

LOCALIZATION PRINCIPLES FOR SCHRÖDINGER OPERATOR WITH A SINGULAR MATRIX POTENTIAL

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ABSTRACT. We study the spectrum of the one-dimensional Schrödinger operator H_0 with a matrix singular distributional potential $q = Q'$ where $Q \in L^2_{loc}(\mathbb{R}, \mathbb{C}^m)$. We obtain generalizations of Ismagilov's localization principles, which give necessary and sufficient conditions for the spectrum of H_0 to be bounded below and discrete.

1. INTRODUCTION

Schrödinger operators occupy a special position in the modern mathematical physics because they have numerous applications to physical problems and other branches of mathematics; see, e.g., [2]. Nowadays the spectral theory of these operators has developed very profoundly and contains a number of fundamental results. Specifically, this concerns the questions about self-adjointness, semiboundedness, and discreteness of the spectrum. These questions are studied in the greatest detail for one-dimensional Schrödinger operators [3, 7, 10, 13, 18], with local integrability being a standard condition on the regularity of the potential. Moreover, in last years of growing interest are problems in which the potential is singular and contains delta-functions supported on a discrete set or contains more general Radon measures [1, 17]. A direct generalization of classical theorems to such operators is associated with serious difficulties. These difficulties become greater if the potentials are matrix-valued and the operator acts on vector-valued functions [8].

The main purpose of our paper is to ground the fundamental localization principles for the most general operators of the mentioned type. In next papers this will allow us to obtain necessary and/or sufficient constructive conditions for these operators to be semibounded and for their spectrum to be discrete provided that we impose additional restrictions on the matrix potential. The proofs of the results given below is based on the regularization of the differential expression with the help of quasiderivatives [4, 5, 6, 9, 12, 14, 16].

The paper consists of five sections and Appendix. Section 1 is an introduction. Section 2 contains the statement of the problem and formulation of our main results, Theorems 1 and 2. They are generalizations of the localization principles of Ismagilov [7] to the case of a matrix distributional potential. These theorems are proved in Section 5. Their proofs are based on the basic Lemma 2 established in Section 4. Beforehand we will obtain some auxiliary results in Section 3.

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

We consider a linear differential expression

$$(1) \quad h(y) := -y'' + qy$$

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in the complex separable Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$, with $m \geq 1$. Here, $y := (y_1, \dots, y_m) \in L^2(\mathbb{R}, \mathbb{C}^m)$, and $q := (q_{i,j})_{i,j=1}^m$ is a matrix potential such that each

$$q_{i,j} = Q'_{i,j} \quad \text{for a certain } Q_{i,j} \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{C}).$$

Throughout the paper, derivatives are understood in the sense of the theory of distributions. Put $Q := (Q_{i,j})_{i,j=1}^m$. In the sequel, the matrix potential Q is supposed to be Hermitian-symmetric, i.e., $Q = Q^*$.

Using the quasiderivatives

$$y^{[1]} := y' - Qy \quad \text{and} \quad y^{[2]} := (y^{[1]})' + Qy^{[1]} + Q^2y$$

(see, e.g., [12]), we write the differential expression (1) in the form $h(y) = -y^{[2]}$. Following [12, Section 1], we associate the maximal, preminimal, and minimal operators with this expression in the following way: the maximal operator

$$(2) \quad Hy := -y^{[2]}$$

is defined on the natural widest domain

$$D(H) := \{y \in L^2(\mathbb{R}, \mathbb{C}^m) : y, y^{[1]} \in AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^m), y^{[2]} \in L^2(\mathbb{R}, \mathbb{C}^m)\}.$$

Here, as usual, $AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^m)$ denotes the set of all vector-valued functions $y : \mathbb{R} \rightarrow \mathbb{C}^m$ that are absolutely continuous on every compact interval $[a, b] \subset \mathbb{R}$. By definition, the preminimal operator H'_0 is the restriction of the maximal operator (2) to the set of all compactly supported functions $y \in D(H)$, and the minimal operator H_0 is the closure of H'_0 . It is known [12, Corollary 2 and Proposition 7] that the domains of H , H'_0 , and H_0 are dense in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$ and that the operators H'_0 and H_0 are symmetric and

$$H = (H'_0)^* = H_0^*.$$

The main results of the paper are generalizations of the localization principles by Ismagilov [7] to the case of a matrix distributional potential.

Let us introduce some designations. Given a nonempty open set $\Omega \subseteq \mathbb{R}$, we put

$$\lambda(\Omega) := \inf \left\{ \frac{\langle H_0 y, y \rangle}{\langle y, y \rangle} : y \in D(H'_0) \setminus \{0\}, \text{supp } y \subset \Omega \right\}.$$

Here and below, $\langle \cdot, \cdot \rangle$ is the inner product in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$. Since the operator H'_0 is symmetric, the inclusion $\langle H'_0 y, y \rangle \in \mathbb{R}$ holds; therefore $\lambda(\Omega)$ is well defined.

