Methods of Functional Analysis and Topology
Vol. 23 (2017), no. 4, pp. 346–366

SPECTRAL PROPERTIES AND STABILITY OF A NONSELFADJOINT EULER–BERNOULLI BEAM

MAHYAR MAHINZAEIM

Abstract. In this note we study the spectral properties of an Euler–Bernoulli beam model with damping and elastic forces applying both at the boundaries as well as along the beam. We present results on completeness, minimality, and Riesz basis properties of the system of eigen- and associated vectors arising from the nonselfadjoint spectral problem. Within the semigroup formalism it is shown that the eigenvectors have the property of forming a Riesz basis, which in turn enables us to prove the uniform exponential decay of solutions of the particular system considered.

This paper was corrected on March 11, 2024. Lemma 5 had the assumption that $\alpha_0, \alpha_1 \geq 0$*. From this assumption it was stated that* A *was boundedly invertible, which is not true. To ensure that* A *has a closed bounded inverse, the lemma now is proven under the stronger assumption that* $\alpha_0, \alpha_1 > 0$*. All results derived in the paper satisfy this assumption*

1. Introduction

The solution of stabilisation problems in infinite dimensions is a difficult mathematical task for two reasons. On the one hand, there is more than one way to extend the ideas of stability and stabilisability from finite-dimensional to infinite-dimensional spaces (depending on the choice of norm induced by the space considered). On the other hand, the spectral mapping theorem does not hold in general for infinite-dimensional systems. This means, more concretely, that infinite-dimensional systems do not necessarily have spectrum-determined growth, and so merely requiring the spectrum of the generating operator for the corresponding semigroup to lie in the open left half-plane does not guarantee that the semigroup is stable, let alone exponentially stable, when it is strongly continuous, in contrast to the case when it is, for example, uniformly continuous. This fact has been known for more than half a century and goes back at least to Hille and Phillips [13, p. 665]. In this respect it should be noted that in addition to being of interest in its own right, the spectrum-determined growth condition – which actually holds for a rather large class of semigroups – is an important tool in major systems problems for infinite-dimensional systems. These include, for example, such concepts as controllability and observability, as well as, in particular, stabilisability. A good treatment of these subjects may be found in the book by Curtain and Zwart [5].

There is one situation, however, where it is justified to relate the spectrum of the infinitesimal generator to that of the semigroup, namely, if it is shown that (in the terminology of N. Dunford [6, Chapters XVIII to XX]) the system operator, that is, the generating operator for the system semigroup, is a discrete spectral operator whose system of eigen- and associated vectors or root vectors form a Riesz basis, that is, a basis equivalent to

2010 *Mathematics Subject Classification.* 34B07, 34L10, 34L20, 35B40.
Key words and phrases. Euler–Bernoulli beam, nonselfadjoint operator, operator pencil, completeness, minimality, Riesz basis, series expansion, exponential stability.

an orthonormal basis for the underlying Hilbert space. Miloslavskii in [24], and redeveloped in [25], and Röh in [33] were among the first to note the connection between the stability properties of infinite-dimensional systems and the Riesz basis properties of the root vectors.

In general, the system operator is not necessarily a discrete spectral operator. In fact, in most of the literature dealing with other approaches to the problems of stability and stabilisability, only very basic mathematical properties of the system operator are verified (or anticipated) for the purposes of unique solvability or well-posedness of such problems: it is closed, densely defined, and dissipative, and usually has a compact resolvent. These preliminary results would also prove to be useful, nevertheless, in obtaining Riesz basis results from that point on. Indeed, we recall from [6, Corollary XVIII.2.33] that, in general, in order that a closed linear operator, A, say, in a Hilbert space be a discrete spectral operator, the following properties are needed: (i) A has a compact resolvent; and (ii) the root vectors of A form a Riesz basis for the underlying Hilbert space. As is well known, it comes down essentially to showing that the spectrum of A is discrete and is separated by a uniform gap, that is,

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0, \quad n, m \in \mathbb{N}. \tag{1.1}$$

Let us describe the system we study in this paper. We consider a damped Euler–Bernoulli beam of unit length on an elastic foundation and with a combination of elastic and viscous damping effects in the boundary conditions. Specifically, we consider the initial/boundary-value problem consisting of the partial differential equation

$$\frac{\partial^4}{\partial s^4} w(s,t) + \gamma^2 w(s,t) + 2\gamma \frac{\partial}{\partial t} w(s,t) + \frac{\partial^2}{\partial t^2} w(s,t) = 0 \tag{1.2}$$

together with boundary conditions

$$\left.\frac{\partial^2}{\partial s^2} w(s,t)\right|_{s=0} = 0, \tag{1.3}$$

$$\left.\left(\frac{\partial^3}{\partial s^3} w(s,t) + \alpha_0 w(s,t) + \beta_0 \frac{\partial}{\partial t} w(s,t)\right)\right|_{s=0} = 0, \tag{1.4}$$

$$\left.\frac{\partial^2}{\partial s^2} w(s,t)\right|_{s=1} = 0, \tag{1.5}$$

$$\left.\left(\frac{\partial^3}{\partial s^3} w(s,t) - \alpha_1 w(s,t) - \beta_1 \frac{\partial}{\partial t} w(s,t)\right)\right|_{s=1} = 0 \tag{1.6}$$

and initial conditions

$$w(s,0) = g(s), \quad \left.\frac{\partial}{\partial t} w(s,t)\right|_{t=0} = h(s), \tag{1.7}$$

where g, h are assumed given and "suitably" smooth (as specified later in Section 6). The transverse displacement of the beam at position s and time t is represented by $w(s,t)$, the parameter $\gamma \geq 0$ determines both the external (viscous) damping and elastic foundation, and the pairs α_0, β_0 and α_1, β_1 represent combinations of elastic and viscous damping effects at $s = 0$ and $s = 1$, respectively. As will be seen, the initial/boundary-value problem (1.2)–(1.7) corresponds to the interesting and relevant case for stability when it can be associated with a dissipative system. (Note that the partial differential equation has a form which closely resembles another wave-like equation, the classical telegrapher's equation [2, p. 192], except for the Euler–Bernoulli elasticity term, which is of fourth order.)

A substantial portion of our work will be devoted, as well as to conclude information about the spectrum of the corresponding spectral problem, but also to addressing the study of the eigenvectors connected with the operator-theoretic or abstract version of (1.2)–(1.7) in state space form. As will be seen, performing this task is not straightforward because the system operator will not be selfadjoint in general. It is well known that the necessary information that a nonselfadjoint linear operator is a discrete spectral operator cannot be proven simply by examining its spectrum (for verification of the spectral gap property (1.1), essentially). Stated another way, once a thorough analysis of the spectrum – that is, existence, location, and asymptotics of eigenvalues – is completed, in actuality there still remains the problem of establishing that the root vectors or eigenvectors are minimal complete and form a basis. We omit here the standard details of the definitions of completeness, minimality, bases or unconditional bases, and so on, which can be found, for example, in [8, 22, 18].

In this paper we follow an argument to prove the Riesz basis property which avoids the need for deriving asymptotic formulae for the eigenvectors as required when working with the theorem, for example, of N. K. Bari [8, Theorem VI.2.3] or variants of it. It is possible to derive asymptotic formulae for the eigenvectors, a line that has been pursued by many investigators – particularly in the fields of stability and control – during the past two decades or so, such as Conrad and Morgül [1], Cox and Zuazua [3, 4], Guo [11, 12], Rao [32], and Xu and Feng [36] (to name only a few that come to mind). However, we feel that it will be interesting to see how we can prove the Riesz basis property not by working with explicit asymptotic expressions for the eigenvectors, but by using knowledge of the asymptotic behaviour of the eigenvalues. Such proof method is indeed possible (in the contexts here relevant), and we demonstrate this by making use of a slight modification of a less well-known theorem, proven in different ways for dissipative bounded linear operators, by I. M. Glazman and, in a paper in Doklady, B. R. Mukminov in the 1950's, as well as by Markus [21]. These, probably being the earliest abstract results on Riesz basis properties, will be seen for the specific system under consideration to be a more direct route to verification of the Riesz basis property, since the system operator naturally possesses a structure that is related directly to the spectral properties of a compact selfadjoint operator in the specialised nondissipative case. (The idea is to some degree, in fact, related to the rationale behind Kato's results [14, Section V.4]. For additional details, see the recent survey [35] of Shkalikov.)

An essential prerequisite for our undertaking is the verification of completeness of the eigenvectors, which usually is the challenging part. Here, however, we essentially exploit the aforementioned specific structure of the system operator to prove completeness, and we will explore in detail exactly how this can be accomplished in Section 5. The key ingredient there is provided by a version of a well-known theorem of M. V. Keldysh which was published in his famous 1951 paper. Properly exploited, it enables us to establish that the eigenvectors are complete in the state space. (An English translation of Keldysh's paper is contained in the Appendix of [22], and further discussion and proofs can be found, for example, in [16] or [31].)

The rest of the paper is structured as follows. Section 2 treats some preliminary material including relevant definitions of spaces, while our true starting point is Section 3, where we pose the abstract spectral problems and define the operators relevant to them. This is an important first step which we take in a way that will have advantages for the study of the spectrum later. For example, instead of working on the spectral problem only in the state space for the system, we choose to additionally work in another Hilbert product space. In doing so, we can invoke some of the theory of quadratic operator pencils in the development in Section 4 of proofs for many results on the spectral properties and asymptotics of eigenvalues. The state space is particularly suited to studying the problems, to be considered in Section 5, of completeness, minimality, and Riesz basis properties, which in turn are used in Section 6 to examine the stability associated with the initial/boundary-value problem (1.2)–(1.7) in its semigroup formulation. So, where it is relevant, we will link the findings of Section 4 to the spectral problem in the state space.

