SPECTRAL PROPERTIES AND STABILITY OF A NONSELFADJOINT EULER–BERNOULLI BEAM

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ABSTRACT. In this note we study the spectral properties of an Euler–Bernoulli beam model with damping and elastic forces applying both at the boundaries as well as along the beam. We present results on completeness, minimality, and Riesz basis properties of the system of eigen- and associated vectors arising from the nonselfadjoint spectral problem. Within the semigroup formalism it is shown that the eigenvectors have the property of forming a Riesz basis, which in turn enables us to prove the uniform exponential decay of solutions of the particular system considered.

1. INTRODUCTION

The solution of stabilisation problems in infinite dimensions is a difficult mathematical task for two reasons. On the one hand, there is more than one way to extend the ideas of stability and stabilisability from finite-dimensional spaces to infinite-dimensional function space (depending on the choice of norm). On the other hand, the spectral mapping theorem does not hold in general for infinite-dimensional systems. This latter fact means, more concretely, that merely requiring the spectrum of the generating operator for the corresponding system semigroup to lie in the open left half-plane does not guarantee that the semigroup is stable, let alone exponentially stable, when it is strongly continuous, in contrast to the case when it is uniformly continuous. This fact has been known for more than half a century and goes back at least to Hille and Phillips [13, p. 665].

There is one situation, however, where we are able to relate the spectrum of the generator to that of the system semigroup, namely, if it is shown that the system operator, that is, the generating operator for the system semigroup, is a discrete spectral operator (in the terminology of N. Dunford, see [6, Chapters XVIII to XX]) whose eigenvectors form a Riesz basis for the underlying Hilbert state space – a Riesz spectral operator to this end. Miloslavskii in [24] (and redeveloped in [25]) and Röh in [33] were among the first to note this connection between the stability properties of an infinite-dimensional system and the Riesz basis property of the eigenvectors.

In general, the generator is not necessarily a spectral operator, which means that it does not need to have a simple spectral decomposition, in much the same way as nonselfadjoint linear operators. Indeed, in most of the stability problems considered in the literature, only few properties of the generator are known, or anticipated, except that it is usually the infinitesimal generator of a strongly continuous semigroup of contractions, and that its eigenvalues are discrete. These preliminary results would prove to be useful, nevertheless, in obtaining basis results from that point on.

We consider in this study an Euler–Bernoulli elastic beam of unit length, with boundary conditions such that it corresponds to the interesting (for the purposes of stabilisation at least) case when it can be associated with a dissipative system. More specifically, we

consider the initial/boundary-value problem consisting of the modified Euler–Bernoulli beam equation

$$\frac{\partial^2}{\partial t^2} w(s,t) + \frac{\partial^4}{\partial s^4} w(s,t) + \gamma^2 w(s,t) + 2\gamma \frac{\partial}{\partial t} w(s,t) = 0,$$

together with the boundary conditions

$$\left. \frac{\partial^2}{\partial s^2} w(s,t) \right|_{s=0} = 0, \quad \left(\frac{\partial^3}{\partial s^3} w(s,t) + \alpha_0 w(s,t) + \beta_0 \frac{\partial}{\partial t} w(s,t) \right) \Bigg|_{s=0} = 0,$$

$$\left. \frac{\partial^2}{\partial s^2} w(s,t) \right|_{s=1} = 0, \quad \left(\frac{\partial^3}{\partial s^3} w(s,t) - \alpha_1 w(s,t) - \beta_1 \frac{\partial}{\partial t} w(s,t) \right) \Bigg|_{s=1} = 0,$$

and given initial conditions

$$w(s,0) = g(s), \quad \left. \frac{\partial}{\partial t} w(s,t) \right|_{t=0} = h(s).$$

We note that the partial differential equation has a form which closely resembles the classical telegrapher's equation (see [2, pp. 192 to 193]), except for the elasticity term, which here is of fourth order.

A substantial portion of our work will be devoted, as well as to conclude information about the spectrum of the corresponding spectral problem, but also to addressing the study of the eigenvectors, and, if any, associated vectors of the system operator connected with the abstract version of the initial/boundary-value problem on the state space. As will be seen, this task is far from straightforward because the system operator will not be selfadjoint in general.

It is a well-known fact from the spectral theory of nonselfadjoint linear operators that the necessary information that a nonselfadjoint system operator is a Riesz spectral operator cannot be proven simply by examining its spectrum and finding a system of eigen- and associated vectors which is minimal complete. Stated another way, once we have knowledge of the completeness and minimality, in actuality there still remains the problem of establishing that the system of eigen- and associated vectors forms a Riesz basis. (We omit here the rather standard details of the definitions of completeness, minimality, and so on, which can be found, for example, in [8, Section VI.1].)

In this paper we follow an argument to prove the Riesz basis property which avoids the need for deriving asymptotic formulae for the eigen- and associated vectors as required when working with the results, for example, of N. K. Bari (see [8, Section VI.2]). It is of course possible to derive asymptotic formulae for the eigen- and associated vectors, a line that has been pursued by many investigators – particularly in the field of infinite-dimensional stabilisation theory – during the past decade or so, such as Conrad and Morgül [1], Cox and Zuazua [3, 4], Guo [11, 12], Rao [32], and Xu and Feng [36]. However, we feel that it will be interesting to see how we can prove the Riesz basis property, for the space considered, not by working with explicit asymptotic expressions for the eigen- and associated vectors, but by using only knowledge of the asymptotic behaviour of the eigenvalues. We shall see that such proof is indeed possible (in the contexts here relevant) if we make use of a slight modification of a less well-known theorem, proven in different ways for dissipative, bounded linear operators, by I. M. Glazman and, in a paper in Doklady, B. R. Mukminov in the 1950's, as well as, later generalised to the case when associated vectors are present, by Markus [21]. These, probably being the earliest results on Riesz basis properties for general operators, will be seen for the specific system under consideration to be a more direct route to verification of the Riesz basis property, since the system operator naturally possesses a structure that is related directly to the spectral properties of a compact dissipative operator. The idea is to some degree, in fact, related to the rationale behind Kato's results in [14, Section V.4]: for operators related

in some sense to perturbed selfadjoint operators we can expect the system of eigen- and associated vectors, corresponding to a properly counted sequence of eigenvalues, to be a basis (at least in some subspace). For additional details, see the recent survey [35] of Shkalikov.

An essential prerequisite for our undertaking is the verification of completeness of the eigen- and associated vectors, which usually is difficult. However, there are a few cases where we can explicitly prove completeness, and we will explore in detail exactly how this can be accomplished in Section 5. The key ingredient there is provided by a version of a well-known theorem of M. V. Keldysh, published in his pioneering 1951 paper, for compact linear operators, which, properly exploited in the factorisation of the system operator, enables us to establish that the eigen- and associated vectors are complete in the state space. (An English translation of Keldysh's paper is contained in the Appendix of [22], and further discussion and proofs can be found in [16] and [31].)

The rest of the paper is structured as follows. Section 2 treats some preliminary material including relevant definitions, while our true starting point is Section 3, where we pose the abstract spectral problems. This is an important first step which we take in a way that will have advantages for the study of the spectrum later. For example, instead of working on the spectral problem only in the state space for the system, we choose to additionally work in a Hilbert space of three-component vectors. In so doing some of the theory of quadratic operator pencils can be invoked in the development in Section 4 of proofs for many results on the location and distribution of the eigenvalues. The state space is particularly suited to studying the problems, to be considered in Section 5, of completeness, minimality, and Riesz basis properties, which in turn are used in Section 6 to examine the nature of the stability associated with the initial/boundary-value problem. So, where it is appropriate, we will link the findings of Section 4 to the spectral problem on the state space.

Also to be explored, in Section 4, is the asymptotic behaviour of the spectrum. There we produce simple asymptotic expressions for the zeros of a set of analytic functions for use in Section 5 in the proof of the Riesz basis property of the system of eigenvectors. The trend of the reasoning is, in its basic idea, close to that of investigating the asymptotic behaviour of the zeros for large values of their modulus. It will be seen that, in a sense to be defined, the indexing of the zeros can take place in a correct fashion, taking their multiplicities into account.

There are a few interesting papers which are close in spirit to this paper. Here, we must particularly note the works of Gomilko and Pivovarchik [9, 10], Möller and Pivovarchik [27], Pivovarchik [30], and a series of works of Möller and Zinsou, many of which are listed in [37]. The main difference between our work and these papers lies in the type of system we are considering. It differs from those considered by the above-mentioned authors due to the combined presence of boundary and distributed viscous damping forces, as well as elastic restoring forces proportional to beam deflection. Although, from a mathematical point of view, this is a rather special system (due primarily to the coefficients in the beam equation being all constants), it has a prominent place in the engineering literature, mainly as the so-called "half-vehicle model". An example is the early paper [19] by the author and others. This paper, however, lacks rigour in that it does not justify mathematically the formal expansion of an arbitrary function in terms of the eigenfunctions, in the manner of an infinite series. Such justification is especially necessary in the consideration of standard approximations to the solutions of the initial/boundary-value problem. It is intuitively obvious that in the end this comes down to establishing conditions for the convergence of the series expansion, and here lies the need for study of the problems of completeness, minimality, and Riesz basis properties. To date, as far as we know, no rigorous mathematical study has been

conducted that adequately addresses these problems in the context of the present system. We shall give such a study in this paper, and we show in Section 6 how to apply our results to prove uniform exponential stability for the system semigroup, a result which we feel to be the first of its kind within this particular system. The result is the culmination of the results of Sections 4 and 5.