We choose a number $\ell > 0$ arbitrarily and put

$$\omega_n^\ell := \left(\frac{n\ell}{2}, \frac{n\ell}{2} + \ell \right) \quad \text{for every } n \in \mathbb{Z}.$$

As in the case of a locally integrable scalar potential, each number $\lambda(\omega_n^\ell)$ coincides with the smallest eigenvalue of the bounded below selfadjoint operator $H^D(\omega_n^\ell)$ generated by the differential expression (1) and the homogeneous boundary conditions $y(n\ell/2) = y(n\ell/2 + \ell) = 0$ in the Hilbert space $L^2(\omega_n^\ell, \mathbb{C}^m)$ (see Appendix below). Therefore these numbers make physical sense.

Theorem 1 (the first localization principle). *The minimal operator H_0 is bounded below and selfadjoint if and only if the sequence of numbers $(\lambda(\omega_n^\ell))_{n=-\infty}^{+\infty}$ is bounded below.*

Theorem 2 (the second localization principle). *The operator H_0 is a bounded below selfadjoint operator with discrete spectrum if and only if*

$$(3) \quad \lambda(\omega_n^\ell) \rightarrow +\infty \quad \text{as } |n| \rightarrow \infty.$$

Remark 1. It follows from Theorems 1 and 2 that if for a certain $\ell > 0$ the sequence $(\lambda(\omega_n^\ell))_{n=-\infty}^{+\infty}$ is bounded below or satisfies (3), this sequence will have the same property for every $\ell > 0$.

Remark 2. In Theorems 1 and 2 it is possible to replace all the intervals ω_n^ℓ , where $n \in \mathbb{Z}$, with their shifts at an arbitrarily chosen number $a \in \mathbb{R}$.

Remark 3. Analogs of Theorems 1 and 2 are true in the case where the differential expression (1) is given on a semiaxis.

Remark 4. Theorem 1 somewhat generalizes and together with Lemma 2 for $\Omega := \mathbb{R}$ supplements the known statement [12] about the self-adjointness of the bounded below operator H_0 . Specifically, it follows from Theorem 1 that the operator H_0 with the periodic matrix potential Q is bounded below and selfadjoint; cf. [11], where the case of $m = 1$ is examined.

3. AUXILIARY RESULTS

Given vector-valued functions $y, z : \mathbb{R} \rightarrow \mathbb{C}^m$, we let (y, z) denote the scalar complex-valued function defined by the formula $(y, z) := y_1 \bar{z}_1 + \dots + y_m \bar{z}_m$ on \mathbb{R} . Note that

$$\langle y, z \rangle = \int_{-\infty}^{\infty} (y, z) dx$$

if $y, z \in L^2(\mathbb{R}, \mathbb{C}^m)$. Throughout the paper all integrals are understood in the sense of Lebesgue, and dx denotes the Lebesgue measure on \mathbb{R} , we omitting the argument x of functions under the integral sign.

We choose a real-valued function $\theta \in C^\infty(\mathbb{R})$ such that $\text{supp } \theta = [0, \ell]$ and

$$(4) \quad \theta^2(x) + \theta^2(x - \ell/2) = 1 \quad \text{for every } x \in [\ell/2, \ell].$$

An example of this function will be given at the end of the present section. Given $k \in \mathbb{Z}$ and $y \in D(H'_0)$, we introduce the functions

$$\theta_k(x) := \theta(x - k\ell/2), \quad u_k(x) := \theta_k^2(x)y(x), \quad \text{and} \quad v_k(x) := \theta_k(x)\theta_{k+1}(x)y(x)$$

of $x \in \mathbb{R}$.

Lemma 1. *Let $y \in D(H'_0)$. Then $u_k, v_k \in D(H'_0)$ for every $k \in \mathbb{Z}$, and we have the equality*

$$(5) \quad \begin{aligned} \langle H_0 y, y \rangle &= \sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle H_0 v_k, v_k \rangle \\ &\quad - 2 \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} (\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2 (y, y) dx. \end{aligned}$$

Proof. We choose $k \in \mathbb{Z}$ arbitrarily and will show that $u_k, v_k \in D(H'_0)$. Since the functions $y \in \text{AC}_{\text{loc}}(\mathbb{R}, \mathbb{C}^m)$ and $\theta_k, \theta_{k+1} \in C^\infty(\mathbb{R}, \mathbb{R})$ are compactly supported, the functions u_k and v_k are also compactly supported and belong to both the spaces $\text{AC}_{\text{loc}}(\mathbb{R}, \mathbb{C}^m)$ and $L^2(\mathbb{R}, \mathbb{C}^m)$. Besides,

$$\begin{aligned} u_k^{[1]} &= u'_k - Qu_k = (\theta_k^2 y)' - Q\theta_k^2 y = (\theta_k^2)' y + \theta_k^2 y' - Q\theta_k^2 y \\ &= (\theta_k^2)' y + \theta_k^2 y^{[1]} \in \text{AC}_{\text{loc}}(\mathbb{R}, \mathbb{C}^m) \end{aligned}$$

and

$$\begin{aligned} u_k^{[2]} &= (u_k^{[1]})' + Qu_k^{[1]} + Q^2 u_k = ((\theta_k^2)' y + \theta_k^2 y^{[1]})' + Q((\theta_k^2)' y + \theta_k^2 y^{[1]}) + Q^2 \theta_k^2 y \\ &= (\theta_k^2)'' y + (\theta_k^2)' y' + (\theta_k^2)' y^{[1]} + \theta_k^2 (y^{[1]})' + (\theta_k^2)' Qy + \theta_k^2 Qy^{[1]} + \theta_k^2 Q^2 y \\ &= \theta_k^2 y^{[2]} + (\theta_k^2)' (y' + y^{[1]} + Qy) + (\theta_k^2)'' y \\ &= \theta_k^2 y^{[2]} + 2(\theta_k^2)' (y^{[1]} + Qy) + (\theta_k^2)'' y \in L_2(\mathbb{R}, \mathbb{C}^m). \end{aligned}$$

