Methods of Functional Analysis and Topology
Vol. 23 (2017), no. 4, pp. 320–345

SOME REMARKS ON OPERATORS OF STOCHASTIC DIFFERENTIATION IN THE LÉVY WHITE NOISE ANALYSIS

M. M. FREI AND N. A. KACHANOVSKY

ABSTRACT. Operators of stochastic differentiation, which are closely related with the extended Skorohod stochastic integral and with the Hida stochastic derivative, play an important role in the Gaussian white noise analysis. In particular, these operators can be used in order to study some properties of the extended stochastic integral and of solutions of so-called normally ordered stochastic equations.

During recent years, operators of stochastic differentiation were introduced and studied, in particular, on spaces of regular and nonregular test and generalized functions of the Lévy white noise analysis, in terms of Lytvynov's generalization of the chaotic representation property. But, strictly speaking, the existing theory in the "regular case" is incomplete without one more class of operators of stochastic differentiation, in particular, the mentioned operators are required in calculation of the commutator between the extended stochastic integral and the operator of stochastic differentiation. In the present paper we introduce this class of operators and study their properties. In addition, we establish a relation between the introduced operators and the corresponding operators on the spaces of nonregular test functions. The researches of the paper can be considered as a contribution to a further development of the Lévy white noise analysis.

INTRODUCTION

Consider a Lévy process $L = (L_t)_{t\in[0,+\infty)}$ (a random process on $[0, +\infty)$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [8, 30, 31] for details) without Gaussian part and drift. As is well known (e.g., [10]), the measure of the white noise of L can be defined as a probability measure μ on the dual Schwartz space $\mathcal{D}'$ with the cylindrical σ-algebra $\mathcal{C}(\mathcal{D}')$, see Definition 1.1 (a notion of a measure that describes a random process is discussed in, e.g., [16]; the case of generalized random processes and its measures is considered in [15]); the space of square integrable random variables (L^2) can be realized as $(L^2) = L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$; and the Lévy process L can be presented now as a generalized pairing (generated by the Lebesgue measure): $L_t(\circ) = \langle \circ, 1_{[0,t)}\rangle \in (L^2)$ (see Subsection 1.1 for details).

In the Gaussian (resp., Poissonian) analysis one can construct the extended Skorohod stochastic integral and the Hida stochastic derivative using a so-called *chaotic representation property* (CRP) of a Gaussian (resp., Poissonian) random process ([20, 34, 19]). This property consists, roughly speaking, in a possibility to decompose a square integrable random variable in a series of repeated stochastic integrals from nonrandom functions. But a general Lévy process has no such a property [36]. So, in order to construct and study the above-mentioned operators in the Lévy analysis "by the classic chart", it is necessary to have an analog (generalization) of the CRP.

One of the most useful and challenging generalizations of the CRP in the Lévy analysis is proposed by E. W. Lytvynov [27], see also [9]. His idea is to decompose elements of

2010 *Mathematics Subject Classification.* 60H05, 60G51, 60H40, 46F05, 46F25.
Key words and phrases. Operator of stochastic differentiation, stochastic derivative, extended stochastic integral, Lévy process.

(L^2) in series of special orthogonal functions (see Subsection 1.2), by analogy with decompositions of square integrable random variables by Hermite (resp., Charlier) polynomials in the Gaussian (resp., Poissonian) analysis. Note that the above-mentioned decompositions in the "classical" cases are equivalent to decompositions by repeated stochastic integrals.

In [23] the extended Skorohod stochastic integral with respect to L and the corresponding Hida stochastic derivative are constructed on (L^2) in terms of Lytvynov's generalization of the CRP, some properties of these operators are established; and it is shown that the above-mentioned integral coincides with the well-known (constructed in terms of Itô's generalization of the CRP [18]) extended stochastic integral with respect to a Lévy process (e.g., [11, 10]). In [21, 14] the stochastic integrals and derivatives are introduced and studied on spaces of regular and nonregular test and generalized functions that belong to so-called regular parametrized (see (1.9), Definitions 1.2, 1.3) and nonregular (see Definition 2.5, Remark 2.8) riggings of (L^2) respectively. This gives a possibility to extend an area of possible applications of the above-mentioned operators. In particular, now it is possible to define the stochastic integral and derivative as linear *continuous* operators. Note that elements of the above-mentioned spaces have orthogonal decompositions similar to Lytvynov's orthogonal decompositions of elements of (L^2). "Prototypes" of these spaces are considered in [5, 3]. Further, (L^2) and the spaces of regular test and generalized functions (Definitions 1.2, 1.3) are isomorphic to so-called *extended symmetric Fock spaces*. In the same way as (L^2) and its riggings, the extended Fock spaces are different in various versions of an analysis, depending on a random process under consideration and the corresponding measure of a white noise; but they have similar structure and properties. Some versions of such spaces are studied in [6]. Stochastic integrals and derivatives in terms of the extended Fock spaces that appears in the so-called Meixner white noise analysis, are considered in [26].

Together with the stochastic integrals and derivatives, it is natural to introduce and to study so-called *operators of stochastic differentiation* in the Lévy white noise analysis, by analogy with the Gaussian analysis [37, 1]. Roughly speaking, the operator of stochastic differentiation acts on Lytvynov's orthogonal decomposition of a random variable in common with an action of the differentiation operator on a Taylor decomposition of a function. In other words, one can understand the stochastic differentiation as a "differentiation" with respect to a "stochastic argument" (we recall that a random variable is a measurable function, and the "stochastic argument" is the argument of this function).

The operators of stochastic differentiation are closely related with the extended Skorohod stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative. They satisfy the Leibniz rule with respect to a so-called Wick product (a natural product in the spaces of generalized functions). By analogy with the Gaussian case, these operators can be used, in particular, in order to study some properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities (normally ordered stochastic equations in another terminology).

In [13, 12] the operators of stochastic differentiation on the spaces that belong to the regular parametrized rigging of (L^2), are introduced and studied; in [25, 24] similar operators are considered on the spaces of nonregular test and generalized functions.

An important property of the operators of stochastic differentiation on different spaces is described in corresponding theorems about the commutator between these operators and the extended stochastic integral. In particular, in the Lévy white noise analysis on the regular rigging of (L^2) such a theorem is proved in [12]. But, strictly speaking, a blemish is allowed: the operators of stochastic differentiation are defined on the spaces that belong to the regular rigging of (L^2) (Definition 2.1), but applied to elements of tensor products of these spaces and the space L^2 of square integrable with respect to the

Lebesgue measure functions. Of course, the sense of such a use of the above-mentioned operators is intuitively clear; nevertheless, it is necessary to correct the inadvertence – this is the main motivation of the present paper. In order to do this, by analogy with an analysis in the nonregular case [25, 24], we introduce a class of new (modified) operators of stochastic differentiation on tensor products of the spaces of regular test and generalized functions and L^2 (Definition 2.2). It is clear that it is natural not to confine ourselves by a theorem about the commutator (Theorem 2.2), but to study more properties of these new operators. In particular, we calculate a superposition of the operators of stochastic differentiation (Proposition 2.1); express kernels from orthogonal decompositions of test and generalized functions via the operators of stochastic differentiation (Proposition 2.2); calculate operators, adjoint to the operators of stochastic differentiation (Proposition 2.3). Further, we introduce *unbounded* operators of stochastic differentiation on the above-mentioned tensor products of spaces (Definition 2.4), and study some their properties. In particular, we show that these operators are closed (Proposition 2.5).

Finally, we establish that the restrictions of the introduced operators of stochastic differentiation to the spaces of nonregular test functions (Definition 2.5) coincide with the corresponding operators (Definition 2.6) on the just now mentioned spaces, this result is presented in Theorem 2.3. For "old" operators of stochastic differentiation the similar result is established in [25].

Note that important auxiliary results, necessary for our considerations, are presented in Lemmas 2.2 and 2.4; the interconnection between "old" and "new" operators of stochastic differentiation and connected objects is described in Remarks 2.3, 2.4 and 2.5.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct convenient for our considerations probability space connected with L; then, following [27, 23, 21], we describe in detail Lytvynov's generalization of the CRP; the regular parametrized rigging of (L^2); and the stochastic integrals and derivatives on the spaces that belong to this rigging. In the second section we deal with the operators of stochastic differentiation: the first subsection is devoted to introduction (including necessary preparations) of the just now mentioned operators and to study of their properties; in the second subsection we consider a relation between the operators of stochastic differentiation on different spaces.

We note that, as distinct from the papers [23, 21, 12, 13, 25, 24], in the present paper we work in the framework of the complex variant of the Lévy white noise analysis. Such an approach does not lead to an essential complication of the presentation and will be useful in forthcoming papers about a Wick calculus.

1. Preliminaries

In this paper we denote by $|\cdot|_H$ or $\|\cdot\|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the scalar product in a space H; and by $\langle\cdot,\cdot\rangle_H$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H. Another notation for norms, scalar products and dual pairings will be introduced when it will be necessary.

1.1. **A Lévy process and its probability space.** Denote $\mathbb{R}_+ := [0,+\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t\in\mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is well known (e.g., [11]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp\Big[t\int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x)\nu(dx)\Big], \tag{1.1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra; $\mathbb{E}$ denotes the expectation. We assume that *ν is a Radon* *measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that* $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty$, *and*

$$\int_{\mathbb{R}} x^2 \nu(dx) = 1. \tag{1.2}$$

Let us define the measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [7]; see also Subsection 2.2). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$. The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of riggings of $L^2(\mathbb{R}_+)$ and in tensor powers of complexifications of such riggings. (The reader can find a detailed information about riggings of a Hilbert space in, e.g., [7, 4].)

Definition 1.1. A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x)\, du\nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{1.3}$$

is called the *measure of a Lévy white noise.*

The existence of μ follows from the Bochner-Minlos theorem (e.g., [17]), see [27]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to* μ, i.e., we take the completion of $\mathcal{C}(\mathcal{D}')$ and preserve for this completion the previous designation. So, now $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote by $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) complex-valued square integrable with respect to μ functions on $\mathcal{D}'$. Let $f \in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k \in \mathcal{D})_{k\in\mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k \to \infty$ (as is well known (e.g., [7]), $\mathcal{D}$ is a dense set in $L^2(\mathbb{R}_+)$). One can show [27, 11, 10, 23] that $\langle \circ, f \rangle := (L^2) - \lim_{k\to\infty}\langle \circ, \varphi_k \rangle$ is well-defined as an element of (L^2).

Denote by 1_A the indicator of a set A. Put $1_{[0,0)} \equiv 0$ and consider $\langle \circ, 1_{[0,t)} \rangle \in (L^2)$, $t \in \mathbb{R}_+$. It follows from (1.1) and (1.3) that $\big(\langle \circ, 1_{[0,t)} \rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ (see, e.g., [11, 10]). So, one can write $L_t = \langle \circ, 1_{[0,t)} \rangle \in (L^2)$ (in the present paper this representation will not be used directly; but it was used for a construction of the stochastic integral with respect to a Lévy process, and is necessary in the theory of normally ordered stochastic equations).

1.2. **Lytvynov's generalization of the CRP.** As is known, some random processes L have a so-called *chaotic representation property* (CRP) that consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L (see, e.g., [28] for a detailed presentation). The CRP plays a very important role in the stochastic analysis (in particular, for processes with the CRP this property can be used in order to construct extended stochastic integrals [20, 34, 19], stochastic derivatives and operators of stochastic differentiation, e.g., [37, 1]), but, unfortunately, the only Lévy processes with the CRP are Wiener and Poisson processes (e.g., [36]).

There are different approaches to a generalization of the CRP for Lévy processes: Itô's approach [18], Nualart-Schoutens' approach [29, 32], Lytvynov's approach [27], Oksendal's approach [11, 10], etc. The interconnections between these generalizations of the CRP are described in, e.g., [27, 2, 38, 11, 35, 10, 23]. In the present paper we deal with Lytvynov's generalization of the CRP that will be described now in detail.

Denote by $\widehat{\otimes}$ a symmetric tensor product, by a subscript $\mathbb{C}$-complexifications of spaces, and put $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \quad N_f \in \mathbb{Z}_+, \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$. Since the measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (1.3) and properties of the measure ν, see also [27]), $\mathcal{P}$ is a dense set in (L^2) [33]. Denote by $\mathcal{P}_n$ the set of polynomials of power smaller or equal to n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \mathop{\oplus}_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\!\langle \circ^{\otimes n}, f^{(n)} \rangle\!:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto $\mathbf{P}_n$. Let us define *real* (i.e., *bilinear*) scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} :\!\langle \omega^{\otimes n}, f^{(n)} \rangle\!::\!\langle \omega^{\otimes n}, g^{(n)} \rangle\!: \mu(d\omega),$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{(f^{(n)}, \overline{f^{(n)}})_{ext}}$ (it easily follows from the definition that $(\cdot,\cdot)_{ext}$ are *quasiscalar* products on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, the fact that these products are *scalar* follows from their explicit formula calculated in [27], see also (1.5) below). Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!: \stackrel{\mathrm{def}}{=} (L^2) - \lim_{k\to\infty} :\!\langle \circ^{\otimes n}, f_k^{(n)} \rangle\!:$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$ (the well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\!\langle \circ^{\otimes n}, f^{(n)} \rangle\!: | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}\}$ is dense in $\mathbf{P}_n$, $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that

$$F = \sum_{n=0}^{\infty} :\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!: \tag{1.4}$$

and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{ext}^2 < \infty.$$

So, for $F, G \in (L^2)$ the real scalar product has a form

$$(F,G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega) G(\omega) \mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n! (F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (1.4) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}\big(:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!:, :\!\langle \circ^{\otimes m}, G^{(m)} \rangle\!:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\!\langle \omega^{\otimes n}, F^{(n)} \rangle\!::\!\langle \omega^{\otimes m}, G^{(m)} \rangle\!: \mu(d\omega) \\ &= \mathbb{E}\big[:\!\langle \circ^{\otimes n}, F^{(n)} \rangle\!::\!\langle \circ^{\otimes m}, G^{(m)} \rangle\!:\big] = \delta_{n,m} n! (F^{(n)}, G^{(n)})_{ext}.\end{aligned}$$

Also we note that in the space (L^2) $:\!\langle \circ^{\otimes 0}, F^{(0)} \rangle\!: = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $:\!\langle \circ, F^{(1)} \rangle\!: = \langle \circ, F^{(1)} \rangle$ [27].

