

## PARAMETRIZATION OF SCALE-INVARIANT SELF-ADJOINT EXTENSIONS OF SCALE-INVARIANT SYMMETRIC OPERATORS

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ABSTRACT. On a Hilbert space  $\mathfrak{H}$ , we consider a symmetric scale-invariant operator with equal defect numbers. It is assumed that the operator has at least one scale-invariant self-adjoint extension in  $\mathfrak{H}$ . We prove that there is a one-to-one correspondence between (generalized) resolvents of scale-invariant extensions and solutions of some functional equation. Two examples of Dirac-type operators are considered.

### 1. INTRODUCTION

This article is a continuation of our previous investigation of scale-invariant symmetric operators [4, 5, 7–10]. At first, we recall some definitions and previous results.

**Definition 1.** Let  $T$  be a densely defined operator on a Hilbert space  $\mathfrak{H}$  with domain  $\mathfrak{D}(T)$ , and let  $q \in (0, \infty) \setminus \{1\}$ . The operator  $T$  is said to be  $q$ -scale-invariant (s-i) (more precisely,  $(q, U_q)$ -scale-invariant) provided there exists a unitary operator  $U_q$  on  $\mathfrak{H}$  such that

1.  $U_q \mathfrak{D}(T) = \mathfrak{D}(T)$ ;
2.  $U_q T f = q T U_q f$ ,  $f \in \mathfrak{D}(T)$ .

It is easily seen that if the operator  $T$  is closable, then its closure  $\bar{T}$  is also a  $q$ -s-i operator. In [8, Lemma 1], it was shown that if  $T$  is a densely defined closed symmetric operator with at least one defect number finite, then the first condition of Definition 1 can be replaced by the weaker condition  $U_q \mathfrak{D}(T) \subset \mathfrak{D}(T)$ .

In [4, 5], by using M. G. Kreĭn's method of the 'real Cayley transform' [18], it was shown that a densely defined symmetric positive operator  $\mathcal{H}$  (i.e.,  $(\mathcal{H}f, f) \geq 0$  for all  $f \in \mathfrak{D}(\mathcal{H})$ ) which is  $q$ -s-i, always admits positive  $q$ -s-i self-adjoint extensions (with the same  $U_q$ !). In particular, the so-called Friedrichs and Kreĭn extensions ('hard' and 'soft' in terminology of M. G. Kreĭn) are always  $q$ -s-i. Moreover, if the index of defect of the operator  $\mathcal{H}$  is  $(1, 1)$ , then only these two extensions are  $q$ -s-i self-adjoint extensions of  $\mathcal{H}$ . This fact was announced in [23] and proved in [4]. In some situations, this fact allows to find Friedrichs and Kreĭn extensions directly from von Neumann formulas (see, for example, [10]). The result [4, Theorem 3.11] also implicitly contains a parametrization of all positive s-i self-adjoint extensions of a positive symmetric s-i operator  $\mathcal{H}$ . This parametrization is given in terms of solutions of some algebraic Riccati equation.

An s-i symmetric operator which is not semi-bounded, generally speaking, does not admit s-i self-adjoint extensions in the same space  $\mathfrak{H}$ . A corresponding example is given in [8]. In order to consider s-i extensions in a larger Hilbert space  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ , it is necessary to modify Definition 1 as follows.

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2010 *Mathematics Subject Classification.* Primary 47B25, 47A20; Secondary 47A10.

*Key words and phrases.* Symmetric operator, scale-invariant operator, self-adjoint extension, generalized resolvents.

**Definition 2.** Let  $\mathcal{H}$  be a densely defined closed symmetric  $(q, U_q)$ -s-i operator on a Hilbert space  $\mathfrak{H}$ , and let  $\tilde{H}$  be its self-adjoint extension in a larger Hilbert space  $\tilde{\mathfrak{H}}$ . The operator  $\tilde{H}$  is called  $q$ -s-i self-adjoint extension of the operator  $\mathcal{H}$  provided there exists a unitary operator  $\tilde{U}_q$  on  $\tilde{\mathfrak{H}}$  such that  $\mathfrak{H}$  reduces  $\tilde{U}_q$ ,  $\tilde{U}_q|_{\mathfrak{H}} = U_q$ , and the operator  $\tilde{H}$  is  $(q, \tilde{U}_q)$ -s-i.

The article [6] considered isometric operators  $V$  defined on a proper subspace  $\mathfrak{D}$  of a Hilbert space  $\mathfrak{H}$ , which are unitarily equivalent to their Möbius transformation, that is,

$$UV = (aV + \bar{b}I)(bV + \bar{a}I)^{-1}U,$$

where  $U$  is a unitary operator on  $\mathfrak{H}$ ,  $a, b \in \mathbb{C}$ , and the matrix  $g = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$  satisfies the condition

$$\det g = 1, \quad g^* J g = J, \quad \text{where } J = \text{diag}\{1, -1\}.$$

Such operators are called automorphic-invariant. In [6], among other results, it was shown that if the defect numbers of the operator  $V$  are finite, then  $V$  admits a maximal contractive automorphic invariant extension [6, Theorem 5.1] and a unitary extension in, generally speaking, a larger Hilbert space  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$  [6, Corollary 5.3]. The article [6] also contains parametrization of generalized resolvents of automorphic-invariant unitary extensions of  $V$  [6, Theorem 5.7]. Since the Cayley transform of a  $q$ -s-i symmetric operator  $\mathcal{H}$  is an automorphic-invariant isometry, where the transformation  $g$  is hyperbolic (having two fixed points on the unit circle), the results of [6] immediately give the following statement:

*Let  $\mathcal{H}$  be a closed symmetric densely defined  $q$ -s-i operator on a Hilbert space  $\mathfrak{H}$  with finite defect numbers. Then  $\mathcal{H}$  admits a self-adjoint or maximal dissipative  $q$ -s-i extension  $H$  in  $\mathfrak{H}$  (i.e.,  $\text{Im}(Hf, f) \geq 0$ ). These possibilities are not mutually exclusive if both defect numbers of  $\mathcal{H}$  are greater than or equal to 2. Consequently, such operator  $\mathcal{H}$  always admits at least one self-adjoint  $q$ -s-i extension  $H$  in a possibly larger Hilbert space  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ .*

The same result, using the same methods as in [6] (except the consideration of extensions in a larger space  $\tilde{\mathfrak{H}}$ ), was recently published in [24]. Note that if the index of defect of the operator  $\mathcal{H}$  is  $(1, 1)$ , then either  $\mathcal{H}$  admits self-adjoint  $q$ -s-i extensions in  $\mathfrak{H}$ , or it admits one maximal dissipative  $q$ -s-i extension in  $\mathfrak{H}$ , or all self-adjoint and maximal dissipative extensions in  $\mathfrak{H}$  are  $q$ -s-i.

In this article, we consider a  $q$ -s-i symmetric operator  $\mathcal{H}$  on a Hilbert space  $\mathfrak{H}$  and assume that it has equal and finite defect numbers. Moreover, we assume that the operator  $\mathcal{H}$  admits at least one  $q$ -s-i self-adjoint extension in  $\mathfrak{H}$ . Then we use M. G. Kreĭn's formula [17], which parametrizes (generalized) resolvents of all self-adjoint extensions of  $\mathcal{H}$ . We show that there exists a one-to-one correspondence between the set of solutions of some functional equation for parameterizing operator-valued function and the set of (generalized) resolvents of  $q$ -s-i self-adjoint extensions of  $\mathcal{H}$ . This material is contained in Section 2. In Section 3, we illustrate our approach with two examples of Dirac-type operators on  $L^2(\mathbb{C}^2, \mathbb{R}_+)$ . In particular, we give an explicit form of the above mentioned functional equation. Those examples may be interesting in themselves.