Here, we use the fact that $y^{[1]} \in AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^m)$, $y^{[2]} \in L^2(\mathbb{R}, \mathbb{C}^m)$, and $Q \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{C}^{m \times m})$. Replacing u_k with v_k and θ_k^2 with $\theta_k \theta_{k+1}$ in the above equalities, we obtain the inclusions $v_k^{[1]} \in AC_{\text{loc}}(\mathbb{R}, \mathbb{C}^m)$ and $v_k^{[2]} \in L^2(\mathbb{R}, \mathbb{C}^m)$. Thus, $u_k, v_k \in D(H'_0)$ by the definition of $D(H'_0)$.

Let us now prove equality (5). Integrating by parts, we write

$$(6) \quad \begin{aligned} \langle H_0 y, y \rangle &= \langle y^{[2]}, y \rangle = - \int_{-\infty}^{\infty} ((y^{[1]})' + Q(y' - Qy) + Q^2 y, y) dx \\ &= \int_{-\infty}^{\infty} ((y^{[1]}, y') - (Qy', y)) dx. \end{aligned}$$

Equality (6) holds true for every function $y \in D(H'_0)$. Since $u_k, v_k \in D(H'_0)$, we may put $y := u_k$ or $y := v_k$ in this equality and write

$$(7) \quad \langle H_0 u_k, u_k \rangle = \int_{k\ell/2}^{k\ell/2+\ell} ((u_k^{[1]}, u'_k) - (Qu'_k, u_k)) dx,$$

$$(8) \quad \langle H_0 v_k, v_k \rangle = \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((v_k^{[1]}, v'_k) - (Qv'_k, v_k)) dx.$$

Here, we use the fact that $\text{supp } u_k \subseteq [k\ell/2, k\ell/2 + \ell]$ and $\text{supp } v_k \subseteq [k\ell/2 + \ell/2, k\ell/2 + \ell]$.

Owing to (7) we write

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2}^{k\ell/2+\ell/2} ((u_k^{[1]}, u'_k) - (Qu'_k, u_k)) dx \\ & \quad + \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((u_k^{[1]}, u'_k) - (Qu'_k, u_k)) dx \\ &= \sum_{j=-\infty}^{\infty} \int_{j\ell/2+\ell/2}^{j\ell/2+\ell} ((u_{j+1}^{[1]}, u'_{j+1}) - (Qu'_{j+1}, u_{j+1})) dx \\ & \quad + \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((u_k^{[1]}, u'_k) - (Qu'_k, u_k)) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((u_k^{[1]}, u'_k) + (u_{k+1}^{[1]}, u'_{k+1}) - (Qu'_k, u_k) - (Qu'_{k+1}, u_{k+1})) dx. \end{aligned}$$

By virtue of this formula and (8), we obtain the equalities

$$\begin{aligned}
 & \sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle H_0 v_k, v_k \rangle \\
 &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((u_k^{[1]}, u'_k) + (u_{k+1}^{[1]}, u'_{k+1}) + 2(v_k^{[1]}, v'_k) \\
 (9) \quad & \quad - (Qu'_k, u_k) - (Qu'_{k+1}, u_{k+1}) - 2(Qv'_k, v_k)) dx \\
 &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} ((u'_k, u'_k) + 2(v'_k, v'_k) + (u'_{k+1}, u'_{k+1}) \\
 & \quad - (Qu_k, u'_k) - 2(Qv_k, v'_k) - (Qu_{k+1}, u'_{k+1}) \\
 & \quad - (Qu'_k, u_k) - 2(Qv'_k, v_k) - (Qu'_{k+1}, u_{k+1})) dx.
 \end{aligned}$$

Let us show that for every $k \in \mathbb{Z}$ the last integrand is equal to

$$(10) \quad (y', y') + 2(\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2 (y, y) - (Qy, y') - (Qy', y).$$

We note beforehand that

$$\begin{aligned}
 (u'_k, u'_k) &= ((\theta_k^2 y)')', (\theta_k^2 y')' = ((2\theta_k \theta'_k y + \theta_k^2 y'), (2\theta_k \theta'_k y + \theta_k^2 y')) \\
 &= 4\theta_k^2 (\theta'_k)^2 (y, y) + 2\theta_k^3 \theta'_k ((y, y') + (y', y)) + \theta_k^4 (y', y')
 \end{aligned}$$

and

$$\begin{aligned}
 (v'_k, v'_k) &= ((\theta_k \theta_{k+1} y)')', (\theta_k \theta_{k+1} y')' \\
 &= ((\theta_k \theta_{k+1})' y + \theta_k \theta_{k+1} y', (\theta_k \theta_{k+1})' y + \theta_k \theta_{k+1} y') \\
 &= ((\theta_k \theta_{k+1})')^2 (y, y) + \theta_k \theta_{k+1} (\theta_k \theta_{k+1})' ((y, y') + (y', y)) \\
 & \quad + \theta_k^2 \theta_{k+1}^2 (y', y').
 \end{aligned}$$