In order to obtain many statements, connected with the spaces $\mathcal{H}_{ext}^{(n)}$, it is necessary to know an explicit formula for the scalar products $(\cdot,\cdot)_{ext}$. Let us write out this formula. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R},\nu)$ of (classes of) square integrable with respect to the Lévy measure ν (see (1.1)) real-valued functions on $\mathbb{R}$. Let

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R}, \quad j \in \{1,\dots,n-1\}, \quad n \in \mathbb{N},$$

be orthogonal in $L^2(\mathbb{R},\nu)$ polynomials, i.e., for natural numbers n,m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then, as it follows from [27], for $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$,

(1.5)

$$\begin{aligned}(F^{(n)},G^{(n)})_{ext} &= (F^{(n)},G^{(n)})_{\mathcal{H}_{ext}^{(n)}}\\ &= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\ &\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} F^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})\\ &\times G^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})\,du_1\cdots du_{s_1+\dots+s_k}.\end{aligned}$$

In particular, for $n=1$

(1.6)

$$(F^{(1)},G^{(1)})_{ext} = (F^{(1)},G^{(1)})_{\mathcal{H}_{ext}^{(1)}} = \|p_1\|_\nu^2 \int_{\mathbb{R}_+} F^{(1)}(u)G^{(1)}(u)\,du = (F^{(1)},G^{(1)})_{L^2(\mathbb{R}_+)_{\mathbb{C}}}$$

(by (1.2) $\|p_1\|_\nu^2 = \int_{\mathbb{R}} x^2\nu(dx) = 1$); in the case $n=2$ we have

$$\begin{aligned}(F^{(2)},G^{(2)})_{ext} &= (F^{(2)},G^{(2)})_{\mathcal{H}_{ext}^{(2)}} = \|p_1\|_\nu^4 \int_{\mathbb{R}_+^2} F^{(2)}(u_1,u_2)G^{(2)}(u_1,u_2)\,du_1du_2\\ &+ \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} F^{(2)}(u,u)G^{(2)}(u,u)\,du\\ &= (F^{(2)},G^{(2)})_{L^2(\mathbb{R}_+)_{\mathbb{C}}^{\otimes 2}} + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} F^{(2)}(u,u)G^{(2)}(u,u)\,du,\end{aligned}$$

etc.

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (this notation will be used very often in the future). It follows from (1.6) that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$; and, as is easily seen, for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k,j \in \{1,\dots,n\}$: $k \neq j$, but $u_k = u_j$). In this sense the *space* $\mathcal{H}_{ext}^{(n)}$ *is an extension of* $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ (this explains why we use the subscript *ext* in the designations $\mathcal{H}_{ext}^{(n)}$, $(\cdot,\cdot)_{ext}$ and $|\cdot|_{ext}$).

1.3. **A regular rigging of (L^2).** Denote

$$\mathcal{P}_W := \Big\{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\Big\} \subset (L^2). \tag{1.7}$$

Accept on default $\beta \in [0,1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0,1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Define *real* (i.e., *bilinear*) scalar products $(\cdot,\cdot)_{q,\beta}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \quad g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W,$$

$$(f,g)_{q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{ext}$$

(it is easy to verify that the axioms of a scalar product are fulfilled). Let $\|\cdot\|_{q,\beta}$ be the corresponding norms, i.e., $\|f\|_{q,\beta} = \sqrt{(f,\overline{f})_{q,\beta}}$. Denote by $(L^2)^\beta_q$ the completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{q,\beta}$; and set $(L^2)^\beta := \mathrm{pr}\lim_{q\to+\infty}(L^2)^\beta_q$ (the projective limit of spaces, see, e.g., [4, 7]).

Definition 1.2. The spaces $(L^2)^\beta_q$ and $(L^2)^\beta$ are called parametrized Kondratiev-type spaces of regular test functions.

As is easy to see, $F \in (L^2)^\beta_q$ if and only if F can be uniquely presented in form (1.4) with $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$ and

$$\|F\|^2_{q,\beta} := \|F\|^2_{(L^2)^\beta_q} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |F^{(n)}|^2_{ext} < \infty; \tag{1.8}$$

and for $F, G \in (L^2)^\beta_q$

$$(F,G)_{(L^2)^\beta_q} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (1.4) for F and G respectively. Further, $F \in (L^2)^\beta$ if and only if F can be uniquely presented in form (1.4) and series (1.8) converges for each $q \in \mathbb{Z}_+$.

Proposition 1.1. ([21]). *For any $\beta \in (0,1]$ and $q \in \mathbb{Z}$ (in the same way as for $\beta = 0$ and any $q \in \mathbb{Z}_+$) the space $(L^2)^\beta_q$ is densely and continuously embedded into (L^2).*

In view of this proposition, consider a chain (a parametrized regular rigging of (L^2))

$$(L^2)^{-\beta} \supset (L^2)^{-\beta}_{-q} \supset (L^2) \supset (L^2)^\beta_q \supset (L^2)^\beta, \tag{1.9}$$

where $(L^2)^{-\beta}_{-q}$ and $(L^2)^{-\beta} = \mathrm{ind}\lim_{q\to+\infty}(L^2)^{-\beta}_{-q}$ (the inductive limit of spaces, see, e.g., [4, 7]) are the spaces dual of $(L^2)^\beta_q$ and $(L^2)^\beta$ respectively with respect to (L^2).

Definition 1.3. The spaces $(L^2)^{-\beta}_{-q}$ and $(L^2)^{-\beta}$ are called parametrized Kondratiev-type spaces of regular generalized functions.

The following statement from the definition of $(L^2)^{-\beta}_{-q}$ and the general duality theory follows.

Proposition 1.2. *1) Any regular generalized function $F \in (L^2)^{-\beta}_{-q}$ can be uniquely presented as formal series* (1.4) *(with coefficients $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$) that converges in $(L^2)^{-\beta}_{-q}$, i.e.,*

$$\|F\|^2_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|^2_{ext} < \infty, \tag{1.10}$$

and, vice versa, any formal series (1.4) *such that series* (1.10) *converges, is a regular generalized function from $(L^2)^{-\beta}_{-q}$ (i.e., now series* (1.4) *converges in $(L^2)^{-\beta}_{-q}$);*

2) for $F, G \in (L^2)^{-\beta}_{-q}$ the scalar product has a form

$$(F,G)_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}(F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (1.4) *for F and G respectively;*

3) the dual pairing between $F \in (L^2)^{-\beta}_{-q}$ and $f \in (L^2)^{\beta}_{q}$ that is generated by the scalar product in (L^2), has a form

$$\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{n=0}^{\infty} n!(F^{(n)}, f^{(n)})_{ext}, \tag{1.11}$$

where $F^{(n)}, f^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (1.4) *for F and f respectively;*

4) $F \in (L^2)^{-\beta}$ if and only if F can be uniquely presented in form (1.4) *and norm* (1.10) *is finite for some $q \in \mathbb{Z}_+$.*

In what follows, it will be convenient to denote the spaces $(L^2)^{\beta}_{q}$, $(L^2) = (L^2)^0_0$, $(L^2)^{-\beta}_{-q}$ from chain (1.9) by $(L^2)^{\beta}_{q}$, $\beta \in [-1,1]$, $q \in \mathbb{Z}$ (we accept this on default). The norms in these spaces are given, obviously, by formula (1.8) (cf. (1.8) and (1.10)).

1.4. **Stochastic integrals and derivatives.** Decomposition (1.4) for elements of $(L^2)^{\beta}_{q}$ defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism)

$$\mathbf{I} : (L^2)^{\beta}_{q} \to \mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}^{(n)}_{ext},$$

where $\mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}^{(n)}_{ext}$ is a weighted extended symmetric Fock space (cf. [26]): for $F \in (L^2)^{\beta}_{q}$ of form (1.4) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \dots) \in \mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}^{(n)}_{ext}$. Let $\mathbf{1} : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ be the identity operator. Then the operator $\mathbf{I}\otimes\mathbf{1} : (L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}} \to \big(\mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}^{(n)}_{ext}\big)\otimes\mathcal{H}_{\mathbb{C}} \cong \mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}})$ is an isometric isomorphism between the spaces $(L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}$ and $\mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}})$. It is clear that for arbitrary $n \in \mathbb{Z}_+$ and $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ a vector $(\underbrace{0,\dots,0}_{n}, F^{(n)}_{\cdot}, 0, \dots)$ belongs to $\mathop{\oplus}_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}})$. Set

$$:\!\langle\circ^{\otimes n}, F^{(n)}_{\cdot}\rangle\!: \stackrel{def}{=} (\mathbf{I}\otimes\mathbf{1})^{-1}(\underbrace{0,\dots,0}_{n}, F^{(n)}_{\cdot}, 0, \dots) \in (L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}. \tag{1.12}$$

By the construction elements $:\!\langle\circ^{\otimes n}, F^{(n)}_{\cdot}\rangle\!:$, $n \in \mathbb{Z}_+$, form orthogonal bases in the spaces $(L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}$ in the sense that any $F \in (L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}$ can be uniquely presented as

$$F(\cdot) = \sum_{n=0}^{\infty} :\!\langle\circ^{\otimes n}, F^{(n)}_{\cdot}\rangle\!:, \quad F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}} \tag{1.13}$$

(the series converges in $(L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}$), with

$$\|F\|^2_{(L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}|F^{(n)}_{\cdot}|^2_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} < \infty. \tag{1.14}$$

Further, consider a chain (cf. (1.9)) $(L^2)^{-\beta}_{-q}\otimes\mathcal{H}_{\mathbb{C}} \supset (L^2)\otimes\mathcal{H}_{\mathbb{C}} \supset (L^2)^{\beta}_{q}\otimes\mathcal{H}_{\mathbb{C}}$, here $\beta \in [0,1]$, $q \in \mathbb{Z}$ if $\beta > 0$ or $q \in \mathbb{Z}_+$ if $\beta = 0$. It is clear that the dual pairing between

$F \in (L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ and $f \in (L^2)^{\beta}_{q} \otimes \mathcal{H}_{\mathbb{C}}$ that is generated by the scalar product in $(L^2)\otimes\mathcal{H}_{\mathbb{C}}$, has a form

$$\langle\langle F, f\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} n!(F^{(n)}_{\cdot}, f^{(n)}_{\cdot})_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}, \tag{1.15}$$

where $F^{(n)}_{\cdot}, f^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decompositions (1.13) for F and f respectively (cf. (1.11)).

Let us describe the construction of an extended stochastic integral that is based on decomposition (1.13) (a detailed presentation is given in [23, 21]). Let $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}^{(n)}_{\cdot} \in F^{(n)}_{\cdot}$ such that

$$\dot{f}^{(n)}_{u}(u_1, \dots, u_n) = 0 \quad \text{if for some} \quad \mathrm{k} \in \{1, \dots, \mathrm{n}\} \quad u = u_k. \tag{1.16}$$

Accept on default $\Delta \in \mathcal{B}(\mathbb{R}_+)$. Let $\widehat{f}^{(n)}_{\Delta}$ be the symmetrization of a function $\dot{f}^{(n)}_{\cdot} 1_{\Delta}(\cdot)$ by $n+1$ variables. Define $\widehat{F}^{(n)}_{\Delta} \in \mathcal{H}^{(n+1)}_{ext}$ as the equivalence class in $\mathcal{H}^{(n+1)}_{ext}$ generated by $\widehat{f}^{(n)}_{\Delta}$ (i.e., $\widehat{f}^{(n)}_{\Delta} \in \widehat{F}^{(n)}_{\Delta}$).