2. Definitions and preliminaries

To begin with, let us proceed formally under the assumption of a separable solution to the initial/boundary-value problem described in the introduction, for some spectral parameter ω. We specify the relationship $\lambda = \omega - i\gamma$ between ω and another spectral parameter λ, and obtain, on making the substitution

$$w(s,t) = e^{i(\lambda+i\gamma)t} w(\lambda, s),$$

the problem

$$(2.1) \quad \begin{cases} w^{(4)}(\lambda, s) - \lambda^2 w(\lambda, s) = 0, \quad s \in [0,1], \\ w^{(2)}(\lambda, 0) = 0, \\ w^{(3)}(\lambda, 0) + (\alpha_0 - \beta_0\gamma + i\beta_0\lambda) w(\lambda, 0) = 0, \\ w^{(2)}(\lambda, 1) = 0, \\ w^{(3)}(\lambda, 1) - (\alpha_1 - \beta_1\gamma + i\beta_1\lambda) w(\lambda, 1) = 0. \end{cases}$$

Sometimes, for convenience, we set

$$(2.2) \qquad \theta_0(\lambda) = \alpha_0 - \beta_0\gamma + i\beta_0\lambda, \quad \theta_1(\lambda) = \alpha_1 - \beta_1\gamma + i\beta_1\lambda,$$

and unless otherwise specified, it is understood that

$$\alpha_0 \geq 0, \quad \beta_0 \geq 0, \quad \alpha_1 \geq 0, \quad \beta_1 \geq 0, \quad \gamma \geq 0.$$

The boundary-eigenvalue problem (2.1) can be cast into the general framework of abstract operator pencils if one considers its operator-theoretic formulation on a Hilbert product space. This is the route that we will ultimately follow, and we wish to do this on two spaces, X and Y, endowed with appropriate topologies for the statement of the abstract spectral problems on them. On Y we will fit the problem into the setting of quadratic nonmonic operator pencils, and the same problem but with λ replaced by $\omega - i\gamma$ will be fit into the setting of linear monic operator pencils on the space X. We define Y as the space $L_2(0,1) \times \mathbb{C}^2$ of three-component vectors, and we define X to be the space of equivalence classes of elements of the space $W_2^2(0,1) \times L_2(0,1)$, modulo the zero elements (the space $W_2^m(0,1)$ is the usual complex Sobolev–Hilbert space of order m). In doing so in the latter case we have taken account of the fact that we have to work with an appropriate quotient space of $W_2^2(0,1) \times L_2(0,1)$, so that the induced norm will not be a seminorm but becomes the norm for X, the so-called state space.

Let us recall now, for completeness, some standard definitions of the spectral theory of unbounded operator pencils as a convenience for the reader. A fine account of the subject, with a useful bibliography, is given by Möller and Pivovarchik in their recent text [26]; refer there and to [22] for equivalent definitions.

Definition 1. *Let $P(\lambda) : D(P(\lambda))(\subseteq X) \to X$ be an operator pencil, supposing that $\lambda \mapsto P(\lambda)$ is a mapping from $\mathbb{C}$ (or some nonempty subset thereof) into the set of closed linear operators in the Hilbert space X. A number λ is said to belong to $\varrho(P(\cdot))$, the resolvent set of $P(\lambda)$, provided $P(\lambda)$ has a closed bounded inverse, that is, provided $P(\lambda)$ is bijective, and $P^{-1}(\lambda) := P(\lambda)^{-1}$. We call the mapping $\lambda \mapsto P^{-1}(\lambda)$ the resolvent of $P(\lambda)$. The complement of $\varrho(P(\cdot))$ is the spectrum of $P(\lambda)$ and is denoted*

by $\sigma\left(P\left(\cdot\right)\right)$. *The set of all eigenvalues of $P\left(\lambda\right)$, or, what is the same, the set of λ such that $\ker P\left(\lambda\right) \neq \{0\}$, is the point spectrum of $P\left(\lambda\right)$, denoted $\sigma_p\left(P\left(\cdot\right)\right)$.*

Definition 2. *The sequence of vectors $\{x_k\left(\lambda_0\right)\}_{k=0}^{m-1}$ in $D\left(P\left(\lambda\right)\right)$ is said to form a chain, of length m, consisting of the eigenvector $x_0\left(\lambda_0\right)\left(\neq 0\right)$ of $P\left(\lambda\right)$ corresponding to an eigenvalue λ_0, and the vectors $x_1\left(\lambda_0\right), x_2\left(\lambda_0\right), \ldots, x_{m-1}\left(\lambda_0\right)$ associated with it, or, for brevity, simply a chain of eigen- and associated vectors corresponding to λ_0, if*

$$\sum_{l=0}^{k} \frac{1}{l!}\left(P^{(l)} x_{k-l}\right)\left(\lambda_0\right) = 0 \quad for \quad k \in \{0, 1, \ldots, m-1\}.$$

The geometric multiplicity of the eigenvalue λ_0 is the number of linearly independent eigenvectors in a system of chains of eigen- and associated vectors of $P\left(\lambda\right)$ corresponding to λ_0 and is defined as $\dim \ker P\left(\lambda_0\right)$. The algebraic multiplicity of λ_0 is the maximum value of the sum of the lengths of chains corresponding to the linearly independent eigenvectors. We call λ_0 simple if its geometric and algebraic multiplicities are equal and, additionally, $\dim \ker P\left(\lambda_0\right) = 1$. Accordingly, we call that part of the spectrum which consists of simple eigenvalues only, simple.

Definition 3. *The set of λ which are isolated points of $\sigma\left(P\left(\cdot\right)\right)$, with a deleted neighbourhood in $\varrho\left(P\left(\cdot\right)\right)$, and which are also eigenvalues, each with finite algebraic multiplicity and such that $P\left(\lambda\right)$ is a Fredholm operator, are called eigenvalues of finite type.*

Definition 4. *A countable sequence of eigenvalues is said to be correctly enumerated if it is a sequence of real or complex numbers which are counted properly – that is, such that*

    (i) *$\lambda_{-j} = -\overline{\lambda_j}$ whenever $\operatorname{Re} \lambda_j \neq 0$;*
    (ii) *$\operatorname{Re} \lambda_j \geq \operatorname{Re} \lambda_k$ for $j > k$;*
    (iii) *their algebraic multiplicities are taken into account; and*
    (iv) *the index set is $\mathbb{Z}$ in case the number of purely imaginary eigenvalues is odd, and $\mathbb{Z}\backslash\{0\}$ in case it is even.*

As we have already mentioned in the introduction, prerequisite conditions for the verification of the Riesz basis property of a system of eigen- and associated vectors are completeness and minimality. These can be deduced readily from the following two theorems, the first of which is due to Keldysh [7, Theorem X.4.1] on the completeness of the eigen- and associated vectors. For a proof of the second theorem, on the minimality of the system of eigen- and associated vectors, see, for example, [20, Lemma 2.4]. We shall invoke both in Section 5.

Theorem 1 (Keldysh). *Let $A : X \to X$, a bounded linear operator on the Hilbert space X, supposing that it is factored such that*

(2.3) $A = K\left(I + S\right),$

where K is a compact selfadjoint operator on X and $\ker K = \{0\}$. Let $\{\lambda_j\}_{-\infty, j\neq 0}^{\infty}$ be the sequence of eigenvalues of K, enumerated correctly according to multiplicity, and such that

(2.4) $$\sum_{-\infty, j\neq 0}^{\infty} |\lambda_j|^n < \infty \quad for \quad n > 0.$$

Suppose further that S is compact and that $I + S$ is invertible. Then the eigen- and associated vectors of A are complete in X.

Theorem 2. *Suppose A is a compact operator on X and $\ker A = \{0\}$. Then the system of eigen- and associated vectors of A is minimal.*

The principal tool in establishing the Riesz basis property of the system of eigenvectors will be the individual theorems of Glazman and Mukminov. The following is a basic version of their result (see [8, Section VI.4]).

Theorem 3 (Glazman and Mukminov). *Consider the operator $A : X \to X$, which is dissipative and whose spectrum consists of a countable sequence $\{\lambda_j\}_{-\infty,\, j\neq 0}^{\infty}$ of simple eigenvalues such that*

$$(2.5) \qquad \sum_{\substack{-\infty,\, j,k\neq 0 \\ j\neq k}}^{\infty} \frac{\operatorname{Im}\lambda_j \operatorname{Im}\lambda_k}{\left|\lambda_j - \overline{\lambda_k}\right|^2} < \infty.$$

Then the corresponding system of normalised eigenvectors in X forms a Riesz basis for its closed linear span.

We close this section with our first result for use later in the paper, which essentially guarantees in the case $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$ the existence of a countable sequence of eigenvalues of the spectral problem described by (2.1). Its proof rests entirely on theorems proven by Mennicken and Möller in [23, Sections 7.2 and 7.3], and the reader is referred there for further information.

Proposition 1. *The problem given by* (2.1), *under the change from λ to μ^2 associated with any nonzero μ, is Birkhoff regular in the sense of* [23, Definition 7.3.1].