The study of  $q$ -s-i operators is closely related to the study of pairs of operators  $A, B$  that satisfy the formal algebraic relation

$$AB = qBA, \quad q \in (0, \infty) \setminus \{1\}.$$

The investigation of such pairs of operators is motivated by the development of the theory of quantum groups and quantum algebras (see, for example, [15, 16, 29]) and, of course, by the development of operator theory (see, for example, [25, 26]). The article [27] considered the case  $B = A^*$ , that is,  $AA^* = qA^*A$ . The corresponding operator  $A$  is

called  $q$ -normal. In [27],  $q$ -normal operators as well as some other classes of  $q$ -deformed operators ( $q$ -quasinormal and  $q$ -hyponormal) were investigated.

In [12], the authors considered a one-parameter family  $\{U_s\}$ ,  $s \in S \subset \mathbb{R}$ , of unitary operators acting in some Hilbert space  $\mathfrak{H}$  and a linear operator  $A$ ,  $A \neq 0$  in  $\mathfrak{H}$  such that

$$U_s A = p(s) A U_s, \quad s \in S,$$

where  $p$  is a real-valued function. In [12], such operators were called  $p(s)$ -homogeneous. In a particular case, when  $S = \mathbb{Z}$ ,  $\{U_s\}$  is a group, and  $p(s) = q^s$ ,  $s \in \mathbb{Z}$ , one obtains that a  $p$ -homogeneous operator is  $q$ -s-i in the sense of Definition 1.

## 2. DESCRIPTION OF SCALE-INVARIANT EXTENSIONS

Let  $\mathcal{H}$  be a densely defined closed prime symmetric operator in a Hilbert space  $\mathfrak{H}$  with domain  $\mathfrak{D}(\mathcal{H})$ . The assumption that  $\mathcal{H}$  is a prime symmetric operator means that there is no proper subspace  $\mathfrak{H}'$  of  $\mathfrak{H}$  which reduces  $\mathcal{H}$  such that  $\mathcal{H}|_{\mathfrak{H}'}$  is self-adjoint. Suppose that the defect numbers of the operator  $\mathcal{H}$  are equal, and denote by  $\overset{\circ}{H}$  a fixed self-adjoint extension of  $\mathcal{H}$  in  $\mathfrak{H}$ . Denote by  $\overset{\circ}{R}(z)$  the resolvent of  $\overset{\circ}{H}$ . Also, put  $\mathfrak{M}_z = (\mathcal{H} - zI)\mathfrak{D}(\mathcal{H})$  and  $\mathfrak{N}_z = \mathfrak{M}_z^\perp$ . For any  $z$  which belongs to the field of regularity of  $\mathcal{H}$ , in particular, for any nonreal  $z$ , the set  $\mathfrak{M}_z$  is a closed subspace of  $\mathfrak{H}$ . Following M. G. Kreĭn [19], for any nonreal  $z$  and  $\zeta$ , we denote by  $U_{z\zeta}$  a bounded operator on  $\mathfrak{H}$  defined by

$$U_{z\zeta} = (\overset{\circ}{H} - zI)(\overset{\circ}{H} - \zeta I)^{-1} = I + (\zeta - z) \overset{\circ}{R}(\zeta).$$

The operator  $U_{z\zeta}$  possesses the following properties:

- (a)  $U_{z\zeta} \mathfrak{H} = \mathfrak{H}$ ;      (b)  $U_{z\zeta}^{-1} = U_{\zeta z}$ ;      (c)  $U_{z\zeta}^* = U_{\bar{z}\bar{\zeta}}$ ;
- (d)  $U_{z\zeta} U_{\zeta\eta} = U_{z\eta}$ ;      (e)  $U_{z\zeta} \mathfrak{M}_\zeta = \mathfrak{M}_z$ ;      (f)  $U_{z\zeta} \mathfrak{N}_z = \mathfrak{N}_\zeta$ .

Let  $z_0$  with  $\text{Im } z_0 \neq 0$  be fixed. Put  $\mathfrak{N} = \mathfrak{N}_{\bar{z}_0}$  and call  $\mathfrak{N}$  a reference subspace. With the subspace  $\mathfrak{N}$ , we associate a holomorphic operator-valued function  $Q_{z_0}(z)$  on  $\mathfrak{N}$  defined by

$$(1) \quad Q_{z_0}(z) = -iy_0 I_{\mathfrak{N}} + (z - \bar{z}_0) P_{\mathfrak{N}} U_{z_0 z} |_{\mathfrak{N}}, \quad y_0 = \text{Im } z_0.$$

Here,  $P_{\mathfrak{N}}$  is the orthogonal projection from  $\mathfrak{H}$  onto  $\mathfrak{N}$ . Using the properties of the operators  $U_{z\zeta}$ , one may check that

$$(Q_{z_0}(z))^* = Q_{z_0}(\bar{z}) \quad \text{and} \quad Q_{z_0}(z) - (Q_{z_0}(z))^* = (z - \bar{z}) P_{\mathfrak{N}} U_{z_0 z} U_{z_0 \bar{z}} |_{\mathfrak{N}}.$$

In particular,  $\text{Im } Q_{z_0}(z) > 0$  for  $\text{Im } z > 0$ .

Let  $H$  be an arbitrary self-adjoint extension of the operator  $\mathcal{H}$ . Self-adjoint extensions of  $\mathcal{H}$  in the same space  $\mathfrak{H}$  are called canonical (orthogonal). We allow  $H$  to act in a Hilbert space  $\tilde{\mathfrak{H}}$  that contains  $\mathfrak{H}$  as a proper subspace. In such a case, we assume that the self-adjoint extension  $H$  is minimal. The last condition means that

$$\text{c.l.h.}\{E(\Delta)\mathfrak{H} : \Delta \in \mathcal{B}(\mathbb{R})\} = \tilde{\mathfrak{H}},$$

where c.l.h. means closed linear hull (see [19]),  $\mathcal{B}(\mathbb{R})$  is the  $\sigma$ -algebra of Borel sets of  $\mathbb{R}$ , and  $E$  is the resolution of identity associated with  $H$ . Let  $R(z) = P(H - zI)^{-1}|_{\mathfrak{H}}$  be the (generalized) resolvent of  $H$ . Here,  $P$  is the orthogonal projection of  $\tilde{\mathfrak{H}}$  onto  $\mathfrak{H}$ . Formulas that describe those resolvents were proved by M. G. Kreĭn [17] for the case of finite defect numbers and by Sh. Saakyan [28] for the case of arbitrary defect numbers.