There are a few interesting papers which are close in spirit to this paper. Here, we must particularly note the works of Gomilko and Pivovarchik [9, 10], Möller and Pivovarchik [27], Pivovarchik [30], and a series of works of Möller and Zinsou, many of which are listed in [37]. The main difference between our work and these papers lies in the type of system we consider. Although, from a mathematical point of view, it is a rather special system (due primarily to the parameters in the partial differential equation governing beam motion being all constants), it has a prominent place in the engineering literature, mainly as the so-called "half-car" or "bridge" models, which describe the structural dynamics of suspended vehicles and bridges. An example is the early paper [19] by the author and others. This paper, however, lacks rigour in that it does not justify mathematically the method used there of Fourier series expansion of the solution of the considered initial/boundary-value problem. It is intuitively obvious that in the end this comes down to establishing conditions for the convergence of the series expansion, and here lies the need for study of the problems of completeness, minimality, and Riesz basis properties. To date, as far as we know, no rigorous mathematical study has been conducted that adequately addresses these problems (and the underlying ones) in the context of the present system. We shall give such a study in this paper, and we show in Section 6 how to apply our results to prove uniform exponential stability for the semigroup, a result which we believe to be the first of its kind for our system. The result is the culmination of the results of the foregoing sections.

2. Definitions and preliminaries

To begin with, let us assume a separable solution to the initial/boundary-value problem (1.2)–(1.7) described in the introduction, for some spectral parameter ω. We specify the relationship $\lambda = \omega - i\gamma$ between ω and another spectral parameter λ, and obtain, on making the substitution

$$w(s,t) = e^{i(\lambda+i\gamma)t} w(\lambda, s),$$

the boundary-eigenvalue problem

$$w^{(4)}(\lambda, s) - \lambda^2 w(\lambda, s) = 0, \tag{2.1}$$

$$w''(\lambda, 0) = 0, \tag{2.2}$$

$$w^{(3)}(\lambda, 0) + (\alpha_0 - \beta_0\gamma + i\beta_0\lambda)\, w(\lambda, 0) = 0, \tag{2.3}$$

$$w''(\lambda, 1) = 0, \tag{2.4}$$

$$w^{(3)}(\lambda, 1) - (\alpha_1 - \beta_1\gamma + i\beta_1\lambda)\, w(\lambda, 1) = 0. \tag{2.5}$$

Sometimes, for convenience, we set

$$\theta_0(\lambda) = \alpha_0 - \beta_0\gamma + i\beta_0\lambda, \quad \theta_1(\lambda) = \alpha_1 - \beta_1\gamma + i\beta_1\lambda \tag{2.6}$$

in the boundary conditions (2.3) and (2.5) and unless otherwise specified, it is understood that

$$\alpha_0, \alpha_1 > 0, \quad \beta_0, \beta_1 \geq 0, \quad \gamma \geq 0, \quad \alpha_0 > \beta_0\gamma, \quad \alpha_1 > \beta_1\gamma$$

throughout.

The boundary-eigenvalue problem (2.1)–(2.5) has eigenvalue-dependent boundary conditions. It is impossible therefore to recast it abstractly as a spectral problem for a linear operator in $L_2(0,1)$. It can, however, be cast into the general framework of abstract operator pencils if one considers its operator-theoretic formulation on a Hilbert product space. This is the route that we will follow, and we wish to do this on two spaces, X and Y, endowed with appropriate topologies. In Y we will fit the problem into the setting of quadratic nonmonic operator pencils, and the same problem but with λ replaced by $\omega - i\gamma$ will be fit into the setting of linear monic operator pencils, or equivalently linear operators in the space X. We define X to be the state space $H^2(0,1) \times L_2(0,1)$ of two-component vectors ($H^m(0,1)$ are the usual Sobolev–Hilbert spaces of order $m \in \mathbb{N}_0$ related to $L_2(0,1)$), and we define Y as the space $L_2(0,1) \times \mathbb{C}^2$ of three-component vectors. The operators basic to the abstract formulations are defined in the next section.

Let us recall now, for completeness, some standard definitions of the spectral theory of unbounded operator pencils as a convenience for the reader. A good account of the spectral theory of operator pencils, with many application examples from mechanics, is given by Möller and Pivovarchik in their recent text [26]; refer there and to [22] for equivalent definitions.

Definition 1. *Let $\lambda \mapsto P(\lambda)$ be a mapping from $\mathbb{C}$ (or some nonempty subset thereof) into the set of closed linear operators in the Hilbert space X. A number $\lambda \in \mathbb{C}$ is said to belong to $\varrho(P)$, the resolvent set of P, provided $P(\lambda)$ has a closed bounded inverse, that is, provided $P(\lambda)$ is boundedly invertible. We call $P^{-1}(\lambda) := (P(\lambda))^{-1}$ the resolvent of $P(\lambda)$. The complement of $\varrho(P)$ is the spectrum of P and is denoted by $\sigma(P)$. If a number $\lambda_0 \in \mathbb{C}$ has the property that* $\ker P(\lambda_0) \neq \{0\}$ *then it is called and eigenvalue of P and there exists an eigenvector $x_0 \neq 0$ corresponding to λ_0 such that $P(\lambda_0)x_0 = 0$.*

Definition 2. *The sequence of vectors $\{x_r\}_{r=0}^{h-1}$ is said to form a chain, of length h, consisting of an eigenvector x_0 of P corresponding to an eigenvalue λ_0 and the vectors $x_1, x_2, \dots, x_{h-1}$ associated with it, or, for brevity, simply a chain of root vectors of P corresponding to λ_0, if*

$$\sum_{l=0}^{r} \frac{1}{l!} \frac{\mathrm{d}^l}{\mathrm{d}\lambda^l} P(\lambda)\bigg|_{\lambda=\lambda_0} x_{r-l} = 0, \quad r = 0, 1, \dots, h-1.$$

The geometric multiplicity of an eigenvalue λ_0 is the number of linearly independent eigenvectors in a system of chains of root vectors of P corresponding to λ_0 and is defined as $\dim\ker P(\lambda_0)$. *The algebraic multiplicity of an eigenvalue is the maximum value of the sum of the lengths of chains corresponding to the linearly independent eigenvectors. We call an eigenvalue λ_0 simple if its geometric and algebraic multiplicities are equal and* $\dim\ker P(\lambda_0) = 1$.

Definition 3. *If an eigenvalue $\lambda_0 \in \sigma(P)$ is an isolated point in $\sigma(P)$ and $P(\lambda_0)$ is a Fredholm operator, then we call λ_0 a normal eigenvalue. The set of all normal eigenvalues is denoted by $\sigma_0(P)$.*

Remark 1. *Obviously the definitions coincide with the definitions for the spectrum of a closed linear operator A in Hilbert space when $P(\lambda) = \lambda I - A$.*

For Riesz basis properties, a suitable indexing of the eigenvalues is crucial. This leads to the following definition.

Definition 4. *An infinite sequence of eigenvalues is said to be properly enumerated if it is a sequence of real or complex numbers which are counted properly – that is, such that*

(i) *their algebraic multiplicities are taken into account;*
(ii) $\lambda_{-n} = -\overline{\lambda_n}$ *when* $\operatorname{Re}\lambda_n \neq 0$; *and*
(iii) $\operatorname{Re}\lambda_{n+1} \geq \operatorname{Re}\lambda_n$.

As we have already mentioned in the Introduction, prerequisite conditions for the verification of the Riesz basis property of root vectors are completeness and minimality. These can be deduced readily from the following two results, the first of which is due to Keldysh [7, Theorem X.4.1] on the completeness property. For a proof of the second result, on the minimality property, see, for example, [20, Lemma 2.4]. We shall invoke both later in Section 5.

Lemma 1 (Keldysh)**.** *Let K be a compact selfadjoint operator on a Hilbert space X with* $\ker K = \{0\}$*. Let the sequence $\{\lambda_n\}_{n=1}^{\infty}$ represent the eigenvalues K, and assume*

$$\sum_{n=0}^{\infty} |\lambda_n|^p < \infty \tag{2.7}$$

for some $p \geq 1$. Suppose further that S is a compact operator such that $I+S$ is invertible. Then the root vectors of the operator

$$A = K\,(I + S) \tag{2.8}$$

are complete in X.

Lemma 2. *Let A be a compact operator on X and* $\ker A = \{0\}$*. Then the root vectors of A are minimal in X.*

The principal tool in establishing the Riesz basis property of the eigenvectors in Section 5 will be the individual theorems of Glazman and Mukminov. The following is a basic version of their result (see [8, p. 328] or [18, p. 213]).

Lemma 3 (Glazman and Mukminov)**.** *Let A be a bounded dissipative operator on X and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of its eigenvalues such that*

$$\sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} \frac{\operatorname{Im}\lambda_n \operatorname{Im}\lambda_m}{\left|\lambda_n - \overline{\lambda_m}\right|^2} < \infty. \tag{2.9}$$

Then the corresponding eigenvectors in X form a Riesz basis (in fact, a Bari basis) for its closed linear span.

We close this section with our first result, which essentially guarantees the existence of an infinite number of normal eigenvalues of the boundary-eigenvalue problem (2.1)–(2.5). Its proof rests entirely on results proven by Mennicken and Möller in [23, Sections 7.2 and 7.3], and the reader is referred there for further information.