Proof. Taking into account the change $\lambda = \mu^2$ (with $\mu \neq 0$), we let $\theta_0(\mu)$ and $\theta_1(\mu)$ stand for the expressions (cf. (2.2))

$$\theta_0(\mu) = \alpha_0 - \beta_0\gamma + i\beta_0\mu^2 \quad \text{and} \quad \theta_1(\mu) = \alpha_1 - \beta_1\gamma + i\beta_1\mu^2.$$

We first of all note that the differential equation in (2.1) has associated with it a characteristic function of degree four, defined by [23, (7.1.4)], which takes here the simple form $\pi(\rho) = \rho^4 - 1$. Its roots are i^{r-1} for $r \in \{1,2,3,4\}$, and it is easily verified that the assumptions for [23, Theorem 7.2.4.A] are satisfied. Thus there exists a 4×4 transformation matrix which we can choose to be

$$C(s,\mu) = \operatorname{diag}\left(1,\mu,\mu^2,\mu^3\right)\left(i^{(r-1)(k-1)}\right)_{r,k=1}^{4}.$$

Consequently, in view of [23, (7.3.1)], we have for the boundary matrices

$$W^{(0)}(\mu) = \begin{pmatrix} \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \theta_0(\mu) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu)$$

and

$$W^{(1)}(\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\theta_1(\mu) & 0 & 0 & 1 \end{pmatrix} C(1,\mu),$$

where $\eta_k := \theta_0(\mu) + (-i)^{k-1}\mu^3$ and $\zeta_k := -\theta_1(\mu) + (-i)^{k-1}\mu^3$. We may now choose $C_2(\mu) = \operatorname{diag}\left(\mu^2,\mu^3,\mu^2,\mu^3\right)$. Then we have, according to the formula in [23, Theorem 7.3.2(i)], that

$$C_2^{-1}(\mu)\, W^{(0)}(\mu) = W_0^{(0)} + O\!\left(\mu^{-1}\right)$$

and

$$C_2^{-1}(\mu)\, W^{(1)}(\mu) = W_0^{(1)} + O\!\left(\mu^{-1}\right),$$

where

$$W_0^{(0)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

In order for the relevant problem to be Birkhoff regular, we must require by [23, Theorem 7.3.2(ii)] the nonsingularity, for $k \in \{1, 2, 3, 4\}$, of the Birkhoff matrices (with I here being the 4×4 identity matrix)

$$W_0^{(0)} \Delta_k + W_0^{(1)} (I - \Delta_k),$$

in which the Δ_k are, according to [23, Definition 7.3.1 and Proposition 4.1.7], four by four diagonal matrices whose diagonal elements consist of two consecutive ones, followed by two consecutive zeros in a cyclic arrangement. These matrices can all be verified to be nonsingular, and therefore the theorem is established. $\qquad\square$

3. The abstract spectral problems

In this section our intention, broadly put, is to recast the boundary-eigenvalue problem (2.1) abstractly as linear operators acting in the spaces X and Y. Returning to the initial/boundary-value problem from the introduction, let us define a mapping $\mathbf{x}(\cdot) : [0, \infty) \to X$ by

$$[\mathbf{x}(t)](s) := x(s, t),$$

and, with $v(s, t) = (\partial w / \partial t)(s, t)$, put $(w(\cdot, t), v(\cdot, t)) =: x(\cdot, t)$, the state at each time $t \geq 0$. The initial state $\mathbf{x}(0) = x$, where

$$(3.1) \qquad\qquad\qquad x := (g(\cdot), h(\cdot)).$$

Having defined the relevant spaces X and Y in the previous section, we proceed to pose our spectral problems on them.

3.1. **The spectral problem on X.** We begin with consideration of the spectral problem on the state space X. This is a Hilbert space under the norm induced by the inner product

$$((w(\cdot), v(\cdot)), (\tilde{w}(\cdot), \tilde{v}(\cdot)))_X := (v(\cdot), \tilde{v}(\cdot))_0 + (w(\cdot), \tilde{w}(\cdot))_2$$

for every $(w(\cdot), v(\cdot))$ and $(\tilde{w}(\cdot), \tilde{v}(\cdot))$ in X, where

$$\begin{aligned} (w(\cdot), \tilde{w}(\cdot))_2 := & \int_0^1 w^{(2)}(s) \overline{\tilde{w}^{(2)}(s)} \, \mathrm{d}s + \gamma^2 \int_0^1 w(s) \overline{\tilde{w}(s)} \, \mathrm{d}s \\ & + \alpha_0 w(0) \overline{\tilde{w}(0)} + \alpha_1 w(1) \overline{\tilde{w}(1)} \quad \text{for} \quad w(\cdot), \tilde{w}(\cdot) \in W_2^2(0, 1), \end{aligned}$$

which is well defined because $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$ by assumption. Throughout we denote by $(\cdot, \cdot)_0$ the inner product on $L_2(0, 1)$ and write $\|\cdot\|_0$ for the resulting norm on $L_2(0, 1)$.

Consider the operator $A : D(A) \to X$ given by

$$(3.2) \qquad\qquad [A\mathbf{x}(\cdot)](s) := i(-v(s), w^{(4)}(s)) \quad \text{for} \quad x(\cdot) \in D(A),$$

with domain $D(A)$ dense in X, defined by

$$(3.3) \quad D(A) := \left\{ x(\cdot) \in X \; \middle| \; \begin{array}{l} w(\cdot) \in W_2^4(0, 1), \; v(\cdot) \in W_2^2(0, 1), \\ w^{(2)}(0) = 0, \; w^{(3)}(0) + \alpha_0 w(0) + \beta_0 v(0) = 0, \\ w^{(2)}(1) = 0, \; w^{(3)}(1) - \alpha_1 w(1) - \beta_1 v(1) = 0 \end{array} \right\}.$$

It will prove to be convenient to work with the operator A as given above, as well as with the operator $B : X \to X$, given by

$$(3.4) \qquad\qquad [B\mathbf{x}(\cdot)](s) := i(0, \gamma^2 w(s) + 2\gamma v(s)) \quad \text{for} \quad x(\cdot) \in X.$$

The operator $A + B$ will be called the system operator, and since B, as thus defined, is bounded,

$$D(A + B) = D(A).$$

So we may regard the system operator for $\gamma > 0$ as a bounded linear perturbation of the operator A.

The formulation of the abstract spectral problem on X now requires that we use the relationship $\omega = \lambda + i\gamma$ in (2.1) to obtain a spectral problem with ω in place of λ. With the monic operator pencil $P(\omega) : D(P(\omega)) \to X$ given by

$$(3.5) \qquad\qquad P(\omega) = \omega I - (A + B),$$

with domain

$$D(P(\omega)) = D(A),$$

we can then study the problem (with $P(\omega) x(\omega, \cdot) = (Px)(\omega, \cdot)$)

$$(3.6) \qquad\qquad (Px)(\omega, s) = 0, \quad x(\omega, \cdot) \in D(A), \quad s \in [0, 1].$$

The following two lemmas will be used to connect the spectral properties of the linear pencil $P(\omega)$ with those of a quadratic pencil (details in the next section) acting in the space Y.

Lemma 1. *The set of eigenvalues, if any, including multiplicities, of the problem* (2.1) *with $\lambda = \omega - i\gamma$ coincides with that of the linear operator pencil $P(\omega)$ given by* (3.5), *where A is defined by* (3.2), (3.3), *and B by* (3.4). *Further, the following relation holds between a chain of eigen- and associated functions for* (2.1) *corresponding to an eigenvalue ω_0 and a chain of eigen- and associated vectors of $P(\omega)$ corresponding to the same eigenvalue:*

$$x_k(\omega_0, \cdot) = \big(w_k(\omega_0, \cdot), v_k(\omega_0, \cdot)\big),$$

for $k \in \{0, 1, \ldots, m - 1\}$, where

$$v_k(\omega_0, \cdot) = i\omega_0 w_k(\omega_0, \cdot) + i w_{k-1}(\omega_0, \cdot).$$

Proof. We follow [34, Lemma 1.4], so it suffices to verify the relations, mentioned in the statement of the lemma, between the eigen- and associated functions and the eigen- and associated vectors (see also [28, pp. 14 to 20]). Let ω_0 be an eigenvalue of $P(\omega)$ with a corresponding chain formed by the eigen- and associated vectors $x_0(\omega_0, \cdot)$, $x_1(\omega_0, \cdot)$, ..., $x_{m-1}(\omega_0, \cdot)$. Recall the spectral problem (3.6), and note that, according to Definition 2,

$$(Px_k)(\omega_0, s) + x_{k-1}(\omega_0, s) = 0 \quad \text{for} \quad k \in \{0, 1, \ldots, m - 1\}.$$

This equation takes the form of a system consisting of

$$w_k^{(4)}(\omega_0, s) + \gamma^2 w_k(\omega_0, s) + 2\gamma v_k(\omega_0, s) + i\omega_0 v_k(\omega_0, s) + i v_{k-1}(\omega_0, s) = 0,$$

$$v_k(\omega_0, s) = i\omega_0 w_k(\omega_0, s) + i w_{k-1}(\omega_0, s),$$

together with the boundary conditions

$$w_k^{(2)}(\omega_0, 0) = 0, \quad w_k^{(3)}(\omega_0, 0) + \alpha_0 w_k(\omega_0, 0) + \beta_0 v_k(\omega_0, 0) = 0,$$

$$w_k^{(2)}(\omega_0, 1) = 0, \quad w_k^{(3)}(\omega_0, 1) - \alpha_1 w_k(\omega_0, 1) - \beta_1 v_k(\omega_0, 1) = 0,$$

which proves the lemma. $\square$

Lemma 2. *Consider the system operator $A + B$, and let $\alpha_0 > 0$ and $\alpha_1 > 0$. The following assertions hold:*

  (i) *$A + B$ is maximal dissipative when there is strict inequality in at least one of the conditions $\beta_0 \geq 0$, $\beta_1 \geq 0$ or $\gamma \geq 0$, and selfadjoint whenever $\beta_0 = 0$, $\beta_1 = 0$ and $\gamma = 0$; and*