Lemma 1.1. ([21, 23]). *For each $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{N}$, the element $\widehat{F}^{(n)}_{\Delta} \in \mathcal{H}^{(n+1)}_{ext}$ is well-defined (in particular, $\widehat{F}^{(n)}_{\Delta}$ does not depend on a choice of a representative $\dot{f}^{(n)}_{\cdot} \in F^{(n)}_{\cdot}$ satisfying* (1.16)*) and*

$$|\widehat{F}^{(n)}_{\Delta}|_{ext} \le |F^{(n)}_{\cdot} 1_{\Delta}(\cdot)|_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \le |F^{(n)}_{\cdot}|_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}. \tag{1.17}$$

Definition 1.4. We define the extended stochastic integral

$$\int_{\Delta} \circ(u)\, \widehat{d}L_u : (L^2)^{\beta}_{q} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}_{q-1} \tag{1.18}$$

by a formula

$$\int_{\Delta} F(u)\, \widehat{d}L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}^{(n)}_{\Delta}\rangle:, \tag{1.19}$$

where $\widehat{F}^{(0)}_{\Delta} := F^{(0)}_{\cdot} 1_{\Delta}(\cdot) \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}^{(1)}_{ext}$; and $\widehat{F}^{(n)}_{\Delta} \in \mathcal{H}^{(n+1)}_{ext}$, $n \in \mathbb{N}$, are constructed by the kernels $F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (1.13) for F.

One can show as in [21] that this integral is a linear *continuous* operator. Moreover, as appears from [23], if F is integrable by Itô (i.e., $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}} = (L^2)^0_0 \otimes \mathcal{H}_{\mathbb{C}}$ and is adapted with respect to the flow of σ-algebras generated by the Lévy process L) then F is integrable in the extended sense and $\int_{\Delta} F(u)\, \widehat{d}L_u = \int_{\Delta} F(u)\, dL_u \in (L^2)$, where $\int_{\Delta} F(u)\, dL_u$ is the Itô stochastic integral (this explains why the integral $\int_{\Delta} \circ(u)\, \widehat{d}L_u$ is called the *extended* one).

Sometimes it can be convenient to define the extended stochastic integral by formula (1.19) as a linear operator

$$\int_{\Delta} \circ(u)\, \widehat{d}L_u : (L^2)^{\beta}_{q} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}_{q}. \tag{1.20}$$

If $\beta = -1$ then this operator is continuous [21], for $\beta \in (-1, 1]$ this is not the case. But if we accept the set

$$\Big\{F \in (L^2)^{\beta}_{q} \otimes \mathcal{H}_{\mathbb{C}} : \Big\| \int_{\Delta} F(u)\, \widehat{d}L_u \Big\|^2_{q,\beta} = \sum_{n=0}^{\infty} ((n+1)!)^{1+\beta} 2^{q(n+1)} |\widehat{F}^{(n)}_{\Delta}|^2_{ext} < \infty \Big\}$$

as the domain of integral (1.20) then the last will be a *closed* operator [21]. Also we note that the extended stochastic integral can be defined by formula (1.19) as a linear continuous operator acting from $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}} := \mathrm{pr}\lim_{q\to+\infty}(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^\beta$, or from $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} := \mathrm{ind}\lim_{q\to+\infty}(L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-\beta}$, here $\beta \in [0,1]$.

At last, we recall briefly a notion of the Hida stochastic derivative in the Lévy white noise analysis, in terms of Lytvynov's CRP (see [23, 21, 14] for a detailed presentation).

Definition 1.5. We define the Hida stochastic derivative $1_\Delta(\cdot)\partial. : (L^2)^{-\beta}_{1-q} \to (L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ as a linear continuous operator adjoint to extended stochastic integral (1.18), i.e., for all $F \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ and $G \in (L^2)^{-\beta}_{1-q}$

$$\langle\!\langle F(\cdot), 1_\Delta(\cdot)\partial.G\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\!\langle \int_\Delta F(u)\,\widehat{d}L_u, G\rangle\!\rangle_{(L^2)}.$$

If instead of integral (1.18) one uses integral (1.20), the corresponding Hida stochastic derivative will be a linear unbounded (except the case $\beta = -1$), but closed operator acting from $(L^2)^{-\beta}_{-q}$ to $(L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ [14]. It is clear also that the Hida stochastic derivative can be defined as a linear continuous operator acting from $(L^2)^\beta$ to $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ ($\beta \in [-1,1]$) that is adjoint to the corresponding extended stochastic integral.

In order to write out an explicit formula for the Hida stochastic derivative in terms of decompositions by the Wick monomials, we need some preparation. Let $G^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in G^{(n)}$ be a representative of $G^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate a one argument of $\dot{g}^{(n)}$, and define $G^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ as the equivalence class in $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ generated by $\dot{g}^{(n)}(\cdot)$ (i.e., $\dot{g}^{(n)}(\cdot) \in G^{(n)}(\cdot)$).

Lemma 1.2. ([23]). *For each $G^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, the element $G^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ is well-defined (in particular, $G^{(n)}(\cdot)$ does not depend on a choice of a representative $\dot{g}^{(n)} \in G^{(n)}$) and*

$$|G^{(n)}(\cdot)|_{\mathcal{H}^{(n-1)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \le |G^{(n)}|_{ext}. \tag{1.21}$$

Note that, in spite of estimate (1.21), the space $\mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}\setminus\{1\}$, can not be considered as a subspace of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ because different elements of $\mathcal{H}^{(n)}_{ext}$ can coincide as elements of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$.

The following statement easily follows from results of [23, 21, 14].

Proposition 1.3. *For a test or square integrable or generalized function G of form* (1.4)

$$1_\Delta(\cdot)\partial.G = \sum_{n=1}^\infty n{:}\langle\circ^{\otimes n-1}, G^{(n)}(\cdot)1_\Delta(\cdot)\rangle{:} \equiv \sum_{n=0}^\infty (n+1){:}\langle\circ^{\otimes n}, G^{(n+1)}(\cdot)1_\Delta(\cdot)\rangle{:}\,. \tag{1.22}$$

Finally, we note that the extended stochastic integral and the Hida stochastic derivative are mutually adjoint operators [23, 21, 14].

2. Operators of stochastic differentiation

2.1. **Stochastic differentiation on spaces of regular test and generalized functions.** In order to define operators of stochastic differentiation on the spaces $(L^2)^\beta_q$, we need some preparation. Let $n, m \in \mathbb{Z}_+$. Consider a function $H : \mathbb{R}^{n+m}_+ \to \mathbb{C}$. Denote

$$\begin{aligned}&\widetilde{H}(u_1,\dots,u_n;u_{n+1},\dots,u_{n+m})\\ &:= \begin{cases} H(u_1,\dots,u_{n+m}), & \text{if for all } i \in \{1,\dots,n\}, j \in \{n+1,\dots,n+m\}\ u_i \neq u_j\\ 0, & \text{in other cases}\end{cases}.\end{aligned} \tag{2.1}$$

Let $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $G^{(m)} \in \mathcal{H}^{(m)}_{ext}$. We select representatives (functions) $\dot{f}^{(n)} \in F^{(n)}$ and $\dot{g}^{(m)} \in G^{(m)}$. Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of $\dot{f}^{(n)} \cdot \dot{g}^{(m)}$ by all variables,

$F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ be the equivalence class in $\mathcal{H}_{ext}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$ (i.e., $\widehat{f^{(n)}g^{(m)}} \in F^{(n)} \diamond G^{(m)}$). In a sense the following statement is a generalization of Lemma 1.1.

Lemma 2.1. ([12]). *The element $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ is well-defined (in particular, $F^{(n)} \diamond G^{(m)}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}$) and*

$$|F^{(n)} \diamond G^{(m)}|_{ext} \le |F^{(n)}|_{ext}|G^{(m)}|_{ext}. \tag{2.2}$$

Remark 2.1. Nonstrictly speaking, $F^{(n)} \diamond G^{(m)}$ is the symmetrization of a function

$$\begin{cases} F^{(n)}(u_1,\dots,u_n)G^{(m)}(u_{n+1},\dots,u_{n+m}), & \text{if } \forall i \in \{1,\dots,n\}, \\ & j \in \{n+1,\dots,n+m\}\ u_i \neq u_j \\ 0, & \text{in other cases} \end{cases} .$$

by all arguments.

Let $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $m > n$. We define a "product" $(f^{(n)}, F^{(m)})_{ext} \in \mathcal{H}_{ext}^{(m-n)}$ by setting for each $g^{(m-n)} \in \mathcal{H}_{ext}^{(m-n)}$

$$(g^{(m-n)}, (f^{(n)}, F^{(m)})_{ext})_{ext} = (f^{(n)} \diamond g^{(m-n)}, F^{(m)})_{ext}. \tag{2.3}$$

Since by the Cauchy-Bunyakovsky inequality and (2.2)

$$|(f^{(n)} \diamond g^{(m-n)}, F^{(m)})_{ext}| \le |f^{(n)} \diamond g^{(m-n)}|_{ext}|F^{(m)}|_{ext} \le |f^{(n)}|_{ext}|g^{(m-n)}|_{ext}|F^{(m)}|_{ext},$$

this definition is well-posed and

$$|(f^{(n)}, F^{(m)})_{ext}|_{ext} \le |f^{(n)}|_{ext}|F^{(m)}|_{ext}. \tag{2.4}$$

Definition 2.1. Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$. We define an operator of stochastic differentiation

$$(D^n\circ)(f^{(n)}) : (L^2)_q^\beta \to (L^2)_{q-1}^\beta \tag{2.5}$$

by setting for $F \in (L^2)_q^\beta$

$$\begin{aligned} (D^nF)(f^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, (f^{(n)}, F^{(m)})_{ext}\rangle: \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, (f^{(n)}, F^{(m+n)})_{ext}\rangle:, \end{aligned} \tag{2.6}$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (1.4) for F.

Using estimate (2.4) one can show [12] that this definition is well-posed, operator (2.5) is a linear *continuous* one; and, in addition, for each $F \in (L^2)_q^\beta$ $(D^nF)(\circ)$ is a linear continuous operator acting from $\mathcal{H}_{ext}^{(n)}$ to $(L^2)_{q-1}^\beta$. Moreover, in the case $\beta = 1$ formula (2.6) defines a linear *continuous* operator $(D^n\circ)(f^{(n)})$ on $(L^2)_q^1$, $q \in \mathbb{Z}$. Finally, as is easily seen, $(D^n\circ)(f^{(n)})$ can be defined by formula (2.6) as a linear continuous operator on $(L^2)^\beta$, $\beta \in [-1, 1]$.

Let us recall main properties of the operators of stochastic differentiation. Denote $D := D^1$, $\partial. := 1_{[0,+\infty)}(\cdot)\partial.$ (see (1.22)).

Theorem 2.1. ([12]). *1) For $k_1, \dots, k_m \in \mathbb{N}$, $f_j^{(k_j)} \in \mathcal{H}_{ext}^{(k_j)}$, $j \in \{1,\dots,m\}$,*

$$(D^{k_m}(\cdots(D^{k_2}((D^{k_1}\circ)(f_1^{(k_1)})))(f_2^{(k_2)})\cdots))(f_m^{(k_m)}) = (D^{k_1+\dots+k_m}\circ)(f_1^{(k_1)} \diamond \dots \diamond f_m^{(k_m)}).$$

2) For each $F \in (L^2)_q^\beta$ the kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, from decomposition (1.4) *can be presented in a mnemonic form*

$$F^{(n)} = \frac{1}{n!}\mathbb{E}(D^n F),$$

i.e., for each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ $(F^{(n)}, f^{(n)})_{ext} = \frac{1}{n!}\mathbb{E}((D^nF)(f^{(n)}))$, here $\mathbb{E}\circ := \langle\!\langle \circ, 1\rangle\!\rangle_{(L^2)}$ is a generalized expectation.

3) The operator adjoint to D^n, $n \in \mathbb{N}$, has a form

$$(D^nG)(f^{(n)})^* = \sum_{m=0}^{\infty} :\langle \circ^{m+n}, f^{(n)} \diamond G^{(m)}\rangle :\in (L^2)_{-q}^{-\beta}, \tag{2.7}$$

here $G \in (L^2)_{1-q}^{-\beta}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$; $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (1.4) *for G.*

4) For all $G \in (L^2)_{1-q}^{-\beta}$ and $f^{(1)} \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$

$$(DG)(f^{(1)})^* = \int_{\mathbb{R}_+} G \cdot f^{(1)}(u)\, \widehat{d}L_u \in (L^2)_{-q}^{-\beta}.$$

5) For all $F \in (L^2)_q^\beta$ and $f^{(1)} \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$

$$(DF)(f^{(1)}) = \int_{\mathbb{R}_+} \partial_u F \cdot f^{(1)}(u)\, du \in (L^2)_{q-1}^{\beta}, \tag{2.8}$$

here the integral in the right hand side is a Pettis one (the weak integral).

Remark 2.2. The Pettis integral $\int_{\mathbb{R}_+} \partial_u F \cdot f^{(1)}(u)du$ from the right hand side of (2.8) is equal to $\langle \partial_\cdot F, f^{(1)}(\cdot)\rangle$ – a partial pairing, i.e., a unique element of $(L^2)_{q-1}^\beta$ such that for each $G \in (L^2)_{1-q}^{-\beta}$

$$\langle\!\langle G, \langle \partial_\cdot F, f^{(1)}(\cdot)\rangle\rangle\!\rangle_{(L^2)} = \langle\!\langle G \otimes f^{(1)}(\cdot), \partial_\cdot F\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$$

(cf. (2.3) and the construction of a "product" $(f^{(n)}, F^{(m)})_{ext}$).

