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LIE DERIVATIONS ON THE ALGEBRAS OF LOCALLY MEASURABLE OPERATORS

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Abstract. We prove that every Lie derivation on a solid $*$-subalgebra in an algebra of locally measurable operators is equal to a sum of an associative derivation and a center-valued trace.

1. Introduction

Let A be an arbitrary associative algebra and let $Z(A)$ be the center of the algebra A. A linear operator $D : A \to A$ is called (an associative) derivation if $D(xy) = D(x)y+xD(y)$ for all $x, y \in A$. For any derivation D, it is always true that $D(Z(A)) \subset Z(A)$. Every element $a \in A$ defines a derivation D_a on A given by $D_a(x) = ax - xa = [a, x]$, $x \in A$. Such a derivation D_a is called the inner derivation.

A linear operator $L : A \to A$ is called a Lie derivation, if $L([x, y]) = [L(x), y]+[x, L(y)]$ for all $x, y \in A$. It is obvious that every associative derivation D on A is a Lie derivation. An example of a non-associative Lie derivation is a nonzero center-valued trace $E : A \to Z(A)$, i.e., a linear map $E : A \to Z(A)$ such that $E(xy) = E(yx)$ for all $x, y \in A$.

It is well known that any Lie derivation L on a C^*-algebra A can be uniquely represented in the form $L = D + E$, where D is an associative derivation and E is a center-valued trace on A [13]. Such representation of the Lie derivation L is called the standard form of L. In case where A is a von Neumann algebra the standard form of a Lie derivation L on A has the form $L = D_a + E$ for some $a \in A$ [15].

Development of the theory of algebras of measurable operators $S(M)$ and of algebras of locally measurable operators $LS(M)$ affiliated with von Neumann algebras or AW^*-algebras M (see for example [6], [17], [18], [20], [21], [23]) provided an opportunity to construct and to study new meaningful examples of $*$-algebras of unbounded operators.

One of the interesting problem is to describe all derivations which act in the algebras $S(M)$ and $LS(M)$. In the case where M is a commutative von Neumann algebra any derivation on $S(M) = LS(M)$ is inner if and only if M is an atomic algebra [3]. For a commutative AW^*-algebra M a criterion for existence of nonzero derivations on $S(M)$ is the lack of the σ-distributive property of Boolean algebra of all projections in M [11].

In the case of type I von Neumann algebra, all associative derivations on the algebras $LS(M)$ (respectively, $S(M)$) are described in [1]. In the case where M is a properly infinite von Neumann algebra, any associative derivation on $LS(M)$ (and $S(M)$) is inner [5].

Following the approach of [15], in this paper we present a standard form of the Lie derivation acting on an arbitrarily solid $*$-subalgebras of $LS(M)$, which contains M.

We use terminology and notations from the von Neumann algebra theory [8], [19], [22] and the theory of locally measurable operators [14], [20], [23].

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2. Preliminaries

Let H be a complex Hilbert space over the field $\mathbb{C}$ of complex numbers and let $B(H)$ be the algebra of all bounded linear operators on H. Let M be a von Neumann subalgebra in $B(H)$ and let $\mathcal{P}(M)$ be the lattice of all projections in M, i.e., $\mathcal{P}(M) = \{p \in M : p^2 = p = p^*\}$. Denote by $\mathcal{P}_{fin}(M)$ the sublattice of all finite projections of $\mathcal{P}(M)$. Let $Z(M)$ be the center of algebra M and $\mathbf{1}$ be the identity in M.

A linear subspace $\mathcal{D}$ of H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subseteq \mathcal{D}$ for every unitary operator u from the commutant M' of the von Neumann algebra M.

A linear subspace $\mathcal{D}$ in H is said to be strongly dense in H with respect to the von Neumann algebra M, if $\mathcal{D}\eta M$ and there exists a sequence of projections $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} := \mathbf{1} - p_n \in \mathcal{P}_{fin}(M)$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers.

A linear operator x on H with a dense domain $\mathcal{D}(x)$ is said to be affiliated with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $ux(\xi) = xu(\xi)$ for all $\xi \in \mathcal{D}(x)$ and for every unitary operator $u \in M^{'}$.

A closed linear operator x acting in the Hilbert space H is said to be measurable with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. By $S(M)$ we denote the set of all measurable operators with respect to M.

The set $S(M)$ is a unital $*$-algebra with respect algebraic operations of strong addition and multiplication and taking the adjoint of an operator (it is assumed that the multiplication by a scalar defined as usual wherein $0 \cdot x = 0$) [21].

A closed linear operator x in H is said to be locally measurable with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1},\ z_n(H) \subset \mathcal{D}(x)$ and $xz_n \in S(M)$ for all $n \in \mathbb{N}$.