  (ii) *$A + B$ has a compact inverse.*

Proof. To prove assertion (i), we must show that the system operator is closed and that $A + B$ is dissipative. To do this, we first observe that, for $x\left(\cdot\right) \in D\left(A\right)$,

$$\left(\left(\left(A + B\right)x\right)\left(\cdot\right), x\left(\cdot\right)\right)_X$$
$$= i\left(w^{(4)}\left(\cdot\right) + \gamma^2 w\left(\cdot\right) + 2\gamma v\left(\cdot\right), v\left(\cdot\right)\right)_0 - i\left(v\left(\cdot\right), w\left(\cdot\right)\right)_2.$$

An elementary calculation shows that

$$i\left(w^{(4)}\left(\cdot\right), v\left(\cdot\right)\right)_0 = i\alpha_0 w\left(0\right)\overline{v\left(0\right)} + i\beta_0 \left|v\left(0\right)\right|^2 + i\alpha_1 w\left(1\right)\overline{v\left(1\right)} + i\beta_1 \left|v\left(1\right)\right|^2$$
$$+ i\int_0^1 w^{(2)}\left(s\right)\overline{v^{(2)}\left(s\right)}\,\mathrm{d}s.$$

Hence, on rearranging,

$$\left(\left(\left(A + B\right)x\right)\left(\cdot\right), x\left(\cdot\right)\right)_X$$
$$= i\left(w\left(\cdot\right), v\left(\cdot\right)\right)_2 - i\left(v\left(\cdot\right), w\left(\cdot\right)\right)_2 + i2\gamma\left\|v\left(\cdot\right)\right\|_0^2 + i\beta_0 \left|v\left(0\right)\right|^2 + i\beta_1 \left|v\left(1\right)\right|^2,$$

and a simple computation reveals that

$$i\left(w\left(\cdot\right), v\left(\cdot\right)\right)_2 - i\left(v\left(\cdot\right), w\left(\cdot\right)\right)_2 = 2\operatorname{Im}\left(v\left(\cdot\right), w\left(\cdot\right)\right)_2.$$

Consequently,

$$(3.7)\qquad \operatorname{Im}\left(\left(\left(A + B\right)x\right)\left(\cdot\right), x\left(\cdot\right)\right)_X = 2\gamma\left\|v\left(\cdot\right)\right\|_0^2 + \beta_0 \left|v\left(0\right)\right|^2 + \beta_1 \left|v\left(1\right)\right|^2 \geq 0,$$

and so we have that $A + B$ is dissipative as long as strict inequality holds in at least one of the conditions $\beta_0 \geq 0$, $\beta_1 \geq 0$ or $\gamma \geq 0$, and symmetric when $\beta_0 = \beta_1 = \gamma = 0$. To finally complete the proof of assertion (i), we now show that $A + B$ under the assumption that $\alpha_0 > 0$ and $\alpha_1 > 0$ is bijective (and thus closed). We do this in two steps. For $\tilde{x}\left(\cdot\right) \in X$ let us first consider the problem

$$\left(Ax\right)\left(s\right) = \tilde{x}\left(s\right), \quad x\left(\cdot\right) \in D\left(A\right), \quad s \in [0, 1].$$

Direct calculations show that

$$\left(A^{-1}\tilde{x}\right)\left(s\right)$$
$$= \left(i\int_s^1 \frac{\left(s - r\right)^3}{6}\tilde{v}\left(r\right)\mathrm{d}r - i\int_0^1 \frac{\left(s - 1\right)\left(s^2 - 2s + r^2\right)r}{6}\tilde{v}\left(r\right)\mathrm{d}r\right.$$
$$\left.- i\int_0^1 \left[\frac{\left(s - 1\right)\left(r - 1\right)}{\alpha_0} + \frac{sr}{\alpha_1}\right]\tilde{v}\left(r\right)\mathrm{d}r + \left[\frac{\beta_0\left(s - 1\right)}{\alpha_0}v\left(0\right) - \frac{\beta_1 s}{\alpha_1}v\left(1\right)\right], v\left(s\right)\right),$$

for all $\tilde{x}\left(\cdot\right) \in X$, and it can be checked that $\left(A^{-1}\tilde{x}\right)\left(\cdot\right) \in D\left(A\right)$. Therefore A is bijective. We now claim that A^{-1} is a Hilbert–Schmidt operator on X and, therefore, compact. Let us write $A^{-1} = H + \left(A^{-1} - H\right)$, where H is the symmetric part of A^{-1} (that is, the operator A^{-1} with $\beta_0 = \beta_1 = 0$). It follows then at once that H is a Hilbert–Schmidt integral operator. Further, since $A^{-1} - H$, being finite-dimensional, is a Hilbert–Schmidt operator, A^{-1} is a Hilbert–Schmidt operator as well. This proves our claim and completes the first step in the proof of assertion (i) (and, in fact, also of assertion (ii)).

The second step involves use of the compactness of A^{-1} to show that the system operator $A + B$ possesses also a compact inverse. So let us write the operator $A + B$ in the form

$$(3.8)\qquad\qquad\qquad A + B = \left(I + BA^{-1}\right)A.$$

Obviously BA^{-1} is a compact operator. Then (see [7, Theorem XI.4.2] or [14, Theorem IV.5.26]) the operator $I + BA^{-1}$ is a Fredholm operator of zero index, and it is readily

verified that $\ker \left(I + BA^{-1} \right) = \{0\}$. So $\left(I + BA^{-1} \right) A$ is bijective since both $I + BA^{-1}$ and A are. Then (3.8) implies

(3.9) $$\left(A + B \right)^{-1} = A^{-1} \left(I + BA^{-1} \right)^{-1}.$$

This proves that $A + B$ is closed. Thus $A + B$ is by (3.7) maximal dissipative when strict inequality holds in at least one of the conditions $\beta_0 \geq 0$, $\beta_1 \geq 0$ or $\gamma \geq 0$, and selfadjoint when $\beta_0 = \beta_1 = \gamma = 0$, proving assertion (i). Now, the operator appearing on the right-hand side of (3.9) is the product of a compact operator with a bounded operator; hence it is also compact. This completes the proof of assertion (ii), and thus of the lemma. $\qquad \square$

The implications of Lemma 2 call for some comments (in preparation for what follows). Firstly, because $P(\omega)$ has a compact inverse, the spectrum of $P(\omega)$ consists only of eigenvalues, $\sigma(P(\cdot)) = \sigma_{\mathrm{p}}(P(\cdot))$, these being eigenvalues of finite type which accumulate, if at all, only at infinity (see [7, Theorem XV.2.3]). Moreover, when $A + B$ is a selfadjoint operator, a version of the spectral theorem (see [7, Theorem XVI.5.1]) yields the existence of an orthonormal basis for X consisting only of eigenvectors of A. However, when $A + B$ is not selfadjoint – we know from Lemma 2 that this is the case when at least one of the conditions $\beta_0 > 0$, $\beta_1 > 0$ or $\gamma > 0$ is satisfied – the direct analogue of this theorem is not true, for it is then generally possible to have a complete minimal system of eigenvectors and, in addition, associated vectors which is not a basis. We shall return to this matter in Section 5 and close this section with the following proposition.

Proposition 2. *Let $\beta_0 > 0$, $\beta_1 > 0$ and $\gamma > 0$. Then there can exist no nonzero purely real eigenvalues of $P(\omega)$.*

Proof. To prove the proposition we show that unless $\beta_0 = \beta_1 = \gamma = 0$ there is no nontrivial element of $\ker P(\omega)$ when ω is purely real. To this end, suppose (to reach a contradiction) there was a nonzero purely real eigenvalue ω_j. Take, then, $x(\omega_j, \cdot)$ to be the corresponding eigenvector, such that

$$\left((Px)(\omega_j, \cdot), x(\omega_j, \cdot) \right)_X = 0.$$

This we can write out as

(3.10) $$\left(\omega_j I - ((A + B) x)(\omega_j, \cdot), x(\omega_j, \cdot) \right)_X = 0,$$

and we have for the imaginary part

$$-\operatorname{Im} \left(((A + B) x)(\omega_j, \cdot), x(\omega_j, \cdot) \right)_X = 0;$$

so, by (3.7),

$$-2\gamma \left\| v(\omega_j, \cdot) \right\|_0^2 - \beta_0 \left| v(\omega_j, 0) \right|^2 - \beta_1 \left| v(\omega_j, 1) \right|^2 = 0.$$

If $\beta_0 > 0$, $\beta_1 > 0$ and $\gamma > 0$, we then have from the above that $v(\omega_j, \cdot) = 0$. Using this in (3.10) along with the fact (see Lemma 2) that $0 \in \varrho(P(\cdot))$ we can infer that $w(\omega_j, \cdot) = 0$ too, and thus have $x(\omega_j, \cdot) = 0$. This is contrary to our assumption that $x(\omega_j, \cdot)$ is an eigenvector. This completes the proof. $\qquad \square$

3.2. **The spectral problem on Y.** Consider now the spectral problem on the space Y. This is a Hilbert space with the inner product

$$\left((w(\cdot), a, c), (\tilde{w}(\cdot), \tilde{a}, \tilde{c}) \right)_Y := \left(w(\cdot), \tilde{w}(\cdot) \right)_0 + a\overline{\tilde{a}} + c\overline{\tilde{c}}$$

for every $(w(\cdot), a, c)$ and $(\tilde{w}(\cdot), \tilde{a}, \tilde{c})$ in Y. Let the operator $G : D(G) \to Y$ be given by

(3.11) $$(Gy)(s) := \left(w^{(4)}(s), w^{(3)}(0) + (\alpha_0 - \beta_0\gamma) w(0), \right.$$
$$\left. - w^{(3)}(1) + (\alpha_1 - \beta_1\gamma) w(1) \right) \quad \text{for} \quad y(\cdot) \in D(G),$$

with

$$(3.12) \qquad D(G) := \left\{ y(\cdot) \in Y \;\middle|\; \begin{array}{c} y(\cdot) = (w(\cdot), w(0), w(1)), \\ w(\cdot) \in W_2^4(0,1), \\ w^{(2)}(0) = 0, \quad w^{(2)}(1) = 0 \end{array} \right\},$$

and let $C : Y \to Y$ and $D : Y \to Y$ be given by

$$(3.13) \qquad (Cy)(s) := \big(0, \beta_0 w(0), \beta_1 w(1)\big) \quad \text{for} \quad y(\cdot) \in Y$$

and

$$(3.14) \qquad (Dy)(s) := \big(w(s), 0, 0\big) \quad \text{for} \quad y(\cdot) \in Y.$$

The operators C and D are both bounded and nonnegative (and hence symmetric), $C \geq 0$ and $D \geq 0$. The operator G will be studied shortly, where we shall show that it is selfadjoint and, for $\alpha_0 > \beta_0 \gamma$ and $\alpha_1 > \beta_1 \gamma$, its inverse is compact.

