi := \chi_Y^{-1} \circ \overline{\Phi} \circ \chi_X$ gives an isometric isomorphism from $(Lip^1_+(X), \oplus, \rho)$ onto $(Lip^1_+(Y), \oplus, \rho)$. Now, it clear that $\Phi(f) = (\overline{f} \circ T^{-1})_{|Y}$ for all $f \in Lip^1_+(X)$. □

Remark 1. (1) *The description of all isomorphisms seems to be more complicated than the representations of the isometric isomorphisms. Here are two examples of isomorphisms which are not isometric.*

(*a*) *The map* $\Phi : Lip^1_+(X) \longrightarrow Lip^1_+(X)$ *defined by* $\Phi(f) = f + \inf_X(f)$ *for all* $f \in Lip^1_+(X)$*, is an isomorphism of monoids which respect the order but is not isometric for* ρ *(the proof is similar to the proof of* [3, Theorem 7]. *Note that we always have* $\inf_X(f \oplus g) = \inf_X(f) + \inf_Y(g)$*).*

(*b*) *The map* $\Phi : Lip^1_+(\mathbb{R}) \longrightarrow Lip^1_+(\mathbb{R})$ *defined by* $\Phi(f)(x) = f(x + \inf_X(f))$ *for all* $f \in Lip^1_+(\mathbb{R})$ *and all* $x \in \mathbb{R}$*, is an isomorphism but not isometric for* ρ.

(2) *Following the proof of Theorem 1 and changing "*1*-Lipschitz function" by "*1*-Lipschitz and convex function", we get a positive answer to the problem 2 in* [3].

2. The group of units

In order to have that the inf-convolution of two functions f and g takes finite values i.e $f \oplus g > -\infty$, we need to assume that f and g are bounded from below. Since, we work with Lipschitz maps, for simplicity, we assume in this section that (X, d) is a bounded invariant metric group. By $Lip^1_0(X)$ we denote the set of all 1-Lipschitz map f from X into $\mathbb{R}$ such that $\inf_X(f) = 0$. By $Lip^1(X)$ (resp. $Lip(X)$,) we denote the set of all 1-Lipschitz map (resp. the set of all Lipschitz map) defined from X to $\mathbb{R}$. We have that

$$Lip^1_0(X) \subset Lip^1_+(X) \subset Lip^1(X) \subset Lip(X).$$

Proposition 1. *Let* (X, d) *be a bounded invariant metric (Abelian) group. Then, the sets* $Lip^1_0(X)$*,* $Lip^1_+(X)$ *and* $Lip^1(X)$ *are (Abelian) monoids having* δ_e *as identity element and* $Lip(X)$ *is a (Abelian) semigroup.*

Proof. The proof is similar to [2, Proposition 1]. □

Note that since (X, d) is bounded, each function $f \in Lip^1(X)$ (resp. $f \in Lip(X)$) is hounded and so $d_\infty(f, g) := \sup_{x \in X} |f(x) - g(x)| < +\infty$ for all $f, g \in Lip^1(X)$ (resp. $f, g \in Lip(X)$). In this case, from [2, Lemma 1], we have that

$$\rho = \frac{d_\infty}{1 + d_\infty}$$

on $Lip(X)$. We also consider the following metric:

$$\theta_\infty(f, g) := d_\infty(f - \inf_X(f), g - \inf_X(g)) + |\inf_X(f) - \inf_X(g)|, \quad \forall f, g \in Lip(X).$$

Proposition 2. *Let* (X, d) *be a bounded invariant metric group. Then, the following map*

$$\begin{array}{rcl} \tau : (Lip^1(X), \theta_\infty) & \longrightarrow & (Lip^1_0(X) \times \mathbb{R}, d_\infty + |.|), \\ f & \mapsto & (f - \inf_X(f), \inf_X(f)) \end{array}$$

is an isomeric isomorphism of monoids, where $Lip^1_+(X) \times \mathbb{R}$ *is equipped with the operation* $\overline{\oplus}$ *defined by* $(f, c)\overline{\oplus}(f', c') := (f \oplus f', c + c')$.

Proof. Clearly, $(Lip^1_+(X) \times \mathbb{R}, \overline{\oplus})$ is a monoid having $(\delta_e, 0)$ as identity element, since $(Lip^1_+(X), \oplus)$ is a monoid having δ_e as identity element. It is also clear that τ is a monoid isomorphism. Now, τ is isometric by the definition of θ_∞. It follows that τ is an isometric isomorphism. □

The following proposition gives an alternative way to consider the group completion of invariant metric groups. Recall that if $(M, \cdot)$ is a monoid having e_M as identity element, the group of units of M is the set

$$\mathcal{U}(M) := \{m \in M / \ \exists m' \in M : m \cdot m' = m' \cdot m = e_M\}.$$

The symbol $\cong$ means isometrically isomorphic as groups. We give below an analogue to [2, Corollary 1], for each of the spaces $Lip^1_0(X), Lip^1(X)$ and $Lip(X)$. Note that in part(1) of the following proposition as in [2, Corollary 1], we do not need to assume that X is bounded.

