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ON SIMILARITY OF UNBOUNDED PERTURBATIONS OF SELFADJOINT OPERATORS

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Abstract. We consider a linear unbounded operator A in a separable Hilbert space with the following property: there is an invertible selfadjoint operator S with a discrete spectrum such that $\|(A-S)S^{-\nu}\| < \infty$ for a $\nu \in [0,1]$. Besides, all eigenvalues of S are assumed to be different. Under certain assumptions it is shown that A is similar to a normal operator and a sharp bound for the condition number is suggested. Applications of that bound to spectrum perturbations and operator functions are also discussed. As an illustrative example we consider a non-selfadjoint differential operator.

1. Introduction and statement of the main result

Let $\mathcal{H}$ be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and the unit operator I. For a linear operator A in $\mathcal{H}$, $Dom(A)$ is the domain, A^* is the operator adjoint to A; $\sigma(A)$ denotes the spectrum of A, and A^{-1} is the inverse of A. Two operators A and M acting in $\mathcal{H}$ are said to be similar if there exists a boundedly invertible bounded operator T such that $A = T^{-1}MT$. The constant $\kappa_T = \|T^{-1}\|\|T\|$ is called the condition number. The condition number is important in applications. We refer the reader to [4], where condition number estimates are suggested for combined potential boundary integral operators in acoustic scattering, and [20], where condition numbers are estimated for second-order elliptic operators. Conditions that provide similarity of various operators to normal and selfadjoint ones were considered by many mathematicians, cf. [1, 3, 6], [12, 13], [15]–[19], and references given therein. In many cases, the condition number must be numerically calculated, e.g. [2, 18]. The interesting generalization of condition numbers of bounded linear operators in Banach spaces were explored in the paper [11]. Bounds for condition numbers of unbounded operators with Hilbert-Schmidt and Shatten-von Neumann Hermitian components have been established in [7] and [9]. The paper [8] deals with bounded perturbations of normal operators.

In the present paper we estimate the condition number of a linear operator A in $\mathcal{H}$ with the following property: there is a positive definite selfadjoint operator S with a discrete spectrum such that $Dom\ (A) = Dom\ (S)$, and for some $\nu \in [0,1]$,

$$q_\nu := \|(A-S)S^{-\nu}\| < \infty. \tag{1.1}$$

Besides, under certain restrictions we considerably generalize the main result of the paper [8]. Our approach in this paper is absolutely different from the one in [7, 9].

Let all the eigenvalues $\lambda_k(S)$ $(k = 1, 2, \ldots)$ of S be simple (i.e., the multiplicities are equal to one), and enumerated in the increasing order. So

$$d_m := \inf_{k \neq m} |\lambda_k(S) - \lambda_m(S)|/2 > 0.$$

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That is, $d_1 = (\lambda_2(S) - \lambda_1(S))/2$ and $d_m = \frac{1}{2}\min\{\lambda_{m+1}(S) - \lambda_m(S), \lambda_m(S) - \lambda_{m-1}(S)\}$ $(m \geq 2)$. It is assumed that

$$2q_\nu \lambda^\nu_{m+1}(S) < d_m \quad (m = 1, 2, \ldots) \tag{1.2}$$

and

$$\zeta_\nu(S) := \sum_{k=1}^\infty \frac{\lambda^{2\nu}_{k+1}(S)}{(d_k - 2q_\nu \lambda^\nu_{k+1}(S))^2} < \infty. \tag{1.3}$$

Under condition (1.2) we have

$$\eta_\nu(S) := \sup_m \frac{d_m}{d_m - q_\nu \lambda^\nu_{m+1}(S)} < \infty. \tag{1.4}$$

Now we are in a position to formulate our main result.