The set $LS(M)$ of all locally measurable operators with respect to M is also a unital $*$-algebra equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator. In addition, $S(M)$ and M are $*$-subalgebras in $LS(M)$ [14, Ch. II, §2.3]. The center $Z(LS(M))$ of the $*$-algebra $LS(M)$ coincides with the $*$-algebra $S(Z(M))$. In the case where M is a factor or M is finite von Neumann algebra, the equality $LS(M) = S(M)$ holds.

Let x be a closed operator with a dense domain $\mathfrak{D}(x)$ in H, let $x = u|x|$ be the polar decomposition of the operator x, where $|x| = (x^*x)^{\frac{1}{2}}$ and u is a partial isometry in $B(H)$ such that u^*u (respectively, uu^*) is the right (left) support $r(x)$ (respectively, $l(x)$) of x. It is known that $x \in LS(\mathcal{M})$ (respectively, $x \in S(\mathcal{M})$) if and only if $|x| \in LS(\mathcal{M})$ (respectively, $|x| \in S(\mathcal{M})$) and $u \in \mathcal{M}$ [14, Ch. II, §§ 2.2, 2.3].

The $*$-subalgebra A in $LS(M)$ is called the solid $*$-subalgebra of $LS(M)$, if $MAM \subset A$ for any $a, b \in M,\ x \in A$. It is known that a $*$-subalgebra A of $LS(M)$ is solid if and only if the conditions $x \in LS(M),\ y \in A,\ |x| \leq |y|$ imply that $x \in A$ (see for example, [2]). The examples of solid $*$-subalgebras of $LS(M)$ are $*$-subalgebras M and $S(M)$.

3. Standard decomposition of Lie derivations

In this section the standard decomposition of a Lie derivation acting in a solid $*$-subalgebra A in $LS(M)$ containing M is established. More precisely, we prove the following result.

Theorem 1. *Let A be a solid $*$-subalgebra in $LS(M)$ containing M, and let L be a Lie derivation on A. Then there exist an associative derivation D on A and a center-valued trace $E : A \to Z(A)$ such that*

$$L(x) = D(x) + E(x) \tag{1}$$

for all $x \in A$.

Remark 2. In the case of algebras $LS(M)$ and $S(M)$, the standard decomposition (1) for the Lie derivation L in general is not unique. For example, if M is a commutative von Neumann algebra without atoms, there are uncountably many distinct non-zero associative derivations $D : S(M) \to S(M)$ [3, 4], which are at the same time $S(M)$-valued traces. Therefore, taking two different nonzero derivations D and D_1 on $S(M)$, we have that for a zero Lie derivation L the equality

$$L = D + (-D) = D_1 + (-D_1)$$

holds, i.e., the standard decomposition for L is not a unique.

Remark 3. If M is a commutative von Neumann algebra, then the $*$-algebra $LS(M)$ is also commutative [14, Ch. II, §2.2], and therefore for any $*$-subalgebra A in $LS(M)$, we have that $Z(A) = A$. Hence, in this case, the class of Lie derivations on A coincides with the class of $Z(A)$-valued trace on A.

To prove Theorem 1, we first consider the case where the von Neumann algebra M has no direct commutative summands, i.e., for any nonzero central projection $z \in \mathcal{P}(Z(M))$ the von Neumann algebra zM is not commutative. In this case, in the von Neumann algebra M there exists a non-zero projection p such that

$$c(p) = c(\mathbf{1} - p) = \mathbf{1} \quad \text{and} \quad p \preceq \mathbf{1} - p, \tag{2}$$

where $c(p) := \mathbf{1} - \sup\{z \in \mathcal{P}(Z(M)) : pz = 0\}$ is the central support of the projection p [10, 6.1.9]. Everywhere below in this section we fix the projector $p \in \mathcal{P}(M)$ satisfying the conditions (2).

Let A be an arbitrary solid $*$-subalgebra in $LS(M)$ such that $M \subset A$. Let $p_1 = p$, $p_2 = \mathbf{1} - p$. Consider the subalgebras $S_{ij} = p_i A p_j = \{p_i x p_j : x \in A\}$ in A, $i, j = 1, 2$. It is clear that $S_{ik} S_{lj} \subset S_{ij}$ and the inclusion $M \subset A$ implies that $p_i M p_j \subset S_{ij}$ for any $i, j, k, l = 1, 2$. In addition, $A = \sum\limits_{i,j=1,2} S_{ij}$. Moreover, for $x \in S_{ik}$, $y \in S_{lj}$, the inclusion $xy \in S_{ij}$ holds, and if $k \neq l$, then $xy = 0$.

To prove Theorem 1 we need some technical lemmas that are similar to the corresponding lemma in [12, 15]. The proofs of these lemmas are similar to those in [12, 15] and are given here for the sake of completeness (in contrast with [12, 15], we also consider the case where $A \neq M$).