Proposition 1. *The boundary-eigenvalue problem* (2.1)–(2.5) *under the spectral transformation $\lambda \mapsto \mu^2$ is Birkhoff regular in the sense of* [23, Definition 7.3.1].

Proof. We assume the change from λ to μ^2 in (2.1)–(2.5) and rewrite (2.6) as

$$\theta_0\,(\mu) = \alpha_0 - \beta_0\gamma + i\beta_0\mu^2, \quad \theta_1\,(\mu) = \alpha_1 - \beta_1\gamma + i\beta_1\mu^2.$$

First of all, note that the differential equation (2.1) has associated with it a characteristic function of degree four, defined by [23, (7.1.4)], which takes here the form $\pi\,(\rho) := \rho^4 - 1$. Its zeros are i^{k-1}, $k = 1, 2, 3, 4$, and it is easily verified that the assumptions for [23, Theorem 7.2.4] are satisfied. Thus there is a 4×4 transformation matrix which we can choose to be

$$C\,(s, \mu) = \operatorname{diag}\,(1, \mu, \mu^2, \mu^3)\,(i^{(k-1)(j-1)})_{k,j=1}^{4}.$$

Consequently, in view of [23, (7.3.1)], we have for the boundary matrices

$$W^{(0)}\,(\mu) = \begin{pmatrix} \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \theta_0\,(\mu) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C\,(0, \mu)$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\theta_1(\mu) & 0 & 0 & 1 \end{pmatrix} C(1,\mu),$$

where

$$\eta_j := \theta_0(\mu) + (-i)^{j-1}\mu^3, \quad \zeta_j := -\theta_1(\mu) + (-i)^{j-1}\mu^3, \quad j = 1,2,3,4.$$

We may now choose $C_2(\mu) = \operatorname{diag}(\mu^2, \mu^3, \mu^2, \mu^3)$. Then we have, according to the formula in [23, Theorem 7.3.2(i)], that

$$C_2^{-1}(\mu) W^{(0)}(\mu) = W_0^{(0)} + O(\mu^{-1})$$

and

$$C_2^{-1}(\mu) W^{(1)}(\mu) = W_0^{(1)} + O(\mu^{-1}),$$

where

$$W_0^{(0)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

For Birkhoff regularity, we must require by [23, Theorem 7.3.2] the nonsingularity of the Birkhoff matrices (I here being the identity matrix)

$$W_0^{(0)}\Delta_j + W_0^{(1)}(I - \Delta_j), \quad j = 1,2,3,4,$$

in which the Δ_j are, according to [23, Definition 7.3.1 and Proposition 4.1.7], 4×4 diagonal matrices whose diagonal elements consist of two consecutive ones, followed by two consecutive zeros in a cyclic arrangement: $\Delta_1 := \operatorname{diag}(1,1,0,0)$, $\Delta_2 := \operatorname{diag}(0,1,1,0)$, $\Delta_3 := \operatorname{diag}(0,0,1,1)$, and $\Delta_4 := \operatorname{diag}(1,0,0,1)$. The Birkhoff matrices can all be verified to be nonsingular, and therefore the theorem is established. □

3. The abstract spectral problems

In this section our intention, broadly put, is to recast the boundary-eigenvalue problem (2.1)–(2.5) abstractly as operator functions acting in the spaces X, Y defined in the previous section as follows:

$$X := H^2(0,1) \times L_2(0,1), \quad Y := L_2(0,1) \times \mathbb{C}^2.$$

We proceed to pose our spectral problems in X and Y.

3.1. **The spectral problem in X.** The state space X is a Hilbert space under the norm induced by the inner product

$$((w,v),(\tilde{w},\tilde{v}))_X := (w,\tilde{w})_2 + (v,\tilde{v})_0$$

for any $(w,v),(\tilde{w},\tilde{v}) \in X$, where

$$(w,\tilde{w})_2 = \int_0^1 w''(s)\overline{\tilde{w}''(s)}\,\mathrm{d}s + \gamma^2 \int_0^1 w(s)\overline{\tilde{w}(s)}\,\mathrm{d}s + \alpha_0 w(0)\overline{\tilde{w}(0)} + \alpha_1 w(1)\overline{\tilde{w}(1)}, \quad w,\tilde{w} \in H^2(0,1),$$

which is well defined because $\alpha_0, \alpha_1 > 0$ and $\gamma \geq 0$ by assumption. Throughout we denote by $(\cdot,\cdot)_0$ the inner product in $L_2(0,1)$ and write, as usual, $\|\cdot\|_0$ for the resulting norm.

Define the linear operator

$$(Ax)(s) := i(-v(s), w^{(4)}(s)), \quad x \in D(A), \quad s \in [0,1], \tag{3.1}$$

with domain

$$D(A) = \left\{ x \in X \;\middle|\; \begin{array}{c} (w,v) \in H^4(0,1) \times H^2(0,1), \\ w''(0) = 0,\ w^{(3)}(0) + \alpha_0 w(0) + \beta_0 v(0) = 0, \\ w''(1) = 0,\ w^{(3)}(1) - \alpha_1 w(1) - \beta_1 v(1) = 0 \end{array} \right\}. \tag{3.2}$$

We further define the operator

$$(Bx)(s) := i\,(0, \gamma^2 w(s) + 2\gamma v(s)), \quad x \in X, \quad s \in [0,1]. \tag{3.3}$$

Since B, as thus defined, is a bounded linear operator on X,

$$D(A+B) = D(A)$$

and we may regard $A+B$ for $\gamma > 0$ as a bounded linear perturbation of A. The operator $i\,(A+B)$ will be called the system operator (to which we will come back in Section 6).

The formulation of the abstract spectral problem in X now requires that we use the relationship $\omega = \lambda + i\gamma$ in (2.1)–(2.5) to obtain the boundary-eigenvalue problem with ω in place of λ. With the operator pencil

$$P(\omega) = \omega I - (A+B), \quad D(P(\omega)) = D(A), \tag{3.4}$$

the boundary-eigenvalue problem can be checked to be equivalent to the spectral problem

$$P(\omega)\,x = [\omega I - (A+B)]\,x = 0, \quad x \in D(A), \quad \omega \in \mathbb{C}. \tag{3.5}$$

The following two lemmas will be used to connect the spectral properties of the linear operator pencil $P(\omega)$ or, equivalently, of $A+B$ with those of a quadratic operator pencil (details in the next section) acting in the space Y.

Lemma 4. *The set of eigenvalues, including multiplicities, of the boundary-eigenvalue problem* (2.1)–(2.5) *with* $\lambda = \omega - i\gamma$ *coincides with that of the linear operator pencil* $P(\omega)$ *given by* (3.4)*, where* A *is defined by* (3.1), (3.2)*, and* B *by* (3.3)*. Further, the following relation holds between a chain of root functions* $w_0, w_1, \ldots, w_{h-1}$ *of* (2.1)–(2.5) *corresponding to an eigenvalue* ω_0 *and a chain of root vectors* $x_0, x_1, \ldots, x_{h-1}$ *of* P *corresponding to the same eigenvalue:*

$$x_r = (w_r, v_r), \quad v_r = i\omega_0 w_r + i w_{r-1}, \quad r = 0, 1, \ldots, h-1.$$

Proof. We follow [34, Lemma 1.4], so it suffices to verify the relations, mentioned in the statement of the lemma, between the root functions and the root vectors (see also [28, Section I.2]). Let ω_0 be an eigenvalue of P with a corresponding chain formed by the root vectors $x_0, x_1, \ldots, x_{h-1}$. Recall the spectral problem (3.5), and note that, according to Definition 2,

$$P(\omega_0)\,x_r + x_{r-1} = 0, \quad r = 0, 1, \ldots, h-1,$$

where $x_{-1} := 0$, or equivalently in coordinates,

$$\begin{aligned} i v_r(\omega_0, s) + \omega_0 w_r(\omega_0, s) + w_{r-1}(\omega_0, s) &= 0, \\ -i w_r^{(4)}(\omega_0, s) - i\gamma^2 w_r(\omega_0, s) - 2i\gamma v_r(\omega_0, s) + \omega_0 v_r(\omega_0, s) + v_{r-1}(\omega_0, s) &= 0 \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} w_r''(\omega_0, 0) &= 0, \\ w_r^{(3)}(\omega_0, 0) + \alpha_0 w_r(\omega_0, 0) + \beta_0 v_r(\omega_0, 0) &= 0, \\ w_r''(\omega_0, 1) &= 0, \\ w_r^{(3)}(\omega_0, 1) - \alpha_1 w_r(\omega_0, 1) - \beta_1 v_r(\omega_0, 1) &= 0 \end{aligned}$$

for $r = 0, 1, \ldots, h-1$. The lemma follows from these relations. $\square$

Lemma 5. *Consider the operator* $A+B$*. The following assertions hold:*

(i) *$A+B$ is maximal dissipative for $\beta_0, \beta_1 > 0$ and $\gamma > 0$, and selfadjoint when $\beta_0 = \beta_1 = \gamma = 0$; and*
(ii) *$A+B$ has a compact inverse.*