In some applications of the Gaussian analysis (in particular, in the Malliavin calculus) an important role belongs to the commutator between the extended stochastic integral and the operator of stochastic differentiation (see, e.g., [1]). In the Lévy analysis on the *regular* rigging of (L^2) such commutator is calculated in [12]; but, strictly speaking, the calculation has a blemish: the operators of stochastic differentiation were defined on the (real) spaces $(L^2)_q^\beta$ (as above), but applied to elements of the spaces $(L^2)_q^\beta \otimes \mathcal{H}$. So, although the sense of such a use of the above-mentioned operators is intuitively clear, it is necessary to correct the inadvertence.

Again we begin with a preparation. Let $n, m \in \mathbb{Z}_+$. Consider a function $H_\cdot : \mathbb{R}_+^{n+m+1} \to \mathbb{C}$. Denote

$$\begin{aligned} &\widetilde{H}_u(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m}) \\ &\quad := \begin{cases} H_u(u_1, \dots, u_{n+m}), & \text{if for all } i \in \{1, \dots, n\}, j \in \{n+1, \dots, n+m\}\ u_i \neq u_j \\ 0, & \text{in other cases} \end{cases} \end{aligned} \tag{2.9}$$

(cf. (2.1)). Let $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$. We select representatives (functions) $\dot{f}^{(n)} \in F^{(n)}$, $\dot{g}_\cdot^{(m)} \in G_\cdot^{(m)}$, and set $f^{(n)}g_\cdot^{(m)} := \dot{f}^{(n)} \cdot \dot{g}_\cdot^{(m)}$. Let $\widehat{f^{(n)}g_\cdot^{(m)}}$ be the symmetrization of $f^{(n)}g_\cdot^{(m)}$ by $n+m$ variables, except the variable '$\cdot$' (to put it in another way, $\widehat{f^{(n)}g_u^{(m)}}(u_1, \dots, u_{n+m})$ is the symmetrization of $f^{(n)}g_u^{(m)}(u_1, \dots, u_{n+m})$ by

$u_1,\dots,u_{n+m})$, $F^{(n)}\Diamond G^{(m)}_{\cdot} \in \mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ be the equivalence class in $\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ that is generated by $\widehat{f^{(n)}g^{(m)}_{\cdot}}$ (i.e., $\widehat{f^{(n)}g^{(m)}_{\cdot}} \in F^{(n)}\Diamond G^{(m)}_{\cdot}$).

Lemma 2.2. (cf. Lemma 2.1). *The element* $F^{(n)}\Diamond G^{(m)}_{\cdot} \in \mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ *is well-defined (in particular,* $F^{(n)}\Diamond G^{(m)}_{\cdot}$ *does not depend on a choice of representatives from* $F^{(n)}$ *and* $G^{(m)}_{\cdot}$*) and*

$$|F^{(n)}\Diamond G^{(m)}_{\cdot}|_{\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \le |F^{(n)}|_{\mathcal{H}^{(n)}_{ext}}|G^{(m)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}. \tag{2.10}$$

Proof. The proof of this statement is quite analogous to the proof of Lemma 2.1 (see [12]), therefore we shall confine ourselves to the short description of main steps.

1) For $n=0$ or $m=0$ the statement of the lemma is, obviously, true, therefore we consider the case $n,m\in\mathbb{N}$. Suppose that $m\ge n$ (the proof in the case $m<n$ is completely analogous). Let $\dot f^{(n)}\in F^{(n)}$, $\dot g^{(m)}_{\cdot}\in G^{(m)}_{\cdot}$. Without loss of generality we can think that $\dot f^{(n)}(u_1,\dots,u_n)$ and $\dot g^{(m)}_u(u_{n+1},\dots,u_{n+m})$ are symmetric functions with respect to arguments $u_1,\dots,u_n$ and $u_{n+1},\dots,u_{n+m}$ respectively. Using this symmetry, by analogy with [12] one can show that

$$\begin{aligned}\widehat{f^{(n)}g^{(m)}_u}(u_1,\dots,u_{n+m}) &= \frac{n!m!}{(n+m)!}\\ \times \sum_{\substack{1\le p_1,\dots,p_n\le n,\, n+1\le q_1,\dots,q_m\le n+m\\ 0\le r\le n,\, p_1<\dots<p_r,\, p_{r+1}<\dots<p_n,\, q_1<\dots<q_{n-r},\, q_{n-r+1}<\dots<q_m}} &\widetilde{f^{(n)}g^{(m)}_u}(u_{p_1},\dots,u_{p_r},u_{q_1},\dots,u_{q_{n-r}};\\ &u_{p_{r+1}},\dots,u_{p_n},u_{q_{n-r+1}},\dots,u_{q_m})\end{aligned} \tag{2.11}$$

(for $r=n$ the argument in the right hand side of (2.11) is '$u_1,\dots,u_n;u_{n+1},\dots,u_{n+m}$'; for $r=0$ this argument is '$u_{q_1},\dots,u_{q_n};u_1,\dots,u_n,u_{q_{n+1}},\dots,u_{q_m}$'). To put it in another way, transcribed in parentheses arguments of $\widetilde{f^{(n)}g^{(m)}_{\cdot}}$ are different combinations of u_j, $j\in\{1,\dots,n+m\}$, in which the indexes of n first and m last arguments (before and after ';') are (independently) ordered in ascending.

2) As appears from (1.5), the norms in the spaces $\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$ have a form

$$\begin{aligned}|H^{(n+m)}_{\cdot}|^2_{\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} &= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{(n+m)!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\ &\times\int_{\mathbb{R}_+^{s_1+\dots+s_k+1}}|H^{(n+m)}_u(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})|^2\\ &\times du_1\cdots du_{s_1+\dots+s_k}du,\quad H^{(n+m)}_{\cdot}\in\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}.\end{aligned}$$

Substituting (2.11) in this formula and evaluating by analogy with [12] we obtain

$$|\widehat{f^{(n)}g^{(m)}_{\cdot}}|_{\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}} \le |\dot f^{(n)}|_{\mathcal{H}^{(n)}_{ext}}|\dot g^{(m)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}. \tag{2.12}$$

So, a function $\widehat{f^{(n)}g^{(m)}_{\cdot}}$ generates an element $F^{(n)}\Diamond G^{(m)}_{\cdot}\in\mathcal{H}^{(n+m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$.

3) Using (2.12) and properties of the operation $\widehat{\circ}$, similarly to [12] one can show that $F^{(n)}\Diamond G^{(m)}_{\cdot}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}_{\cdot}$. Estimate (2.10) follows from (2.12). □

Remark 2.3. Let $F^{(n)}\in\mathcal{H}^{(n)}_{ext}$, $G^{(m)}\in\mathcal{H}^{(m)}_{ext}$, $n,m\in\mathbb{Z}_+$, $H^{(1)}\in\mathcal{H}_{\mathbb{C}}$. We select representatives (functions) $\dot f^{(n)}\in F^{(n)}$, $\dot g^{(m)}\in G^{(m)}$ and $\dot h^{(1)}\in H^{(1)}$. It is clear that $\dot g^{(m)}\cdot\dot h^{(1)}$ is a representative of $G^{(m)}\otimes H^{(1)}\in\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}$. It follows from (2.1) and (2.9) that $\dot{f}^{(n)} \cdot (\widetilde{\dot{g}^{(m)} \cdot \dot{h}^{(1)}(\cdot)}) = (\widetilde{\dot{f}^{(n)} \cdot \dot{g}^{(m)}}) \cdot \dot{h}^{(1)}(\cdot))$ and therefore $\dot{f}^{(n)} \cdot (\widehat{\dot{g}^{(m)} \cdot \dot{h}^{(1)}(\cdot)}) = (\widehat{\dot{f}^{(n)} \cdot \dot{g}^{(m)}}) \cdot \dot{h}^{(1)}(\cdot))$, whence

$$F^{(n)} \lozenge (G^{(m)} \otimes H^{(1)}(\cdot)) = (F^{(n)} \diamond G^{(m)}) \otimes H^{(1)}(\cdot). \tag{2.13}$$

Let $F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $m \geq n$. We define a "product" $(f^{(n)}, F^{(m)}_{\cdot})_{EXT} \in \mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ by setting for each $g^{(m-n)}_{\cdot} \in \mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$

$$(g^{(m-n)}_{\cdot}, (f^{(n)}, F^{(m)}_{\cdot})_{EXT})_{\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} = (f^{(n)} \lozenge g^{(m-n)}_{\cdot}, F^{(m)}_{\cdot})_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}. \tag{2.14}$$

Since by the Cauchy-Bunyakovsky inequality and (2.10)

$$\begin{aligned} |(f^{(n)} \lozenge g^{(m-n)}_{\cdot}, F^{(m)}_{\cdot})_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}| &\leq |f^{(n)} \lozenge g^{(m-n)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} |F^{(m)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \\ &\leq |f^{(n)}|_{\mathcal{H}^{(n)}_{ext}} |g^{(m-n)}_{\cdot}|_{\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} |F^{(m)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}, \end{aligned}$$

this definition is well-posed and

$$|(f^{(n)}, F^{(m)}_{\cdot})_{EXT}|_{\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \leq |f^{(n)}|_{\mathcal{H}^{(n)}_{ext}} |F^{(m)}_{\cdot}|_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}. \tag{2.15}$$

Remark 2.4. For $n, m \in \mathbb{Z}_+$, $m \geq n$, let $F^{(m)} \in \mathcal{H}^{(m)}_{ext}$, $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$; $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$; $g^{(m-n)} \in \mathcal{H}^{(m-n)}_{ext}$, $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$. By (2.14), (2.13) and (2.3) we obtain

$$\begin{aligned} &(g^{(m-n)} \otimes h^{(1)}(\cdot), (f^{(n)}, F^{(m)} \otimes H^{(1)}(\cdot))_{EXT})_{\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= (f^{(n)} \lozenge (g^{(m-n)} \otimes h^{(1)}(\cdot)), F^{(m)} \otimes H^{(1)}(\cdot))_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= ((f^{(n)} \diamond g^{(m-n)}) \otimes h^{(1)}(\cdot), F^{(m)} \otimes H^{(1)}(\cdot))_{\mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= (f^{(n)} \diamond g^{(m-n)}, F^{(m)})_{\mathcal{H}^{(m)}_{ext}} (h^{(1)}, H^{(1)})_{\mathcal{H}_{\mathbb{C}}} \\ &= (g^{(m-n)}, (f^{(n)}, F^{(m)})_{ext})_{\mathcal{H}^{(m-n)}_{ext}} (h^{(1)}, H^{(1)})_{\mathcal{H}_{\mathbb{C}}} \\ &= (g^{(m-n)} \otimes h^{(1)}(\cdot), (f^{(n)}, F^{(m)})_{ext} \otimes H^{(1)}(\cdot))_{\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}. \end{aligned}$$

Since the set $\{g^{(m-n)} \otimes h^{(1)} : g^{(m-n)} \in \mathcal{H}^{(m-n)}_{ext}, h^{(1)} \in \mathcal{H}_{\mathbb{C}}\}$ is total in $\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, we can conclude that

$$(f^{(n)}, F^{(m)} \otimes H^{(1)}(\cdot))_{EXT} = (f^{(n)}, F^{(m)})_{ext} \otimes H^{(1)}(\cdot) \tag{2.16}$$

in $\mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$. As a corollary from this formula one can obtain the following intuitively clear result. Let $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $n, m \in \mathbb{Z}_+$, $m \geq n$. Denote $G^{(m-n)}_{\cdot} := (f^{(n)}, F^{(m)}_{\cdot})_{EXT} \in \mathcal{H}^{(m-n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$. Let $\dot{F}^{(m)}_{\cdot} \in F^{(m)}_{\cdot}$. Then $\dot{G}^{(m-n)}_{\cdot} := (f^{(n)}, \dot{F}^{(m)}_{\cdot})_{EXT}$ is a representative of the equivalence class $G^{(m-n)}_{\cdot}$, and for $u \in \mathbb{R}_+$ such that $\dot{F}^{(m)}_u$ is well-defined, $\dot{G}^{(m-n)}_u = (f^{(n)}, \dot{F}^{(m)}_u)_{ext}$ (roughly speaking, substituting in $(f^{(n)}, F^{(m)}_{\cdot})_{EXT}$ a number u on the place of '$\cdot$', we obtain $(f^{(n)}, F^{(m)}_u)_{ext}$).

Definition 2.2. Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$. We define a linear continuous operator

$$(\mathbf{D}^n \circ)(f^{(n)}) : (L^2)^{\beta}_q \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}_{q-1} \otimes \mathcal{H}_{\mathbb{C}} \tag{2.17}$$

by setting for $F \in (L^2)^{\beta}_q \otimes \mathcal{H}_{\mathbb{C}}$

$$\begin{aligned} (\mathbf{D}^n F(\cdot))(f^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, (f^{(n)}, F^{(m)}_{\cdot})_{EXT} \rangle: \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, (f^{(n)}, F^{(m+n)}_{\cdot})_{EXT} \rangle: \end{aligned} \tag{2.18}$$

(cf. (2.6)), where $F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.13) for F.