Lemma 4. (cf. [15, Lemma 1]). *If $x \in S_{ij}$ and $xy = 0$ for all $y \in S_{jk}$, then $x = 0$.*

Proof. If $j = k$, then for $p_j \in S_{jj}$ we have that $x = xp_j = 0$. Let $i = j = 1, k = 2$, $x \in S_{11}$ and $xy = 0$ for all $y \in S_{12}$. Since $p_1 = p \preceq \mathbf{1} - p = p_2$ there exists a projection $q_1 \leq p_2$ such that $p_1 \sim q_1$, i.e., $u^*u = p_1$, $uu^* = q_1$ for some partial isometry $u \in M$. Taking into account that $u^* q_1 u = p_1$, $q_1 = q_1 p_2$, $x = p_1 x p_1$ and the inclusion $M \subset A$, we have that $p_1 u^* q_1 p_2 \in S_{12}$ and

$$x = p_1 x p_1 u^* u = x(p_1 u^* q_1 p_2) u = 0.$$

For other indices i, j the proof of Lemma 4 is similar. □

Lemma 5. *If L is a Lie derivation on A then $L(p) = [p, a] + z$ for some $a \in A$ and $z \in Z(A)$.*

The proof is exactly the same as the proof of Lemma 5 in [12] and Lemma 4 in [15].

Lemma 6. *If $L : A \to A$ is a Lie derivation and $L(p) \in Z(A)$ then*
(i). $L(S_{ij}) \subset S_{ij}$ *for* $i \neq j$*;*
(ii). $L(S_{ii}) \subset S_{ii} + Z(A)$, $i = 1, 2$.

Proof. The proof of (i) is analogous to the proof of Lemma 6 in [12].

(ii). If $a \in S_{11}$ and $L(a) = \sum\limits_{i,j=1,2} x_{ij}$, where $x_{ij} \in S_{ij}$, then

$$a = pap, \ [p, a] = pa - ap = 0,$$

and therefore

$$0 = L([p, a]) = [L(p), a] + [p, L(a)] = [p, L(a)] = x_{12} - x_{21}.$$

Consequently, $x_{12} = px_{12} = p(x_{12} - x_{21}) = 0$ and $x_{21} = x_{12} - (x_{12} - x_{21}) = 0$, i.e., $L(a) \in S_{11} + S_{22}$. Similarly, if $a \in S_{22}$, then $L(a) \in S_{11} + S_{22}$.

Let now $a \in S_{11}$, $b \in S_{22}$, $L(a) = a_{11} + a_{22}$, $L(b) = b_{11} + b_{22}$, where $a_{ii}, b_{ii} \in S_{ii}$, $i = 1, 2$. From the equation $0 = L([a, b]) = [a_{22}, b] + [a, b_{11}]$ and the inclusions $[a_{22}, b] \in S_{22}$, $[a, b_{11}] \in S_{11}$ it follows that $[a_{22}, b] = 0 = [a, b_{11}]$. Thus $a_{22}b = ba_{22}$ for all $b \in (\mathbf{1} - p)A(\mathbf{1} - p)$, i.e., $a_{22} \in Z((\mathbf{1} - p)A(\mathbf{1} - p))$.

Let us show that $Z((\mathbf{1} - p)A(\mathbf{1} - p)) = (\mathbf{1} - p)Z(A)$. It is clear that

$$(\mathbf{1} - p)Z(A) \subset Z((\mathbf{1} - p)A(\mathbf{1} - p)).$$

Let $0 \le x \in Z((\mathbf{1} - p)A(\mathbf{1} - p))$, $z = x + (\mathbf{1} - p)$ and let y be an inverse of z in the algebra $(\mathbf{1} - p)LS(M)(\mathbf{1} - p)$. Since $0 \le y \le \mathbf{1} - p$ and $y \in Z((\mathbf{1} - p)A(\mathbf{1} - p))$, it follows that $y \in Z((\mathbf{1} - p)M(\mathbf{1} - p))$. According to [8, Part I, Ch. 2, §1], the equality $Z((\mathbf{1} - p)M(\mathbf{1} - p)) = (\mathbf{1} - p)Z(M)$ holds, and therefore $y = (\mathbf{1} - p)d$, where $d \in Z(M) \subset Z(A)$. Consequently, $y \in (\mathbf{1} - p)Z(A)$ and the operator z also belongs to the algebra $(\mathbf{1} - p)Z(A)$, which implies $x \in (\mathbf{1} - p)Z(A)$.

Thus, $Z((\mathbf{1} - p)A(\mathbf{1} - p)) = (\mathbf{1} - p)Z(A)$ and $a_{22} = (\mathbf{1} - p)\, z$, for some $z \in Z(A)$, i.e.,

$$L(a) = a_{11} + (\mathbf{1} - p)\, z = (a_{11} - pz) + z \in S_{11} + Z(A).$$

The inclusion $L(S_{22}) \subset S_{22} + Z(A)$ is established similarly. □

The Lemma 6 implies that in the case where $L(p) \in Z(A)$, we have that $L(x) \in S_{ij}$ for $x \in S_{ij}$, $i \neq j$ and that $L(x) = d(x) + z(x)$ for $x \in S_{ii}$, where $d(x) \in S_{ii}$ and $z(x) \in Z(A)$.