In order to formulate the Kreĭn–Saakyan theorem (Theorem 1 below), we need the notions of a proper and an improper  $R_{\mathfrak{N}}$ -operator function. In our definition, we follow [20, 21]. For a more modern terminology, we refer to [11]. For information about scalar  $R$ -functions, we refer to [14], and regarding operator  $R$ -functions, to [30]. Let  $V(z)$  be a function holomorphic in the upper half-plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ , whose values are contractive operators on the Hilbert space  $\mathfrak{N}$ . Since we assume that our  $q$ -s-i

operator has finite defect numbers, we consider only the case  $\dim \mathfrak{N} < \infty$ . Using the maximum principle, it is easily seen that the set  $\mathfrak{F}(V)$  of all fixed vectors of  $V(z)$ , i.e.,  $\{f : V(z)f = f\}$ , does not depend on  $z$  and is given by

$$\mathfrak{F}(V) = \mathfrak{N} \ominus \overline{(V(z) - I_{\mathfrak{N}})\mathfrak{N}}.$$

Since  $\mathfrak{F}(V) = \mathfrak{F}(V^*)$ , the subspace  $\mathfrak{F}(V)$  reduces  $V(z)$ . Denote by  $\hat{P}$  the orthogonal projection onto  $\overline{(V(z) - I_{\mathfrak{N}})\mathfrak{N}}$ . Evidently,

$$V(z) = V(z)\hat{P} + (I_{\mathfrak{N}} - \hat{P}) = \hat{P}V(z)\hat{P} + (I_{\mathfrak{N}} - \hat{P}) = \hat{V}(z) + (I_{\mathfrak{N}} - \hat{P}),$$

where  $\hat{V}(z) = \hat{P}V(z)\hat{P}$  is a contractive operator on  $\hat{\mathfrak{N}} = \mathfrak{N} \ominus \mathfrak{F}(V)$ . If  $\mathfrak{F}(V) = (0)$  (i.e., 1 does not belong to the spectrum of  $V(z)$ ), then the operator-valued function  $\tau$  given by

$$(2) \quad \tau(z) = i(I_{\mathfrak{N}} + V(z))(I_{\mathfrak{N}} - V(z))^{-1}$$

is holomorphic in  $\mathbb{C}_+$  and  $\text{Im} \langle \tau(z)f, f \rangle \geq 0$  for all  $f \in \mathfrak{N}$ . Such an operator-valued function  $\tau$  is called a *proper*  $R_{\mathfrak{N}}$ -operator-valued function. If, however,  $\mathfrak{F}(V) \neq (0)$ , then we define an *improper*  $R_{\mathfrak{N}}$ -operator-valued function  $\tau$  by the formal equality

$$(3) \quad \tau(z) = \hat{\tau}(z)\hat{P} + \infty(I_{\mathfrak{N}} - \hat{P}),$$

where  $\hat{\tau}$  is a proper  $R_{\hat{\mathfrak{N}}}$ -operator-valued function defined by (2) with  $\hat{\mathfrak{N}}$  and  $\hat{V}(z)$  instead of  $\mathfrak{N}$  and  $V(z)$ , respectively. All formulas that contain an improper operator-valued function  $\tau$  are understood in the sense that at first  $\tau(z)$  is replaced by  $\tau_n(z) = \hat{\tau}(z)\hat{P} + n(I_{\mathfrak{N}} - \hat{P})$  and then the limit is taken as  $n \rightarrow \infty$ . In particular, if  $B$  is an operator on  $\mathfrak{N}$  such that the operator  $(\hat{P}B|_{\hat{\mathfrak{N}}} + \hat{\tau}(z))^{-1}$  exists, then  $(B + \tau(z))^{-1} = \hat{P}(\hat{P}B|_{\hat{\mathfrak{N}}} + \hat{\tau}(z))^{-1}\hat{P}$ . An improper scalar-valued function assumes the value  $\infty$  identically.

**Theorem 1.** *There is a one-to-one correspondence between the set of all resolvents (canonical and generalized) of the operator  $\mathcal{H}$  and the set of all (proper and improper)  $R_{\mathfrak{N}}$ -operator-valued functions. This correspondence is established by the formula*

$$(4) \quad R(z) = \overset{\circ}{R}(z) - U_{\bar{z}_0 z} [\tau_{z_0}(z) + Q_{z_0}(z)]^{-1} P_{\mathfrak{N}} U_{z_0 z},$$

where  $\tau_{z_0}(z)$  is an arbitrary (proper or improper)  $R_{\mathfrak{N}}$ -function. Resolvents of canonical self-adjoint extensions are obtained when  $\tau$  is a constant self-adjoint operator in  $\mathfrak{N}$ .

**Lemma 1.** *Let  $z_0$  and  $z'_0$  be two distinct nonreal points and  $\mathfrak{N} = \mathfrak{N}_{\bar{z}_0}$  and  $\mathfrak{N}' = \mathfrak{N}_{\bar{z}'_0}$  be corresponding reference subspaces. Then the functions  $\tau_{z_0}(z)$  and  $\tau_{z'_0}(z)$  that correspond to the same (generalized) resolvent according to the formula (4) are related by the expression*

$$(5) \quad \tau_{z'_0}(z) = -Q_{z'_0}(z) + P_{\mathfrak{N}'} U_{z'_0 z_0} [\tau_{z_0}(z) + Q_{z_0}(z)] U_{z_0 \bar{z}'_0} |_{\mathfrak{N}'}$$

*Remark 1.* It is clear that  $\tau_{z_0}$  and  $\tau_{z'_0}$  are both proper or both improper  $R_{\mathfrak{N}}$ -operator-valued functions and  $R_{\mathfrak{N}'}$ -operator-valued functions, respectively.

*Remark 2.* Formula (5) assumes that  $\tau_{z_0}$  and  $\tau_{z'_0}$  are both proper  $R_{\mathfrak{N}}$ -operator-valued functions and  $R_{\mathfrak{N}'}$ -operator-valued functions. If they are both improper functions, then a formula similar to (5) is valid. We need to replace  $\tau_{z_0}$  and  $\tau_{z'_0}$  by  $\hat{\tau}_{z_0}$  and  $\hat{\tau}_{z'_0}$ ,  $\mathfrak{N}$  and  $\mathfrak{N}'$  by  $\hat{\mathfrak{N}}$  and  $\hat{\mathfrak{N}}'$ , and  $Q_{z_0}$  and  $Q_{z'_0}$  by  $\hat{P}\hat{Q}_{z_0}|_{\hat{\mathfrak{N}}}$  and  $\hat{P}'\hat{Q}_{z'_0}|_{\hat{\mathfrak{N}'}}$ , where  $\hat{P}$  and  $\hat{P}'$  are orthogonal projections onto  $\hat{\mathfrak{N}}$  and  $\hat{\mathfrak{N}}'$ , respectively.

*Proof of Lemma 1.* Writing down the formula (4) for the same resolvent  $R(z)$  and two reference subspaces  $\mathfrak{N}$  and  $\mathfrak{N}'$ , one obtains

$$U_{\bar{z}_0 z} [\tau_{z_0}(z) + Q_{z_0}(z)]^{-1} P_{\mathfrak{N}} U_{z_0 z} = U_{\bar{z}'_0 z} [\tau_{z'_0}(z) + Q_{z'_0}(z)]^{-1} P_{\mathfrak{N}'} U_{z'_0 z}.$$

Using properties of the operators  $U_{z\zeta}$ , we get

$$[\tau_{z_0}(z) + Q_{z_0}(z)]^{-1} P_{\mathfrak{N}} = U_{z_0 \bar{z}_0} [\tau_{z'_0}(z) + Q_{z'_0}(z)]^{-1} P_{\mathfrak{N}'} U_{z'_0 z_0}.$$