Theorem 2. *Let (X, d) be a bounded invariant metric group. Then, we have that*

(1) $(\mathcal{U}(Lip^1_0(X)), d_\infty) = (\mathcal{U}(Lip^1_+(X)), d_\infty) \cong (\overline{X}, d)$,

(2) $(\mathcal{U}(Lip^1(X)), \theta_\infty) \cong (\overline{X} \times \mathbb{R}, d + |.|)$.

(3) *The group $\mathcal{U}(Lip^1(X))$ is the maximal subgroup of the semigroup $Lip(X)$, having δ_e as identity element.*

Proof. (1) The fact that $(\mathcal{U}(Lip^1_+(X)), d_\infty) \cong (\overline{X}, d)$, is given in [2, Corollary 1]. On the other hand, since, $\mathcal{G}(X) \subset \mathcal{U}(Lip^1_0(X)) \subset \mathcal{U}(Lip^1_+(X))$ and since $\overline{\mathcal{G}(X)} \cong \overline{X}$ (see [2, Lemma 2]) we get that $\mathcal{U}(Lip^1_0(X)) = \mathcal{U}(Lip^1_+(X))$. Let us prove part(2). Indeed, since τ (Proposition 2) is an isometric isomorphism, it sends isometrically the group of units onto the group of units. Hence, from Proposition 2 we have

$$(\mathcal{U}(Lip^1(X)), \oplus, \theta_\infty) \cong (\mathcal{U}(Lip^1_0(X) \times \mathbb{R}), \overline{\oplus}, d_\infty + |.|).$$

Since $\mathcal{U}(Lip^1_0(X) \times \mathbb{R}) = \mathcal{U}(Lip^1_0(X)) \times \mathbb{R}$, the conclusion follows from the first part. For part(3), let f be an element of the maximal group having δ_e as identity element. Then, $f \oplus \delta_e = f$ and so it follows that f is 1-Lipschitz map i.e $f \in Lip^1(X)$. Thus, $f \in \mathcal{U}(Lip^1(X))$. □

3. Examples

Let G be an algebraic group having e as identity element and let $f : G \longrightarrow \mathbb{R}^+$ be a function, we denote $Osc(f) := \sup_{t,t' \in G} |f(t) - f(t')|$ and by G^* we denote the following set :

$$G^* := \{f : G \longrightarrow \mathbb{R}^+ / Osc(f) \leq 1\}.$$

Note that the set G^* is just the set $Lip^1_+(G)$ where $(G, disc)$ is equipped with the discrete metric "*disc*", which is an invariant complete metric. So, $(G^*, \oplus)$ is a monoid having δ_e as identity element, where $\delta_e(\cdot) := disc(\cdot, e)$ i.e. $\delta_e(e) = 0$ and $\delta_e(t) = 1$ for all $t \neq e$. Observe also that two algebraic groups G and G' are isomorphic if and only they are isometrically isomorphic when equipped respectively with the discrete metric. Thus, we obtain that the algebraic group structure of any group G is completely determined by the algebraic monoid structure of $(G^*, \oplus)$.

Corollary 2. *Let G and G' be two groups. Then the following assertions are equivalent:*

(1) *the groups G and G' are isomorphic,*

(2) *the monoids $(G^*, \oplus, \rho)$ and $({G'}^*, \oplus, \rho)$ are isometrically isomorphic,*

(3) *the monoids $(G^*, \oplus, d_\infty)$ and $({G'}^*, \oplus, d_\infty)$ are isometrically isomorphic (where $d_\infty(f, g) := \sup_{t \in G} |f(t) - g(t)| < +\infty$, for all $f, g \in G^*$),*

(4) *the monoids $(G^*, \oplus)$ and $({G'}^*, \oplus)$ are isomorphic.*

Moreover, $\Phi : (G^*, \oplus, \rho) \longrightarrow ({G'}^*, \oplus, \rho)$ (resp. $\Phi : (G^*, \oplus, d_\infty) \longrightarrow ({G'}^*, \oplus, d_\infty)$) is an isometric isomorphism of monoids, if and only if there exists an isomorphism of groups $T : G \longrightarrow G'$ such that $\Phi(f) = f \circ T^{-1}$ for all $f \in G^*$.