Proof. To prove assertion (i), we show that $A+B$ is dissipative and boundedly invertible. To do this, we first observe that, for $x \in D(A)$,

$$((A+B)\,x,x)_X = -i\,(v,w)_2 + i\,(w^{(4)} + \gamma^2 w + 2\gamma v, v)_0.$$

An elementary calculation using integration by parts shows that

$$\begin{aligned} i\,(w^{(4)} + \gamma^2 w, v)_0 &= i\int_0^1 w''(s)\,\overline{v''(s)}\;\mathrm{d}s + i\gamma^2\int_0^1 w(s)\,\overline{v(s)}\;\mathrm{d}s + i\alpha_0 w(0)\,\overline{v(0)} \\ &\quad + i\alpha_1 w(1)\,\overline{v(1)} + i\beta_0\,|v(0)|^2 + i\beta_1\,|v(1)|^2 \\ &= i\,(w,v)_2 + i\beta_0\,|v(0)|^2 + i\beta_1\,|v(1)|^2. \end{aligned}$$

Hence, on rearranging,

$$((A+B)\,x,x)_X = i\,(w,v)_2 - i\,(v,w)_2 + 2i\gamma\,\|v\|_0^2 + i\beta_0\,|v(0)|^2 + i\beta_1\,|v(1)|^2,$$

and a simple computation reveals that

$$i\,(w,v)_2 - i\,(v,w)_2 = 2\,\mathrm{Im}\,(v,w)_2.$$

Consequently,

$$\mathrm{Im}\,((A+B)\,x,x)_X = 2\gamma\,\|v\|_0^2 + \beta_0\,|v(0)|^2 + \beta_1\,|v(1)|^2. \tag{3.6}$$

By assumption $\beta_0, \beta_1 \geq 0$ and $\gamma \geq 0$ so that $\mathrm{Im}\,((A+B)\,x,x)_X \geq 0$. So we have that $A+B$ is dissipative. Obviously it is symmetric when $\beta_0 = \beta_1 = \gamma = 0$. To finally complete the proof of assertion (i), we now show that $A+B$ is boundedly invertible (and thus closed). We do this in two steps.

Step 1. For $\tilde{x} \in X$, $x \in D(A)$ let us consider the problem

$$Ax = \tilde{x}.$$

Equivalently in coordinates,

$$-iv(s) = \tilde{w}(s), \tag{3.7}$$
$$iw^{(4)}(s) = \tilde{v}(s), \tag{3.8}$$
$$w''(0) = 0, \tag{3.9}$$
$$w^{(3)}(0) + \alpha_0 w(0) + \beta_0 v(0) = 0, \tag{3.10}$$
$$w''(1) = 0, \tag{3.11}$$
$$w^{(3)}(1) - \alpha_1 w(1) - \beta_1 v(1) = 0. \tag{3.12}$$

By formally integrating the differential equation (3.8) four times we obtain

$$w(s) = w(0) + sw'(0) + \frac{s^2}{2!}w''(0) + \frac{s^3}{3!}w^{(3)}(0) - i\int_0^s \frac{(s-r)^3}{3!}\tilde{v}(r)\,\mathrm{d}r.$$

The boundary conditions (3.9)–(3.12) together with (3.7) imply

$$\begin{aligned} w''(0) &= 0, \\ \alpha_0 w(0) + w^{(3)}(0) &= -i\beta_0 \tilde{w}(0), \\ w''(0) + w^{(3)}(0) &= i\int_0^1 (1-r)\,\tilde{v}(r)\,\mathrm{d}r, \\ -\alpha_1 w(0) - \alpha_1 w'(0) - \frac{\alpha_1}{2!} w''(0) + \left(1 - \frac{\alpha_1}{3!}\right) w^{(3)}(0) &= i\int_0^1 \left[1 - \frac{\alpha_1}{3!}(1-r)^3\right] \tilde{v}(r)\,\mathrm{d}r \\ &\quad - i\beta_1 \tilde{w}(1). \end{aligned}$$

The coefficients of $w(0)$, $w'(0)$, $w''(0)$, $w^{(3)}(0)$ form the coefficient matrix of this system of algebraic equations and a direct calculation shows that (where we recall that, by assumption, $\alpha_0, \alpha_1 > 0$)

$$\det \begin{pmatrix} 0 & 0 & 1 & 0 \\ \alpha_0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -\alpha_1 & -\alpha_1 & -\frac{\alpha_1}{2!} & 1 - \frac{\alpha_1}{3!} \end{pmatrix} = \alpha_0 \alpha_1 \neq 0.$$

So we have that there is a unique solution which is given by the column vector formed by $w(0)$, $w'(0)$, $w''(0)$, $w^{(3)}(0)$. Thus we have obtained the uniquely-determined nonzero function $w \in H^4(0,1)$. In particular, we observe that $x = A^{-1}\tilde{x}$, x being a nontrivial element of $D(A)$, so A is bijective and it is obviously closed. A well-known consequence of the closed graph theorem is that A^{-1} is a bounded operator. So $0 \in \varrho(A)$. We know by Sobolev's embedding theorem that $H^4(0,1) \times H^2(0,1)$ is compactly embedded in $H^2(0,1) \times L_2(0,1)$ and therefore A^{-1} is compact. This completes the first step in the proof of assertion (i) (and, in fact, also of assertion (ii)).

Step 2. The second step involves use of the compactness of A^{-1} to show that the operator $A + B$ possesses also a compact inverse. So let us write $A + B$ in the form

$$A + B = (I + BA^{-1})\,A.$$

Obviously BA^{-1} is a compact operator (recall $A + B$ is for $\gamma > 0$ a bounded linear perturbation of the Fredholm operator A). Therefore, using the compactness of A^{-1} together with that of BA^{-1},

$$(A+B)^{-1} = A^{-1}(I + BA^{-1})^{-1}$$

exists and is compact, proving assertion (ii). Thus, $A + B$ is maximal dissipative. The selfadjointness of $A + B$ for $\beta_0 = \beta_1 = \gamma = 0$ then follows immediately from the surjectivity of A. This completes the proof of assertion (i), and thus of the lemma.

□

The implications of Lemma 5 call for some natural comments (in preparation for what follows). Firstly, because $0 \in \varrho(A+B)$, the spectrum of P or, equivalently, of $A + B$ is discrete and consists only of normal eigenvalues, $\sigma(A+B) = \sigma_0(A+B)$, which accumulate only at infinity (see [7, Theorem XV.2.3] or [14, Theorem III.6.29]). Moreover, when $A+B$ is selfadjoint – we know from Lemma 5 that this is the case when $\beta_0 = \beta_1 = \gamma = 0$ – the spectral theorem [7, Theorem XVI.5.1] yields the existence of an orthonormal basis for X consisting only of eigenvectors of A corresponding to real eigenvalues. However, when $A+B$ is not selfadjoint, the direct analogue of this theorem is not true, for it is then generally possible to have root vectors which are minimal complete but which are not a basis (in Schauder's sense). We shall return to this matter in Section 5 and close this section with the following obvious proposition.

Proposition 2. *Let $\beta_0, \beta_1 > 0$ and $\gamma > 0$. Then there can exist no purely real eigenvalues of $A + B$.*

Proof. To prove the proposition we show that unless $\beta_0 = \beta_1 = \gamma = 0$ there is no nontrivial element of $\ker P(\omega)$ when ω is purely real. To this end, suppose (to reach a contradiction) there is an eigenvalue $\omega_0 \in \mathbb{R}$ and take x_0 to be the corresponding eigenvector, such that

$$(P(\omega_0)\,x_0, x_0)_X = (\omega_0 I - (A+B)\,x_0, x_0)_X = 0. \tag{3.13}$$

We have for the imaginary part

$$\operatorname{Im}((A+B)\,x_0, x_0)_X = 0;$$

so, by (3.6),

$$2\gamma\,\|v_0\|_0^2 + \beta_0\,|v_0(0)|^2 + \beta_1\,|v_0(1)|^2 = 0.$$

If $\beta_0, \beta_1 > 0$ and $\gamma > 0$, we then have from the above that $v_0 = 0$. Using this along with the fact that $0 \in \varrho(A+B)$ we can infer that $w_0 = 0$, and thus have $x_0 = 0$. This is contrary to our assumption that x_0 is an eigenvector. The proof is complete. □

3.2. **The spectral problem in Y.** We consider now the spectral problem in the space Y. This is a Hilbert space with the inner product

$$((w,a,c),(\tilde{w},\tilde{a},\tilde{c}))_Y := (w,\tilde{w})_0 + a\overline{\tilde{a}} + c\overline{\tilde{c}}$$

for any $(w,a,c),(\tilde{w},\tilde{a},\tilde{c}) \in Y$. Define the linear operator

$$\begin{aligned}(Gy)(s) := (&w^{(4)}(s), w^{(3)}(0) + (\alpha_0 - \beta_0\gamma)\,w(0), \\ &- w^{(3)}(1) + (\alpha_1 - \beta_1\gamma)\,w(1)), \quad y \in D(G), \quad s \in [0,1],\end{aligned} \tag{3.14}$$

with

$$D(G) = \left\{ y \in Y \;\middle|\; \begin{array}{c} w \in H^4(0,1), \\ a = w(0),\ c = w(1),\ w''(0) = 0,\ w''(1) = 0 \end{array} \right\}, \tag{3.15}$$

and define

$$(Cy)(s) := (0, \beta_0 w(0), \beta_1 w(1)), \quad y \in Y, \quad s \in [0,1], \tag{3.16}$$

and

$$(Dy)(s) := (w(s), 0, 0), \quad y \in Y, \quad s \in [0,1]. \tag{3.17}$$

The operators C, D are bounded linear operators on Y and selfadjoint, $C \geq 0$, $D \geq 0$, and the operator G will be studied shortly, where we shall see that it is selfadjoint and its inverse exists and is compact.