The operators on both sides of the last expression annihilate  $\mathfrak{M}_{z_0}$ , and their ranges are in  $\mathfrak{N}$ . Thus, they may be considered as operators on  $\mathfrak{N}$ . Therefore,

$$[\tau_{z_0}(z) + Q_{z_0}(z)]^{-1} = U_{z_0 \bar{z}_0} [\tau_{z'_0}(z) + Q_{z'_0}(z)]^{-1} P_{\mathfrak{N}'} U_{z'_0 z_0} |_{\mathfrak{N}},$$

i.e.,

$$[\tau_{z'_0}(z) + Q_{z'_0}(z)] U_{z_0 \bar{z}_0} = P_{\mathfrak{N}'} U_{z'_0 z_0} |_{\mathfrak{N}} [\tau_{z_0}(z) + Q_{z_0}(z)].$$

Consequently,

$$\tau_{z'_0}(z) + Q_{z'_0}(z) = P_{\mathfrak{N}'} U_{z'_0 z_0} [\tau_{z_0}(z) + Q_{z_0}(z)] U_{z_0 \bar{z}_0} |_{\mathfrak{N}'},$$

proving (5).  $\square$

**Theorem 2.** *Let  $\mathcal{H}$  be a  $q$ -s-i densely defined closed symmetric operator in a Hilbert space  $\mathfrak{H}$ , and let  $H$  be its self-adjoint extension in a possibly larger Hilbert space  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ . The extension  $H$  is  $q$ -s-i if and only if its (generalized) resolvent  $R(z) = P(H - zI)^{-1}|_{\mathfrak{H}}$ , where  $P$  as before is the orthogonal projection of  $\tilde{\mathfrak{H}}$  onto  $\mathfrak{H}$ , satisfies the relation*

$$(6) \quad U_q R(z) = \frac{1}{q} R(z/q) U_q.$$

*Proof.* Since for the extension  $H$  in  $\mathfrak{H}$  the statement is clear, we will prove it only for the extension in  $\tilde{\mathfrak{H}}$ .

Suppose at first that  $H$  is a  $q$ -s-i extension in the sense of Definition 2. Since  $\mathfrak{H}$  reduces  $\tilde{U}_q$  and  $\tilde{U}_q|_{\mathfrak{H}} = U_q$ , we have  $U_q P = P \tilde{U}_q$ . Therefore,

$$\begin{aligned} U_q R(z) &= U_q P(H - zI)^{-1}|_{\mathfrak{H}} = P \tilde{U}_q (H - zI)^{-1}|_{\mathfrak{H}} \\ &= \frac{1}{q} P(H - z/qI)^{-1} \tilde{U}_q|_{\mathfrak{H}} \\ &= \frac{1}{q} P(H - z/qI)^{-1}|_{\mathfrak{H}} U_q = \frac{1}{q} R(z/q) U_q. \end{aligned}$$

Suppose now that the generalized resolvent  $R(z)$  of the operator  $H$  satisfies (6), and denote by  $F(\lambda)$  the spectral function of  $H$ . Recall that  $F(\lambda) = P E(\lambda)|_{\mathfrak{H}}$ , where  $E(\lambda)$  is the orthogonal resolution of identity of  $H$  in the Hilbert space  $\tilde{\mathfrak{H}}$ . Without loss of generality, we may assume that  $H$  is a minimal self-adjoint extension of  $\mathcal{H}$  in  $\tilde{\mathfrak{H}}$ . Then

$$R(z) = \int_{-\infty}^{\infty} \frac{dF(\lambda)}{\lambda - z}$$

(convergence of the integral is understood in the sense of the weak operator topology). From the last expression, using the Stieltjes inversion formula [3, Section 69], one obtains that (6) is equivalent to the relation

$$U_q F(\delta) U_q^* = F(\delta/q)$$

for an arbitrary Borel set  $\delta \subset \mathbb{R}$ . Now, from a well-known theorem of M. A. Najmark (see, for example, [1, Section 110]) and the minimality of  $H$ , it follows that orthogonal resolutions of identities that correspond to  $F(\lambda)$  and  $F(\lambda/\delta)$  are unitarily equivalent, i.e., in a larger Hilbert space  $\tilde{\mathfrak{H}}$ , there exists a unitary operator  $\tilde{U}_q$ ,  $\tilde{U}_q|_{\mathfrak{H}} = U_q$ , such that  $\tilde{U}_q E(\delta) \tilde{U}_q^* = E(\delta/q)$ . Therefore, for the canonical resolvent  $\tilde{R}(z)$  of the operator  $H$ , one has  $\tilde{U}_q \tilde{R}(z) \tilde{U}_q^* = \frac{1}{q} \tilde{R}(z/q)$ , from which the statement follows.  $\square$

Suppose that  $\overset{\circ}{H}$  is a  $q$ -s-i self-adjoint extension of a  $q$ -s-i symmetric operator  $\mathcal{H}$ . Then one has

$$U_q \mathfrak{M}_z = \mathfrak{M}_{z/q}, \quad U_q \mathfrak{N}_z = \mathfrak{N}_{z/q}, \quad U_q U_{z\zeta} = U_{z/q\zeta/q} U_q.$$

For the operator  $Q_{z_0}(z)$  defined by (1), one has

$$(7) \quad U_q Q_{z_0}(z) = q Q_{z_0/q}(z/q) U_q.$$

Indeed,

$$\begin{aligned} U_q Q_{z_0}(z) &= -iy_0 U_q I_{\mathfrak{N}} + (z - \bar{z}_0) U_q P_{\mathfrak{N}} U_{z_0 z} |_{\mathfrak{N}} \\ &= -iy_0 I_{\mathfrak{N}_{\bar{z}_0/q}} U_q + (z - \bar{z}_0) P_{\mathfrak{N}_{\bar{z}_0/q}} U_{z_0/q z/q} |_{\mathfrak{N}_{\bar{z}_0/q}} U \\ &= q Q_{z_0/q}(z/q) U_q. \end{aligned}$$

Now (4) and Theorem 2 gives

$$\begin{aligned} U_q R(z) &= \frac{1}{q} \overset{\circ}{R}(z/q) U_q \\ &\quad - \frac{1}{q} U_{\bar{z}_0/q z/q} [\hat{\tau}_{z_0/q}(z/q) + Q_{z_0/q}(z/q)]^{-1} P_{\mathfrak{N}_{\bar{z}_0/q}} U_{z_0/q z/q} U_q \\ &= \frac{1}{q} \hat{R}(z/q) U_q, \end{aligned}$$

where the  $R_{\mathfrak{N}_{\bar{z}_0/q}}$ -function  $\hat{\tau}_{z_0/q}(z/q)$  is defined by the formula

$$(8) \quad \hat{\tau}_{z_0/q}(z/q) = \frac{1}{q} U_q \tau_{z_0}(z) U_q^*,$$

and  $\hat{R}(z/q)$  is the generalized resolvent of an another extension of  $\mathcal{H}$  and the reference subspace  $\mathfrak{N}_{\bar{z}_0/q}$ .

The resolvent  $\hat{R}(z/q)$  coincides with  $R(z/q)$  if and only if the parameter  $\hat{\tau}_{z_0/q}(z/q)$  coincides with  $\tau_{z_0/q}(z/q)$  given by the right-hand side of (5) with  $z'_0 = z_0/q$  and  $z/q$  instead of  $z$ . Thus, we have obtained the following result.