Now we prove the identity $pAp \cap Z(A) = \{0\}$. If there is $0 \neq a \in pAp \cap Z(A)$, then $0 \neq |a| \in pAp \cap Z(A)$ and therefore the operator $|a|$ has non-zero spectral projection $q = \{|a| \ge \epsilon\}$ such that $q \le p$ and $q \in Z(A)$. Hence, $q(\mathbf{1} - p) = 0$, which is impossible, since $c(\mathbf{1} - p) = \mathbf{1}$.

If $x \in S_{ii}$ and $L(x) = d_1(x) + z_1(x)$, where $d_1(x) \in S_{ii}, z_1(x) \in Z(A)$, then

$$d(x) - d_1(x) = z(x) - z_1(x) \in S_{ii} \cap Z(A) = (p_i A p_i) \bigcap Z(A) = \{0\}.$$

Consequently, for each $x \in S_{ii}$ the element $L(x)$ is uniquely represented in the form $L(x) = d(x) + z(x)$, where $d(x) \in S_{ii}, z(x) \in Z(A)$.

Thus, in the case $L(p) \in Z(A)$, the map $D : A \to A$ is correctly defined by

$$D(x) = d(x_{11}) + d(x_{22}) + L(x_{12} + x_{21}), \tag{3}$$

where $x \in A$, $x_{ij} = p_i x p_j$, $i, j = 1, 2$. If $x, y \in S_{11}$, then

$$L(x + y) = d(x + y) + z(x + y); \ L(x) = d(x) + z(x); \ L(y) = d(y) + z(y).$$

Since L is a linear map, it follows that

$$d(x + y) + z(x + y) = d(x) + d(y) + z(x) + z(y),$$
$$d(\lambda x) + z(\lambda x) = \lambda(d(x) + z(x)), \quad \lambda \in \mathbb{C}.$$

Using the identity $pAp \cap Z(A) = \{0\}$ we have that

$$d(x + y) = d(x) + d(y), \quad z(x + y) = z(x) + z(y),$$
$$d(\lambda x) = \lambda d(x), \quad z(\lambda x) = \lambda z(x).$$

This means that the mappings $d : S_{11} \to S_{11}$ and $z : S_{11} \to Z(A)$ are linear. For the same reasons, these mappings are linear on S_{22}. Thus the map D constructed in (3) is a linear operator acting on the solid $*$-subalgebra A. In this case, the map E from A to $Z(A)$, defined by

$$E(x) = L(x) - D(x) = z(x_{11}) + z(x_{22}), \quad x \in A,$$

is also linear. In additional, for $x \in S_{ij},\ i \neq j$, we have that $E(x) = 0$, i.e., $L(x) = D(x)$.

Lemma 7. *If $x \in S_{ij},\ i \neq j,\ y \in A$, then*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x).$$

The proof is exactly the same as the proof of Lemma 9 in [12].

Lemma 8. (cf. [12, Lemma 10, 11]). *If $x \in S_{ii}$ and $y \in A$, or $x \in A$ and $y \in S_{ii}$, $i = 1, 2$, then*

$$D(xy) = D(x)y + xD(y). \tag{4}$$

Proof. If $x \in S_{11}$ and $y \in S_{12}$, then $[x, y] = xy$ and

$$\begin{aligned} D(xy) = L(xy) = L([x, y]) &= [D(x) + E(x), y] + [x, D(y) + E(y)] \\ &= [D(x), y] + [x, D(y)] = D(x)y + xD(y). \end{aligned}$$

For $x \in S_{11}$ and $y \in S_{21}$ we have that $D(x) \in S_{11}$ and $D(y) \in S_{21}$, which implies (4). For the cases $x \in S_{22},\ y \in S_{ij},\ i \neq j$, the identity (4) is established similarly.

Now let $x, y \in S_{11},\ z \in S_{12}$. According to the arguments above it follows that

$$\begin{aligned} D(xy)z &= D(xyz) - xyD(z) = D(x)yz + xD(yz) - xyD(z) \\ &= D(x)yz + x(D(y)z + yD(z)) - xyD(z) = (D(x)y + xD(y))z. \end{aligned}$$

Consequently, $(D(xy) - D(x)y - xD(y))z = 0$ for all $z \in S_{12}$ and from Lemma 4 we have that $D(xy) - D(x)y - xD(y) = 0$.

In the case where $x \in S_{11},\ y \in S_{22}$, or $x \in S_{22},\ y \in S_{11}$, we have that $xy = 0$ and $D(x)y + xD(y) = 0$. The case $x, y \in S_{22}$ can be treated similarly to that of $x, y \in S_{11}$.