It follows directly from formula (4) and the definition of θ_k that

$$(11) \quad \begin{aligned} \theta_k^2(x) + \theta_{k+1}^2(x) &= 1 \quad \text{and} \quad \theta_k(x)\theta'_k(x) + \theta_{k+1}(x)\theta'_{k+1}(x) = 0 \\ &\text{for every } x \in [k\ell/2 + \ell/2, k\ell/2 + \ell]. \end{aligned}$$

Therefore we have the following equalities on $[k\ell/2 + \ell/2, k\ell/2 + \ell]$:

$$\begin{aligned}
 & (u'_k, u'_k) + 2(v'_k, v'_k) + (u'_{k+1}, u'_{k+1}) \\
 &= 2(y, y)(2\theta_k^2 (\theta'_k)^2 + ((\theta_k \theta_{k+1})')^2 + 2\theta_{k+1}^2 (\theta'_{k+1})^2) \\
 & \quad + 2((y, y') + (y', y))(\theta_k^3 \theta'_k + \theta_k \theta_{k+1} (\theta_k \theta_{k+1})' + \theta_{k+1}^3 \theta'_{k+1}) \\
 & \quad + (y', y')(\theta_k^4 + 2\theta_k^2 \theta_{k+1}^2 + \theta_{k+1}^4) \\
 &= 2(y, y)(2\theta_k^2 (\theta'_k)^2 + (\theta'_k)^2 \theta_{k+1}^2 + 2\theta_k \theta'_k \theta_{k+1} \theta'_{k+1} + \theta_k^2 (\theta'_{k+1})^2 + 2\theta_{k+1}^2 (\theta'_{k+1})^2) \\
 & \quad + 2((y, y') + (y', y))(\theta_k^3 \theta'_k + \theta_k \theta'_k \theta_{k+1}^2 + \theta_k^2 \theta_{k+1} \theta'_{k+1} + \theta_{k+1}^3 \theta'_{k+1}) \\
 & \quad + (y', y')(\theta_k^2 + \theta_{k+1}^2)^2 \\
 &= 2(y, y)(2\theta_k^2 (\theta'_k)^2 + 4\theta_k \theta'_k \theta_{k+1} \theta'_{k+1} + 2\theta_{k+1}^2 (\theta'_{k+1})^2) \\
 & \quad + (\theta'_k)^2 \theta_{k+1}^2 - 2\theta_k \theta'_k \theta_{k+1} \theta'_{k+1} + \theta_k^2 (\theta'_{k+1})^2) \\
 & \quad + 2((y, y') + (y', y))(\theta_k^2 (\theta_k \theta'_k + \theta_{k+1} \theta'_{k+1}) + \theta_{k+1}^2 (\theta_k \theta'_k + \theta_{k+1} \theta'_{k+1})) + (y', y') \\
 &= 2(y, y)(2(\theta_k \theta'_k + \theta_{k+1} \theta'_{k+1})^2) + (\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2 + (y', y') \\
 &= 2(y, y)(\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2 + (y', y'),
 \end{aligned}$$

i.e.,

$$(12) \quad (u'_k, u'_k) + 2(v'_k, v'_k) + (u'_{k+1}, u'_{k+1}) = (y', y') + 2(\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2(y, y).$$

Besides,

$$(Qu_k, u'_k) = (Q(\theta_k^2 y), (\theta_k^2 y)') = \theta_k^2 (Qy, 2\theta_k \theta'_k y + \theta_k^2 y') = 2\theta_k^3 \theta'_k (Qy, y) + \theta_k^4 (Qy, y')$$

and

$$\begin{aligned} (Qv_k, v'_k) &= (Q(\theta_k \theta_{k+1} y), (\theta_k \theta_{k+1} y)') = \theta_k \theta_{k+1} (Qy, (\theta_k \theta_{k+1})' y + \theta_k \theta_{k+1} y') \\ &= \theta_k \theta_{k+1} (\theta_k \theta_{k+1})' (Qy, y) + \theta_k^2 \theta_{k+1}^2 (Qy, y'). \end{aligned}$$

Hence, in view of (11), we have the following equalities on the compact interval $[k\ell/2 + \ell/2, (k+1)\ell/2 + \ell/2]$:

$$\begin{aligned} &(Qu_k, u'_k) + 2(Qv_k, v'_k) + (Qu_{k+1}, u'_{k+1}) \\ &= 2(Qy, y)(\theta_k^3 \theta'_k + \theta_k \theta'_k \theta_{k+1}^2 + \theta_k^2 \theta_{k+1} \theta'_{k+1} + \theta_{k+1}^3 \theta'_{k+1}) \\ &\quad + (Qy, y')(\theta_k^4 + 2\theta_k^2 \theta_{k+1}^2 + \theta_{k+1}^4) \\ &= 2(Qy, y)(\theta_k^2 (\theta_k \theta'_k + \theta_{k+1} \theta'_{k+1}) + \theta_{k+1}^2 (\theta_k \theta'_k + \theta_{k+1} \theta'_{k+1})) + (Qy, y')(\theta_k^2 + \theta_{k+1}^2)^2 \\ &= (Qy, y'), \end{aligned}$$