**Theorem 3.** *Let  $\mathcal{H}$  be a densely defined closed  $q$ -s-i symmetric operator in a Hilbert space  $\mathfrak{H}$ . Suppose that the operator  $\mathcal{H}$  admits a  $q$ -scale-invariant self-adjoint extension  $\overset{\circ}{H}$  in the space  $\mathfrak{H}$ . A (generalized) resolvent  $R(z)$  is the resolvent of a  $q$ -s-i self-adjoint extension of  $\mathcal{H}$  if and only if the parameterizing operator function  $\tau_{z_0}(z)$  satisfies the relation*

$$(9) \quad \begin{aligned} &\frac{1}{q} U_q \tau_{z_0}(z) U_q^* \\ &= -Q_{z_0/q}(z/q) + P_{\mathfrak{N}_{\bar{z}_0/q}} U_{z_0/q z_0} [\tau_{z_0}(z/q) + Q_{z_0}(z/q)] U_{\bar{z}_0/q \bar{z}_0} |_{\mathfrak{N}_{\bar{z}_0/q}}. \end{aligned}$$

### 3. DIRAC-TYPE SCALE-INVARIANT OPERATORS

In this section, we consider two examples of Dirac-type symmetric  $q$ -s-i operators, and in both cases, we write down the corresponding realization of (9). For completeness, in each case, we also provide formulas for canonical resolvents of self-adjoint extensions.

#### 3.1. Symmetric operators generated by Hermitian matrix-valued functions.

In this subsection, we recall a general construction of a Dirac-type symmetric operator associated with a  $2 \times 2$  Hermitian matrix-valued function. This construction can be found in many books (see, for example, [31]). Deficiency indices of operators and linear relations associated with first-order linear matrix differential equations, in particular, deficiency indices of the Dirac type operators, were studied in [22] (see also extensive list

of references there). Let  $\mathcal{H}(t)$ ,  $t > 0$ , be a  $2 \times 2$  Hermitian ( $\mathcal{H}^*(t) = \mathcal{H}(t)$ ) matrix-valued function (m.-f.) whose entries are measurable functions bounded on each compact subset of  $(0, \infty)$ . Suppose also that

$$(10) \quad \lim_{t \rightarrow \infty} \mathcal{H}(t) = 0$$

componentwise or in norm. Thus,  $\mathcal{H}(t)$  is bounded on  $[b, \infty)$  for any  $b > 0$ .

Denote by  $\mathcal{D}$  the differential expression defined by

$$(11) \quad \mathcal{D}x = J \frac{dx}{dt} + \mathcal{H}(t)x, \quad t > 0,$$

where  $x \in \mathbb{C}^2$  and  $J$  is the  $2 \times 2$  symplectic matrix of the form

$$(12) \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we define an operator  $D_0$  in a Hilbert space  $\mathfrak{H} = L^2(\mathbb{C}^2, \mathbb{R}_+)$  as follows. Its domain  $\mathfrak{D}(D_0)$  consists of  $C^\infty$  vector-valued functions with values in  $\mathbb{C}^2$  which have compact support within  $\mathbb{R}_+$ . On such a function, the operator  $D_0$  is defined by

$$(13) \quad D_0x = \mathcal{D}x, \quad x \in \mathfrak{D}(D_0).$$

Therefore,  $D_0$  is densely defined. It is easily seen that

$$(D_0x, y) = (x, D_0y), \quad x, y \in \mathfrak{D}(D_0)$$

where  $(\cdot, \cdot)$  denotes the inner product in  $\mathfrak{H}$ . Therefore, the operator  $D$ , which is the closure of  $D_0$ , is symmetric and possibly self-adjoint.

The domain  $\mathfrak{D}(D^*)$  of the operator  $D^* = D_0^*$  consists of absolutely continuous functions  $x \in \mathfrak{H}$  for which  $\mathcal{D}x \in \mathfrak{H}$ . Consequently,  $dx/dt \in L^2(\mathbb{C}^2, [b, \infty))$  for any  $b > 0$ ,  $x \in \mathfrak{D}(D^*)$ , from which it follows that for such  $x$  one has  $\lim_{t \rightarrow \infty} x(t) = 0$ . Now integration by parts yields the following description of the operator  $D$ :

The domain  $\mathfrak{D}(D)$  consists of functions  $x \in \mathfrak{H}$  such that

1.  $x$  is absolutely continuous;
2.  $\mathcal{D}x \in \mathfrak{H}$ ;
3.  $\lim_{t \rightarrow 0^+} \langle Jx(t), y(t) \rangle = 0$  for any  $y \in \mathfrak{D}(D^*)$ ,

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{C}^2$ . The operator  $D$  is defined by

$$(14) \quad Dx = \mathcal{D}x, \quad x \in \mathfrak{D}(D).$$

Suppose now that the matrix-valued function  $\mathcal{H}$  satisfies the condition

$$(15) \quad \mathcal{H}(qt) = \mathcal{H}(t)/q$$

where  $q > 1$  is a fixed number. Clearly,  $\lim_{t \rightarrow \infty} \mathcal{H}(t) = 0$ .

Matrix-valued functions  $\mathcal{H}$  which satisfy (15) are closely related to periodic matrix-valued functions. Indeed, consider an  $2 \times 2$  m.-f.  $\tilde{\mathcal{H}}(\tau)$ ,  $\tau \in \mathbb{R}$ , defined by

$$(16) \quad \tilde{\mathcal{H}}(\tau) = e^\tau \mathcal{H}(e^\tau).$$

Then  $\tilde{\mathcal{H}}(\tau)$  is a Hermitian periodic m.-f.,  $\tilde{\mathcal{H}}(\tau+T) = \tilde{\mathcal{H}}(\tau)$ , where  $T = \ln q$ . Conversely, any periodic matrix-valued function  $\tilde{\mathcal{H}}(\tau)$ ,  $\tau \in \mathbb{R}$ , by means of (16), defines a matrix-valued function  $\mathcal{H}(t)$ ,  $t \in \mathbb{R}_+$ , which satisfies (15).

Denote by  $U_q$  a unitary operator on  $\mathfrak{H}$  defined as

$$(17) \quad (U_q x)(t) = \frac{1}{\sqrt{q}} x(t/q).$$

It is easily seen that

$$(18) \quad (U_q^* x)(t) = \sqrt{q} x(qt).$$

It is also evident that  $U_q \mathfrak{D}(D_0) = \mathfrak{D}(D_0)$  and  $U_q D_0 x = q D_0 U_q x$ ,  $x \in \mathfrak{D}(D_0)$ , that is, the operator  $D_0$  is  $q$ -s-i (see Definition 1 above). Consequently, the operators  $D^*$  and  $D$  are also  $(q, U_q)$ -scale-invariant.

Let  $\mathcal{H}$  be a densely defined closed symmetric operator on a Hilbert space  $\mathfrak{H}$ . Denote by  $\mathfrak{D}(\mathcal{H})$  the domain of  $\mathcal{H}$  and suppose that the index of defect of  $\mathcal{H}$  is  $(1, 1)$ . Also denote by  $\varphi(z)$ ,  $\text{Im } z \neq 0$ , the normalized ( $\|\varphi(z)\| = 1$ ) defect vectors of  $\mathcal{H}$  ( $\mathcal{H}^* \varphi(z) = z \varphi(z)$ ). Recall that according to von Neumann formulas (see, for example [3]), the domain  $\mathfrak{D}(H)$  of an arbitrary self-adjoint extension  $H_\rho$  of  $\mathcal{H}$  in  $\mathfrak{H}$  is described by

$$(19) \quad \mathfrak{D}(H_\rho) = \{f = f_0 + \xi(\varphi(i) + \rho\varphi(-i)) : f_0 \in \mathfrak{D}(\mathcal{H}), \xi \in \mathbb{C}, |\rho| = 1\},$$

and for  $f \in \mathfrak{D}(H_\rho)$ ,

$$(20) \quad H_\rho f = \mathcal{H}f_0 + i\xi(\varphi(i) - \rho\varphi(-i)).$$

Later on we use the following theorem (see [8, Theorem 1, Remark 1, Remark 2]).