Thus, identity (4) holds for any $x \in S_{ii},\ y \in S_{kl}$ for all $i, k, l = 1, 2$. Using the equality $A = \sum\limits_{i,j=1,2} S_{ij}$ we obtain that the equality (4) holds for any $x \in S_{ii},\ y \in A,\ i = 1, 2$. For the case $x \in A$ and $y \in S_{ii},\ i = 1, 2$ the equality (4) is established similarly. □

Lemma 9. (cf. [16, Lemma 10]). *The linear mapping D defined by (3) is an associative derivation on A.*

Proof. In the case where $x \in S_{ii},\ i = 1, 2,\ y \in A$, Lemma 9 is proved in Lemma 8. If $x \in S_{12}$ and $y \in S_{21}$, then

$$\begin{aligned} E([x, y]) = L([x, y]) - D([x, y]) &= [L(x), y] + [x, L(y)] - D([x, y]) \\ &= [D(x), y] + [x, D(y)] - D(xy) + D(yx). \end{aligned}$$

Consequently, for $z = E([x, y]) \in Z(A)$, we have that

$$z = D(x)y + xD(y) - D(xy) + D(yx) - D(y)x - yD(x). \tag{5}$$

Since $(D(x)y + xD(y) - D(xy)) \in S_{11},\ (D(y)x + yD(x) - D(yx)) \in S_{22}$ and $S_{11} \cap S_{22} = \{0\}$, it follows in the case $z = 0$ that equation (5) implies $D(xy) = D(x)y + xD(y)$.

Let us show that $z = 0$. Suppose that $z \neq 0$. Multiplying the left hand-side of (5) by x and then by y, we have that

$$xz = xD(yx) - xD(y)x - xyD(x), \quad yz = yD(x)y + yxD(y) - yD(xy). \tag{6}$$

Since $yx \in S_{22}$, it follows from Lemma 8 that $D(xyx) = D(x)yx + xD(yx)$. Now, using the equality (6), we have that

$$xz = D(xyx) - D(x)yx - xD(y)x - xyD(x)$$

and by Lemma 7, $xz = 0$.

Due to $xy \in S_{11}$ and Lemma 8, we have $D(yxy) = D(y)xy + yD(xy)$.

Therefore, from (6) and Lemma 6 it follows that

$$\begin{aligned} yz &= yD(x)y + yxD(y) - D(yxy) + D(y)xy \\ &= yD(x)y + yxD(y) - D(y)xy - yD(x)y - yxD(y) + D(y)xy = 0. \end{aligned}$$

Let $x = u|x|$ be the polar decomposition of the operator $x \in A$. Since $xz = 0,\ M \subset A$, it follows that $|x| = u^*x \in A$ and $|x|z = u^*xz = 0$. Hence $z^*|x| = 0$ and $xz^* = z^*x = 0$. Similarly, using the equality $yz = 0$, we have that $yz^* = z^*y = 0$. Multiplying now the left hand-side and the right hand-side of (5) by z^*, we have that

$$z^*zz^* = z^*(D(yx) - D(xy))z^*.$$

Since $yx \in S_{22}$ and $uxz^* = 0 = z^*y$, it follows by Lemma 8 that

$$0 = D(yx(\mathbf{1} - p)z^*) = D(yx)(\mathbf{1} - p)z^* + yxD((\mathbf{1} - p)z^*). \tag{7}$$

Using $D(yx) \in S_{22}$ and equality (7), we have that

$$D(yx)z^* = D(yx)(\mathbf{1} - p)z^* = -yxD((\mathbf{1} - p)z^*).$$

Similarly, $D(xy)z^* = -yxD(pz^*)$. Since $z^*x = 0 = z^*y$, it follows that

$$z^*zz^* = z^*xyD(pz^*) - z^*yxD((\mathbf{1} - p)z^*) = 0,$$

i.e., $z = 0$. In the case $x \in S_{21}$ and $y \in S_{12}$ the statement of Lemma 9 is proved similarly. □

In particular, Lemma 9 implies the following result.

Corollary 10. $E([x, y]) = 0$ *for all* $x, y \in A$.

Proof. Since the D is an associative derivation it follows that D is also a Lie derivation. Thus for any $x, y \in A$ we have

$$L([x, y]) = [D(x) + E(x), y] + [x, D(y) + E(y)] = [D(x), y] + [x, D(y)] = D([x, y]),$$

which implies that $E([x, y]) = 0$. □

Now we are ready to establish the standard decomposition for the Lie derivation L in the case where the von Neumann algebra M has no direct commutative summands.