Theorem 1.1. *Let conditions (1.1)–(1.3) hold. Then there are a bounded and boundedly invertible operator T, and a normal operator M, acting in $\mathcal{H}$, such that*

$$TAx = MTx \quad (x \in Dom(A)). \tag{1.5}$$

Moreover,

$$\kappa_T \leq \eta_\nu(S)\left(1 + 2q_\nu\sqrt{\zeta_\nu(S)}\right)^2.$$

The proof of this theorem is presented in the next section. The theorem is sharp: if $A = S$ is selfadjoint, then $q_\nu = 0$ and $\eta_\nu(S) = 1$; we thus obtain $\kappa_T = 1$.

2. Proof of Theorem 1.1

Let $\{e_k\}$ be the set of all normed eigenvectors of S, such that

$$S = \sum_{k=1}^\infty \lambda_k(S) P_k, \quad \text{where} \quad P_k = (., e_k)e_k.$$

Put $\Omega(c, r) := \{z \in \mathbb{C} : |z - c| \leq r\}$ $(c \in \mathbb{C}, r > 0)$. Due to [10, Theorem 1.1], if the conditions (1.1) and (1.2) are satisfied, then A has in $\Omega(\lambda_m(S), d_m)$ a simple eigenvalue, say $\lambda_m(A)$. Let $\{g_k\}$ be the set of all eigenvectors of A and $\{h_k\}$ the corresponding biorthogonal sequence: $(g_k, h_j) = 0, k \neq j$, $(g_k, h_k) = 1$. Then $Q_k = (\cdot, h_k)g_k$ are the eigenprojections of A and

$$A = \sum_{k=1}^\infty \lambda_k(A) Q_k.$$

By Lemma 3.1 from [10], under conditions (1.1) and (1.2) one has

$$\|P_m - Q_m\| \leq \frac{q_\nu \lambda^\nu_{m+1}(S)}{d_m - q_\nu \lambda^\nu_{m+1}(S)}. \tag{2.1}$$

Put

$$T = \sum_{k=1}^\infty (., h_k)e_k. \tag{2.2}$$

Simple calculations show that the inverse operator is defined by

$$T^{-1} = \sum_{k=1}^\infty (., e_k)g_k. \tag{2.3}$$

Below we check that T and T^{-1} are bounded.

Lemma 2.1. *Let conditions (1.1), (1.2) hold and T be defined by (2.2). Then (1.5) is valid with*

$$M = \sum_{k=1}^\infty \lambda_k(A) P_k. \tag{2.4}$$

Proof. Indeed,

$$AT^{-1}f = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_k(A)(f,e_j)(g_j,h_k)g_k = \sum_{k=1}^{\infty}\lambda_k(A)(f,e_k)g_k$$
$$(f\in\mathcal{H}, T^{-1}f\in Dom(A))$$

and

$$TAT^{-1}f = \sum_{k=1}^{\infty}\lambda_k(A)\sum_{j=1}^{\infty}(g_k,h_j)e_j(f,e_k) = \sum_{k=1}^{\infty}\lambda_k(A)(f,e_k)e_k = Mf,$$

as claimed. □

Introduce the operator

$$J = \sum_{k=1}^{\infty}\|h_k\|(.,e_k)e_k.$$

Then

$$Tf - Jf = \sum_{k=1}^{\infty}\|h_k\|(f,\hat{h}_k - e_k)e_k, \quad \text{where} \quad \hat{h}_k = \frac{h_k}{\|h_k\|}.$$

Hence,

$$\|Tf - Jf\|^2 = \sum_{k=1}^{\infty}\|h_k\|^2|(f,\hat{h}_k - e_k)|^2 \le \|f\|^2\sum_{k=1}^{\infty}\|h_k\|^2\|\hat{h}_k - e_k\|^2. \tag{2.5}$$

It is clear the h_k are eigenvectors of A^*. Besides, $\|(A^*-S)S^{-\nu}\| = \|(A-S)S^{-\nu}\| = q_\nu$.