i.e.,

$$(13) \quad (Qu_k, u'_k) + 2(Qv_k, v'_k) + (Qu_{k+1}, u'_{k+1}) = (Qy, y').$$

Since $Q = Q^*$, the equalities

$$\begin{aligned} &(Qu'_k, u_k) + 2(Qv'_k, v_k) + (Qu'_{k+1}, u_{k+1}) \\ (14) \quad &= (u'_k, Qu_k) + 2(v'_k, Qv_k) + (u'_{k+1}, Qu_{k+1}) \\ &= \overline{(Qy, y')} = (Qy', y) \end{aligned}$$

hold on the same interval.

Owing to (12)–(14) we conclude that the last integrand in (9) equals (10) for every $k \in \mathbb{Z}$. Hence, according to (9) and (6), we have the equalities

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle H_0 v_k, v_k \rangle \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{(k+1)\ell/2+\ell/2} ((y', y') + 2(\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2(y, y) - (Qy, y') - (Qy', y)) dx \\ &= \int_{-\infty}^{\infty} ((y' - Qy, y') - (Qy', y)) dx + 2 \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{(k+1)\ell/2+\ell/2} (\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2(y, y) dx \\ &= \langle H_0 y, y \rangle + 2 \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{(k+1)\ell/2+\ell/2} (\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2(y, y) dx. \end{aligned}$$

This immediately implies the required formula (5). □

Example. Let us give an example of a real-valued function $\theta \in C^\infty(\mathbb{R})$ that satisfies the equality $\text{supp } \theta = [0, \ell]$ and condition (4). Recall that this function is used in Lemma 1. We choose a function $\eta_0 \in C^\infty(\mathbb{R})$ such that $\text{supp } \eta_0 = [0, \ell]$ and $\eta_0(x) > 0$ for every

$x \in (0, \ell)$. Let η denote the ℓ -periodic extension of the function η_0^2 over the whole \mathbb{R} . We introduce the real-valued function

$$\theta(x) := \frac{\eta_0(x)}{\sqrt{\eta(x) + \eta(x - h/2)}} \quad \text{of } x \in \mathbb{R}.$$

Since $\eta(x) + \eta(x - h/2) > 0$ for every $x \in \mathbb{R}$, this function is well defined and satisfies the conditions $\theta \in C^\infty(\mathbb{R})$ and $\text{supp } \theta = [0, \ell]$. It also satisfies condition (4). Indeed, given $x \in [\ell/2, \ell]$, we obtain the equalities

$$\begin{aligned} \theta^2(x) + \theta^2(x - \ell/2) &= \frac{\eta_0^2(x)}{\eta(x) + \eta(x - \ell/2)} + \frac{\eta_0^2(x - \ell/2)}{\eta(x - \ell/2) + \eta(x - \ell)} \\ &= \frac{\eta_0^2(x) + \eta_0^2(x - \ell/2)}{\eta(x) + \eta(x - \ell/2)} = 1 \end{aligned}$$

in view of the definition of η .

4. BASIC LEMMA

We put

$$\varkappa := \max\{|\theta'(x)| : 0 \leq x \leq \ell\}.$$

Lemma 2. *Let Ω be an nonempty open subset of \mathbb{R} . Then there exists $n \in \mathbb{Z}$ such that $\omega_n^\ell \cap \Omega \neq \emptyset$ and*

$$(15) \quad \nu(\omega_n^\ell) \leq \lambda(\Omega) + 8\varkappa^2.$$

Proof. It follows from property (4) and the definition of \varkappa that $0 \leq \theta_k(x) \leq 1$ and $|\theta'_k(x)| \leq \varkappa$ for arbitrary $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Therefore $(\theta'_k \theta_{k+1} - \theta_k \theta'_{k+1})^2(x) \leq 4\varkappa^2$; hence, owing to Lemma 1, the inequality

$$(16) \quad \langle H_0 y, y \rangle \geq \sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle H_0 v_k, v_k \rangle - 8\varkappa^2 \langle y, y \rangle$$

holds true for every $y \in D(H'_0)$.

Note that the equality

$$(17) \quad \langle y, y \rangle = \sum_{k=-\infty}^{\infty} \langle u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle v_k, v_k \rangle$$

is valid for every $y \in D(H'_0)$. Indeed,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \langle u_k, u_k \rangle &= \sum_{k=-\infty}^{\infty} \langle \theta_k^2 y, \theta_k^2 y \rangle = \sum_{k=-\infty}^{\infty} \int_{k\ell/2}^{k\ell/2+\ell} (\theta_k^2 y, \theta_k^2 y) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2}^{k\ell/2+\ell/2} \theta_k^4(y, y) dx + \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} \theta_k^4(y, y) dx \\ &= \sum_{j=-\infty}^{\infty} \int_{j\ell/2+\ell/2}^{j\ell/2+\ell} \theta_{j+1}^4(y, y) dx + \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{\ell/2+\ell} \theta_k^4(y, y) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} (\theta_{k+1}^4 + \theta_k^4)(y, y) dx. \end{aligned}$$