Consider the operator pencil

$$L(\lambda) = \lambda^2 D - i\lambda C - G, \quad D(L(\lambda)) = D(G). \tag{3.18}$$

Then the boundary-eigenvalue problem (2.1)–(2.5) takes the abstract form

$$L(\lambda)\,y = (\lambda^2 D - i\lambda C - G)\,y = 0, \quad y \in D(G), \quad \lambda \in \mathbb{C}. \tag{3.19}$$

Obviously, if there is a chain of root vectors $y_0, y_1, \ldots, y_{h-1}$ of L corresponding to an eigenvalue λ_0, then, because the domain of $L(\lambda)$ is independent of the spectral parameter, it is in one-to-one correspondence with a chain formed by the root functions $w_0, w_1, \ldots, w_{h-1}$ of (2.1)–(2.5) corresponding to the same eigenvalue. Thus (3.19) holds if and only if (2.1)–(2.5) holds.

The next proposition is important and will be needed later, in Section 4, in the study of the spectral properties of $L(\lambda)$.

Proposition 3. *Consider the operators C and D, defined by* (3.16) *and* (3.17)*, respectively. The operator D is positive definite when restricted to the domain $D(G)$, that is $D|_{D(G)} > 0$. If $\beta_0, \beta_1 > 0$, then $C|_{D(G)} > 0$ too.*

Proof. It is obvious that we need only show that we have

$$(Cy, y)_Y > 0, \quad (Dy, y)_Y > 0,$$

provided we restrict $y\,(\neq 0)$ to $D(G)$, which we assume henceforth. It follows that if we have $w = 0$, then $y = 0$. Therefore, when $y \neq 0$, then $w \neq 0$. So

$$(Dy, y)_Y = \|w\|_0^2 > 0,$$

proving the first statement. The key step in the proof of the second statement is to notice that $w(0) = w(1) = 0$ implies $w = 0$, and thus $y = 0$. That this is true is proven as follows. We take $\{w_k\}_{k=1}^4$ to be a fundamental system of solutions to (2.1) satisfying

$$w_k^{(m)}(0) = \delta_{k,m+1}, \quad m = 0, 1, 2, 3.$$

Then, because of the boundary condition (2.2), there are three linearly independent solutions, namely w_1, w_2, and w_4. With

$$w(\mu, s) = \frac{1}{2\mu^3}\sinh \mu s - \frac{1}{2\mu^3}\sin \mu s$$

where $\mu = \sqrt{\lambda}$, $\lambda \neq 0$, let $w_k(s) = w^{(4-k)}(\mu, s)$, $k = 1, 2, 3, 4$. Now, suppose $w(\lambda, 0) = w(\lambda, 1) = 0$. Then the eigenvalue-dependent boundary conditions (2.3) and (2.5) imply $w^{(3)}(\lambda, 0) = w^{(3)}(\lambda, 1) = 0$. We also have that $w''(\lambda, 0) = w''(\lambda, 1) = 0$, and we note that the function w_2 is the only solution satisfying $w(\lambda, 0) = w''(\lambda, 0) = w^{(3)}(\lambda, 0) = 0$ and $w'(\lambda, 0) = 1$. Also, it must satisfy $w_2(1) = w_2''(1) = w_2^{(3)}(1) = 0$, which implies that

$$\frac{1}{\mu}\sinh \mu + \frac{1}{\mu}\sin \mu = \mu \sinh \mu - \mu \sin \mu = \mu^2 \cosh \mu - \mu^2 \cos \mu = 0.$$

This is impossible unless the solution is the trivial one. It follows therefore that with $\beta_0, \beta_1 > 0$, for any nonzero $y \in D(G)$,

$$(Cy, y)_Y = \beta_0 |w(0)|^2 + \beta_1 |w(1)|^2 > 0,$$

and the lemma follows. □

4. Spectral properties of $L(\lambda)$ and eigenvalue asymptotics

In Section 3.1 we have verified that since $0 \in \varrho(A + B)$ and $(A + B)^{-1}$ is compact, the spectrum of $A + B$ consists only of normal eigenvalues. In the following result in Theorem 1 we deduce, for the sake of completeness, a closely related result in the context of $L(\lambda)$ (which in fact already follows from Proposition 1). First we prove the following lemmas.

Lemma 6. *For the operator G as defined by* (3.14), (3.15)*, the following assertions hold:*

(i) *G is selfadjoint; and*
(ii) *G has a compact inverse.*

Proof. We only establish the symmetry of G as the rest of the proof parallels the proof of Lemma 5. Let us begin by showing that G is densely defined. Assume there is an element $\tilde{y} \in Y$ such that for any $y \in D(G)$

$$(y, \tilde{y})_Y = (w, \tilde{w})_0 + w(0)\,\overline{\tilde{a}} + w(1)\,\overline{\tilde{c}} = 0.$$

Let w be a smooth function such that $w^{(m)}(0) = w^{(m)}(1) = 0$, $m \in \mathbb{N}_0$. It follows from the above that

$$(w, \tilde{w})_0 = 0.$$

Consequently, $\tilde{w} = 0$. Consider the polynomial

$$w(s) = \frac{s^3}{2}(s-2) + \frac{s}{2} + 1,$$

which satisfies $w''(0) = w''(1) = 0$ and $w(0) = w(1) = 1$. Clearly $y \in D(G)$, and, since $\tilde{w} = 0$,

$$(y, \tilde{y})_Y = w(0)\,\overline{\tilde{a}} + w(1)\,\overline{\tilde{c}} = 0;$$

but his implies $\tilde{a} = \tilde{c} = 0$ since $w(0) = w(1) = 1$. Thus $\tilde{y} = 0$, and we conclude $D(G)^{\perp} = \{0\}$ and G is densely defined in Y.

Now let $y, \tilde{y} \in D(G)$, and note that

$$\begin{aligned}(Gy, \tilde{y})_Y &= (w^{(4)}, \tilde{w})_0 + [(w^{(3)}(0) + (\alpha_0 - \beta_0\gamma)\,w(0)]\,\overline{\tilde{w}(0)} \\ &\quad - [w^{(3)}(1) - (\alpha_1 - \beta_1\gamma)\,w(1)]\,\overline{\tilde{w}(1)} \\ &= (w^{(4)}, \tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 + (\alpha_0 - \beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} \\ &\quad + (\alpha_1 - \beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)}.\end{aligned}$$

Since $y \in D(G)$, we calculate that

$$(w^{(4)}, \tilde{w})_0 = (w'', \tilde{w}'')_0 + w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1.$$

Hence

$$(w'', \tilde{w}'')_0 = (w^{(4)}, \tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1. \tag{4.1}$$

Now, the left side of (4.1) is symmetric, so

$$(w'', \tilde{w}'')_0 = (w, \tilde{w}^{(4)})_0 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1.$$

Equating this to (4.1) gives

$$(w^{(4)}, \tilde{w})_0 = (w, \tilde{w}^{(4)})_0 + w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1.$$

Combining the results, we obtain

$$\begin{aligned}(Gy, \tilde{y})_Y &= (w^{(4)}, \tilde{w})_0 - w^{(3)}(s)\,\overline{\tilde{w}(s)}\Big|_0^1 + (\alpha_0 - \beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} + (\alpha_1 - \beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)} \\ &= (w, \tilde{w}^{(4)})_0 - w(s)\,\overline{\tilde{w}^{(3)}(s)}\Big|_0^1 + (\alpha_0 - \beta_0\gamma)\,w(0)\,\overline{\tilde{w}(0)} + (\alpha_1 - \beta_1\gamma)\,w(1)\,\overline{\tilde{w}(1)}.\end{aligned}$$

Thus

$$(Gy, \tilde{y})_Y = (y, G\tilde{y})_Y\,,$$

and the denseness of the domain $D(G)$ ensures that G is symmetric. □

Lemma 7. *Consider the operator pencils*

$$L(\lambda) = -G - i\lambda C + \lambda^2 D$$

and

$$\tilde{L}(\lambda) = I + i\lambda C G^{-1} - \lambda^2 D G^{-1}$$

with domains $D(L(\lambda)) = D(G)$ and $D(\tilde{L}(\lambda)) = Y$, respectively. Then

$$\varrho(L) = \varrho(\tilde{L}), \quad \sigma(L) = \sigma(\tilde{L}),$$

and we have that $y_0, y_1, \ldots, y_{h-1}$ is a chain of root vectors of L corresponding to an eigenvalue λ_0, provided $-Gy_0, -Gy_1, \ldots, -Gy_{h-1}$ is a chain of root vectors of $\tilde{L}$ corresponding to the same eigenvalue.

Proof. The proof is a straightforward modification of the proof of Lemma 4 and is, therefore, omitted (see also [22, Lemma 20.1]). □

Theorem 1. *The spectrum of L given by* (3.18) *consists of an infinite number of normal eigenvalues.*

Proof. Consider

$$L(\lambda) = -G - i\lambda C + \lambda^2 D.$$

Since G, by virtue of Lemma 6, has a compact inverse we observe that

$$-L(\lambda)\, G^{-1} = I + i\lambda C G^{-1} - \lambda^2 D G^{-1}$$

where the operator on the right-hand side is a Fredholm operator for each fixed value of λ. Putting $-L(\lambda)\, G^{-1} =: \tilde{L}(\lambda)$, the theorem then follows from Lemma 7 and a more general perturbation result [7, Corollary XI.8.4] for holomorphic Fredholm operator functions. □

Remark 2. *To summarise, we have that the spectra of $A + B$ and L, including their algebraic and geometric multiplicities, coincide when $\gamma = 0$, and for $\gamma > 0$ there is a direct correspondence between the two.*

We analyse now the spectrum of L in more detail. We begin with its location by showing that all the eigenvalues are located symmetrically with respect to the imaginary axis in the closed upper half-plane, excluding the origin, and that when $\beta_0, \beta_1 > 0$, then they are confined to the open upper half-plane. (Throughout the remainder of this section it will be understood that the operators C, D, G are defined as in Section 3.2.)