Due to [10, Theorem 1.1], if the conditions (1.1), (1.2) are fulfilled, then the eigenvector $v_m(A)$ of A corresponding to $\lambda_m(A)$ and the eigenvector e_m of S corresponding to $\lambda_m(S)$ with $\|v_m(A)\| = \|e_m\| = 1$ satisfy the inequality

$$\|v_m(A) - e_m\| \le \frac{2q_\nu\lambda_{m+1}^\nu(S)}{d_m - 2q_\nu\lambda_{m+1}^\nu(S)}.$$

Applying this result with A^* instead of A, and $\hat{h}_m$ instead of $v_m(A)$, we can write

$$\|e_m - \hat{h}_m\| \le \frac{2q_\nu\lambda_{m+1}^\nu(S)}{d_m - 2q_\nu\lambda_{m+1}^\nu(S)}. \tag{2.6}$$

Now (2.5) implies

$$\|T - J\|^2 \le (2q_\nu)^2\sum_{k=1}^{\infty}\frac{\|h_k\|^2\lambda_{k+1}^{2\nu}(S)}{(d_k - 2q_\nu\lambda_{k+1}^\nu(S))^2}. \tag{2.7}$$

We always can take h_k and g_k in such a way that $\|h_k\| = \|g_k\|$. Clearly, $Q_kh_k = (h_k,h_k)g_k$. So

$$(Q_kh_k,g_k) = (h_k,h_k)(g_k,g_k) = \|h_k\|^4 = \|g_k\|^4.$$

Hence, $\|h_k\|^4 \le \|Q_k\|\|h_k\|\|g_k\| = \|Q_k\|\|h_k\|^2$. Thus $\|h_k\|^2 \le \|Q_k\|$ and $\|g_k\|^2 \le \|Q_k\|$. Therefore, (2.6) gives us the inequality

$$\|T - J\|^2 \le (2q_\nu)^2\sum_{k=1}^{\infty}\frac{\|Q_k\|\lambda_{k+1}^{2\nu}(S)}{(d_k - 2q_\nu\lambda_{k+1}^\nu(S))^2}. \tag{2.8}$$

Moreover, since $\|P_m\| = 1$ by (2.1) we get

$$\|Q_m\| \le \sup_m\,(1 + \frac{q_\nu\lambda_{m+1}^\nu(S)}{d_m - q_\nu\lambda_{m+1}^\nu(S)}) = \sup_m\frac{d_m}{d_m - q_\nu\lambda_{m+1}^\nu(S)} = \eta_\nu(S)$$
$$(m = 1,2,\ldots). \tag{2.9}$$

Consequently,

$$\|T-J\|^2 \le (2q_\nu)^2 \eta_\nu(S) \sum_{k=1}^{\infty} \frac{\lambda_{k+1}^{2\nu}(S)}{(d_k - 2q_\nu \lambda_{k+1}^{\nu}(S))^2} = (2q_\nu)^2 \eta_\nu(S) \zeta_\nu(S).$$

Due to (2.9),

$$\|Jf\|^2 = \sum_{k=1}^{\infty} \|h_k\|^2 |(f,e_k)|^2 \le \sum_{k=1}^{\infty} \|Q_k\| |(f,e_k)|^2 \le \eta_\nu(S) \sum_{k=1}^{\infty} |(f,e_k)|^2 = \|f\|^2 \eta_\nu(S)$$
$$(f \in \mathcal{H}).$$

Thus we obtain

$$\|T\|^2 = \|J + (T-J)\|^2 \le (\|J\| + \|T-J\|)^2 \le \eta_\nu(S)(1 + 2q_\nu \sqrt{\zeta_\nu(S)})^2.$$

The same arguments give us the inequality

$$\|T^{-1}\|^2 \le \eta_\nu(S)(1 + 2q_\nu \sqrt{\zeta_\nu(S)})^2.$$

This proves the theorem.