Besides,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \langle v_k, v_k \rangle &= \sum_{k=-\infty}^{\infty} \langle \theta_k \theta_{k+1} y, \theta_k \theta_{k+1} y \rangle \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} (\theta_k \theta_{k+1} y, \theta_k \theta_{k+1} y) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} \theta_k^2 \theta_{k+1}^2 (y, y) dx. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \langle u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle v_k, v_k \rangle &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} (\theta_k^2 + \theta_{k+1}^2)^2 (y, y) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{k\ell/2+\ell/2}^{k\ell/2+\ell} (y, y) dx = \langle y, y \rangle. \end{aligned}$$

It follows from the definition of $\lambda(\Omega)$ that for every number $\delta > 0$ there exists a function $y \in D(H'_0)$ such that $\text{supp } y \subset \Omega$ and

$$\langle H_0 y, y \rangle < (\lambda(\Omega) + \delta) \langle y, y \rangle.$$

Applying (16) and (17) to the last formula, we obtain the inequality

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \langle H_0 u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle H_0 v_k, v_k \rangle \\ - (8\mathcal{I}^2 + \lambda(\Omega) + \delta) \left(\sum_{k=-\infty}^{\infty} \langle u_k, u_k \rangle + 2 \sum_{k=-\infty}^{\infty} \langle v_k, v_k \rangle \right) < 0. \end{aligned}$$

Grouping summands, we write this inequality in the form

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (\langle H_0 u_k, u_k \rangle - (\lambda(\Omega) + 8\mathcal{I}^2 + \delta) \langle u_k, u_k \rangle) \\ + 2 \sum_{k=-\infty}^{\infty} (\langle H_0 v_k, v_k \rangle - (\lambda(\Omega) + 8\mathcal{I}^2 + \delta) \langle v_k, v_k \rangle) < 0. \end{aligned}$$

Here, at least one of the summands is less than zero. Let a negative summand have an index $k = k_0$. Then $u_{k_0} \neq 0$ or $v_{k_0} \neq 0$ for otherwise this summand would equal to zero. Hence,

$$\emptyset \neq (\text{supp } u_{k_0}) \cup (\text{supp } v_{k_0}) \subset \Omega \cap \omega_{k_0}^\ell,$$

i.e., $\omega_{k_0}^\ell \cap \Omega \neq \emptyset$. Besides,

$$\frac{\langle H_0 w_{k_0}, w_{k_0} \rangle}{\langle w_{k_0}, w_{k_0} \rangle} < \lambda(\Omega) + 8\mathcal{I}^2 + \delta,$$

with $w_{k_0} := u_{k_0}$ or $w_{k_0} := v_{k_0}$. It follows from this inequality and the inclusion $\text{supp } w_{k_0} \subset \omega_{k_0}^\ell$ that

$$\lambda(\omega_{k_0}^\ell) < \lambda(\Omega) + 8\mathcal{I}^2 + \delta.$$

Passing here to the limit as $\delta \rightarrow 0+$, we obtain the required inequality (15). \square

5. PROOFS OF THE MAIN RESULTS

We will prove Theorems 1 and 2 with the help of Lemma 2.

Proof of Theorem 1. Sufficiency. Assume that there exists a number $\alpha \in \mathbb{R}$ such that $\lambda(\omega_n^\ell) \geq \alpha$ for every $n \in \mathbb{Z}$. Then, according to Lemma 2 for $\Omega := \mathbb{R}$, we have the inequalities

$$\lambda(\mathbb{R}) \geq \lambda(\omega_n^\ell) - 8\kappa^2 \geq \alpha - 8\kappa^2.$$

Hence, $H'_0 \geq (\alpha - 8\kappa^2)I$, where I is the identity operator. Then the operator H_0 is also bounded below so that it is selfadjoint due to [12, Corollary 2]. Sufficiency is proved.

Necessity is obvious. Indeed, if $H_0 \geq \beta I$ for certain $\beta \in \mathbb{R}$, then $\nu(\omega_n^\ell) \geq \beta$ for every $n \in \mathbb{Z}$. □

Proof of Theorem 2. Sufficiency. Assume that $\lambda(\omega_n^\ell) \rightarrow +\infty$ as $|n| \rightarrow \infty$. Then, owing to Theorem 1, the operator H_0 is bounded below and selfadjoint; hence, $H_0 = H$. Let us prove that its spectrum is discrete, i.e., $\sigma_{\text{ess}}(H_0) = \emptyset$. We arbitrarily choose a number $r > 0$. By our assumption, there exists a number $n_r \in \mathbb{N}$ such that

$$\lambda(\omega_n^\ell) > r \quad \text{whenever} \quad |n| \geq n_r.$$

Let us use Lemma 2 for the open set

$$\Omega := \left(-\infty, -\frac{n_r \ell}{2} - \ell\right) \cup \left(\frac{n_r \ell}{2} + \ell, \infty\right).$$

Observe that $\omega_n^\ell \cap \Omega \neq \emptyset \Rightarrow |n| \geq n_r$. Therefore it follows from this lemma that

$$(18) \quad \langle H_0 y, y \rangle \geq (r - 8\kappa^2) \langle y, y \rangle \quad \text{whenever} \quad y \in D(H'_0) \text{ and } \text{supp } y \subset \Omega.$$