Theorem 2. *The spectrum of L is symmetric with respect to the imaginary axis and is located in the closed upper half-plane but excluding the origin. In the case when $\beta_0, \beta_1 > 0$, the spectrum is confined to the open upper half-plane.*

Proof. Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of L with corresponding eigenvector y_0. Then

$$L(-\overline{\lambda_0})\, \overline{y_0} = (\overline{\lambda_0^2} D + i\overline{\lambda_0} C - G)\, \overline{y_0} = \overline{(\lambda_0^2 D - i\lambda_0 C - G)\, y_0} = \overline{L(\lambda_0)\, y_0} = 0,$$

and we have that $\overline{y_0}$ is an eigenvector corresponding to an eigenvalue $-\overline{\lambda_0}$. This proves that the spectrum of L is symmetric with respect to the imaginary axis. We take now the inner product with the corresponding y_0 and obtain

$$(L(\lambda_0)\, y_0, y_0)_Y = ((\lambda_0^2 D - i\lambda_0 C - G)\, y_0, y_0)_Y = 0.$$

This we can write out in terms of real and imaginary parts:

$$((\operatorname{Re}\lambda_0)^2 - (\operatorname{Im}\lambda_0)^2)\,(Dy_0, y_0)_Y + \operatorname{Im}\lambda_0\,(Cy_0, y_0)_Y - (Gy_0, y_0)_Y = 0 \tag{4.2}$$

and

$$\operatorname{Re}\lambda_0\,(2\operatorname{Im}\lambda_0\,(Dy_0, y_0)_Y - (Cy_0, y_0)_Y) = 0. \tag{4.3}$$

Consider the case $\operatorname{Re}\lambda_0 = 0$. Then from (4.2),

$$(\operatorname{Im}\lambda_0)^2\,(Dy_0, y_0)_Y + (Gy_0, y_0)_Y = \operatorname{Im}\lambda_0\,(Cy_0, y_0)_Y\,,$$

and since $C \geq 0$, $D \geq 0$, and $G \geq 0$, we find that $\operatorname{Im}\lambda_0 \geq 0$. If $\operatorname{Re}\lambda_0 \neq 0$, then we have from (4.3) that

$$\operatorname{Im}\lambda_0 = \frac{(Cy_0, y_0)_Y}{2\,(Dy_0, y_0)_Y} \geq 0. \tag{4.4}$$

This proves that the spectrum of L lies in the closed upper half-plane. That the spectrum does not include the origin is obvious in view of the previous results. In fact, suppose the origin belongs to the spectrum. Then from (4.2) we obtain

$$(Gy_0, y_0)_Y = 0,$$

and consequently $Gy_0 = 0$; but then $y_0 = 0$ by Lemma 6, and we have a contradiction (since y_0 is an eigenvector). Let now $\beta_0, \beta_1 > 0$, and suppose λ_0 is a purely real eigenvalue, $0 \neq \lambda_0 \in \mathbb{R}$. From (4.3) it follows at once that

$$(Cy_0, y_0)_Y = 0,$$

which is a contradiction by Proposition 3. This proves the last statement of the theorem. □

Remark 3. *It follows from* (4.4) *in the proof of the theorem that, in the case* $\operatorname{Re}\lambda_0 \neq 0$, *there are constants* a, b *such that* $\operatorname{Im}\lambda_0 \in [a, b]$, *where*

$$a := \inf_{y(\neq 0)\in D(G)} \frac{(Cy_0, y_0)_Y}{2\,(Dy_0, y_0)_Y}, \quad b := \sup_{y(\neq 0)\in D(G)} \frac{(Cy_0, y_0)_Y}{2\,(Dy_0, y_0)_Y}$$

which, using the result of Proposition 3, immediately yields that $b < \infty$, *and if* $\beta_0, \beta_1 > 0$, *then* $a > 0$. *(See also* [26, Lemma 1.4.2]*.)*

4.1. **Asymptotics of eigenvalues.** We pass now to the final task of this section, which is to obtain asymptotic expressions (for large $|\lambda|$) for the eigenvalues of L (or equivalently, of the boundary-eigenvalue problem (2.1)–(2.5)). This is best done by deriving a few intermediate results.

According to Theorem 2 all eigenvalues lie in the upper half-plane and those with nonzero real part occur in pairs λ, $-\overline{\lambda}$. So we need only consider the boundary-eigenvalue problem in the first quadrant of the complex plane (corresponding to eigenvalues with $\operatorname{Re}\lambda \geq 0$). The sector $0 \leq \arg\mu \leq \pi/4$ corresponds to this quadrant under the spectral transformation $\lambda \mapsto \mu^2$. Now let us use μ^2 in place of λ in (2.1)–(2.5), and let $\{w_k\}_{k=1}^4$ be a fundamental system of (2.1), as in the proof of Proposition 3. We have noted in the proof of Proposition 3 that the boundary condition (2.2) demands we consider w_1, w_2, w_4, that is,

$$w_1(s) = \frac{1}{2}\cosh\mu s + \frac{1}{2}\cos\mu s, \quad w_2(s) = \frac{1}{2\mu}\sinh\mu s + \frac{1}{2\mu}\sin\mu s,$$
$$w_4(s) = \frac{1}{2\mu^3}\sinh\mu s - \frac{1}{2\mu^3}\sin\mu s.$$

Applying the remaining three boundary conditions (2.3)–(2.5) to these yields the reduced characteristic matrix

$$M := \begin{pmatrix} \theta_0(\mu) & 0 & 1 \\ \frac{\mu^2}{2}(\cosh\mu - \cos\mu) & \frac{\mu}{2}(\sinh\mu - \sin\mu) & \frac{1}{2\mu}(\sinh\mu + \sin\mu) \\ \frac{\mu^3}{2}(\sinh\mu + \sin\mu) - \frac{\theta_1(\mu)}{2}(\cosh\mu + \cos\mu) & \frac{\mu^2}{2}(\cosh\mu - \cos\mu) - \frac{\theta_1(\mu)}{2\mu}(\sinh\mu + \sin\mu) & \frac{1}{2}(\cosh\mu + \cos\mu) - \frac{\theta_1(\mu)}{2\mu^3}(\sinh\mu - \sin\mu) \end{pmatrix},$$

wherein $\theta_0(\mu)$, $\theta_1(\mu)$ are as given in the proof of Proposition 1. Using a determinantal calculation we can write down explicitly the characteristic equation $2\det M = 0$ by defining

$$\phi_0(\mu) := \mu^4(1 - \cos\mu\cosh\mu), \quad \phi_1(\mu) := -\mu(\sin\mu\cosh\mu - \cos\mu\sinh\mu),$$
$$\phi_2(\mu) := \frac{2}{\mu^2}\sin\mu\sinh\mu,$$

giving

$$\phi(\mu) := \phi_0(\mu) + (\theta_0(\mu) + \theta_1(\mu))\,\phi_1(\mu) + \theta_0(\mu)\,\theta_1(\mu)\,\phi_2(\mu) = 0.$$

Clearly we are justified in referring to the squares of the zeros of $\phi(\mu)$ as the eigenvalues of L. We examine now the asymptotic behaviour of the roots of $\phi(\mu) = 0$, and hence of the eigenvalues of L.

Theorem 3. *The spectrum of L consists of an infinite number of normal eigenvalues, these being symmetric about the imaginary axis, and which accumulate only at infinity. The eigenvalues have asymptotic form*

$$\lambda_n = \mu_n^2, \quad \mu_n = \left(n - \frac{1}{2}\right)\pi + i(\beta_0 + \beta_1)\left[\left(n - \frac{1}{2}\right)\pi\right]^{-1} + O(n^{-2}), \tag{4.5}$$

as $n \to \infty$.