**Theorem 4.** *Let  $\mathcal{H}$  be a symmetric  $q$ -s-i operator in a Hilbert space  $\mathfrak{H}$  with inner product  $(\cdot, \cdot)$ . Suppose that the index of defect of  $\mathcal{H}$  is  $(1, 1)$ . Using the notations above, define*

$$(21) \quad \begin{aligned} \mathcal{A} &= (q+1)(\varphi(-i/q), \varphi(-i)), \\ \mathcal{B} &= (q-1)(\varphi(i/q), \varphi(-i)), \\ \mathcal{C} &= (q-1)(\varphi(-i/q), \varphi(i)), \\ \mathcal{D} &= (q+1)(\varphi(i/q), \varphi(i)), \end{aligned}$$

and put

$$\Gamma(\rho) = \frac{\mathcal{A}\rho + \mathcal{B}}{\mathcal{C}\rho + \mathcal{D}}.$$

Then the transformation  $\rho \rightarrow \Gamma(\rho)$  maps the unit circle onto itself and the interior of the unit disk onto itself. Each fixed point  $\rho$  of the transformation  $\Gamma$  located on the unit circle defines a  $q$ -s-i self-adjoint extension of the operator  $\mathcal{H}$  in  $\mathfrak{H}$  according to (19) and (20).

Consequently, a  $q$ -s-i symmetric operator  $\mathcal{H}$  with index of defect  $(1, 1)$  may have either no self-adjoint  $q$ -s-i extensions in  $\mathfrak{H}$ , or one  $q$ -s-i extension, or two  $q$ -s-i extensions, or each extension of  $\mathcal{H}$  is  $q$ -s-i self-adjoint. In the last case, the transformation  $\Gamma$  is identity,  $\Gamma\rho = \rho$  for all  $\rho$  such that  $|\rho| \leq 1$  (see Example 1 below). For examples of  $q$ -s-i symmetric operators which do not admit any  $q$ -s-i extension in  $\mathfrak{H}$  or admit only one extension, we refer to [8].

**3.2. Example 1.** We assume that the matrix-valued function  $\mathcal{H}(t)$  is of the form

$$(22) \quad \mathcal{H}(t) = \begin{pmatrix} \frac{\gamma}{t} & i \frac{\varphi(\ln(t))}{t} \\ -i \frac{\varphi(\ln(t))}{t} & \frac{\gamma}{t} \end{pmatrix},$$

where  $\gamma \in \mathbb{R}$  and  $\varphi$  is a periodic and bounded real-valued measurable function with period  $T$ . Clearly, (15) is satisfied with  $q = e^T$ .

**Theorem 5.** *Let  $D$  be the operator in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  defined by (14), where the matrix-valued function  $\mathcal{H}(t)$  is given by (22). Then, for any  $\gamma \in \mathbb{R}$ , the operator  $D$  is symmetric with index of defect  $(1, 1)$ .*

*Proof.* By direct calculations, one may check that linearly independent solutions of the differential equation

$$\mathcal{D}x = zx, \quad \text{Im } z \neq 0$$

for  $\mathcal{H}(t)$  given by (22) are

$$(23) \quad x_z^+(t) = e^{izt} e^{-i\omega(t) - i\gamma \ln t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and

$$(24) \quad x_z^-(t) = e^{-izt} e^{-i\omega(t) + i\gamma \ln t} \begin{pmatrix} 1 \\ i \end{pmatrix},$$

where  $\omega(t) = \int_1^t (\varphi(\ln s)/s) ds$ . From (23) and (24), it follows that for  $\text{Im } z > 0$ , only the vector  $x_z^+$  belongs to  $L^2(\mathbb{R}_+)$ , while for  $\text{Im } z < 0$ , only the vector  $x_z^-$  is in  $L^2(\mathbb{R}_+)$ . This proves the theorem and shows that  $x_z^+$  and  $x_z^-$  are defect vectors of the operator  $D$ . Moreover,  $\|x_z^+\| = \|x_z^-\| = |\text{Im } z|^{-1/2}$ .  $\square$

**Proposition 1.** *The operator  $D$  is prime.*

*Proof.* The statement that the operator  $D$  is prime is equivalent to the statement that c.l.h.  $\{\mathfrak{N}_z : \text{Im } z \neq 0\} = \mathfrak{H}$ . Suppose that a vector  $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathfrak{H}$  is orthogonal to  $x_z^+$  for all  $z \in \mathbb{C}_+$  and  $x_z^-$  for all  $z \in \mathbb{C}_-$ , that is ,

$$\int_0^\infty e^{izt} e^{-i\omega(t) - i\gamma \ln t} (\overline{g_1(t)} - i\overline{g_2(t)}) dt = 0 \quad \text{for all } z \in \mathbb{C}_+$$

and

$$\int_0^\infty e^{-izt} e^{-i\omega(t) + i\gamma \ln t} (\overline{g_1(t)} + i\overline{g_2(t)}) dt = 0 \quad \text{for all } z \in \mathbb{C}_-.$$

The integral on the left-hand side of the first expression is a function which is holomorphic in the upper half-plane  $\mathbb{C}_+$  and which is the Fourier transform of the function  $e^{-i\omega(t) - i\gamma \ln t} (g_1(t) - ig_2(t)) \in L^2(\mathbb{R}_+)$ . Therefore, according to the Paley–Wiener theorem [13], the left-hand side of the first expression is an element of the Hardy space  $H_+^2$  in the upper half-plane. Since that function is equal to zero identically, one concludes that  $g_1(t) + ig_2(t) = 0$  a.e. on  $\mathbb{R}_+$ . Similar arguments applied to the second integral, with replacing upper half-plane by the lower half-plane  $\mathbb{C}_-$ , yield  $g_1(t) - ig_2(t) = 0$  a.e. on  $\mathbb{R}_+$ . Thus  $g_1(t) = 0$  a.e. and  $g_2(t) = 0$  a.e., that is,  $g(t) = 0$  a.e. This concludes the proof.  $\square$

We use now Theorem 4. Calculating the coefficients  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  according to (21), one obtains  $\mathcal{B} = \mathcal{C} = 0$  and  $\mathcal{A} = \mathcal{D}$ . Therefore, the transformation  $\Gamma$  acts as identity, and we obtain the following result.

**Theorem 6.** *Let  $D$  be the operator in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  defined by (14), where the m.-f.  $\mathcal{H}(t)$  is given by (22). Then all self-adjoint extensions of  $D$  in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  are  $q$ - $s$ - $i$ .*

We use now von Neumann formulas (19) and (20) and present the expression for the resolvent  $R_\rho(z) = (H_\rho - zI)^{-1}$ , where  $H_\rho$  is a self-adjoint extension of  $D$  which corresponds to a value  $\rho$  ( $|\rho| = 1$ ) in those formulas. The cases  $\text{Im } z > 0$  and  $\text{Im } z < 0$  are considered separately. In the formulas below,  $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^2(\mathbb{C}^2, \mathbb{R}_+)$  and  $f = R_\rho(z)g$ .