For each fixed value of λ consider now the nonmonic operator pencil $L(\lambda) : D(L(\lambda)) \to Y$ given by

$$(3.15) \qquad L(\lambda) = \lambda^2 D - i\lambda C - G,$$

with domain

$$D(L(\lambda)) = D(G).$$

Then (2.1) takes the abstract form

$$(3.16) \qquad (Ly)(\lambda, s) = 0, \quad y(\lambda, \cdot) \in D(G), \quad s \in [0,1].$$

Obviously, if there exists a chain formed by the eigen- and associated vectors $y_0(\lambda_0, \cdot), y_1(\lambda_0, \cdot), \ldots, y_{m-1}(\lambda_0, \cdot)$ of $L(\lambda)$ corresponding to an eigenvalue λ_0, then, because the domain $D(G)$ of $L(\lambda)$ is independent of the spectral parameter, it is in one-to-one correspondence with a chain formed by the eigen- and associated functions $w_0(\lambda_0, \cdot), w_1(\lambda_0, \cdot), \ldots, w_{m-1}(\lambda_0, \cdot)$ corresponding to the same eigenvalue. Thus (3.16) holds if and only if (2.1) holds.

The next proposition will be needed later, in Section 4, in the study of the spectral properties of $L(\lambda)$.

Proposition 3. *Consider the operators C and D, defined by (3.13) and (3.14), respectively. The operator D is positive when restricted to the domain $D(G)$, that is $D|_{D(G)} > 0$. Suppose $\beta_0 > 0$ and $\beta_1 > 0$; then $C|_{D(G)} > 0$ too.*

Proof. It is obvious that we need only show that

$$\big((Cy)(\cdot), y(\cdot)\big)_Y > 0 \quad \text{and} \quad \big((Dy)(\cdot), y(\cdot)\big)_Y > 0$$

for all nonzero $y(\cdot) \in D(G)$. Assume there is an element $y(\cdot) \in D(G)$ such that $y(\cdot) \neq 0$. Clearly, if we have $w(\cdot) = 0$, then $y(\cdot) = 0$. Therefore, when $y(\cdot) \neq 0$, then $w(\cdot) \neq 0$. So

$$\big((Dy)(\cdot), y(\cdot)\big)_Y = \big\|w(\cdot)\big\|_0^2 > 0,$$

proving the first statement. The key step in the proof of the second statement is to notice that $w(0) = w(1) = 0$ implies $w(\cdot) = 0$, and thus $y(\cdot) = 0$. That this is true is proven as follows. We take $\{w_r(\lambda, s)\}_1^4$ to be the fundamental system of solutions to the differential equation in (2.1) satisfying (here δ is Kronecker's delta)

$$w_r^{(n)}(\lambda, 0) = \delta_{r, n+1}$$

for $n \in \{0, 1, 2, 3\}$ (see [28, p. 13]). Then, because of the boundary condition $w^{(2)}(\lambda, 0) = 0$, there are three linearly independent nontrivial solutions, namely $w_1(\lambda, \cdot)$, $w_2(\lambda, \cdot)$ and $w_4(\lambda, \cdot)$. Now, suppose $w(\lambda, 0) = w(\lambda, 1) = 0$. Then the eigenvalue-dependent boundary conditions in (2.1) imply $w^{(3)}(\lambda, 0) = w^{(3)}(\lambda, 1) = 0$. We also have that $w^{(2)}(\lambda, 0) = w^{(2)}(\lambda, 1) = 0$, and we note then that the function $s \mapsto w_2(\lambda, s)$ is the

only solution satisfying $w(\lambda, 0) = w^{(2)}(\lambda, 0) = w^{(3)}(\lambda, 0) = 0$, and $w^{(1)}(\lambda, 0) = 1$. Also, it must satisfy $w(\lambda, 1) = w^{(2)}(\lambda, 1) = w^{(3)}(\lambda, 1) = 0$; but this is impossible unless $w_2(\lambda, \cdot) = 0$. We have seen above that this in turn implies $y(\lambda, \cdot) = 0$. It follows therefore that $w(\lambda, 0) \neq 0$ and $w(\lambda, 1) \neq 0$ when $y(\lambda, \cdot) \neq 0$. So, for all nonzero $y(\cdot) \in D(G)$,

$$((Cy)(\cdot), y(\cdot))_Y = \beta_0 |w(0)|^2 + \beta_1 |w(1)|^2 > 0,$$

and the lemma follows.   □

In Section 3.1 we argued that since the resolvent of $P(\omega)$ was compact for $\alpha_0 > 0$ and $\alpha_1 > 0$, because $0 \in \varrho(P(\cdot))$, the spectrum of $P(\omega)$ consisted only of eigenvalues of finite type, constituting an at most countable set with no finite point of accumulation in the extended complex plane. Similarly we can deduce the same result in the context of $L(\lambda)$. First we prove the following lemmas.

Lemma 3. *For the operator G as defined by (3.11), (3.12), assuming that $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$, the following assertions hold:*

   (i) G is selfadjoint; and
   (ii) G has a compact inverse.

Proof. We only establish the symmetry of G as the rest of the proof exactly parallels the proof of Lemma 2. Let us begin by showing that G is densely defined. Assume there is an element $\tilde{y}(\cdot) \in Y$ such that $(y(\cdot), \tilde{y}(\cdot))_Y = 0$ for all $y(\cdot) \in D(G)$, that is

$$(w(\cdot), \tilde{w}(\cdot))_0 + w(0)\,\overline{\tilde{a}} + w(1)\,\overline{\tilde{c}} = 0.$$

Let $w(\cdot) \in C_0^\infty([0,1])$. Then $w(0) = w(1) = 0$, and it follows that

$$(w(\cdot), \tilde{w}(\cdot))_0 = 0$$

for all $w(\cdot) \in C_0^\infty([0,1])$. Consequently, $\tilde{w}(\cdot) = 0$. Consider the polynomial

$$w(s) = s^3(s-1)^3 + 1,$$

which satisfies $w^{(2)}(0) = w^{(2)}(1) = 0$, and $w(0) = w(1) = 1$. This yields

$$(y(\cdot), \tilde{y}(\cdot))_Y = \overline{\tilde{a}} + \overline{\tilde{c}},$$

which implies $\tilde{a} = \tilde{c} = 0$. Thus $\tilde{y}(\cdot) = 0$, and we conclude $D(G)^\perp = \{0\}$, that is, the orthogonal complement of $D(G)$ in Y consists of the zero element only. Hence G is densely defined. Now let $y(\cdot), \tilde{y}(\cdot) \in D(G)$, and note that

$$
\begin{aligned}
((Gy)(\cdot), \tilde{y}(\cdot))_Y &= (w^{(4)}(\cdot), \tilde{w}(\cdot))_0 + \left[(w^{(3)}(0) + (\alpha_0 - \beta_0\gamma)w(0)\right]\overline{\tilde{w}(0)} \\
&\quad - \left[w^{(3)}(1) - (\alpha_1 - \beta_1\gamma)w(1)\right]\overline{\tilde{w}(1)} \\
&= (w^{(4)}(\cdot), \tilde{w}(\cdot))_0 - \left. w^{(3)}(s)\overline{\tilde{w}(s)}\right|_0^1 + (\alpha_0 - \beta_0\gamma)w(0)\overline{\tilde{w}(0)} \\
&\quad + (\alpha_1 - \beta_1\gamma)w(1)\overline{\tilde{w}(1)}.
\end{aligned}
$$

Since $y(\cdot) \in D(G)$, we calculate that

$$\left(w^{(4)}(\cdot), \tilde{w}(\cdot)\right)_0 = \left. w^{(3)}(s)\overline{\tilde{w}(s)}\right|_0^1 + \int_0^1 w^{(2)}(s)\overline{\tilde{w}^{(2)}(s)}\,\mathrm{d}s.$$

Hence

(3.17) $$\left(w^{(2)}(\cdot), \tilde{w}^{(2)}(\cdot)\right)_0 = \left(w^{(4)}(\cdot), \tilde{w}(\cdot)\right)_0 - \left. w^{(3)}(s)\overline{\tilde{w}(s)}\right|_0^1.$$

Now, the left side of (3.17) is symmetric, so

$$\left(w^{(2)}(\cdot), \tilde{w}^{(2)}(\cdot)\right)_0 = \left(w(\cdot), \tilde{w}^{(4)}(\cdot)\right)_0 - \left. w(s)\overline{\tilde{w}^{(3)}(s)}\right|_0^1.$$