Proof. The first statement is immediate from Proposition 1 or Theorem 1, and Theorem 2. Now, note that

$$\begin{aligned}
2\mu^{-4}e^{-\mu}\phi_0(\mu) &= 2e^{-\mu} - \cos\mu\,(1 + e^{-2\mu}),\\
2\mu^{-4}e^{-\mu}\phi_1(\mu) &= -\frac{\sin\mu - \cos\mu}{\mu^3} - \frac{\sin\mu + \cos\mu}{\mu^3}e^{-2\mu},\\
2\mu^{-4}e^{-\mu}\phi_2(\mu) &= \frac{2}{\mu^6}\sin\mu\,(1 - e^{-2\mu}).
\end{aligned}$$

With these it is easily checked, by calculating the asymptotic form of $2\mu^{-4}e^{-\mu}\phi(\mu) = 0$, that

$$\cos\mu + \frac{i(\beta_0 + \beta_1)(\sin\mu - \cos\mu)}{\mu} + O(\mu^{-2}) = 0 \tag{4.6}$$

or

$$\cos\mu + O(\mu^{-1}) = 0. \tag{4.7}$$

This means the characteristic equation takes either of the asymptotic forms (4.6) or (4.7), valid for all values of μ in small neighbourhoods around $(n - 1/2)\pi$. Define

$$f(\mu) := \cos\mu, \quad g(\mu) := -O(\mu^{-1})$$

and consider $f(\mu)$ on a closed contour around $\mu = (n - 1/2)\pi$, arbitrarily large n, so that $|f(\mu)| \geq 1$ on the contour and the estimate $|g(\mu)| < |f(\mu)|$ holds. Within the contour, by Rouche's theorem, there is precisely one zero of $f(\mu)$, just as there is precisely one zero of $f(\mu) - g(\mu)$, namely

$$\mu_n = \left(n - \frac{1}{2}\right)\pi + \nu_n. \tag{4.8}$$

We note that we can specify

$$\cos\mu_n = \cos\left(n - \frac{1}{2}\right)\pi\cos\nu_n - \sin\left(n - \frac{1}{2}\right)\pi\sin\nu_n = -(-1)^{n-1}\sin\nu_n, \tag{4.9}$$

$$\sin\mu_n = \sin\left(n - \frac{1}{2}\right)\pi\cos\nu_n + \cos\left(n - \frac{1}{2}\right)\pi\sin\nu_n = (-1)^{n-1}\cos\nu_n. \tag{4.10}$$

Writing μ_n in place of μ in (4.6), substitution of (4.9) and (4.10) in (4.6) yields

$$\sin\nu_n = i(\beta_0 + \beta_1)\frac{\cos\nu_n}{\mu_n} + O(\mu_n^{-2}).$$

For large n, $\sin\nu_n \sim \nu_n$ and $\cos\nu_n \sim 1$, and we have

$$\nu_n = i(\beta_0 + \beta_1)\left[\left(n - \frac{1}{2}\right)\pi\right]^{-1} + O(n^{-2}).$$

The substitution of this into (4.8) leads to the asymptotic form (4.5) for the μ_n in the statement of the theorem. The proof is complete. $\square$

Remark 4. *From Theorem 3 and its proof we can draw the following conclusions:*

(1) $\lambda = \mu^2 \in \mathbb{C}$ *is an eigenvalue of the boundary-eigenvalue problem* (2.1)–(2.5) *if and only if* $\phi(\mu) = 0$. *Hence,*

$$\sigma(L) = \left\{\lambda = \mu^2 \in \mathbb{C} \mid \phi(\mu) = 0\right\}.$$

(2) *Let the* λ_n *be given as in* (4.5). *Then*

$$\sigma(L) = \left\{\lambda_n, -\overline{\lambda_n} \mid n \in \mathbb{N}\right\}.$$

(3) *For large* n, *e.g.* $n \geq n_0$ *for some* $n_0 < \infty$, *the* λ_n *are not purely imaginary and are simple eigenvalues of* L. *(It is possible to have only finitely many purely imaginary eigenvalues.)*

(4) *Using* (4.5) *one can verify that the eigenvalues of* L *satisfy*

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2 + 2i(\beta_0 + \beta_1) + O(n^{-1}), \quad n \to \infty.$$

Thus for the eigenvalues of $A + B$

$$\omega_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2 + i\left[2(\beta_0 + \beta_1) + \gamma\right] + O(n^{-1}), \quad n \to \infty,$$

and $\operatorname{Im}\omega \sim 2(\beta_0 + \beta_1) + \gamma$ *is the horizontal asymptote of* $\sigma(A + B)$.

(5) *For large* n, *the eigenvalues of* $A + B$ *have a uniform asymptotic gap, i.e.* $\inf_{n \neq m} |\omega_n - \omega_m| > 0$ *for* $n \geq n_0$, *and can be properly enumerated in the sense of Definition 4. The enumeration is such that in the sequence* $\{\omega_{\pm n}\}_{n=1}^{\infty}$, $\omega_{-n} = -\overline{\omega_n}$, *we have*

$$\sup \operatorname{Im}\omega_{\pm n} < \infty, \quad \operatorname{Re}\omega_{\pm n} \to \pm\infty, \quad n \to \infty.$$

5. Completeness, minimality, and Riesz basis property

In this section we return to the state space setting in X where, in the light of the results obtained so far, we turn our focus to the question of the Riesz basis property of the root vectors or eigenvectors of the operator $A + B$. Indeed, it follows directly from Theorem 3 and the remark following it, that the spectrum of $A+B$ consists of a sequence $\{\omega_{\pm n}\}_{n=1}^{\infty}$ of normal eigenvalues, and that all eigenvalues (except for a finite number of them) are simple. Let $\{x_{\pm n}\}_{n=1}^{\infty}$ be the sequence of eigenvectors corresponding to the $\omega_{\pm n}$. The crucial step now in proving the Riesz basis property is to establish that all of the conditions of Lemmas 1 to 3 listed in Section 2 are satisfied, and hence there is in X a unique sequence of vectors, $\{z_{\pm n}\}_{n=1}^{\infty}$, say, such that $x_{\pm n}$, $z_{\pm n}$ are biorthogonal pairs (up to normalisation) in X. This will be the main part of the work in this section, and we start with addressing the questions of completeness and minimality.

Theorem 4. *The eigenvectors of* $A + B$, *with* A *defined by* (3.1), (3.2) *and* B *by* (3.3), *are minimal complete in* X.

Proof. Recall from Lemma 5 that $0 \in \varrho(A + B)$ and $(A + B)^{-1}$ is compact. Hence, by Lemma 2, the eigenvectors of $(A + B)^{-1}$ are minimal. Let now $A = H + (A - H)$, where H is the selfadjoint part of A (that is, A with $\beta_0 = \beta_1 = 0$), and write

$$(A + B)^{-1} = (H + (A - H) + B)^{-1}.$$

This may be written equivalently in the form (the asterisk denotes the adjoint)

$$\left[\left[(A + B)^*\right]^{-1}\right]^* = \left[\left[(H + (A - H) + B)^*\right]^{-1}\right]^*.$$

The selfadjoint operator H has a compact inverse. Thus

$$\begin{aligned}\left[(H+(A-H)+B)^*\right]^{-1} &= \left(H+(A-H)^*+B^*\right)^{-1}\\ &= \left(I+(A-H)^*H^{-1}+B^*H^{-1}\right)^{-1}H^{-1}.\end{aligned}$$

Set

$$\left(I+(A-H)^*H^{-1}+B^*H^{-1}\right)^{-1}=I+T^*.$$

A straightforward computation shows that

$$T^*=-\left((A-H)^*H^{-1}+B^*H^{-1}\right)\left(I+(A-H)^*H^{-1}+B^*H^{-1}\right)^{-1},$$

which is compact. So we have demonstrated that $(A+B)^{-1}$ can be factored in the form

$$(A+B)^{-1}=H^{-1}(I+T)$$

and that the conditions imposed in Lemma 1 on the operators involved in the factorisation given in (2.8) hold if we identify K and S with H^{-1} and T, respectively. The condition (2.7) is also satisfied (identify the λ_n in (2.7) with λ_n^{-1} using the relevant formula in Remark 4 with $\beta_0=\beta_1=0$). This proves the completeness property of the eigenvectors of $(A+B)^{-1}$. The proof of the theorem is then complete, because the eigenvectors of $(A+B)^{-1}$ are also eigenvectors of $A+B$. □

Having proven the completeness and minimality, it now remains only to prove that the eigenvectors of $A+B$ have the property of being a Riesz basis for X. Here is where we use Lemma 3. We can now state and prove the main result of this section.

Theorem 5. *There exists a sequence of eigenvectors of $A+B$ which forms a Riesz basis for X.*

Proof. To obtain that there exists a sequence of eigenvectors of $A+B$ in X which forms a Riesz basis for its closed linear span, we first note that $A+B$ is dissipative if and only if the operator $-(A+B)^{-1}$ is. Indeed, we know from Lemma 5 that $A+B$ is dissipative and boundedly invertible. Thus, with $(A+B)x=\tilde{x}$ for $\tilde{x}\in X$, $x\in D(A)$,

$$\begin{aligned}&\operatorname{Im}\left(-(A+B)^{-1}\tilde{x},\tilde{x}\right)_X\\ &\qquad=-\operatorname{Im}\left((A+B)^{-1}\tilde{x},\tilde{x}\right)_X=\operatorname{Im}\left(\tilde{x},(A+B)^{-1}\tilde{x}\right)_X=\operatorname{Im}\left((A+B)x,x\right)_X\geq 0.\end{aligned}$$

So $-(A+B)^{-1}$ is a bounded dissipative operator on X, which we identify with A in Lemma 3. We can immediately verify by making use of the formula in Remark 4 for the ω_n that the condition in (2.9), wherein we identify λ_n with ω_n^{-1}, is satisfied. So all conditions of Lemma 3 are satisfied, and there exists a sequence of eigenvectors of $-(A+B)^{-1}$ in X which forms a Riesz basis for its closed linear span. Hence, by means of Theorem 4, we can infer that there exists a sequence of eigenvectors of $A+B$ which forms a Riesz basis for X. □

Remark 5. *The same result as obtained in Theorem 5 could have been achieved if we had invoked a theorem proven by Katsnelson in* [15]*, which states the theorem to be true also under somewhat weaker conditions; more details can be found in* [29, Lecture X]*. This was done by Miloslavskii in* [25] *for an application example.*

6. Well-posedness and exponential stability

All the necessary ingredients for the problem of stability analysis of (solutions of) the initial/boundary-value problem (1.2)–(1.7) stated in the introduction are now available. In particular, by Lemma 5, the operator $A+B$ is a closed, densely defined, and maximal dissipative operator in X. Using this, or Theorem 5 combined with Remark 4, we have that an abstract or semigroup formulation of the problem will be well posed. This will be our first result.