We put $\gamma := n_r \ell / 2 + 2\ell$ and consider the decomposition of the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$ in the orthogonal sum of its subspaces

$$L^2(\mathbb{R}) = L^2(-\infty, -\gamma] \oplus L^2[-\gamma, \gamma] \oplus L^2[\gamma, \infty).$$

For the sake of brevity of formulas in the proof, we omit the expression \mathbb{C}^m and exterior parentheses in designations of spaces of vector-valued functions. For example, $L^2(-\infty, -\gamma]$ stands for the space $L^2((-\infty, -\gamma], \mathbb{C}^m)$. Besides, we identify vector-valued functions given on an interval $G \subset \mathbb{R}$ with their extensions by zero over the whole \mathbb{R} . In this sense, $L^2(G)$ is considered as a subspace of $L^2(\mathbb{R})$. With the operator H_0 and this decomposition we associate three unbounded operators $H_j y := -y^{[2]}$, where $j \in \{1, 2, 3\}$. They are defined respectively on the linear manifolds

$$\begin{aligned} D(H_1) &:= \{y \in L^2(-\infty, -\gamma] : y, y^{[1]} \in \text{AC}_{\text{loc}}(-\infty, -\gamma], \\ &\quad y(-\gamma) = y^{[1]}(-\gamma) = 0, y^{[2]} \in L^2(-\infty, -\gamma]\}, \\ D(H_2) &:= \{y \in L^2[-\gamma, \gamma] : y, y^{[1]} \in \text{AC}[-\gamma, \gamma], \\ &\quad y(-\gamma) = y^{[1]}(-\gamma) = y(\gamma) = y^{[1]}(\gamma) = 0, y^{[2]} \in L^2[-\gamma, \gamma]\}, \\ D(H_3) &:= \{y \in L^2[\gamma, \infty) : y, y^{[1]} \in \text{AC}_{\text{loc}}[\gamma, \infty), \\ &\quad y(\gamma) = y^{[1]}(\gamma) = 0, y^{[2]} \in L^2[\gamma, \infty)\}. \end{aligned}$$

Each operator H_j is closed and a restriction of H_0 . This follows from the fact that $(O_j y)^{[1]} = O_j(y^{[1]})$ and $(O_j y)^{[2]} = O_j(y^{[2]})$ for every $y \in D(H_j)$, where O_j is the operator of the extension of a function by zero from the corresponding set onto the whole \mathbb{R} . Hence, H_0 is an extension of the orthogonal sum $H_1 \oplus H_2 \oplus H_3$ of these operators. Since H_0 is bounded below, all H_1, H_2, H_3 are also bounded below.

We let H_j^{F} denote the selfadjoint Friedrichs extension of the semibounded operator H_j , with $j \in \{1, 2, 3\}$. The spectrum of H_2^{F} is discrete [16]. Owing to property (18) and the definition of γ , the operators H_1^{F} and H_3^{F} are bounded below by the number $r - 8\kappa^2$.

The resolvents of the selfadjoint operators H_0 and $H_1^F \oplus H_2^F \oplus H_3^F$ in $L^2(\mathbb{R})$ differ in an operator with finite rank. Hence,

$$\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H_1^F) \cup \sigma_{\text{ess}}(H_3^F),$$

which yields the equality

$$\sigma_{\text{ess}}(H_0) \cap (-\infty, r - 8\epsilon^2) = \emptyset.$$

Thus, $\sigma_{\text{ess}}(H_0) = \emptyset$ because the number $r > 0$ is arbitrarily chosen. Sufficiency is proved.

Necessity. Assume that H_0 is a bounded below selfadjoint operator with discrete spectrum. Let us deduce property (3) by means of proof by contradiction. Suppose the contrary, i.e., there exists a number $r > 0$ and sequence $(n_k)_{k=1}^\infty \subset \mathbb{Z}$ such that $|n_k| \rightarrow \infty$ as $k \rightarrow \infty$ and that $\lambda(\omega_{n_k}^\ell) < r$. Passing to a subsequence, we may suppose that $\omega_{n_k}^\ell \cap \omega_{n_p}^\ell = \emptyset$ whenever $k \neq p$. It follows from the definition of $\lambda(\omega_{n_k}^\ell)$ that for every integer $k \geq 1$ there exists a vector-valued function $y_k \in D(H_0') \setminus \{0\}$ such that $\text{supp } y_k \subset \omega_{n_k}^\ell$ and $\langle H_0' y_k - r y_k, y_k \rangle < 0$. Let G be a linear span of $\{y_k : 1 \leq k \in \mathbb{Z}\}$. Since $\text{supp } y_k \cap \text{supp } y_p = \emptyset$ whenever $k \neq p$, we deduce the properties $\dim G = \infty$ and

$$\langle H_0 y - r y, y \rangle < 0 \quad \text{for every } y \in G \setminus \{0\}.$$

Therefore, applying [3, Chapter 1, Theorem 13] to the selfadjoint operator H_0 , we conclude that the set $\sigma(H_0) \cap (-\infty, r)$ is infinite. This contradicts our assumption, according to which the spectrum of H_0 is bounded below and does not contain any limit points. Necessity is proved. \square

APPENDIX

Let $n \in \mathbb{Z}$. In the complex separable Hilbert space $L^2(\omega_n^\ell, \mathbb{C}^m)$ we consider the operator $H^D(\omega_n^\ell)$ which is the restriction of the maximal operator H on the set of all vector-valued functions $y \in D(H)$ that satisfy the boundary conditions $y(n\ell/2) = 0$ and $y(n\ell/2 + \ell) = 0$. The operator $H^D(\omega_n^\ell)$ is selfadjoint and bounded below [9].