In the case  $\text{Im } z > 0$ , we have

$$\begin{aligned} f(t) &= \frac{\xi}{\rho} x_z^+ - x_z^+ \frac{1}{2i} \int_0^t \exp[i\omega(s) + i\gamma \ln s - izs] (g_1 + ig_2) ds \\ &\quad + x_z^- \frac{1}{2i} \int_t^\infty \exp[i\omega(s) - i\gamma \ln s + izs] (-g_1 + ig_2) ds, \end{aligned}$$

where

$$\xi = \frac{1}{2i} \int_0^\infty \exp[i\omega(s) - i\gamma \ln s + izs] (-g_1 + ig_2) ds$$

and  $\omega(s) = \int_1^s \psi(v)dv$ .

In the case  $\text{Im } z < 0$ , we have

$$\begin{aligned} f(t) &= \rho \xi x_z^- - x_z^- \frac{1}{2i} \int_0^t \exp [i\omega(s) - i\gamma \ln s + izs](-g_1 + ig_2)ds \\ &\quad + x_z^+ \frac{1}{2i} \int_t^\infty \exp [i\omega(s) + i\gamma \ln s - izs](g_1 + ig_2)ds, \end{aligned}$$

where

$$\xi = \frac{1}{2i} \int_0^\infty \exp [i\omega(s) + i\gamma \ln s - izs](g_1 + ig_2)ds.$$

Now we select the self-adjoint extension of  $D$  which corresponds to the value  $\rho = 1$  in (19) and (20) and denote it by  $\overset{\circ}{H}$ . According to Theorem 6, the operator  $\overset{\circ}{H}$  is  $q$ -s-i. Denote the corresponding resolvent by  $\overset{\circ}{R}(z)$ . In particular, for  $\rho = 1$ , we have the formulas

$$(25) \quad \overset{\circ}{R}(z)x_\zeta^+ = \begin{cases} \frac{x_z^+ - x_\zeta^+}{z - \zeta}, & \text{Im } z > 0, \\ -\frac{x_z^- + x_\zeta^+}{z - \zeta}, & \text{Im } z < 0 \end{cases}$$

and

$$(26) \quad \overset{\circ}{R}(z)x_\zeta^- = \begin{cases} -\frac{x_z^+ + x_\zeta^-}{z - \zeta}, & \text{Im } z > 0, \\ \frac{x_z^- - x_\zeta^-}{z - \zeta}, & \text{Im } z < 0. \end{cases}$$

Since c.l.h.  $\{\mathfrak{N}_\zeta : \text{Im } \zeta \neq 0\} = \mathfrak{H}$  and  $R(\bar{z}) = R(z)^*$ , it is sufficient to evaluate  $R(z)$  only on defect vectors and assume that  $\text{Im } z > 0$ . Applying (4), one obtains the following statement.

**Theorem 7.** *All resolvents of the operator  $D$  are given by the formulas*

$$R(z)x_\zeta^+ = \overset{\circ}{R}(z)x_\zeta^+ = \frac{x_z^+ - x_\zeta^+}{z - \zeta}$$

and

$$R(z)x_\zeta^- = \frac{x_z^+ v(z) - x_\zeta^-}{z - \zeta},$$

where

$$v(z) = \frac{i - \tau(z)}{i + \tau(z)}$$

and  $\tau(z)$  is an arbitrary scalar-valued  $R$ -function.

We now use Theorem 3 and obtain the following result.

**Theorem 8.** *In Theorem 7, the resolvent  $R(z)$  corresponds to a  $q$ -s-i extension if and only if the parameterizing function  $\tau$  satisfies the relation*

$$(27) \quad \tau(z/q) = \tau(z), \quad \text{Im } z > 0.$$

Since any real constant satisfies (27), we again obtained the statement of Theorem 6. Equation (27) also has nonconstant solutions. For example, any function of the form

$$\tau(z) = F(\log z),$$

where  $F(z)$  is a  $T$ -periodic scalar-valued  $R$ -function ( $F(z+T) = F(z)$ ), satisfies (27). Therefore, the operator  $D$  has infinitely many scale-invariant extensions in larger Hilbert spaces  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ . Below we show how to construct a  $T$ -periodic  $R$ -function  $F$ . Let  $p(\lambda)$ ,  $\lambda \in \mathbb{R}$ , be a nonnegative measurable and bounded  $T$ -periodic function. Put

$$d\sigma(\lambda) = \frac{p(\lambda)}{1 + \lambda^2} d\lambda.$$

Without loss of generality, we may assume that  $\int_{\mathbb{R}} d\sigma(\lambda) = 1$ . Then the function

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda)$$

is an  $R$ -function which is  $T$ -periodic.

**3.3. Example 2.** Now we assume that the matrix-valued function  $\mathcal{H}(t)$  is given by

$$(28) \quad \mathcal{H}(t) = \begin{pmatrix} \frac{\gamma}{t} & i \frac{\varphi(\ln(t))}{t} \\ -i \frac{\varphi(\ln(t))}{t} & -\frac{\gamma}{t} \end{pmatrix},$$

where  $\gamma$  and  $\varphi$  satisfy the same condition as before.

**Theorem 9.** *Let  $D$  be the operator in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  defined by (14), where the matrix-valued function  $\mathcal{H}(t)$  is given by (28). Then the operator  $D$  is symmetric with index of defect  $(1, 1)$  for  $|\gamma| < 1/2$  and self-adjoint for  $|\gamma| \geq 1/2$ .*

*Proof.* Denote by  $W(t)$  a m.-f. defined by

$$(29) \quad W(t) = \begin{pmatrix} \cosh(\gamma \ln t) & -\sinh(\gamma \ln t) \\ -\sinh(\gamma \ln t) & \cosh(\gamma \ln t) \end{pmatrix}.$$

One may check that  $e^{-i \int_1^t \psi(s) ds} W(t)$  is the fundamental matrix for the differential equation

$$J \frac{dy}{dt} + \mathcal{H}(t)y = 0.$$

Since all entries of  $W$  are in  $L^2([0, b])$  for any  $b > 0$  if and only if  $|\gamma| < 1/2$ , the statement follows (see, for example, [31]).  $\square$

It is also possible to give explicit formulas for the solution of the equation

$$J \frac{dy}{dt} + \mathcal{H}(t)y = zy, \quad \text{Im } z \neq 0.$$

Those solutions are given by

$$y_z^+(t) = W(t)\kappa_z^+(t)$$

and

$$y_z^-(t) = W(t)\kappa_z^-(t),$$

where

$$\kappa_z^+(t) = \begin{pmatrix} t^{\gamma+1/2} H_{\gamma+1/2}^{(1)}(zt) + t^{-\gamma+1/2} H_{\gamma-1/2}^{(1)}(zt) \\ t^{\gamma+1/2} H_{\gamma+1/2}^{(1)}(zt) - t^{-\gamma+1/2} H_{\gamma-1/2}^{(1)}(zt) \end{pmatrix}$$

and

$$\kappa_z^-(t) = \begin{pmatrix} t^{\gamma+1/2} H_{\gamma+1/2}^{(2)}(zt) + t^{-\gamma+1/2} H_{\gamma-1/2}^{(2)}(zt) \\ t^{\gamma+1/2} H_{\gamma+1/2}^{(2)}(zt) - t^{-\gamma+1/2} H_{\gamma-1/2}^{(2)}(zt) \end{pmatrix},$$

and  $H_\nu^{(1)}$  and  $H_\nu^{(2)}$  are Hankel functions of order  $\nu$  of first and second kind respectively. Explicitly,