So let us return to the initial/boundary-value problem (1.2)–(1.7) and define

$$[\mathbf{x}(t)](s) := x(s,t).$$

With $v(s,t) = (\partial w/\partial t)(s,t)$, let $\mathbf{x}(t) = (w(\cdot,t), v(\cdot,t))$, the state at each time $t \geq 0$. The initial state $\mathbf{x}(0) = x$, where

$$x = (g,h),$$

is assumed suitably smooth in the sense that $x \in D(A)$. We can rewrite (1.2)–(1.7) as an abstract first-order Cauchy problem in X in the form

$$\dot{\mathbf{x}}(t) = i(A+B)\,\mathbf{x}(t), \quad \mathbf{x}(t) = (w(\cdot,t), v(\cdot,t)), \quad \mathbf{x}(0) = x, \tag{6.1}$$

The next theorem is a direct consequence of the familiar Lumer–Phillips theorem [17, Theorem I.4.2], or [5, Theorem 2.3.5].

Theorem 6. *The Cauchy problem* (6.1) *is well posed in the sense that it has for any* $x \in D(A)$ *a unique solution* $\mathbf{x} \in C^1((0,\infty); X) \cap C([0,\infty); D(A))$ *given by*

$$\mathbf{x}(t) = T(t)\,x, \quad t \geq 0, \tag{6.2}$$

where $T(t)$ *is a strongly continuous contraction semigroup on* X *with infinitesimal generator* $i(A+B)$*.*

We are interested now in what we can infer in regard to the stability of the solutions to the Cauchy problem (6.1). Indeed, with the results now in hand we are ready to establish – in the sense of the norm in X – the uniform exponential decay of the solution (6.2). The following theorem constitutes the final result of the paper which follows directly from Lemma 5, Proposition 2, Theorem 5, and Remark 4.

Theorem 7. *Suppose that the eigenvalues of* $A+B$ *are simple. Given* $x \in D(A)$*, the solution* (6.2) *of* (6.1) *can be represented as norm-convergent series of the form*

$$T(t)\,x = \sum_{n=1}^{\infty} e^{i\omega_n t}\,(x, z_n)_X\, x_n + \sum_{n=1}^{\infty} e^{i\omega_{-n} t}\,(x, z_{-n})_X\, x_{-n}, \quad t \geq 0,$$

where $\{z_{\pm n}\}_{n=1}^{\infty}$ *is (up to normalisation) a biorthogonal sequence for* $\{x_{\pm n}\}_{n=1}^{\infty}$*, and* $T(t)$ *is, by Theorem 6, the strongly continuous contraction semigroup generated by the system operator* $i(A+B)$*. When* $\beta_0, \beta_1 > 0$ *and* $\gamma > 0$*, the semigroup is exponentially stable with uniform decay rate* $\delta > 0$ *determined by the spectrum of* $A+B$*, in the sense that* $\sup\{\operatorname{Re} i\omega \mid \omega \in \sigma(A+B)\} \leq -\delta$ *and there exists a constant* $M \geq 1$ *such that*

$$\|T(t)\|_X \leq M e^{-\delta t}, \quad t \geq 0.$$

References

1. F. Conrad and Ö. Morgül, *On the stabilization of a flexible beam with a tip mass*, SIAM Journal on Control and Optimization **36** (1998), no. 6, 1962–1986.
2. R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. II: Partial Differential Equations*. John Wiley & Sons, New York, NY, USA, 1962.
3. S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Communications in Partial Differential Equations **19** (1994), no. 1-2, 213–243.
4. S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana University Mathematics Journal **44** (1995), no. 2, 545–573.
5. R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21 of Texts in Applied Mathematics. Springer-Verlag, New York, NY, USA, 1995.
6. N. Dunford and J. T. Schwartz, *Linear Operators. Part III: Spectral Operators*, vol. VII of Pure and Applied Mathematics. John Wiley & Sons, New York, NY, USA, 1971.
7. I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators. Vol. I*, vol. 49 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, Switzerland, 1990.

8. I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, vol. 18 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1969.
9. A. Gomilko and V. Pivovarchik, *On bases of eigenfunctions of boundary problem associated with small vibrations of damped nonsmooth inhomogeneous string*, Asymptotic Analysis **20** (1999), no. 3-4, 301–315.
10. A. Gomilko and V. Pivovarchik, *On basis properties of a part of eigenfunctions of the problem of vibrations of a smooth inhomogeneous string damped at the midpoint*, Mathematische Nachrichten **245** (2002), no. 1, 72–93.
11. B.-Z. Guo, *Riesz basis approach to the stabilization of a flexible beam with a tip mass*, SIAM Journal on Control and Optimization **39** (2001), no. 6, 1736–1747.
12. B.-Z. Guo, *Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equations with variable coefficients*, SIAM Journal on Control and Optimization **40** (2002), no. 6, 1905–1923.
13. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, vol. 31 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, USA, 1957.
14. T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics. Springer-Verlag, Berlin, Germany, 1995.
15. V. E. Katsnel′son, *Conditions under which systems of eigenvectors of some classes of operators form a basis*, Functional Analysis and Its Applications **1** (1967), no. 2, 122–132.
16. M. V. Keldysh, *On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators*, Russian Mathematical Surveys **26** (1971), no. 4, 15–44.
17. S. G. Kreĭn, *Linear Differential Equations in Banach Space*, vol. 29 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1971.
18. Yu. I. Lyubich, *Linear Functional Analysis.* In *Functional Analysis I*, vol. 19 of Encyclopaedia of Mathematical Sciences, pages 1–283. Springer-Verlag, Berlin, Germany, 1992.
19. M. Mahinzaeim, D. C. Swailes, and J. M. Hale, *Bounded vibration control of distributed parameter systems, with application to a beam on viscoelastic supports*, Proceedings of the 4th ECCOMAS Thematic Conference on Smart Structures and Materials, 2009. Held in Porto, Portugal, July 13–15 (CD-ROM).
20. M. M. Malamud and L. L. Oridoroga, *On the completeness of root subspaces of boundary value problems for first order systems of ordinary differential equations*, Journal of Functional Analysis **263** (2012), no. 7, 1939–1980.
21. A. S. Markus, *A basis of root vectors of a dissipative operator*, Soviet Math. Dokl. **1** (1960), no. 1-3, 599–602.
22. A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, vol. 71 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, USA, 1988.
23. R. Mennicken and M. Möller, *Non-Self-Adjoint Boundary Eigenvalue Problems*, vol. 192 of North-Holland Mathematics Studies. Elsevier/North-Holland, Amsterdam, The Netherlands, 2003.
24. A. I. Miloslavskii, *Foundation of the spectral approach in nonconservative problems of the theory of elastic stability*, Functional Analysis and Its Applications **17** (1983), no. 3, 233–235.
25. A. I. Miloslavskii, *Stability of certain classes of evolution equations*, Siberian Mathematical Journal **26** (1985), no. 5, 723–735.
26. M. Möller and V. Pivovarchik, *Spectral Theory of Operator Pencils, Hermite–Biehler Functions, and their Applications*, vol. 246 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, Switzerland, 2015.
27. M. Möller and V. N. Pivovarchik, *Spectral properties of a fourth order differential equation*, Zeitschrift für Analysis und ihre Anwendungen **25** (2006), no. 3, 341–366.
28. M. A. Naimark, *Linear Differential Operators. Part I: Elementary Theory of Linear Differential Operators.* Frederick Ungar Publishing, New York, NY, USA, 1967.
29. N. K. Nikol′skiĭ, *Treatise on the Shift Operator. Spectral Function Theory*, vol. 273 of Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Germany, 1986.
30. V. N. Pivovarchik, *On spectra of a certain class of quadratic operator pencils with one-dimensional linear part.*, Ukrainian Mathematical Journal **59** (2007), no. 5, 766–781.
31. G. V. Radzievskii, *The problem of the completeness of root vectors in the spectral theory of operator-valued functions*, Russian Mathematical Surveys **37** (1982), no. 2, 91–164.
32. B. Rao, *Optimal energy decay rate in a damped Rayleigh beam*, Discrete and Continuous Dynamical Systems **4** (1998), no. 4, 721–734.

33. H. Röh, *Dissipative operators with finite dimensional damping*, Proceedings of the Royal Society of Edinburgh. Section A: Mathematics **91** (1982), no. 3-4, 243–263.
34. A. A. Shkalikov, *Boundary problems for ordinary differential equations with parameter in the boundary conditions*, Journal of Mathematical Sciences **33** (1986), no. 6, 1311–1342.
35. A. A. Shkalikov, *Perturbations of self-adjoint and normal operators with discrete spectrum*, Russian Mathematical Surveys **71** (2016), no. 5, 907–964.
36. G.-Q. Xu and D.-X. Feng, *The Riesz basis property of a Timoshenko beam with boundary feedback and application*, IMA Journal of Applied Mathematics **67** (2002), no. 4, 357–370.
37. B. Zinsou, *Fourth-order Birkhoff regular problems with eigenvalue parameter dependent boundary conditions*, Turkish Journal of Mathematics **40** (2016), no. 4, 864–873.

School of Mechanical and Systems Engineering, Newcastle University, Stephenson Building, Newcastle upon Tyne, NE1 7RU, United Kingdom

Current address: Vogelhofstr. 28, 73525 Schwäbisch Gmünd, Germany

E-mail address: `mahyar.mahinzaeim@alumni2005.newcastle.ac.uk`

Received 02/05/2017; Revised 27/06/2017.