3. Example

Consider in $L^2(0,1)$ the problem

$$u^{(4)}(x) + a(x)u'(x) = \lambda u(x) \quad (\lambda \in \mathbb{C};\ 0 < x < 1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $a(x)$ $(0 \le x \le 1)$ is a bounded complex valued function. Take $S = d^4/dx^4$ with

$$Dom\ (S) = \{v \in L^2(0,1) : v^{(4)} \in L^2(0,1),\ v(0) = v(1) = v''(0) = v''(1) = 0\},$$

and $\nu = 1/4$. Define A by

$$(Au)(x) = u^{(4)}(x) + a(x)u'(x) \tag{3.1}$$

with $Dom\ (A) = Dom\ (S)$. Obviously, $\lambda_j(S) = \pi^4 j^4$ $(j = 1, 2, \dots)$ and

$$q_{1/4} = \|(S-A)S^{-1/4}\| = \sup_x |a(x)| \sup_{f \in Dom(S), \|f\|=1} \|(S^{-1/4}f)'_x\|.$$

But

$$(S^{-1/4}f)(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_j^{1/4}(S)} (f,e_k)e_k(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{j} (f,e_k)e_k(x),$$

where $e_k(x) = \sqrt{2}\ sin\ \pi(kx)$. Thus

$$\frac{d}{dx}(S^{-1/4}f)(x) = \frac{d}{dx} \sum_{k=1}^{\infty} \frac{1}{\lambda_j^{1/4}(S)} (f,e_k)e_k(x) = \sqrt{2} \sum_{k=1}^{\infty} (f,e_k) cos\ \pi(kx).$$

Hence,

$$\|(S^{-1/4}f)'_x\|^2 = \sum_{k=1}^{\infty} |(f,e_k)|^2 = \|f\|^2.$$

Therefore, $q_{1/4} = |a|_C$, where $|a|_C := \sup_{0 \le x \le 1} |a(x)|$. Moreover,

$$d_j = (\lambda_{j+1}(S) - \lambda_j(S))/2 = \pi^4[(j+1)^4 - j^4]/2$$
$$= \pi^4(2j^3 + 3j^2 + 2j + 1/2) \ge 2\pi^4 j^3.$$

Condition (1.2) is provided by the inequality

$$|a|_C(j+1) < \pi^3 j^3 \quad (j = 1, 2, \dots).$$

Since

$$\min_j \frac{j^3}{j+1} \ge 1/2,$$

condition (1.2) obviously holds, if

$$|a|_C < \pi^3/2. \tag{3.2}$$

Besides,

$$\eta_{1/4}(S) \le \sup_m \frac{d_m}{d_m - q_{1/4}\lambda_{m+1}^{1/4}(S)} \le \hat{\eta}_{1/4},$$

where

$$\hat{\eta}_{1/4} = \frac{7\pi^3}{7\pi^3 - 4|a|_C}.$$

In addition,

$$\zeta_{1/4}(S) = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(\pi^3(2k^3 + 3k^2 + 2k + 1/2) - 2|a|_C(k+1))^2}.$$

Evidently, $\zeta_{1/4}(S) \le \hat{\zeta}$, where

$$\hat{\zeta} = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(2\pi^3 k^3 - 2|a|_C(k+1))^2} < \infty.$$

Consequently, by Theorem 1.1, there are a bounded and boundedly invertible operator T, and a normal operator M, acting in $\mathcal{H}$, such that (1.5) holds and

$$\kappa_T \le \hat{\eta}_{1/4}\left(1 + 2|a_C|\sqrt{\hat{\zeta}}\right)^2,$$

provided condition (3.2) holds.

4. Applications of Theorem 1.1 and concluding remarks

4.1. **Some applications of Theorem 1.1.** For brevity, put

$$\gamma_\nu(S) = \eta_\nu(S)\left(1 + 2q_\nu\sqrt{\zeta_\nu(S)}\right)^2.$$

Let $f(z)$ be a scalar-valued function defined and uniformly bounded on the spectrum of A. Put

$$f(A) = \sum_{k=1}^{\infty} f(\lambda_k(A))Q_k.$$

Recall that Q_k $(k = 1, 2, \ldots)$ are eigenprojections of A. Due to (1.5) Theorem 1.1 immediately implies the following.