Theorem 11. *Let M be a von Neumann algebra without direct commutative summands, let A be a solid $*$-subalgebra in $LS(M)$, containing M, and let L be a Lie derivation on A. Then there exist an associative derivation D on A and a center-valued trace $E : A \to Z(A)$ such that $L(x) = D(x) + E(x)$ for all $x \in A$.*

Proof. Since the von Neumann algebra M has no direct commutative summands, there exists a projection $p \in \mathcal{P}(M)$ such that $c(p) = c(\mathbf{1} - p) = \mathbf{1}$ ([10, 6.1.9]). Assume first that $L(p) \in Z(A)$. In this case, according to Lemma 8 and Corollary 10, we have that $L = D + E$, where D is an associative derivation on A and E is a trace on A with values in $Z(A)$.

Now let $L(p) \bar{\in} Z(A)$. By Lemma 5, the equality $L(p) = [p, a] + z$ holds for some $a \in A$ and $z \in Z(A)$. Consider on A the inner derivation $D_a(x) = [a, x]$, and let $L_1 = L + D_a$. It is clear that L_1 is a Lie derivation on A, moreover,

$$L_1(p) = L(p) + D_a(p) = [p, a] + [a, p] + z = z \in Z(A).$$

By Lemma 9 and Corollary 10 the Lie derivation L_1 has the standard form, i.e., $L_1 = D_1 + E_1$, where D_1 is an associative derivation on A and E_1 is a $Z(A)$-valued trace on A. Therefore,

$$L = L_1 - D_a = D_1 - D_a + E_1.$$

It is clear that $D = D_1 - D_a$ is an associative derivation on A, and consequently the equality $L = D + E_1$ completes the proof of Theorem 11. □

We now consider the case of an arbitrary von Neumann algebra M. We need the following

Lemma 12. *Let M be an arbitrary von Neumann, and let A be a $*$-subalgebra in $LS(M)$. If L is a Lie derivation on A then $L(Z(A)) \subset Z(A)$.*

Proof. If $z \in Z(A),\ a \in A$, then $[z, a] = 0 = [z, L(a)]$, and consequently $[L(z), a] = L([z, a]) - [z, L(a)] = 0$, i.e., $L(z)a = aL(z)$, which implies that $L(z) \in Z(A)$. □

If $z_0 = sup\{z \in \mathcal{P}(Z(M)) : zM \subset Z(M)\}$ then $M_0 := z_0M = z_0Z(M)$ is a commutative von Neumann algebra, and the von Neumann algebra $M_1 = (\mathbf{1} - z_0)M$ has no direct commutative summands, moreover $M = M_1 \bigoplus M_0$.

Let A be a solid $*$-subalgebra in $LS(M)$, containing M, $z_1 = \mathbf{1} - z_0,\ A_1 = z_1A$, and let L be a Lie derivation on A.

Lemma 13. (i). *$L_1(x) := z_1L(x)$ is a Lie derivation on the solid $*$-subalgebra A_1 in $LS(M_1)$;*

(ii). *The linear mappings $F_1(x) := z_0L(z_1x),\ F_2(x) := z_1L(z_0x)$ and $F_3(x) := z_0L(z_0x)$ are $Z(A)$-valued traces on A;*

(iii). *If E_1 is a $Z(A_1)$-valued traces on A_1, then $E(x) := E_1(z_1x)$ is a $Z(A)$-valued traces on A;*

(iv). *If D_1 is an associative derivation on A_1, then $D(x) := D_1(z_1x)$ is an associative derivation on A.*

Proof. (i). If $x, y \in A_1$, then

$$L_1([x, y]) = z_1L([x, y]) = z_1([L(x), z_1y] + [z_1x, L(y)]) = [L_1(x), y] + [x, L_1(y)].$$

(ii). It is clear that $F_1(x) \in z_0A \subset Z(A)$. If $x, y \in A$, then $z_1[x, y] = [z_1x, z_1y]$ and from equality $z_0z_1 = 0$ it follows that

$$F_1([x, y]) = z_0(L([z_1x, z_1y]) = z_0([L(z_1x),\ z_1y] + [z_1x, L(z_1y)]) = 0,$$

i. e., $F_1(xy) = F_1(yx)$.

Since $z_0x \in Z(A)$, by Lemma 12, we have that $F_2(x) = z_1L(z_0x) \in Z(A)$. Moreover,

$$F_2([x, y]) = z_1L(z_0[x, y]) = z_1L([z_0x, z_0y] = 0,$$

i. e., $F_2(xy) = F_2(yx)$. Similarly we can show that $F_3(xy) = F_3(yx)$.