$$y_z^+ = e^{-i\omega(t)} t^{1/2} \begin{pmatrix} H_{\gamma+1/2}^{(1)}(zt) + H_{\gamma-1/2}^{(1)}(zt) \\ H_{\gamma+1/2}^{(1)}(zt) - H_{\gamma-1/2}^{(1)}(zt) \end{pmatrix}$$

and

$$y_z^+ = e^{-i\omega(t)} t^{1/2} \begin{pmatrix} H_{\gamma+1/2}^{(2)}(zt) + H_{\gamma-1/2}^{(2)}(zt) \\ H_{\gamma+1/2}^{(2)}(zt) - H_{\gamma-1/2}^{(2)}(zt) \end{pmatrix},$$

where, as before,  $\omega(t) = \int_1^t \psi(s) ds$ . Taking into account the asymptotic behavior of Hankel functions (see, for example [1]), one obtains that for  $\text{Im } z > 0$ , only the vector  $y_z^+$  belongs to the space  $L^2(\mathbb{C}^2, \mathbb{R}_+)$ , while for  $\text{Im } z < 0$ , only the vector  $y_z^-$  is in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$ , that is,  $y_z^+$  and  $y_z^-$  are defect vectors of the operator  $D$ .

Note that  $W^*(t) = W(t)$ ,  $W(t)W(s) = W(ts)$ , and the vector-valued functions  $\kappa_z^\pm$  are solutions of the differential equation

$$J \frac{d\kappa}{dt} = zW(t^2)\kappa, \quad t > 0.$$

For the solutions  $h_z(t)$  of the last equation, one has

$$\begin{aligned} (\lambda - \bar{\mu}) \int_a^b \langle W(t)h_\lambda(t), W(t)h_\mu(t) \rangle dt \\ = \langle Jh_\lambda(b), h_\mu(b) \rangle - \langle Jh_\lambda(a), h_\mu(a) \rangle. \end{aligned}$$

Here,  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{C}^2$ . From the previous expressions, it follows that for the  $L^2$ -norm of the defect vectors  $y_z^\pm$ , one has

$$(30) \quad \|y_z^\pm\|^2 = -\frac{1}{2i \text{Im } z} \lim_{t \downarrow 0} \langle J\kappa_z^\pm(t), \kappa_z^\pm(t) \rangle$$

because  $y_z^\pm(t) \rightarrow 0$  as  $t \rightarrow \infty$ , and the same is true for  $\kappa_z^\pm(t)$ .

Taking into account the asymptotic behavior of the functions  $H_\nu^{(i)}(\xi)$  ( $i = 1, 2$ ) as  $\xi \rightarrow 0$ , one obtains

$$(31) \quad \|y_z^+\|^2 = \frac{4 \cos[\gamma(\pi - 2\alpha)]}{\pi|z| \text{Im } z \cos \pi\gamma}, \quad \text{Im } z > 0, \quad 0 < \alpha < \pi,$$

where  $\alpha = \arg z$ . Similarly,

$$(32) \quad \|y_z^-\|^2 = -\frac{4 \cos[\gamma(\pi + 2\alpha)]}{\pi|z| \text{Im } z \cos \pi\gamma}, \quad \text{Im } z < 0, \quad -\pi < \alpha < 0.$$

In particular,  $\|y_z^+\| = \|y_{\bar{z}}^-\|$ .

*Remark 3.* We consider the branch of the function  $\ln z$  with the cut along the negative real semi-axis.

It is possible to prove that the operator  $D$  is prime, in a way similar to that used in the previous subsection. This time, it is necessary to use properties of Hankel transforms instead of Fourier transforms. Regarding properties of Hankel transforms, we refer to [2, Chapter 9].

Evaluating the coefficients  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  according to (21) and using the properties of Hankel functions, one obtains the following result.

**Theorem 10.** *Let  $D$  be the operator in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  defined by (14), where the matrix-valued function  $\mathcal{H}(t)$  is given by (28). Then the operator  $D$  has two  $q$ - $s$ - $i$  self-adjoint extensions in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$ . They correspond to the values of the parameter  $\rho$  equal to  $\rho_1 = e^{i\pi(\gamma-1/2)}$  and  $\rho_2 = -\rho_1$ .*

Indeed, the linear fractional transformation  $\rho \rightarrow \Gamma(\rho)$  (see Theorem 4 above) has two fixed points  $\rho_1 = e^{i\pi(\gamma-1/2)}$  and  $\rho_2 = -\rho_1$  on the unit circle.

As before, using von Neumann formulas (19) and (20), it is possible to give explicit expression for the resolvent of an arbitrary orthogonal self-adjoint extension of the operator  $D$ . We consider only a  $q$ -s-i extension which corresponds to the value  $\rho = \rho_1 = e^{i\pi(\gamma-1/2)}$ . We denote it by  $\overset{\circ}{H}$ , and the corresponding resolvent is denoted by  $\overset{\circ}{R}(z)$ . For  $\overset{\circ}{R}(z)$ , one has

$$(33) \quad \overset{\circ}{R}(z)y_{\zeta}^+ = \begin{cases} \frac{1}{z-\zeta} \left( \frac{z^{1/2-\gamma}}{\zeta^{1/2-\gamma}} y_z^+ - y_{\zeta}^+ \right), & \text{Im } z > 0, \\ -\frac{1}{z-\zeta} \left( \frac{z^{1/2-\gamma}}{\zeta^{1/2-\gamma}} y_z^- + y_{\zeta}^+ \right), & \text{Im } z < 0 \end{cases}$$

and

$$(34) \quad \overset{\circ}{R}(z)y_{\zeta}^- = \begin{cases} -\frac{1}{z-\zeta} \left( \frac{z^{1/2-\gamma}}{\zeta^{1/2-\gamma}} y_z^+ + y_{\zeta}^- \right), & \text{Im } z > 0, \\ \frac{1}{z-\zeta} \left( \frac{z^{1/2-\gamma}}{\zeta^{1/2-\gamma}} y_z^- - y_{\zeta}^- \right), & \text{Im } z < 0. \end{cases}$$

**Theorem 11.** *Let  $D$  be the operator in  $L^2(\mathbb{C}^2, \mathbb{R}_+)$  defined by (14), where the matrix-valued function  $\mathcal{H}(t)$  is given by (28). Then, (4) gives the resolvent of a  $q$ -s-i extension if and only if the function  $\tau$  satisfies the relation*

$$(35) \quad \tau(z/q) = q^{2\gamma} \tau(z), \quad \text{Im } z \neq 0.$$

The proof of Theorem 11 is obtained by direct calculations using (9), (33), and (34).

From (35), it follows that the only constant  $R$ -functions that satisfy that expression are the functions  $\tau(z) \equiv 0$  and  $\tau(z) \equiv \infty$ . Therefore, we obtained the result from Theorem 10: The operator  $D$  has only two  $q$ -s-i extensions in  $\mathfrak{H}$ .

Equation (35) has also nonconstant solutions. For example, any function of the form

$$\tau(z) = Ci \frac{e^{i\pi\gamma}}{z^{2\gamma}},$$

where  $C$  is an arbitrary positive real constant, is an  $R$ -function and satisfies (35). Therefore, the operator  $D$  has infinitely many  $q$ -s-i extensions in larger Hilbert spaces  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ .

*Acknowledgments.* The authors would like to thank a referee for giving extremely useful suggestions.

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Received 11/08/2017; Revised 03/11/2017