Since (see (1.14), (2.18) and (2.15))

(2.19)

$$\begin{aligned}
&\|(\mathbf{D}^n F(\cdot))(f^{(n)})\|^2_{(L^2)^\beta_{q-1}\otimes\mathcal{H}_{\mathbb{C}}}\\
&= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{(q-1)m} \Big(\frac{(m+n)!}{m!}\Big)^2 |(f^{(n)}, F^{(m+n)}_{\cdot})_{EXT}|^2_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}\\
&= 2^{-qn} \sum_{m=0}^{\infty} ((m+n)!)^{1+\beta} 2^{q(m+n)} \Big[2^{-m}\Big(\frac{(m+n)!}{m!}\Big)^{1-\beta}\Big] |(f^{(n)}, F^{(m+n)}_{\cdot})_{EXT}|^2_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}\\
&\le 2^{-qn} |f^{(n)}|^2_{\mathcal{H}^{(n)}_{ext}} c(n) \sum_{m=0}^{\infty} ((m+n)!)^{1+\beta} 2^{q(m+n)} |F^{(m+n)}_{\cdot}|^2_{\mathcal{H}^{(m+n)}_{ext}\otimes\mathcal{H}_{\mathbb{C}}}\\
&\le 2^{-qn} |f^{(n)}|^2_{\mathcal{H}^{(n)}_{ext}} c(n) \|F\|^2_{(L^2)^\beta_q\otimes\mathcal{H}_{\mathbb{C}}},
\end{aligned}$$

where $c(n) := \max_{m\in\mathbb{Z}_+}\Big[2^{-m}\Big(\frac{(m+n)!}{m!}\Big)^{1-\beta}\Big]$, this definition is well-posed. In addition, for each $F \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ $(\mathbf{D}^n F(\cdot))(\circ)$ is a linear continuous operator acting from $\mathcal{H}^{(n)}_{ext}$ to $(L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}$. Moreover, as is easily seen, for each $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, $(\mathbf{D}^n\circ)(f^{(n)})$ can be defined by formula (2.18) as a linear continuous operator on $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$, $\beta \in [-1,1]$. In the case $\beta = 1$ formula (2.18) defines a linear *continuous* operator $(\mathbf{D}^n\circ)(f^{(n)})$ on $(L^2)^1_q \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{Z}$, this can be proved by analogy with calculation (2.19).

Remark 2.5. In the same way as "products" $(\cdot,\cdot)_{ext}$ and $(\cdot,\cdot)_{EXT}$, the operators D^n and $\mathbf{D}^n$, $n \in \mathbb{N}$, are closely connected. In fact, let $F \in (L^2)^\beta_q$, $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$, $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$. Using (2.18), (2.16) and (2.6) one can easily show that

$$(\mathbf{D}^n F \otimes H^{(1)}(\cdot))(f^{(n)}) = (D^n F)(f^{(n)}) \otimes H^{(1)}(\cdot) \in (L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}.$$

For general $F \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ set $g(\cdot) := (\mathbf{D}^n F(\cdot))(f^{(n)})$, and let $\dot F \in F$. Then $\dot g(\cdot) := (\mathbf{D}^n \dot F(\cdot))(f^{(n)})$ is a representative of the equivalence class $g(\cdot) \in (L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}$ and, as follows from Remark 2.4, if $u \in \mathbb{R}_+$ is such that $\dot F(u)$ is well-defined then $\dot g(u) = (D^n \dot F(u))(f^{(n)})$ (roughly speaking, considering F as a function on $\mathbb{R}_+$ with values in $(L^2)^\beta_q$ and substituting in $(\mathbf{D}^n F(\cdot))(f^{(n)})$ a number u on the place of '$\cdot$', we obtain $(D^n F(u))(f^{(n)})$).

Now it is natural to describe properties of the operators $\mathbf{D}^n$, $n \in \mathbb{N}$, using as a pattern Theorem 2.1 about properties of D^n.

An analog of property '1)' is described in the following

Proposition 2.1. *For $k_1,\dots,k_m \in \mathbb{N}$, $f_j^{(k_j)} \in \mathcal{H}^{(k_j)}_{ext}$, $j \in \{1,\dots,m\}$,*

$$(\mathbf{D}^{k_m}(\cdots(\mathbf{D}^{k_2}((\mathbf{D}^{k_1}\circ)(f_1^{(k_1)})))(f_2^{(k_2)})\cdots))(f_m^{(k_m)}) = (\mathbf{D}^{k_1+\cdots+k_m}\circ)(f_1^{(k_1)} \diamond \cdots \diamond f_m^{(k_m)}).$$

Proof. It follows from the construction of the "products" $\diamond$ and $\Diamond$ that for $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $G^{(m)} \in \mathcal{H}^{(m)}_{ext}$ and $H^{(k)}_{\cdot} \in \mathcal{H}^{(k)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $n,m,k \in \mathbb{Z}_+$,

$$F^{(n)} \Diamond (G^{(m)} \Diamond H^{(k)}_{\cdot}) = (F^{(n)} \diamond G^{(m)}) \Diamond H^{(k)}_{\cdot} \in \mathcal{H}^{(n+m+k)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}.$$

Using this formula one can easily prove the result by the mathematical induction method. □

In order to write out an analog of property '2)' we need a small preparation. For $F \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ denote by $\mathbf{E}F := \langle\!\langle F, 1\rangle\!\rangle_{(L^2)} \in \mathcal{H}_{\mathbb{C}}$ a partial pairing, i.e., a unique element of $\mathcal{H}_{\mathbb{C}}$ such that for each $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$(\langle\!\langle F, 1\rangle\!\rangle_{(L^2)}, h^{(1)})_{\mathcal{H}_{\mathbb{C}}} = \langle\!\langle F, 1 \otimes h^{(1)}\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}. \tag{2.20}$$

Since by the generalized Cauchy-Bunyakovsky inequality and (1.14)

$$|\langle\langle F, 1\otimes h^{(1)}\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}| \le \|F\|_{(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}}\|1\otimes h^{(1)}\|_{(L^2)_{-q}^{-\beta}\otimes\mathcal{H}_{\mathbb{C}}} = \|F\|_{(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}}|h^{(1)}|_{\mathcal{H}_{\mathbb{C}}},$$

this definition is well-posed and $|\mathbf{E}F|_{\mathcal{H}_{\mathbb{C}}} = |\langle\langle F, 1\rangle\rangle_{(L^2)}|_{\mathcal{H}_{\mathbb{C}}} \le \|F\|_{(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}}$. Now for each $F\in (L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}$, $n\in\mathbb{N}$, $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$ and $h^{(1)}\in\mathcal{H}_{\mathbb{C}}$ by (2.20), (2.18) and (1.15)

$$\begin{aligned}(\langle\langle(\mathbf{D}^nF(\cdot))(f^{(n)}),1\rangle\rangle_{(L^2)},h^{(1)})_{\mathcal{H}_{\mathbb{C}}} &= \langle\langle(\mathbf{D}^nF(\cdot))(f^{(n)}),1\otimes h^{(1)}\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}\\ &= n!((f^{(n)},F_\cdot^{(n)})_{EXT},h^{(1)})_{\mathcal{H}_{\mathbb{C}}},\end{aligned}$$

here $F_\cdot^{(n)}\in\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$ is the kernel from decomposition (1.13) for F. So, we proved

Proposition 2.2. *For each $F\in(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}$ the kernels $F_\cdot^{(n)}\in\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$, $n\in\mathbb{N}$, from decomposition* (1.13) *can be presented in a mnemonic form*

$$F_\cdot^{(n)} = \frac{1}{n!}\mathbf{E}\big(\mathbf{D}^nF(\cdot)\big),$$

i.e., for each $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$

$$\mathbf{E}\big((\mathbf{D}^nF(\cdot))(f^{(n)})\big) = \langle\langle(\mathbf{D}^nF(\cdot))(f^{(n)}),1\rangle\rangle_{(L^2)} = n!(f^{(n)},F_\cdot^{(n)})_{EXT}.$$

An analog of property '3)' is described in the following

Proposition 2.3. *The operator adjoint to $\mathbf{D}^n$, $n\in\mathbb{N}$, has a form*

$$(\mathbf{D}^nG(\cdot))(f^{(n)})^* = \sum_{m=0}^\infty :\langle\circ^{m+n}, f^{(n)}\Diamond G_\cdot^{(m)}\rangle:\in(L^2)_{-q}^{-\beta}\otimes\mathcal{H}_{\mathbb{C}}, \tag{2.21}$$

here $G\in(L^2)_{1-q}^{-\beta}\otimes\mathcal{H}_{\mathbb{C}}$, $f^{(n)}\in\mathcal{H}_{ext}^{(n)}$; $G_\cdot^{(m)}\in\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.13) *for G.*

Proof. For arbitrary $F\in(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}$, using (2.18), (1.13), (1.15) and (2.14) we obtain

$$\begin{aligned}&\langle\langle F(\cdot),(\mathbf{D}^nG(\cdot))(f^{(n)})^*\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\langle(\mathbf{D}^nF(\cdot))(f^{(n)}),G(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}\\ &= \langle\langle\sum_{m=0}^\infty\frac{(m+n)!}{m!}:\langle\circ^{\otimes m},(f^{(n)},F_\cdot^{(m+n)})_{EXT}\rangle:,\sum_{k=0}^\infty:\langle\circ^{\otimes k},G_\cdot^{(k)}\rangle:\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}\\ &= \sum_{m=0}^\infty(m+n)!((f^{(n)},F_\cdot^{(m+n)})_{EXT},G_\cdot^{(m)})_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}\\ &= \sum_{m=0}^\infty(m+n)!(F_\cdot^{(m+n)},f^{(n)}\Diamond G_\cdot^{(m)})_{\mathcal{H}_{ext}^{(m+n)}\otimes\mathcal{H}_{\mathbb{C}}} \qquad (2.22)\\ &= \langle\langle\sum_{k=0}^\infty:\langle\circ^{\otimes k},F_\cdot^{(k)}\rangle:,\sum_{m=0}^\infty:\langle\circ^{m+n},f^{(n)}\Diamond G_\cdot^{(m)}\rangle:\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}\\ &= \langle\langle F(\cdot),\sum_{m=0}^\infty:\langle\circ^{m+n},f^{(n)}\Diamond G_\cdot^{(m)}\rangle:\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}},\end{aligned}$$

whence the result follows. □

Finally, in order to formulate analogs of properties '4)' and '5)' of D^n it is necessary to introduce a stochastic integral and a Hida derivative on the spaces $(L^2)_{1-q}^{-\beta}\otimes\mathcal{H}_{\mathbb{C}}\otimes\mathcal{H}_{\mathbb{C}}$ and $(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}$ respectively, but this is not a subject of the present paper.

Now we pass to a revised statement about the commutator between the extended stochastic integral and the operator of stochastic differentiation. Denote $\mathbf{D}:=\mathbf{D}^1$.

Theorem 2.2. *For arbitrary $F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$, $f^{(1)} \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$ and $\Delta \in \mathcal{B}(\mathbb{R}_+)$*

$$(2.23)\quad (D\int_\Delta F(u)\,\widehat{d}L_u)(f^{(1)}) = \int_\Delta (\mathbf{D}F(u))(f^{(1)})\,\widehat{d}L_u + \int_\Delta F(u)f^{(1)}(u)\,du \in (L^2)_{q-1}^\beta,$$

where $\int_\Delta (\mathbf{D}F(u))(f^{(1)})\,\widehat{d}L_u := \int_\Delta g(u)\,\widehat{d}L_u$ with $g(\cdot) := (\mathbf{D}F(\cdot))(f^{(1)}) \in (L^2)_{q-1}^\beta \otimes \mathcal{H}_{\mathbb{C}}$; $\int_\Delta F(u)f^{(1)}(u)\,du \in (L^2)_q^\beta \subset (L^2)_{q-1}^\beta$ is a Pettis integral (cf. (2.8)).

Note that $\int_\Delta F(u)f^{(1)}(u)\,du = \langle F(\cdot), f^{(1)}(\cdot)1_\Delta(\cdot)\rangle$ – a partial pairing.

Proof. Actually this statement is proved in [12] (up to some nuances connected with a notation). Nevertheless we give here the short description of main steps for convenience of a reader.

1) Decomposing all terms of (2.23) by Wick monomials one can conclude that in order to prove this equality it is sufficient to prove that for each $m \in \mathbb{Z}_+$ and $g^{(m)} \in \mathcal{H}_{ext}^{(m)}$

$$(2.24)\quad \begin{aligned} &(m+1)((f^{(1)}, \widehat{F}_\Delta^{(m)})_{ext}, g^{(m)})_{\mathcal{H}_{ext}^{(m)}} \\ &= m((f^{(1)}, \widehat{F_\cdot^{(m)}})_{EXT\,\Delta}, g^{(m)})_{\mathcal{H}_{ext}^{(m)}} + \Big(\int_\Delta F_u^{(m)} f^{(1)}(u)\,du, g^{(m)}\Big)_{\mathcal{H}_{ext}^{(m)}}. \end{aligned}$$

Here $\widehat{F}_\Delta^{(m)} \in \mathcal{H}_{ext}^{(m+1)}$ and $(f^{(1)}, \widehat{F_\cdot^{(m)}})_{EXT\,\Delta} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decompositions (1.4) for $\int_\Delta F(u)\,\widehat{d}L_u$ and $\int_\Delta (\mathbf{D}F(u))(f^{(1)})\,\widehat{d}L_u$ respectively, these kernels constructed by $F_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ and $(f^{(1)}, F_\cdot^{(m)})_{EXT} \in \mathcal{H}_{ext}^{(m-1)} \otimes \mathcal{H}_{\mathbb{C}}$ (see (2.14)) respectively as described before Lemma 1.1 in Subsection 1.4, $F_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.13) for F; $\int_\Delta F_u^{(m)} f^{(1)}(u)du \in \mathcal{H}_{ext}^{(m)}$ is a Pettis integral (this integral is equal to a partial product $(F_\cdot^{(m)}, f^{(1)}(\cdot)1_\Delta(\cdot))_{\mathcal{H}_{\mathbb{C}}}$ – a unique element of $\mathcal{H}_{ext}^{(m)}$ such that for each $g^{(m)} \in \mathcal{H}_{ext}^{(m)}$ $\big((F_\cdot^{(m)}, f^{(1)}(\cdot)1_\Delta(\cdot))_{\mathcal{H}_{\mathbb{C}}}, g^{(m)}\big)_{\mathcal{H}_{ext}^{(m)}} = \big(F_\cdot^{(m)}, g^{(m)} \otimes (f^{(1)}(\cdot)1_\Delta(\cdot))\big)_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}\big)$.