Proof. Since $G^* = Lip^1_+(G)$, where G is equipped with the discrete metric and since G and G' are isomorphic if and only if $(G, disc)$ and $(G', disc)$ are isometrically isomorphic, then part (1) $\iff$ (2) is a direct consequence of Theorem 1. Part (2) $\Longrightarrow$ (3), follows from the fact that $\rho = \frac{d_\infty}{1+d_\infty}$ by using [2, Lemma 1]. Part (3) $\Longrightarrow$ (4) is trivial. Let us prove (4) $\Longrightarrow$ (1). Since an isomorphism of monoids sends the group of units onto the group of units, and since the group of units of G^* (resp. of ${G'}^*$) is isomorphic to G (resp. to G') by Theorem 2, we get that G and G' are isomorphic. The last assertion is given by Theorem 1. □

As mentioned in Remark 1, if $T : G \longrightarrow G'$ is an isomorphism, then $\Phi(f) := f \circ T^{-1} + \inf_G(f)$ for all $f \in G^*$ gives an isomorphism of monoids between G^* and ${G'}^*$ which is not isometric.

In the following example, we treat the case where G is a finite group.

Examples 1. *Let $n \geq 1$ and $(\mathbb{R}^n, d_\infty)$ the usual n-dimensional space equipped with the max-distance. The subsets M_+^n and M^n of $\mathbb{R}^n$ are defined as follows:*

$$M_+^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n / |x_i - x_j| \leq 1, \qquad 1 \leq i, j \leq n\},$$
$$M^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n / |x_i - x_j| \leq 1, \qquad 1 \leq i, j \leq n\}.$$

Let $G := \{g_1, g_2, \dots g_n\}$, be a group of cardinal n, where g_1 is the identity of G. We define the law $\star_G$ on $\mathbb{R}^n$ depending on G as follows: for all $x = (x_1, x_2, \dots, x_n)$ and all $y = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$,

$$x \star_G y = (z_1, z_2, \dots, z_n),$$

where for each $1 \leq k \leq n$,

$$z_k := \min\{x_i + y_j / g_i \cdot g_j = g_k, 1 \leq i, j \leq n\}.$$

Then

(1) *The set $(\mathbb{R}^n, \star_G)$ has a semigroup structure (and is Abelian if G is Abelian).*

(2) *The sets $(M_+^n, \star_G)$ and $(M^n, \star_G)$ are monoids having $e = (0, 1, 1, 1, \dots, 1)$ as identity element.*

(3) *Let G and G' be two groups of cardinal n. The monoids $(M_+^n, \star_G)$ and $(M_+^n, \star_{G'})$ are isomorphic if and only if the groups G and G' are isomorphic.*

(4) *We have that*

$$\mathcal{U}(M_+^n) \simeq G,$$
$$\mathcal{U}(M^n) \simeq G \times \mathbb{R}.$$

Moreover, the maximal subgroup of $(\mathbb{R}^n, \star_G)$ having e as identity element is isomorphic to the group $G \times \mathbb{R}$.

(5) *We have that*

$$Is_m(M_+^n) \simeq Aut(G).$$

The properties $(1) - (5)$ follows easily from the results of this note. It suffices to see that the semigroup $(\mathbb{R}^n, \star_G, d_\infty)$ can be identified isometrically to the semigroup $(Lip(G), \oplus, d_\infty)$ of all real-valued Lipschitz map on $(G, disc)$. Indeed, the map

$$\begin{aligned} i : (Lip(G), \oplus, d_\infty) &\longrightarrow (\mathbb{R}^n, \star_G, d_\infty), \\ f &\mapsto (f(g_1), \dots, f(g_n)) \end{aligned}$$

is an isometric isomorphism of semigroups. On the other hand, the subset M_+^n is identified to $Lip_+^1(G)$ and M^n is identified to $Lip^1(G)$.

References

1. B. G. Averbukh, *On unitary Cauchy filters on topological monoids*, Topol. Algebra Appl. **1** (2013), 46–59.
2. M. Bachir, *A Banach-Stone type theorem for invariant metric groups*, Topology Appl. **209** (2016), 189–197.
3. M. Bachir, *The inf-convolution as a law of monoid. An analogue to the Banach-Stone theorem*, J. Math. Anal. Appl. **420** (2014), no. 1, 145–166.
4. J. H. Carruth, J. A. Hildebrant and R. J. Koch, *The Theory of Topological Semigroups*, Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1983.
5. M. I. Garrido, J. A. Jaramillo, *Variations on the Banach-Stone theorem*, Extracta Math. **17** (2002), no. 3, 351–383.
6. M. I. Garrido, J. A. Jaramillo, *Lipschitz-type functions on metric spaces*, J. Math. Anal. Appl. **340** (2008), 282–290.
7. A. Jiménez-Vargas, M. Villegas-Vallecillos, *2-local isometries on spaces of Lipschitz functions*, Canad. Math. Bull. **54** (2011), 680–692.
8. V. L. Klee, *Invariant metrics in groups (solution of a problem of Banach)*, Proc. Amer. Math. Soc. **3** (1952), 484–487.

9. A. D. Wallace *On the structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112.
10. N. Weaver *Isometries of noncompact Lipschitz spaces*, Canad. Math. Bull. **38** (1995), 242–249.

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