Theorem A. *The operator $H^D(\omega_n^\ell)$ is the Friedrichs extension of the minimal operator $H_0(\omega_n^\ell)$ generated by the differential expression (1) on the interval ω_n^ℓ .*

Since the spectrum of $H^D(\omega_n^\ell)$ is discrete [16], it follows from Theorem A and properties of the Friedrichs extension that the first eigenvalue of $H^D(\omega_n^\ell)$ coincides with the number $\lambda(\omega_n^\ell)$ introduced in Section 2.

Let us outline the proof of Theorem A. This theorem is known in the case where $q = 0$ and $m = 1$. If $q = 0$ and $m \geq 2$, the minimal operator $H_0(\omega_n^\ell)$ is the orthogonal sum of m scalar minimal operators. It follows then from the construction of the Friedrichs extension (see, e.g., [15, Section 124]) that the operator $H^D(\omega_n^\ell)$ is the orthogonal sum of m Friedrichs extensions of the scalar minimal operators. If $q \neq 0$ and $m \geq 1$, then, using the reasoning from [14], we may show that the operator $H^D(\omega_n^\ell)$ is the form-sum of the free Hamiltonian ($q \equiv 0$) and the quadratic form corresponding to the singular potential. The latter is zero relative form-bounded. This implies Theorem A in view of KLMN theorem.

REFERENCES

1. Sergio Albeverio, Aleksey Kostenko, and Mark Malamud, *Spectral theory of semibounded Sturm-Liouville operators with local interactions on a discrete set*, J. Math. Phys. **51** (2010), no. 10, 102102, 24 pp.
2. H. L. Cycon, R. G. Froese, W Kirsch, and B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin, 1987.
3. I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Daniel Davey & Co., Inc., New York, 1966.

4. A. S. Goriunov and V. A. Mikhailets, *Regularization of two-term differential equations with singular coefficients by quasiderivatives*, Ukrainian Math. J. **63** (2012), no. 9, 1361–1378.
5. Andrii Goriunov, Vladimir Mikhailets, and Konstantin Pankrashkin, *Formally self-adjoint quasi-differential operators and boundary-value problems*, Electron. J. Differential Equations (2013), no. 101, 1–16.
6. A. S. Goryunov and V. A. Mikhailets, *Resolvent convergence of Sturm–Liouville operators with singular potentials*, Math. Notes **87** (2010), no. 1-2, 287–292.
7. R. S. Ismagilov, *Conditions for the semiboundedness and discreteness of the spectrum in the case of one-dimensional differential operators*, Dokl. Akad. Nauk SSSR **140** (1961), no. 1, 33–36. (Russian)
8. R. S. Ismagilov and A. G. Kostyuchenko, *On the spectrum of a vector Schrödinger operator*, Funct. Anal. Appl. **41** (2007), no. 1, 31–41.
9. O. O. Konstantinov, *Two-term differential equations with matrix distributional coefficients*, Ukrainian Math. J. **67** (2015), no. 5, 625–634.
10. V. V. Martynov, *The conditions of discreteness and continuity of the spectrum in case of a self-conjugate system of even order differential equations*, Differentsial'nye Uravneniya **1** (1965), no. 12, 1578–1591. (Russian)
11. Vladimir Mikhailets and Volodymyr Molyboga, *One-dimensional Schrödinger operators with singular periodic potentials*, Methods Funct. Anal. Topology **14** (2008), no. 2, 184–200.
12. Vladimir Mikhailets and Volodymyr Molyboga, *Remarks on Schrödinger operators with singular matrix potentials*, Methods Funct. Anal. Topology **19** (2013), no. 2, 161–167.
13. A. M. Molčanov, *On conditions for discreteness of the spectrum of self-adjoint differential equations of the second order*, Trudy Moskov. Mat. Obšč. **2** (1953), 169–199. (Russian)
14. V. M. Molyboga, *Schrödinger operators with distributional matrix potentials*, Ukrainian Math. J. **67** (2015), no. 5, 748–763.
15. Frigyes Riesz and Béla Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.
16. A. M. Savchuk and A. A. Shkalikov, *Sturm–Liouville operators with singular potentials*, Math. Notes **66** (1999), no. 5-6, 741–753.
17. Jun Yan and Guoliang Shi, *Spectral properties of Sturm–Liouville operators with local interactions on a discrete set*, J. Differential Equations **257** (2014), no. 9, 3423–3447.
18. L. B. Zelenko, *Discreteness conditions for the spectrum of the Sturm–Liouville operator in a space of vector functions*, Mat. Zametki **1** (1967), no. 5, 583–588. (Russian)

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