2) Using (2.3), (2.14), an equality

$$(\widehat{F}_\Delta^{(m)}, h^{(m+1)})_{\mathcal{H}_{ext}^{(m+1)}} = (F_\cdot^{(m)}1_\Delta(\cdot), h^{(m+1)}(\cdot))_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}} = \int_\Delta (F_u^{(m)}, h^{(m+1)}(u))_{\mathcal{H}_{ext}^{(m)}} du,$$

$h^{(m+1)} \in \mathcal{H}_{ext}^{(m+1)}$, $h^{(m+1)}(\cdot) \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ (see Lemma 1.2), which is proved in [23], the construction of "products" $\diamond$ and $\lozenge$, the symmetry of $g^{(m)}$, and the non-atomicity of the Lebesgue measure, one can obtain equality (2.24) by direct calculation.

3) Now it remains to prove that $(D\int_\Delta F(u)\widehat{d}L_u)(f^{(1)}) \in (L^2)_{q-1}^\beta$ if $F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ (it follows directly from the definitions of the extended stochastic integral and of the operators of stochastic differentiation that $(D\int_\Delta F(u)\widehat{d}L_u)(f^{(1)}) \in (L^2)_{q-2}^\beta$, but this statement can be amplified). By direct calculation with use (1.8), (2.4) and (1.17) one can show that $\|(D\int_\Delta F(u)\widehat{d}L_u)(f^{(1)})\|_{(L^2)_{q-1}^\beta} \le \frac{3}{2}|f^{(1)}|_{\mathcal{H}_{\mathbb{C}}}\|F\|_{(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}}$, this completes the proof. □

As is easily seen, the above-presented results hold true (up to obvious modifications) if we consider the operators $(D^n\circ)(f^{(n)})$, $(\mathbf{D}^n\circ)(f^{(n)})$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, on the spaces $(L^2)^\beta$, $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ respectively, here $\beta \in [-1,1]$.

Sometimes it can be useful to consider $(D^n\circ)(f^{(n)})$, $(\mathbf{D}^n\circ)(f^{(n)})$ as operators acting in $(L^2)_q^\beta$, $(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ respectively. If $\beta = 1$ then these operators can be defined by formulas (2.6), (2.18) respectively as linear *continuous* operators (it was explained above), but for $\beta \in [-1,1)$ this is not the case. Let us consider the case $\beta \in [-1,1)$ in detail.

Definition 2.3. Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$. We define an operator

$$(D^n \circ)(f^{(n)}) : (L^2)_q^\beta \to (L^2)_q^\beta \tag{2.25}$$

with the domain

$$\begin{aligned}\operatorname{dom}\big((D^n \circ)(f^{(n)})\big) &= \big\{F \in (L^2)_q^\beta : \|(D^n F)(f^{(n)})\|_{q,\beta}^2 \\ &= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |(f^{(n)}, F^{(m+n)})_{ext}|_{ext}^2 < \infty\big\}\end{aligned} \tag{2.26}$$

(here $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (1.4) for F) by formula (2.6).

It is clear that an operator

$$(D^n \circ)(f^{(n)})^* : (L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta}, \tag{2.27}$$

adjoint to operator (2.25), is well-defined and can be calculated by formula (2.7); the domain of this operator is

$$\begin{aligned}\operatorname{dom}\big((D^n \circ)(f^{(n)})^*\big) &= \big\{G \in (L^2)_{-q}^{-\beta} : \|(D^n G)(f^{(n)})^*\|_{(L^2)_{-q}^{-\beta}}^2 \\ &= \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-q(m+n)} |f^{(n)} \diamond G^{(m)}|_{ext}^2 < \infty\big\}\end{aligned} \tag{2.28}$$

(see [12]).

Proposition 2.4. ([12]). *Operator* (2.25) *with domain* (2.26) *and operator* (2.27) *with domain* (2.28) *are mutually adjoint. In particular, these operators are closed.*

Remark 2.6. The domain of operator (2.25) depends on a "coefficient" $f^{(n)}$, this can restrict an area of applications of this operator. The problem can be solved in the following way. Let

$$A_n := \big\{F \in (L^2)_q^\beta : \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F^{(m+n)}|_{ext}^2 < \infty\big\}, \quad n \in \mathbb{N},$$

here $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (1.4) for F. For each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ we define an operator

$$(\widetilde{D}^n \circ)(f^{(n)}) : (L^2)_q^\beta \to (L^2)_q^\beta \tag{2.29}$$

with the domain A_n by formula (2.6). It is shown in [12] that this definition is well-posed and for each $F \in A_n$ the operator $(\widetilde{D}^n F)(\circ) : \mathcal{H}_{ext}^{(n)} \to (L^2)_q^\beta$ is a linear bounded (and, therefore, continuous) operator. Moreover, it follows from Proposition 2.4 that operator (2.29) is closable (its closure is operator (2.25) with domain (2.26)).

Now we pass to operators $\mathbf{D}^n$.

Definition 2.4. Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$. We define an operator

$$(\mathbf{D}^n \circ)(f^{(n)}) : (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \tag{2.30}$$

with the domain

$$\begin{aligned}\operatorname{dom}\big((\mathbf{D}^n \circ)(f^{(n)})\big) &= \big\{F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} : \|(\mathbf{D}^n F(\cdot))(f^{(n)})\|_{(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}}^2 \\ &= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |(f^{(n)}, F_\cdot^{(m+n)})_{EXT}|_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}^2 < \infty\big\}\end{aligned} \tag{2.31}$$

(here $F_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.13) for F) by formula (2.18).

It is clear that an operator

$$(\mathbf{D}^n\circ)(f^{(n)})^* : (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}, \tag{2.32}$$

adjoint to operator (2.30), is well-defined and can be calculated by formula (2.21); below we will show that the domain of this operator is

$$\begin{aligned} \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})^*\big) = \big\{G \in (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} : \|(\mathbf{D}^n G(\cdot))(f^{(n)})^*\|^2_{(L^2)_{-q}^{-\beta}\otimes\mathcal{H}_{\mathbb{C}}} \\ = \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-q(m+n)} |f^{(n)} \Diamond G_{\cdot}^{(m)}|^2_{\mathcal{H}_{ext}^{(m+n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty\big\} \end{aligned} \tag{2.33}$$

(actually this result in the same way as (2.28) from a general duality theory follows).

Proposition 2.5. *Operator* (2.30) *with domain* (2.31) *and operator* (2.32) *with domain* (2.33) *are mutually adjoint. In particular, these operators are closed.*

Proof. We have to show that there exists the second adjoint to $(\mathbf{D}^n\circ)(f^{(n)})$ operator $(\mathbf{D}^n\circ)(f^{(n)})^{**} = (\mathbf{D}^n\circ)(f^{(n)})$. Since, obviously, the domain of operator (2.30) is a dense set in $(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$, the adjoint operator $(\mathbf{D}^n\circ)(f^{(n)})^* : (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ is well-defined. By definition, $G \in \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})^*\big)$ if and only if

$$(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \supset \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})\big) \ni F \mapsto \langle\langle (\mathbf{D}^n F)(f^{(n)}), G\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \in \mathbb{C}$$

is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $K \in (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ such that $\langle\langle (\mathbf{D}^n F)(f^{(n)}), G\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\langle F, K\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$. But by calculation (2.22) K has form (2.21), therefore the domain of operator (2.32) has form (2.33). This set is, obviously, dense in $(L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$, hence $(\mathbf{D}^n\circ)(f^{(n)})^{**} : (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ is well-defined. Now it remains to show that

$$\operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})^{**}\big) = \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})\big). \tag{2.34}$$

By analogy with the consideration above, $F \in \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})^{**}\big)$ if and only if

$$(L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} \supset \operatorname{dom}\big((\mathbf{D}^n\circ)(f^{(n)})^*\big) \ni G \mapsto \langle\langle F, (\mathbf{D}^n G)(f^{(n)})^*\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \in \mathbb{C}$$

is a linear continuous functional. The last is possible if and only if there exists $H \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ such that $\langle\langle F, (\mathbf{D}^n G)(f^{(n)})^*\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\langle H, G\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$. It is clear that H has form (2.18), therefore equality (2.34) follows from (2.31).

The last statement of the proposition follows from the well-known fact for operators acting on Hilbert spaces: an adjoint operator is closed. □

Remark 2.7. As in the case of operators of stochastic differentiation on the space $(L^2)_q^\beta$, one can define an operator

$$(\widetilde{\mathbf{D}}^n\circ)(f^{(n)}) : (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}, \tag{2.35}$$

$f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, with the independent on $f^{(n)}$ domain

$$B_n := \Big\{F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} : \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F_{\cdot}^{(m+n)}|^2_{\mathcal{H}_{ext}^{(m+n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty\Big\},$$

where $F_{\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.13) for F, by formula (2.18). Since by (2.18), (1.14) and (2.15)

$$\begin{aligned} \|(\widetilde{\mathbf{D}}^n F(\cdot))(f^{(n)})\|^2_{(L^2)_q^\beta\otimes\mathcal{H}_{\mathbb{C}}} &= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |(f^{(n)}, F_{\cdot}^{(m+n)})_{EXT}|^2_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}} \\ &\le |f^{(n)}|^2_{\mathcal{H}_{ext}^{(m)}} \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F_{\cdot}^{(m+n)}|^2_{\mathcal{H}_{ext}^{(m+n)}\otimes\mathcal{H}_{\mathbb{C}}}, \end{aligned}$$

this definition is well-posed and for each $F \in B_n$ the operator $(\widetilde{\mathbf{D}}^n F)(\circ) : \mathcal{H}_{ext}^{(n)} \to (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ is a linear bounded (and, therefore, continuous) operator. Moreover, it follows from Proposition 2.5 that operator (2.35) is closable (its closure is operator (2.30) with domain (2.31)).

2.2. **Interconnection between operators of stochastic differentiation on different spaces.** In the paper [25] analogs of operators $(D^n\circ)(f^{(n)})$ and $(\mathbf{D}^n\circ)(f^{(n)})$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$ (denote these analogs by $(\widehat{D}^n\circ)(f^{(n)})$ and $(\widehat{\mathbf{D}}^n\circ)(f^{(n)})$ respectively), on the spaces of nonregular test functions of the Lévy white noise analysis, were introduced and studied. In particular, it was shown that the restrictions of $(D^n\circ)(f^{(n)})$ from $(L^2) = (L^2)_0^0$ to the above-mentioned spaces coincide with $(\widehat{D}^n\circ)(f^{(n)})$. Taking into account this result and the construction of all mentioned operators, it is natural to expect that the restrictions of $(\mathbf{D}^n\circ)(f^{(n)})$ from $(L^2) \otimes \mathcal{H}_{\mathbb{C}} = (L^2)_0^0 \otimes \mathcal{H}_{\mathbb{C}}$ to the corresponding spaces of nonregular test functions coincide with $(\widehat{\mathbf{D}}^n\circ)(f^{(n)})$. In fact, such a result takes place. In the present subsection we will explain this in detail.

Denote by T the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is the completion of the set $\mathcal{D}$ of all real-valued infinite differentiable functions on $\mathbb{R}_+$ with compact supports with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)\, du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. As we said above, it is well known (e.g., [7]) that $\mathcal{D}$ can be turned into a linear topological space with the projective limit topology: $\mathcal{D} = \mathrm{pr}\lim_{\tau\in T} \mathcal{H}_\tau$ (moreover, for each $n \in \mathbb{N}$ $\mathcal{D}^{\widehat{\otimes} n} = \mathrm{pr}\lim_{\tau\in T} \mathcal{H}_\tau^{\widehat{\otimes} n}$, see, e.g., [4] for details), and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} = L^2(\mathbb{R}_+)$. Therefore one can consider a chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem (e.g., [7]) $\mathcal{D}' = \bigcup_{\tau\in T} \mathcal{H}_{-\tau}$. By analogy with [22] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each* $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.

Denote the norms in tensor powers of complexifications of $\mathcal{H}_\tau$ by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)}, \overline{f^{(n)}})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}}$ (as above, in complexifications of Hilbert spaces we consider *real*, i.e., *bilinear* scalar products).