(iii). For any $x, y \in A$ we have $E(x) \in Z(A_1) \subset Z(A)$ and

$$E(xy) = E_1(z_1xy) = E_1((z_1x)(z_1y)) = E_1((z_1y)(z_1x)) = E(yx).$$

(iv). Since $z_1D_1(a) = D_1(a)$ for any $a \in A_1$, it follows that

$$\begin{aligned} D(xy) &= D_1(z_1xy) = D_1((z_1x)(z_1y)) = D_1(z_1x)(z_1y) + z_1xD_1(z_1y) \\ &= D_1(z_1x)y + xD_1(z_1y) = D(x)y + xD(y) \end{aligned}$$

for all $x, y \in A$. □

Now we will prove Theorem 1. If $x \in A$ then using notations of Lemma 10, we obtain

$$\begin{aligned} L(x) &= z_1 L(z_1 x) + z_1 L(z_0 x) + z_0 L(z_1 x) + z_0 L(z_0 x) \\ &= L_1(z_1 x) + F_2(x) + F_1(x) + F_3(x). \end{aligned}$$

By Theorem 11 and Lemma 13 (i) there exist an associative derivation D_1 on A_1 and a $Z(A_1)$-valued trace E_1 on A_1 such that $L_1(z_1 x) = D_1(z_1 x) + E_1(z_1 x)$ for all $x \in A$. By Lemma 13 (iii), (iv), the mapping $D(x) = D_1(z_1 x)$ is an associative derivation on A and the mapping $E(x) = E_1(z_1 x)$ is a $Z(A)$-valued trace on A. Thus,

$$L(x) = D(x) + E(x) + F_1(x) + F_2(x) + F_3(x) \tag{8}$$

for all $x \in A$. Since $E + F_1 + F_2 + F_3$ is a $Z(A)$-valued trace on A (see Lemma 13 (ii)), the equality (8) completes the proof of Theorem 1.

4. Lie derivations on $LS(M)$ in the case of type I von Neumann algebra

Let Z be a commutative von Neumann algebra, let H_n be an n-dimensional complex Hilbert space and let $M_n = B(H_n) \bigotimes Z$ be a homogeneous von Neumann algebra of type I_n. The von Neumann algebra M_n is $*$-isomorphic to the $*$-algebra $Mat(n, Z)$ of all $n \times n$-matrix $(a_{ij})_{i,j=1}^n$ with entries $a_{ij} \in Z$. Since M_n is a finite von Neumann algebra, it follows that $LS(M_n) = S(M_n)$, in addition, $*$-algebra $S(M_n)$ is identified with the $*$-algebra $Mat(n, S(Z))$ of all $n \times n$-matrix with entries from $S(M)$ (see [1]).

If e_{ij}, $i, j = 1, \ldots, n$, is the matrix unit of $Mat(n, S(Z))$, then every element $x \in Mat(n, S(Z))$ has the form

$$x = \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}, \quad \lambda_{ij} \in S(Z), \quad i, j = 1, \ldots, n.$$

For any derivation $\delta : S(Z) \to S(Z)$ the linear operator

$$D_\delta \Bigg(\sum_{i,j=1}^{n} \lambda_{ij} e_{ij} \Bigg) = \sum_{i,j=1}^{n} \delta(\lambda_{ij}) e_{ij} \tag{9}$$

is a derivation on $S(M_n) = Mat(n, S(Z))$. Since the algebras $Z(S(M_n))$ and $S(Z)$ are isomorphic, we can identify them. The restriction of D_δ to the center $S(Z)$ coincides with δ. It is know that in the case where a commutative von Neumann algebra Z has no atoms there exists an uncountable set of mutually different derivations $\delta : S(Z) \to S(Z)$ (see [4]). Consequently, in this case, there exists an uncountable family of mutually different derivations on the algebra $Mat(n, S(Z)) = S(M)$ of the form D_δ.

Now let M be an arbitrary finite von Neumann algebra of type I with center Z. There exists a family $\{z_n\}_{n \in F}$, $F \subseteq \mathbb{N}$, of mutually orthogonal central projections from M with $\sup\limits_{n \in F} z_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic to the direct sum of the von Neumann algebras $z_n M$ of type I_{k_n}, $n \in F$, i.e.,

$$M \cong \sum_{n \in F} \bigoplus z_n M,$$

and

$$\begin{aligned} z_n M = B(H_{k_n}) \bigotimes L^\infty(\Omega_n, \Sigma_n, \mu_n) &\cong Mat(k_n, L^\infty(\Omega_n, \Sigma_n, \mu_n)) \\ = \{(a_{ij})_{i,j=1}^{k_n} : a_{ij} \in L^\infty(\Omega_n, \Sigma_n, \mu_n)\}, \end{aligned}$$

where $k_n = dim(H_n) < \infty$, $(\Omega_n, \Sigma_n, \mu_n)$ is a Maharam measure space, $n \in \mathbb{N}$. According to Proposition 1.1 from [1], we have that

$$LS(M) \cong \prod_{n \in F} LS(z_n M) = \prod_{n \in F} S(z_n M)$$

and

$$LS(z_nM) = S(z_nM) \cong Mat(k_n, L(\Omega_n, \Sigma_n, \mu_n)) \\ = \{(\lambda_{ij})_{i,j=1}^{k_n} : \lambda_{ij} \in L(\Omega_n, \Sigma_n, \mu_n)\},$$

where $L(\Omega_n, \Sigma_n, \mu_n)$ is an $*$-algebra of all complex measurable functions on the measure space $(\Omega_n, \Sigma_n, \mu_n)$ (functions that are equal almost everywhere are identified).