Equating this to (3.17) gives

$$\left(w^{(4)}\left(\,\cdot\,\right),\tilde{w}\left(\,\cdot\,\right)\right)_0 = \left(w\left(\,\cdot\,\right),\tilde{w}^{(4)}\left(\,\cdot\,\right)\right)_0 + w^{(3)}\left(s\right)\overline{\tilde{w}\left(s\right)}\Big|_0^1 - w\left(s\right)\overline{\tilde{w}^{(3)}\left(s\right)}\Big|_0^1.$$

Combining the results, we obtain

$$\left(\left(Gy\right)\left(\,\cdot\,\right),\tilde{y}\left(\,\cdot\,\right)\right)_Y = \left(w^{(4)}\left(\,\cdot\,\right),\tilde{w}\left(\,\cdot\,\right)\right)_0 - w^{(3)}\left(s\right)\overline{\tilde{w}\left(s\right)}\Big|_0^1 + \left(\alpha_0 - \beta_0\gamma\right)w\left(0\right)\overline{\tilde{w}\left(0\right)}$$
$$+ \left(\alpha_1 - \beta_1\gamma\right)w\left(1\right)\overline{\tilde{w}\left(1\right)}$$
$$= \left(w\left(\,\cdot\,\right),\tilde{w}^{(4)}\left(\,\cdot\,\right)\right)_0 - w\left(s\right)\overline{\tilde{w}^{(3)}\left(s\right)}\Big|_0^1 + \left(\alpha_0 - \beta_0\gamma\right)w\left(0\right)\overline{\tilde{w}\left(0\right)}$$
$$+ \left(\alpha_1 - \beta_1\gamma\right)w\left(1\right)\overline{\tilde{w}\left(1\right)}.$$

Thus

$$\left(\left(Gy\right)\left(\,\cdot\,\right),\tilde{y}\left(\,\cdot\,\right)\right)_Y = \left(y\left(\,\cdot\,\right),\left(G\tilde{y}\right)\left(\,\cdot\,\right)\right)_Y,$$

which in view of the denseness of $D\left(G\right)$ shows that G is symmetric. $\square$

Lemma 4. *Consider the operator pencils*

$$L\left(\lambda\right) = -G - i\lambda C + \lambda^2 D \quad and \quad \tilde{L}\left(\lambda\right) = I + i\lambda CG^{-1} - \lambda^2 DG^{-1}$$

with domains $D\left(L\left(\,\cdot\,\right)\right) = D\left(G\right)$ and $D\left(\tilde{L}\left(\,\cdot\,\right)\right) = Y$, respectively. Then

$$\varrho\left(L\left(\,\cdot\,\right)\right) = \varrho\left(\tilde{L}\left(\,\cdot\,\right)\right), \quad \sigma\left(L\left(\,\cdot\,\right)\right) = \sigma\left(\tilde{L}\left(\,\cdot\,\right)\right), \quad \sigma_p\left(L\left(\,\cdot\,\right)\right) = \sigma_p\left(\tilde{L}\left(\,\cdot\,\right)\right),$$

and we have that $\{y_k\left(\lambda_0, \cdot\right)\}_{k=0}^{m-1}$ forms a chain of eigen- and associated vectors of $L\left(\lambda\right)$ corresponding to an eigenvalue λ_0, provided $\{-Gy_k\left(\lambda_0, \cdot\right)\}_{k=0}^{m-1}$ forms a chain of eigen- and associated vectors of $\tilde{L}\left(\lambda\right)$ corresponding to the same eigenvalue.

Proof. The proof is a straightforward modification of the proof of Lemma 1 and is, therefore, omitted (see also [22, Lemma 20.1]). $\square$

Theorem 4. *The spectrum of the operator pencil $L\left(\lambda\right)$ given by (3.15) consists purely of eigenvalues, these being eigenvalues of finite type which form a countable set, and which accumulate only at infinity.*

Proof. We know from Proposition 1 that the eigenvalues form a countable set. Consider (cf. (3.15))

$$L\left(\lambda\right) = -G - i\lambda C + \lambda^2 D.$$

Since G, by virtue of Lemma 3, has a compact inverse for $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$, we observe that

$$-L\left(\lambda\right)G^{-1} = I + i\lambda CG^{-1} - \lambda^2 DG^{-1},$$

wherein the operator on the right-hand side is a Fredholm operator of zero index for each fixed value of λ; it is bijective, for example, for $\lambda = 0$. Putting $-L\left(\lambda\right)G^{-1} =: \tilde{L}\left(\lambda\right)$, the theorem then follows from [7, Corollary XI.8.4] and Lemma 4. $\square$

On noting now that the spectra of $P\left(\omega\right)$ and $L\left(\lambda\right)$, and therefore the sets of eigenvalues of the corresponding spectral problems, including their algebraic and geometric multiplicities, coincide when $\gamma = 0$, and that for $\gamma > 0$ there is a direct correspondence between the two spectra, it is immediately clear from the theorem that the eigenvalues of $P\left(\omega\right)$ also constitute a countable set.

4. Spectral properties of $L(\lambda)$ and spectral asymptotics

We analyse in this section the spectrum of $L(\lambda)$ as defined in Section 3.2 in more detail. We begin with its location by showing in essence that all the eigenvalues are located symmetrically with respect to the imaginary axis in the closed upper half-plane, excluding the origin when $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$, and that when strict inequality holds in the conditions $\beta_0 \geq 0$ and $\beta_1 \geq 0$, then they are confined to the open upper half-plane. The treatment is along the general lines of that given in [26, Chapter 1].

Throughout the section it will be understood that the operators C, D and G are defined as in Section 3.2, and for clarity we will suppress any reference to s as the second argument of the eigen- and associated vectors of $L(\lambda)$.

Lemma 5. *The spectrum of the operator pencil $L(\lambda)$ given by (3.15) is symmetric with respect to the imaginary axis.*

Proof. This is an easy consequence of the fact that, for any $\lambda \in \mathbb{C}$,

$$L(-\overline{\lambda}) = \overline{\lambda}^2 D + i\overline{\lambda}C - G = \left(\lambda^2 D - i\lambda C - G\right)^* = L(\lambda)^*.$$

where $L(\lambda)^*$ is the adjoint of $L(\lambda)$. □

Lemma 6. *The spectrum of $L(\lambda)$ lies in the closed upper half-plane but excluding the origin when $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$. In the case when additionally the inequalities $\beta_0 > 0$ and $\beta_1 > 0$ are satisfied, the spectrum is confined to the open upper half-plane.*

Proof. Let $y(\lambda_j)$ be the eigenvector corresponding to the eigenvalue λ_j, so

$$\left(Ly(\lambda_j), y(\lambda_j)\right)_Y = 0.$$

This we can write out in terms of real and imaginary parts:

(4.1)
$$\begin{aligned}\left((\operatorname{Re}\lambda_j)^2 - (\operatorname{Im}\lambda_j)^2\right)&\left(Dy(\lambda_j), y(\lambda_j)\right)_Y \\ &+ \operatorname{Im}\lambda_j\left(Cy(\lambda_j), y(\lambda_j)\right)_Y - \left(Gy(\lambda_j), y(\lambda_j)\right)_Y = 0,\end{aligned}$$

and

(4.2)
$$\operatorname{Re}\lambda_j\left[2\operatorname{Im}\lambda_j\left(Dy(\lambda_j), y(\lambda_j)\right)_Y - \left(Cy(\lambda_j), y(\lambda_j)\right)_Y\right] = 0.$$

Consider the case $\operatorname{Re}\lambda_j = 0$. Then from (4.1),

$$(\operatorname{Im}\lambda_j)^2\left(Dy(\lambda_j), y(\lambda_j)\right)_Y + \left(Gy(\lambda_j), y(\lambda_j)\right)_Y = \operatorname{Im}\lambda_j\left(Cy(\lambda_j), y(\lambda_j)\right)_Y,$$

and since $C \geq 0$, $D \geq 0$ and $G \geq 0$, we find that $\operatorname{Im}\lambda_j \geq 0$. If $\operatorname{Re}\lambda_j \neq 0$, then we have from (4.2) that $\operatorname{Im}\lambda_j \geq 0$. This proves that the spectrum of $L(\lambda)$ lies in the closed upper half-plane. That, when $\alpha_0 > \beta_0\gamma$ and $\alpha_1 > \beta_1\gamma$, the spectrum does not include the origin is obvious. In fact, suppose the origin belongs to the spectrum. Then from (4.1) we obtain

$$\left(Gy(\lambda_j), y(\lambda_j)\right)_Y = 0,$$

and consequently $Gy(\lambda_j) = 0$; but then $y(\lambda_j) = 0$ by Lemma 3, and we have a contradiction (since $y(\lambda_j)$ is an eigenvector). Let now further $\beta_0 > 0$ and $\beta_1 > 0$, and suppose λ_j was a nonzero purely real eigenvalue. From (4.2) it follows at once that

$$\left(Cy(\lambda_j), y(\lambda_j)\right)_Y = 0,$$

which is a contradiction by Proposition 3. □

Remark 1. *It follows from (4.2) in the proof of the lemma that, in the case $\operatorname{Re}\lambda_j \neq 0$, we have $\operatorname{Im}\lambda_j \in [a_0, a_1]$, where*

$$a_0 = \frac{1}{2}\inf_{y(\neq 0)\in D(G)}\frac{(Cy, y)_Y}{(Dy, y)_Y} \quad \text{and} \quad a_1 = \frac{1}{2}\sup_{y(\neq 0)\in D(G)}\frac{(Cy, y)_Y}{(Dy, y)_Y}.$$

Using the result of Proposition 3 immediately yields that $a_1 < \infty$, since C is bounded and $D|_{D(G)} > 0$, and if the inequalities $\beta_0 > 0$ and $\beta_1 > 0$ are satisfied, then $a_0 > 0$ since $C|_{D(G)} > 0$. If $\alpha_0 > \beta_0 \gamma$ and $\alpha_1 > \beta_1 \gamma$, then we obtain for the case when $\mathrm{Re}\,\lambda_j = 0$ that $\mathrm{Im}\,\lambda_j \in [b_0, b_1]$, where

$$b_0 = \sqrt{|c|} \quad and \quad b_1 = a_1 + \sqrt{a_1^2 + |c|},$$

with

$$c = \inf_{y(\neq 0) \in D(G)} \frac{(Gy, y)_Y}{(Dy, y)_Y}.$$

By similar reasoning, using (4.1), it is clear that $b_1 < \infty$ and $b_0 > 0$.