Lemma 2.3. ([21]). *There exists* $\tau' \in T$ *such that if for some* $\tau \in T$ *the space* $\mathcal{H}_\tau$ *is continuously embedded into the space* $\mathcal{H}_{\tau'}$ *then for each* $n \in \mathbb{N}$ *the space* $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ *is densely and continuously embedded into the space* $\mathcal{H}_{ext}^{(n)}$*, and there exists* $c(\tau) > 0$ *such that for all* $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ $|f^{(n)}|_{ext}^2 \leq n!c(\tau)^n |f^{(n)}|_\tau^2$.

In what follows, it will be convenient to assume that the *indexes* τ *such that* $\mathcal{H}_\tau$ *is not continuously embedded into* $\mathcal{H}_{\tau'}$*, are removed from* T.

In this subsection *accept on default* $q \in \mathbb{Z}_+$, $\tau \in T$. Set $\mathcal{H}_\tau^{\widehat{\otimes}0} := \mathbb{R}$ (hence $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}0} = \mathbb{C}$). Define *real* scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ (see (1.7)) by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \quad g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W,$$
$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}}.$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,\overline{f})_{\tau,q}}$. The well-posedness of this definition is proved in [25].

Denote by $(\mathcal{H}_\tau)_q$ the completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$; and set $(\mathcal{H}_\tau) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+}(\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+,\tau\in T}(\mathcal{H}_\tau)_q$.

Definition 2.5. The spaces $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ are called Kondratiev spaces of nonregular test functions.

As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be uniquely presented in a form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \quad f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \tag{2.36}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|^2_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|^2_\tau < \infty; \tag{2.37}$$

and for $f,g \in (\mathcal{H}_\tau)_q$ $(f,g)_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^\infty (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}}$, here $f^{(n)}, g^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ are the kernels from decompositions (2.36) for f and g respectively (since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is a well-defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be uniquely presented in form (2.36) and norm (2.37) is finite for each $q \in \mathbb{Z}_+$ (for each $q \in \mathbb{Z}_+$ and each $\tau \in T$).

Remark 2.8. It is proved in [21] that for each $\tau \in T$ there exists $q_0 = q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \{q_0, q_0+1, \dots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2). In view of this statement one can consider spaces dual of the Kondratiev spaces of nonregular test functions with respect to (L^2). Such dual spaces are called *Kondratiev spaces of nonregular generalized functions* and have many applications in the Lévy analysis (in particular, in the theory of normally ordered stochastic equations). The interested reader can find more information about the spaces of nonregular test and generalized functions of the Lévy analysis in [21, 25, 24].

Consider chains

$$\mathcal{D}'^{(m)}_{\mathbb{C}} \supset \mathcal{H}^{(m)}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{(m)}_{ext} \supset \mathcal{H}^{\widehat{\otimes}m}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes}m}_{\mathbb{C}},$$
$$\mathcal{D}'^{\widehat{\otimes}m}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes}m}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes}m}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes}m}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes}m}_{\mathbb{C}},$$

$m \in \mathbb{Z}_+$, where $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$, $\mathcal{D}'^{(m)}_{\mathbb{C}} = \bigcup_{\tau\in T} \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ and $\mathcal{H}^{\widehat{\otimes}m}_{-\tau,\mathbb{C}}$, $\mathcal{D}'^{\widehat{\otimes}m}_{\mathbb{C}} = \bigcup_{\tau\in T} \mathcal{H}^{\widehat{\otimes}m}_{-\tau,\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes}m}_{\tau,\mathbb{C}}$, $\mathcal{D}^{\widehat{\otimes}m}_{\mathbb{C}} = \mathrm{pr}\lim_{\tau\in T} \mathcal{H}^{\widehat{\otimes}m}_{\tau,\mathbb{C}}$ with respect to $\mathcal{H}^{(m)}_{ext}$ and $\mathcal{H}^{\widehat{\otimes}m}_{\mathbb{C}}$ respectively (for $m = 0$ all spaces of these chains are equal to $\mathbb{C}$). Since the spaces of test functions in both chains coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)}_{\mathbb{C}} \to \mathcal{D}'^{\widehat{\otimes}m}_{\mathbb{C}}$$

such that for all $F_{ext}^{(m)} \in \mathcal{D}'^{(m)}_{\mathbb{C}}$ and $f^{(m)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$

$$\langle F_{ext}^{(m)}, f^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}} = \langle U_m F_{ext}^{(m)}, f^{(m)}\rangle_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} m}}. \tag{2.38}$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ and $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m}$ (we preserve for these restrictions the designation U_m). Note that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$ and therefore $U_1 = \mathbf{1}$ is the identity operator on $\mathcal{D}'^{(1)}_{\mathbb{C}} = \mathcal{D}'_{\mathbb{C}}$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{C}$.

Let $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $G_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, $n, m \in \mathbb{Z}_+$. We define

$$\begin{aligned} &F_{ext}^{(n)} \widehat{\Diamond} G_{ext,\cdot}^{(m)} \\ &\quad := (U_{n+m}^{-1} \otimes \mathbf{1})\{(Pr \otimes \mathbf{1})[(U_n F_{ext}^{(n)}) \otimes ((U_m \otimes \mathbf{1}) G_{ext,\cdot}^{(m)})]\} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}, \end{aligned} \tag{2.39}$$

where $Pr \otimes \mathbf{1}$ is the orthoprojector acting from $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ to $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ (of course, this operator depends on τ, n and m, but we simplify the notation). As is easy to see,

$$\begin{aligned} &|F_{ext}^{(n)} \widehat{\Diamond} G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} = |(Pr \otimes \mathbf{1})[(U_n F_{ext}^{(n)}) \otimes ((U_m \otimes \mathbf{1}) G_{ext,\cdot}^{(m)})]|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} \\ &\quad \leq |U_n F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}} |(U_m \otimes \mathbf{1}) G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} = |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}} |G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}}. \end{aligned} \tag{2.40}$$

Let $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $f_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$, $m \geq n$. We define a generalized partial pairing $\langle F_{ext}^{(n)}, f_\cdot^{(m)}\rangle_{EXT} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ by setting for arbitrary $G_{ext,\cdot}^{(m-n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m-n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$

$$\langle G_{ext,\cdot}^{(m-n)}, \langle F_{ext}^{(n)}, f_\cdot^{(m)}\rangle_{EXT}\rangle_{\mathcal{H}_{ext}^{(m-n)} \otimes \mathcal{H}_{\mathbb{C}}} = \langle F_{ext}^{(n)} \widehat{\Diamond} G_{ext,\cdot}^{(m-n)}, f_\cdot^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}. \tag{2.41}$$

By the generalized Cauchy-Bunyakovsky inequality and (2.40)

$$\begin{aligned} |\langle F_{ext}^{(n)} \widehat{\Diamond} G_{ext,\cdot}^{(m-n)}, f_\cdot^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}| &\leq |F_{ext}^{(n)} \widehat{\Diamond} G_{ext,\cdot}^{(m-n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} |f_\cdot^{(m)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}} \\ &\leq |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}} |G_{ext,\cdot}^{(m-n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} |f_\cdot^{(m)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}}, \end{aligned}$$

which implies that this definition is well-posed and

$$|\langle F_{ext}^{(n)}, f_\cdot^{(m)}\rangle_{EXT}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau,\mathbb{C}}} \leq |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}} |f_\cdot^{(m)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}}.$$

In order to define analogs of operators $\mathbf{D}^n$, $n \in \mathbb{N}$, on the spaces of nonregular test functions $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$, we need orthogonal bases in these spaces. Since, as is easily seen, the restriction of the generalized Wiener-Itô-Sigal isomorphism $\mathbf{I}$ (see Subsection 1.4) from $(L^2) = (L^2)_0^0$ to $(\mathcal{H}_\tau)_q$ is an isometric isomorphism between $(\mathcal{H}_\tau)_q$ and a weighted symmetric Fock space $\bigoplus_{m=0}^{\infty} (m!)^2 2^{qm} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ (cf. [26]); and, of course, the restriction of the identity operator on $\mathcal{H}_{\mathbb{C}}$ to the space $\mathcal{H}_{\tau,\mathbb{C}}$ is the identity operator on $\mathcal{H}_{\tau,\mathbb{C}}$, for arbitrary $m \in \mathbb{Z}_+$ and $f_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ we have $:\langle \circ^{\otimes m}, f_\cdot^{(m)}\rangle: \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ (see (1.12)). It is clear that such elements form an orthogonal basis in each $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$, i.e., any $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ can be uniquely presented in a form

$$f(\cdot) = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, f_\cdot^{(m)}\rangle:, \quad f_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}} \tag{2.42}$$

(the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$), and

$$\|f\|^2_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}} = \sum_{m=0}^{\infty} (m!)^2 2^{qm} |f_\cdot^{(m)}|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}} < \infty.$$

Definition 2.6. Let $n \in \mathbb{N}$, $F_{ext}^{(n)} \in \mathcal{H}_{-\tau}^{(n)}$. We define a linear continuous operator

$$(\widehat{\mathbf{D}}^n \circ)(F_{ext}^{(n)}) : (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}} \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}} \tag{2.43}$$

by setting for $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$

$$\begin{aligned}(\widehat{\mathbf{D}}^n f(\cdot))(F_{ext}^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} {:}\langle \circ^{\otimes m-n}, \langle F_{ext}^{(n)}, f_\cdot^{(m)} \rangle_{EXT} \rangle{:} \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} {:}\langle \circ^{\otimes m}, \langle F_{ext}^{(n)}, f_\cdot^{(m+n)} \rangle_{EXT} \rangle{:}\end{aligned} \tag{2.44}$$

(cf. (2.6), (2.18)), where $f_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ are the kernels from decomposition (2.42) for f.

The well-posedness of this definition is proved in [25].

Note that, as is easily seen, the restriction of $(\widehat{\mathbf{D}}^n \circ)(F_{ext}^{(n)})$ to the space $(\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}} :=$ pr $\lim_{q \in \mathbb{Z}_+} (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ or to the space $(\mathcal{D}) \otimes \mathcal{D}_{\mathbb{C}} :=$ pr $\lim_{q \in \mathbb{Z}_+, \tau \in T} (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ is a linear continuous operator on the corresponding space.

Comparing the construction of operators (2.17) and (2.43), one can conclude that for a study of an interconnection between these operators it is necessary to study an interconnection between "products" $\lozenge$ and $\widehat{\lozenge}$.

Lemma 2.4. *Let $n, m \in \mathbb{Z}_+$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and $G_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$*

$$F^{(n)} \widehat{\lozenge} G_\cdot^{(m)} = F^{(n)} \lozenge G_\cdot^{(m)} \in \mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \tag{2.45}$$

(more exactly, $F^{(n)} \widehat{\lozenge} G_\cdot^{(m)} = OF^{(n)} \lozenge G_\cdot^{(m)}$, where $O : \mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ is the embedding operator).

Proof. For $n = 0$ or $m = 0$ (2.45) is, obviously, fulfilled, therefore we consider the case $n, m \in \mathbb{N}$. At first we establish that for each $\lambda \in \mathcal{D}_{\mathbb{C}}$

$$\langle F^{(n)} \widehat{\lozenge} G_\cdot^{(m)}, \lambda^{\otimes n+m+1} \rangle_{\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}} = (F^{(n)} \lozenge G_\cdot^{(m)}, \lambda^{\otimes n+m+1})_{\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}}. \tag{2.46}$$

In fact, by (2.39) and (2.38)

$$\begin{aligned}&\langle F^{(n)} \widehat{\lozenge} G_\cdot^{(m)}, \lambda^{\otimes n+m+1} \rangle_{\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= \langle (U_{n+m}^{-1} \otimes \mathbf{1}) \{ (Pr \otimes \mathbf{1}) [(U_n F^{(n)}) \otimes ((U_m \otimes \mathbf{1}) G_\cdot^{(m)})] \}, \lambda^{\otimes n+m+1} \rangle_{\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}} \\ &= \langle (U_n F^{(n)}) \otimes ((U_m \otimes \mathbf{1}) G_\cdot^{(m)}), \lambda^{\otimes n+m+1} \rangle_{\mathcal{H}_{\mathbb{C}}^{\otimes n+m+1}} \\ &= \langle U_n F^{(n)}, \lambda^{\otimes n} \rangle_{\mathcal{H}_{\mathbb{C}}^{\otimes n}} \langle (U_m \otimes \mathbf{1}) G_\cdot^{(m)}, \lambda^{\otimes m+1} \rangle_{\mathcal{H}_{\mathbb{C}}^{\otimes m+1}} \\ &= \langle F^{(n)}, \lambda^{\otimes n} \rangle_{\mathcal{H}_{ext}^{(n)}} \langle G_\cdot^{(m)}, \lambda^{\otimes m+1} \rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}.\end{aligned} \tag{2.47}$$

On the other hand, using the notation accepting with the definition of $F^{(n)}\Diamond G^{(m)}_{\cdot}$, by (1.5) we obtain

$$
\begin{aligned}
&(F^{(n)}\Diamond G^{(m)}_{\cdot}, \lambda^{\otimes n+m+1})_{\mathcal{H}_{ext}^{(n+m)}\otimes\mathcal{H}_{\mathbb{C}}} = (\widehat{f^{(n)}g^{(m)}_{\cdot}}, \lambda^{\otimes n+m+1})_{\mathcal{H}_{ext}^{(n+m)}\otimes\mathcal{H}_{\mathbb{C}}}\\
&= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\cdots>l_k,\\ l_1s_1+\cdots+l_ks_k=n+m}} \frac{(n+m)!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\
&\times\int_{\mathbb{R}_+^{s_1+\cdots+s_k+1}} \widehat{f^{(n)}g^{(m)}_u}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\cdots+s_k},\dots,u_{s_1+\cdots+s_k}}_{l_k})\\
&\times\lambda^{l_1}(u_1)\cdots\lambda^{l_k}(u_{s_1+\cdots+s_k})\,du_1\cdots du_{s_1+\cdots+s_k}du.
\end{aligned}
$$

Then, operating by analogy with [25, Subsection 2.2], one can show that the right hand side of this equality is equal to

$$(F^{(n)},\lambda^{\otimes n})_{\mathcal{H}_{ext}^{(n)}}(G^{(m)}_{\cdot},\lambda^{\otimes m+1})_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}} = \langle F^{(n)},\lambda^{\otimes n}\rangle_{\mathcal{H}_{ext}^{(n)}}\langle G^{(m)}_{\cdot},\lambda^{\otimes m+1}\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}.$$

Comparing this result with (2.47), we obtain (2.46).