Suppose that D is a derivation on $LS(M)$ and δ is the restriction of D onto the center $S(LS(M)) = S(Z(M))$. The restriction of the derivation δ onto $z_nS(Z(M))$ defines a derivation δ_n on $z_nS(Z(M))$ for each $n \in F$.

Let D_{δ_n} be a derivation on the matrix algebra $Mat(k_n, z_nS(Z(M))) \cong S(z_nM)$, defined by the equation (9). Set

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\}, \quad \{x_n\}_{n\in F} \in \prod_{n\in F} S(z_nM) = LS(M). \tag{10}$$

It is clear that D_δ is a derivation on the algebra $LS(M)$. If M is an arbitrary von Neumann algebra of type I, then there exists a central projection $z_0 \in M$ such that z_0M is a properly infinite algebra and $z_0^\perp M$ is a finite von Neumann algebra.

Consider a derivation D on $LS(M)$ and by δ denote the restriction of D onto the center $Z(LS(M))$. By Theorem 2.7 [1], the derivation z_0D is an inner derivation on $z_0LS(M) = LS(z_0M)$, moreover, $z_0\delta = 0$, i.e. $\delta = z_0^\perp\delta$.

Let D_δ be a derivation on $z_0^\perp LS(M)$ of the form (10). Consider the extension of D_δ onto $LS(M) = z_0LS(M) \bigoplus z_0^\perp LS(M)$, defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_2), \quad x_1 \in z_0LS(M), \quad x_2 \in z_0^\perp LS(M). \tag{11}$$

By [1, Theorems 2.8, 3.6], any derivation D on the algebra $LS(M)$ (respectively, on the algebra $S(M)$) can be uniquely represented as the sum $D = D_a + D_\delta$, where a is an element in $LS(M)$ (respectively, in $S(M)$). Thus by Theorem 1 we have the following.

Theorem 14. *If M is a type I von Neumann algebra. Then any Lie derivation on the algebra $LS(M)$ (respectively, on $S(M)$) has the form*

$$L = D_a + D_\delta + E,$$

where D_a is an inner derivation, D_δ is a derivation given by (11), generated by the derivation δ on the center of $LS(M)$ (respectively, on $S(M)$) and E is a center-valued trace on $LS(M)$ (respectively, on $S(M)$).

5. Lie derivations on EW^* algebras

In this section we give applications of Theorem 1 to a description of Lie derivations on EW^*-algebras. The class of EW^*-algebras (extended W^*-algebras) was introduced in [9] for the purpose of description of $*$-algebras of unbounded closed operators, which are "similar" to W^*-algebras by their algebraic and order properties.

Let A be a set of closed, densely defined operators on a Hilbert space H, which is an $*$-algebra with the identity $\mathbf{1}$ equipped with the strong sum, strong product, the scalar multiplication and the usual adjoint of operators.

An $*$-algebra A is said to be EW^*-algebra if the following conditions hold:

(i). $(\mathbf{1} + x^*x)^{-1} \in A$ for every $x \in A$;

(ii). A $*$-subalgebra A_b of bounded operators in A is a von Neumann subalgebra in $B(H)$.

In this case, an $*$-algebra A is said to be an EW^*-algebra over von Neumann algebra A_b. It is clear that every solid $*$-subalgebras A in $LS(M)$ with $M \subset A$ is an EW^*-algebra and $A_b = M$. The converse implication is given in [7], where it is established that every EW^*-algebra A with the bounded part $A_b = M$ is a solid $*$-subalgebra in the $*$-algebra

$LS(M)$. Thus $LS(M)$ is the greatest EW^*-algebra of EW^*-algebras with the bounded part coinciding with M.

Using Theorem 1, we get the following description Lie derivation on an EW^*-algebra.

Theorem 15. *Any Lie derivation L, defined on an EW^*-algebra A has a standard form $L = D + E$, where $D : A \to A$ is an associative derivation, and $E : A \to Z(A)$ is a center-valued trace on A.*

In the case where the bounded part A_b of an EW^*-algebra A is a properly infinite W^*-algebra, any associative derivation $D : A \to LS(A_b)$ is inner [5, Theorem 5.1(ii)].

Thus Theorem 15 implies the following.

Theorem 16. *If the bounded part A_b of an EW^*-algebra A is a properly infinite von Neumann algebra, then every Lie derivation L on A is equal to $D_a + E$, where $a \in A$ and $E : A \to Z(A)$ is a central-valued trace.*

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