We pass now to the final, rather important task of this section, which is to obtain asymptotic estimates for the eigenvalues. This is best done by deriving a few intermediate results. Let us proceed formally, making the change in (2.1) from λ to μ^2, and take $\{w_r(\mu, s)\}_1^4$ to be the fundamental system of solutions to the differential equation in (2.1) with $\lambda = \mu^2$. Set

$$w(\mu, s) = \frac{1}{2\mu^3} \sinh(\mu s) - \frac{1}{2\mu^3} \sin(\mu s),$$

and let $w_r(\mu, s) = w^{(4-r)}(\mu, s)$ for $r \in \{1, 2, 3, 4\}$. We noted in the proof of Proposition 3 that the boundary condition $w^{(2)}(\mu, 0) = 0$ demands we consider $w_1(\mu, s)$, $w_2(\mu, s)$ and $w_4(\mu, s)$, that is,

$$w_1(\mu, s) = \frac{1}{2}\cosh(\mu s) + \frac{1}{2}\cos(\mu s), \quad w_2(\mu, s) = \frac{1}{2\mu}\sinh(\mu s) + \frac{1}{2\mu}\sin(\mu s),$$

$$w_4(\mu, s) = \frac{1}{2\mu^3}\sinh(\mu s) - \frac{1}{2\mu^3}\sin(\mu s).$$

Applying the remaining three boundary conditions in (2.1) to these and using a determinantal calculation we can write down explicitly the characteristic equation by defining

$$\phi_0(\mu) := \mu^4 \cos\mu \cosh\mu, \quad \phi_1(\mu) := -\mu^4,$$

(4.3)

$$\phi_2(\mu) := -\mu(\sin\mu \cosh\mu - \cos\mu \sinh\mu), \quad \phi_3(\mu) := -\frac{2}{\mu^2}\sin\mu \sinh\mu,$$

giving

(4.4) $\phi(\mu) = \phi_0(\mu) + \phi_1(\mu) + (\theta_0(\mu) + \theta_1(\mu))\phi_2(\mu) + \theta_0(\mu)\,\theta_1(\mu)\,\phi_3(\mu) = 0,$

where $\theta_0(\mu)$ and $\theta_1(\mu)$ are as given in the proof of Proposition 1. We are justified (cf. Lemma 1) in referring to the squares of the zeros of the function $\mu \mapsto \phi(\mu)$ as the eigenvalues of $L(\lambda)$. Let us consider the zeros of $\phi_0(\cdot)$.

Lemma 7. *The function $\mu \mapsto \phi_0(\mu)$ defined in (4.3) has a zero of algebraic multiplicity four at 0, which can be associated with $\mu_1^{\pm}$ and $\mu_{-1}^{\pm}$. All other zeros are simple and different from zero, and they can be associated with $\mu_j^{\pm}$ and $\mu_{-j}^{\pm}$ for $j \in \mathbb{Z}\backslash\{0, 1\}$, where*

$$\mu_j^{\pm} = \begin{cases} \pm(2j - 3)\dfrac{\pi}{2} & for\ j = 2, 3, \ldots, \\[2mm] \pm i(2|j| - 3)\dfrac{\pi}{2} & for\ j = -2, -3, \ldots. \end{cases}$$

Proof. The proof is immediate. First of all, it is easily seen that the function $\mu \mapsto \mu^4$ has a zero of multiplicity four at 0. All other zeros are of the function $\mu \mapsto \cos\mu \cosh\mu$. The lemma then follows on noting that $\cosh i\mu = \cos\mu$ and $\cos i\mu = \cosh\mu$. $\square$

Next we examine the asymptotic behaviour of the zeros of $\phi(\cdot)$, and hence of the eigenvalues of $L(\lambda)$.

Theorem 5. *The spectrum of the operator pencil $L(\lambda)$ consists of a countable sequence $\{\lambda_j\}_{-\infty,\, j\neq 0}^{\infty}$ of eigenvalues, these being eigenvalues of finite type which are restricted to a strip of finite height parallel to the real axis in the closed upper half-plane, and which only have two points of accumulation in the extended complex plane, namely $+\infty$ and $-\infty$. The sequence of eigenvalues can be correctly enumerated in the sense of Definition 4, such that for $|j| < j_0$, where j_0 is some positive integer, the eigenvalues are purely imaginary, and $\lambda_{-j} = -\overline{\lambda_j}$ for $|j| \geq j_0$, where*

$$(4.5) \qquad \lambda_j = \left(\mu_j^{\pm}\right)^2$$

for $j \in \mathbb{Z}\backslash\{0\}$, with the $\mu_j^{\pm}$ being the zeros of the function $\mu \mapsto \phi(\mu)$ given in (4.4), whose asymptotic behaviour is specified by

$$\mu_j^{\pm} = \begin{cases} \pm(2j-3)\dfrac{\pi}{2} + i\,(\beta_0 + \beta_1)\left[(2j-3)\dfrac{\pi}{2}\right]^{-1} + O\!\left(j^{-2}\right) & \text{for } j > 0, \\[2ex] \pm i\,(2\,|j|-3)\dfrac{\pi}{2} + (\beta_0 + \beta_1)\left[(2\,|j|-3)\dfrac{\pi}{2}\right]^{-1} + O\!\left(|j|^{-2}\right) & \text{for } j < 0. \end{cases}$$

In particular, there is an even number of purely imaginary eigenvalues.

Proof. That all eigenvalues lie in a strip of finite height parallel to the real axis in the upper half-plane follows from Remark 1. Let us verify the asymptotic formulae in the statement of the theorem. To begin, it suffices, in view of Lemma 5, to confine the solutions of the characteristic equation $\phi(\mu) = 0$ to zeros of $\phi(\cdot)$ lying in the first quadrant. In fact, it is an elementary task to verify that the functions $\mu \mapsto \phi_0(\mu)$, $\mu \mapsto \phi_1(\mu)$, $\mu \mapsto \phi_2(\mu)$ and $\mu \mapsto \phi_3(\mu)$ defined in (4.3) are all even for any $\mu \in \mathbb{C}$. So $\mu \mapsto \phi(\mu)$ is an even function too because $\theta_0(-\mu) = \theta_0(\mu)$ and $\theta_1(-\mu) = \theta_1(\mu)$. Now, set $\phi_1(\mu) = O\!\left(|\mu|^4\right)$, and note that

$$2e^{-\mu}\phi_0(\mu) = \mu^4 \cos\mu\left(1 + e^{-2\mu}\right),$$
$$2e^{-\mu}\phi_2(\mu) = \mu\left(\sin\mu - \cos\mu\right) + \mu e^{-2\mu}\left(\sin\mu + \cos\mu\right),$$
$$2e^{-\mu}\phi_3(\mu) = -\frac{2}{\mu^2}\sin\mu\left(1 - e^{-2\mu}\right).$$

With these it is easily checked that the characteristic equation takes the form, as $|\mu| \to \infty$ (equivalently, as $\operatorname{Re}\mu \to \infty$),

$$(4.6) \qquad \mu\cos\mu + i\,(\beta_0 + \beta_1)\,(\sin\mu - \cos\mu) + O\!\left(|\mu|^{-1}\right) = 0.$$

We consider (4.6) for all values of μ in a small neighbourhood of $(2j-3)\pi/2$. Let us write it as

$$(4.7) \qquad \cos\mu + O\!\left(|\mu|^{-1}\right) = 0,$$

and consider $\cos\mu$ on a simple, closed contour in the first quadrant encircling $(2j-3)\pi/2$ for $j = 2, 3, \ldots$ so that $|\cos\mu| \geq 1$ on the contour. There is exactly one simple zero of the function $\mu \mapsto \cos\mu$ within the contour by Lemma 7. So Rouché's theorem applies to show that the function $\mu \mapsto \phi(\mu)$ has, asymptotically, the same number of zeros as the function $\mu \mapsto \cos\mu$, namely one, inside the contour. Therefore, for $j \to \infty$, it is clear that

$$(4.8) \qquad \mu_j^{\pm} = \pm(2j-3)\frac{\pi}{2} + O\!\left(j^{-1}\right),$$

and we note that we can specify

$$\sin\mu_j = (-1)^{j+1}\cos O\!\left(j^{-1}\right), \quad \cos\mu_j = (-1)^{j}\sin O\!\left(j^{-1}\right).$$

Substituting μ_j for μ in (4.6), and developing $\cos O(j^{-1})$ and $\sin O(j^{-1})$ in Taylor series, we obtain that

$$(-1)^j O(j^{-1}) \left[(2j-3)\frac{\pi}{2} + O(j^{-1}) \right]$$
$$= -i(\beta_0 + \beta_1) \left[(-1)^{j+1} \left(1 + O(j^{-1}) \right) \right] + O(j^{-1}),$$

implying the relation

$$O(j^{-1}) = i(\beta_0 + \beta_1) \left[2(j-3)\frac{\pi}{2} \right]^{-1} + O(j^{-2}).$$

The substitution of this into (4.8) leads to the first expression for $\mu_j^{\pm}$ in the statement of the theorem. That the number of purely imaginary eigenvalues is even follows from the fact (see Lemma 5) that the eigenvalues with nonzero real parts occur in pairs λ_j, $-\overline{\lambda_j}$, so $\lambda_{-j} = -\overline{\lambda_j}$, and their indexing is symmetric with respect to the imaginary axis.   □

5. Completeness, minimality, and Riesz basis property

In this section we return to the state space setting where, in the light of the results obtained so far, we turn our focus to the question of the Riesz basis property of the system of eigenvectors of the system operator $A + B$ as defined in Section 3.1. We shall require $\alpha_0 > 0$ and $\alpha_1 > 0$ in what follows (here and in the next section), so $0 \in \varrho(P(\cdot))$ and the spectral problem (3.6) is nondegenerate.