Further, $F^{(n)}\Diamond G^{(m)}_{\cdot}\in\mathcal{H}_{ext}^{(n+m)}\otimes\mathcal{H}_{\mathbb{C}}\subset\mathcal{D}'^{(n+m)}_{\mathbb{C}}\otimes\mathcal{D}'_{\mathbb{C}}=\bigcup_{\tau\in T}\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$ generates a linear continuous functional on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}\otimes\mathcal{D}_{\mathbb{C}}=\mathrm{pr}\lim_{\tau\in T}\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n+m}\otimes\mathcal{H}_{\tau,\mathbb{C}}$ by a formula $l(\circ):=(F^{(n)}\Diamond G^{(m)}_{\cdot},\circ)_{\mathcal{H}_{ext}^{(n+m)}\otimes\mathcal{H}_{\mathbb{C}}}$. On the other hand, $F^{(n)}\widehat{\Diamond}G^{(m)}_{\cdot}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)}\otimes\mathcal{H}_{-\tau,\mathbb{C}}\subset\mathcal{D}'^{(n+m)}_{\mathbb{C}}\otimes\mathcal{D}'_{\mathbb{C}}$ generates a linear continuous functional on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}\otimes\mathcal{D}_{\mathbb{C}}$ by a formula $\widehat{l}(\circ):=\langle F^{(n)}\widehat{\Diamond}G^{(m)}_{\cdot},\circ\rangle_{\mathcal{H}_{ext}^{(n+m)}\otimes\mathcal{H}_{\mathbb{C}}}$. By (2.46) $\widehat{l}=l$ on a total in $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}\otimes\mathcal{D}_{\mathbb{C}}$ set $\{\lambda^{\otimes n+m+1}:\lambda\in\mathcal{D}_{\mathbb{C}}\}$, therefore these *linear continuous* functionals coincide on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n+m}\otimes\mathcal{D}_{\mathbb{C}}$, whence (2.45) follows. □

As a corollary of this lemma we obtain the following statement.

Theorem 2.3. *For arbitrary $n\in\mathbb{N}$ and $f^{(n)}\in\mathcal{H}_{ext}^{(n)}\subset\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ the restriction of the operator $(\mathbf{D}^n\circ)(f^{(n)})$ from the space $(L^2)\otimes\mathcal{H}_{\mathbb{C}}=(L^2)_0^0\otimes\mathcal{H}_{\mathbb{C}}$ to the space $(\mathcal{H}_\tau)_q\otimes\mathcal{H}_{\tau,\mathbb{C}}$ coincides with the operator $(\widehat{\mathbf{D}}^n\circ)(f^{(n)})$.*

Proof. By (2.18) and (2.44) it is sufficient to show that for arbitrary $m\geq n$ and $F^{(m)}_{\cdot}\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}m}\otimes\mathcal{H}_{\tau,\mathbb{C}}\subset\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}$ $(f^{(n)},F^{(m)}_{\cdot})_{EXT}=\langle f^{(n)},F^{(m)}_{\cdot}\rangle_{EXT}$ in $\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}_{\mathbb{C}}$. In fact, by (2.14), (2.45) and (2.41) for arbitrary $G^{(m-n)}_{\cdot}\in\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}_{\mathbb{C}}\subset\mathcal{H}_{-\tau,\mathbb{C}}^{(m-n)}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$ we obtain

$$
\begin{aligned}
&(G^{(m-n)}_{\cdot},(f^{(n)},F^{(m)}_{\cdot})_{EXT})_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}_{\mathbb{C}}} = (f^{(n)}\Diamond G^{(m-n)}_{\cdot},F^{(m)}_{\cdot})_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}\\
&= \langle f^{(n)}\widehat{\Diamond}G^{(m-n)}_{\cdot},F^{(m)}_{\cdot}\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}} = \langle G^{(m-n)}_{\cdot},\langle f^{(n)},F^{(m)}_{\cdot}\rangle_{EXT}\rangle_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}_{\mathbb{C}}}\\
&= (G^{(m-n)}_{\cdot},\langle f^{(n)},F^{(m)}_{\cdot}\rangle_{EXT})_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}_{\mathbb{C}}},
\end{aligned}
$$

whence the result follows. □

Remark 2.9. As it was noted above, there is an analog of Theorem 2.3 for the operators D^n: for arbitrary $n\in\mathbb{N}$ and $f^{(n)}\in\mathcal{H}_{ext}^{(n)}\subset\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ the restriction of the operator $(D^n\circ)(f^{(n)})$ from the space $(L^2)=(L^2)_0^0$ to the space $(\mathcal{H}_\tau)_q$ coincides with the operator of stochastic differentiation on $(\mathcal{H}_\tau)_q$. The interested reader can find a detailed presentation in [25].

Acknowledgments. We are very grateful to Professor Yu. M. Berezansky and Professor E. W. Lytvynov for helpful advices.

References

1. F. E. Benth, *The Gross derivative of generalized random variables*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 3, 381–396.
2. F. E. Benth, G. Di Nunno, A. Lokka, B. Oksendal, and F. Proske, *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*, Math. Finance **13** (2003), no. 1, 55–72.
3. Yu. M. Berezansky, *Spectral theory of commutative Jacobi fields: direct and inverse problems*, Fields Institute Communications **25** (2000), 211–224.
4. Yu. M. Berezansky, Yu. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Vols. 1,2/1. Kluwer Academic Publishers, Dordrecht—Boston—London, 1995. (Russian edition: Naukova Dumka, Kiev, 1988. – 680 p.)
5. Yu. M. Berezansky, V. O. Livinsky, E. W. Lytvynov, *A generalization of Gaussian white noise analysis*, Methods Funct. Anal. Topology **1** (1995), no. 1, 28–55.
6. Yu. M. Berezansky, D. A. Merzejewski, *The structure of extended symmetric Fock space*, Methods Funct. Anal. Topology **6** (2000), no. 4, 1–13.
7. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis*, Vols. 1,2/2. Birkhäuser Verlag, Basel—Boston—Berlin, 1996. (Russian edition: Vyshcha shkola, Kyiv, 1990. – 600 p.)
8. J. Bertoin, *Lévy Processes.* Cambridge University Press, Cambridge, 1996.
9. M. Bożejko, E. W. Lytvynov, and I. V. Rodionova, *An extended anyon Fock space and noncommutative Meixner-type orthogonal polynomials in infinite dimensions*, Russian Math. Surveys **70** (2015), no. 5, 857–899.
10. G. Di Nunno, B. Oksendal, and F. Proske, *Malliavin Calculus for Lévy Processes with Applications to Finance.* Universitext. Springer-Verlag, Berlin, 2009.
11. G. Di Nunno, B. Oksendal, and F. Proske, *White noise analysis for Lévy processes*, J. Funct. Anal. **206** (2004), no. 1, 109–148.
12. M .M. Dyriv, N. A. Kachanovsky, *On operators of stochastic differentiation on spaces of regular test and generalized functions of Lévy white noise analysis*, Carpathian Mathematical Publications **6** (2014), no. 2, 212–229.
13. M .M. Dyriv, N. A. Kachanovsky, *Operators of stochastic differentiation on spaces of regular test and generalized functions in the Lévy white noise analysis*, Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" (2014), no. 4, 36–40.
14. M .M. Dyriv, N. A. Kachanovsky, *Stochastic integrals with respect to a Levy process and stochastic derivatives on spaces of regular test and generalized functions*, Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" (2013), no. 4, 27–30.
15. I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions, Vol. IV: Applications of Harmonic Analysis.* Academic Press, New York—London, 1964.
16. I. I. Gihman, A. V. Skorohod, *The Theory of Stochastic Processes*, Vols. 1,2,3/2. Springer-Verlag, Berlin—Heidelberg—New York, 1975. (Russian edition: Nauka, Moscow, 1973. – 640 p.)
17. H. Holden, B. Oksendal, J. Uboe, and T.-S. Zhang, *Stochastic Partial Differential Equations. A Modeling, White Noise Functional Approach.* Birkhäuser, Boston—Basel—Berlin, 1996.
18. K. Itô, *Spectral type of the shift transformation of differential processes with stationary increments*, Trans. Amer. Math. Soc. **81** (1956), 253–263.
19. Yu. M. Kabanov, *Extended stochastic integrals*, Teor. Veroyatnost. i Primenen. **20** (1975), no. 4, 725–737. (Russian); English transl. Theory Probab. Appl. **20** (1976), no. 4, 710-722.
20. Yu. M. Kabanov, A. V. Skorohod, *Extended stochastic integrals.* In: Proc. School-Symposium "Theory Stoch. Proc."(Druskininkai, Lietuvos Respublika, November 25–30, 1974). Inst. Phys. Math., Vilnus, 1975, pp. 123–167.
21. N. A. Kachanovsky, *Extended stochastic integrals with respect to a Lévy process on spaces of generalized functions*, Mathematical Bulletin of Taras Shevchenko Scientific Society **10** (2013), 169–188.
22. N. A. Kachanovsky, *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*, Methods Funct. Anal. Topology **13** (2007), no. 4, 338–379.
23. N. A. Kachanovsky, *On extended stochastic integrals with respect to Lévy processes*, Carpathian Mathematical Publications **5** (2013), no. 2, 256–278.
24. N. A. Kachanovsky, *Operators of stochastic differentiation on spaces of nonregular generalized functions of Lévy white noise analysis*, Carpathian Mathematical Publications **8** (2016), no. 1, 83–106.
25. N. A. Kachanovsky, *Operators of stochastic differentiation on spaces of nonregular test functions of Lévy white noise analysis*, Methods Funct. Anal. Topology **21** (2015), no. 4, 336–360.

26. N. A. Kachanovsky, V. A. Tesko, *Stochastic integral of HitsudaSkorokhod type on the extended Fock space*, Ukrainian Math. J. **61** (2009), no. 6, 873–907.
27. E. W. Lytvynov, *Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), no. 1, 73–102.
28. P. A. Meyer, *Quantum Probability for Probabilists.* In: Lecture Notes in Mathematics, Vol. 1538. Springer–Verlag, Berlin, 1993.
29. D. Nualart, W. Schoutens, *Chaotic and predictable representations for Lévy processes*, Stochastic Process. Appl. **90** (2000), no. 1, 109–122.
30. P. Protter, *Stochastic Integration and Differential Equations.* Springer, Berlin—New York—Heidelberg, 1990.
31. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, In: Cambridge University Studies in Advanced Mathematics, Vol. 68. Cambridge University Press, Cambridge, 1999.
32. W. Schoutens, *Stochastic Processes and Orthogonal Polynomials*, In: Lecture Notes in Statistics, Vol. 146. Springer-Verlag, New York, 2000.
33. A. V. Skorohod, *Integration in Hilbert Space*, Springer-Verlag, Berlin—New York—Heidelberg, 1974. (Russian edition: Nauka, Moscow, 1974. – 232 p.)
34. A. V. Skorohod, *On a generalization of a stochastic integral*, Teor. Veroyatnost. i Primenen. **20** (1975), no. 2, 223–238.
35. J. L. Solé, F. Utzet, and J. Vives, *Chaos expansions and Malliavin calculus for Lévy processes*, In: Stochastic Analysis and Applications, Vol. 2 of Abel Symposium. Springer, Berlin, 2007, pp. 595–612.
36. D. Surgailis, *On L^2 and non-L^2 multiple stochastic integration*, In: Lecture Notes in Control and Information Sciences, Vol. 36. Springer-Verlag, New York, 1981, pp. 212–226.
37. A. S. Ustunel, *An Introduction to Analysis on Wiener Space*, In: Lecture Notes in Mathematics, Vol. 1610. Springer-Verlag, Berlin, 1995.
38. A.M. Vershik, N.V. Tsilevich, *Fock factorizations and decompositions of the L^2 spaces over general Lévy processes*, Russian Math. Surveys **58** (2003), no. 3, 427–472.

Precarpathian National University, 57 Shevchenka str., Ivano-Frankivs'k, 76025, Ukraine
E-mail address: mashadyriv@ukr.net

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka str., Kyiv, 01601, Ukraine
E-mail address: nkachano@gmail.com

Received 17/02/2017; Revised 27/08/2017