Up to this point we have shown that the operator pencil $P(\omega)$ has pure point spectrum, consisting of a countable sequence $\{\omega_j\}_{-\infty, j \neq 0}^{\infty}$ of possibly complex eigenvalues of finite type accumulating only at $+\infty$ and $-\infty$, and that all eigenvalues with sufficiently large modulus are simple. Indeed, this follows directly from Theorems 4 and 5 (bearing in mind that $\lambda_j = \omega_j - i\gamma$). Let $\{x(\lambda_j)\}_{-\infty, j \neq 0}^{\infty}$ be the system of eigenvectors in X, corresponding to the λ_j. The crucial step now in proving the Riesz basis property is to establish that there exists in X a unique system of vectors, $\{z(\omega_j)\}_{-\infty, j \neq 0}^{\infty}$, say, such that $\{x(\omega_j)\}_{-\infty, j \neq 0}^{\infty}$, $\{z(\omega_j)\}_{-\infty, j \neq 0}^{\infty}$ is a biorthogonal pair in X. This will be the main part of the work in this section, and we start with addressing the questions of completeness and minimality.

Theorem 6. *The system operator $A + B$, with A defined by (3.2), (3.3) and B by (3.4), has a complete minimal system of eigen- and associated vectors in X.*

Proof. We begin with the proof of the minimality property. Recall from Lemma 2 that the operator $(A + B)^{-1}$ exists and is compact. It is not difficult to check that it is injective. Hence, by Theorem 2, the system of eigen- and associated vectors of $(A + B)^{-1}$ is minimal. The proof of the minimality property of the system of eigenvectors $A + B$ is then complete, because the eigen- and associated vectors of $(A + B)^{-1}$ are also eigen- and associated vectors of $A + B$.

To now prove completeness, it will suffice to demonstrate that $(A + B)^{-1}$ can be factored in the form given in (2.3), and that the conditions imposed in Theorem 1 on the operators involved in the factorisation hold. The theorem then follows, by virtue of Theorem 5, if we identify λ_j in (2.4) with the inverse of (4.5), the reciprocal of λ_j, setting $\beta_0 = \beta_1 = 0$ in their asymptotic expressions, and note that, as a result, the condition (2.4) is satisfied. To see that the factorisation is possible, let $A = H + (A - H)$, where H is the selfadjoint part of A, and write

$$(A + B)^{-1} = (H + (A - H) + B)^{-1}.$$

This may be written equivalently in the form

$$\left[\left[(A + B)^* \right]^{-1} \right]^* = \left[\left[(H + (A - H) + B)^* \right]^{-1} \right]^*,$$

where $(A + B)^*$ denotes, as usual, the (formal) adjoint of $A + B$. The operator H has a trivial kernel, and its inverse is compact, as we see from Lemma 2. Thus

$$\left[(H + (A - H) + B)^* \right]^{-1} = \left(H + (A - H)^* + B^* \right)^{-1}$$
$$= \left(I + (A - H)^* H^{-1} + B^* H^{-1} \right)^{-1} H^{-1}.$$

Now set

$$\left(I + (A - H)^* H^{-1} + B^* H^{-1} \right)^{-1} = I + S^*.$$

A straightforward computation shows that

$$S^* = - \left[(A - H)^* H^{-1} + B^* H^{-1} \right] \left(I + (A - H)^* H^{-1} + B^* H^{-1} \right)^{-1},$$

which, in view of the proof of Lemma 2, is obviously compact. As this, combined with the results above, yields (2.3) with $K = H^{-1}$, the proof is complete. $\qquad \square$

Having proven the completeness and minimality, it now remains only to prove, with the proviso that all eigenvalues are simple, that the system of eigenvectors of the operator $A + B$ has the property of being a Riesz basis for X. Here is where Theorem 3 comes into the picture. We can now state and prove the main result of this section.

Theorem 7. *Suppose the sequence $\{\omega_j\}_{-\infty,\, j \neq 0}^{\infty}$ of eigenvalues of $A + B$ is simple. Then the corresponding system of normalised eigenvectors of $A + B$ forms a Riesz basis for X.*

Proof. We see from Theorem 5 that we are justified in supposing the eigenvalues are simple for $|j| \geq j_0$ (for some $j_0 < \infty$). To obtain that the system of eigenvectors of $A + B$, when normalised, forms a Riesz basis for its closed linear span, we first note that $A + B$ is dissipative if and only if the operator $-(A + B)^{-1}$ is. From Lemma 2 we recall that $A + B$ is dissipative and has a compact inverse. Thus $-(A + B)^{-1}$ is dissipative, and its eigenvectors correspond to the negative inverse of the ω_j, that is to

$$(5.1) \qquad\qquad -\omega_j^{-1} = -\left(\lambda_j + i\gamma \right)^{-1},$$

with λ_j given by (4.5). We can immediately verify by making use of the asymptotic formulae in Theorem 5 for the $\mu_j^{\pm}$ that the condition in (2.5), wherein we identify λ_j with (5.1), is satisfied. So all conditions of Theorem 3 are satisfied, and the system of normalised eigenvectors of $A + B$ forms a Riesz basis for its closed linear span. Hence, by means of Theorem 6, we can infer that the system of normalised eigenvectors of $A + B$ forms in fact a Riesz basis for X. $\qquad \square$

Remark 2. *The same result as obtained in Theorem 7 could have been achieved if we had invoked a theorem proven by Katsnelson in [15], which states the theorem to be true also when condition (2.5) is weakened to some corresponding Carleson-type conditions; more details can be found in [29, Lectures VI to X]. This was done by Miloslavskii in [25] for an application example.*

The above theorem is an important result which plays an essential role in the spectral decomposition of the infinitesimal generator, as well as opening the door to a range of further investigations of major systems problems for infinite-dimensional systems. These include, for example, such concepts as controllability, observability, and, in particular, stabilisability, and a good treatment of these subjects may be found in the book by Curtain and Zwart [5].

6. Exponential stability of the system semigroup

As has been mentioned, the results in the last section play an important role in the problem of stability associated with the initial/boundary-value problem stated in the introduction. With the definitions and notation from Section 3 we can write the initial/boundary-value problem in abstract form as

$$(6.1) \qquad \dot{\mathbf{x}}(t) = i(A+B)\mathbf{x}(t), \quad \mathbf{x}(0) = x, \quad t > 0,$$

with x as given by (3.1). The initial-value problem (6.1) is uniquely solvable for $x \in D(A)$, and we are interested now in what we can infer in regard to the decay of the solutions. Indeed, with the results now in hand we are ready to establish – in the sense of the norm on X – the uniform exponential decay of the solutions to the initial-value problem. The following theorem, which constitutes the final result of the paper, is a direct consequence of Lemma 2, Remark 1 and Theorem 7.

Theorem 8. *Consider the system operator $A+B$, with A defined by (3.2), (3.3) and B by (3.4), and let $\alpha_0 > 0$ and $\alpha_1 > 0$. The operator $i(A+B)$ is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ of contractions, where $T(t)$ is a bounded linear operator on X for each $t \geq 0$. Supposing the sequence of eigenvalues of $i(A+B)$ is simple, we have for each $t \geq 0$ and $x \in D(A)$,*

$$(6.2) \qquad T(t)x = \sum_{-\infty,\, j\neq 0}^{\infty} e^{i\omega_j t} (x, z(\omega_j))_X x(\omega_j),$$

convergent in X. The semigroup is uniformly exponentially stable when there is strict inequality in the conditions $\beta_0 \geq 0$, $\beta_1 \geq 0$ or $\gamma \geq 0$, in the sense that

$$(6.3) \qquad \left\| T(t)x \right\|_X \leq M e^{\varepsilon t} \left\| x \right\|_X,$$

where M is some positive real number, and ε is a negative real number determined by the supremum of the real parts of the eigenvalues of the generator.

Proof. From Lemma 2 we know that the operator $i(A+B)$ is maximal dissipative when strict inequality holds in at least one of the conditions $\beta_0 \geq 0$, $\beta_1 \geq 0$ or $\gamma \geq 0$. Thus, by [17, Theorem I.4.5], the operator $i(A+B)$ generates a strongly continuous semigroup of contractions on X, and so (see [17, Section I.1]) the initial-value problem (6.1) is correct in the sense that for $x \in D(A)$ it has the unique, continuously differentiable solution $\mathbf{x}(\cdot) \in C^1((0,\infty); X) \cap C([0,\infty); D(A))$, given by

$$\mathbf{x}(t) = T(t)x.$$

In view of Theorem 7, it is obvious that the expansion (6.2) holds because the spectral mapping theorem holds. A standard argument using [8, Section VI.2.(2.4)] then applies to show that (6.3) holds. That $\varepsilon < 0$ follows from Remark 1. The proof is complete. $\square$

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