Corollary 4.1. *Let conditions (1.1)–(1.3) hold. Then $f(A) = T^{-1}f(M)T$ and therefore,*

$$\|f(A)\| \le \gamma_\nu(S) \sup_k |f(\lambda_k(A))|.$$

In particular, we have

$$\|e^{-At}\| \le \gamma_\nu(S)e^{-\beta(A)t} \quad (t \ge 0),$$

where $\beta(A) = \inf_k \Re\ \lambda_k(A)$. In addition,

$$\|R_\lambda(A)\| \le \frac{\gamma_\nu(S)}{\rho(A,\lambda)} \quad \text{where} \quad \rho(A,\lambda) = \inf_{s\in\sigma(A)} |\lambda - s| \ \ (\lambda \notin \sigma(A)). \tag{4.1}$$

Let A and $\tilde{A}$ be linear operators. Then the quantity

$$sv_A(\tilde{A}) := \sup_{t\in\sigma(\tilde{A})} \inf_{s\in\sigma(A)} |t - s|$$

is said to be the variation of $\tilde{A}$ with respect to A.

Let $\tilde{A}$ be a linear operator in $\mathcal{H}$ with $Dom(A) = Dom(\tilde{A})$ and

$$\xi := \|A - \tilde{A}\| < \infty. \tag{4.2}$$

From (4.1) it follows that $\lambda \notin \sigma(\tilde{A})$, provided $\xi\gamma_\nu(S) < \rho(A,\lambda)$. So for any $\mu \in \sigma(\tilde{A})$ we have $\xi\gamma_\nu(S) \geq \rho(A,\mu)$. This inequality implies our next result.

Corollary 4.2. *Let conditions (1.1)–(1.3) and (4.2) hold. Then* $sv_A(\tilde{A}) \leq \xi\gamma_\nu(S)$.

4.2. **α-dependence.** For a parameter $\alpha > 0$ put $A_\alpha = A+\alpha I$, $S_\alpha = S+\alpha I$ and consider the dependence of our results on α. Clearly, A_α is similar to a normal operator if and only if so is A. On the other hand,

$$q_\nu(\alpha) := \|(A_\alpha - S_\alpha)(S+\alpha I)^{-\nu}\| = \|(A-S)(S+\alpha I)^{-\nu}\| \leq q_\nu(0) = q_\nu$$

and $\lambda_k(S_\alpha) - \lambda_m(S_\alpha) = \lambda_k(S) - \lambda_m(S)$ $(k \neq m)$. If $\nu > 0$, then the condition

$$2q_\nu(\alpha)(\lambda_k(S)+\alpha)^\nu < d_k \tag{4.3}$$

is not applicable for all $k \geq 1$, provided α is sufficiently large. In our opinion, *this is because we do not have an explicit dependence of $q_\nu(\alpha)$ on α.* Note that $(\lambda_k(S) + \alpha)/\lambda_k(S) \to 1$ as $k \to \infty$, and due to (1.2), for any $\alpha > 0$ there is a sufficiently large integer k_0 such that (4.3) holds for all $k \geq k_0$.

If $\nu = 0$, then taking into account that $(\lambda_k(S)+\alpha)^0 = 1$, $(S+\alpha I)^0 = I$, we get $q_0(\alpha) = q_0$,

$$\zeta_0(S_\alpha) := \sum_{k=1}^\infty \frac{1}{(d_k - 2q_0)^2} = \zeta_0(S), \eta_0(S_\alpha) := \sup_m \frac{d_m}{d_m - q_0} = \eta_0(S_0)$$

and condition (4.3) takes the form $2q_0 < d_k$. So *if $\nu = 0$, then the conditions of Theorem 1.1 do not depend on α.*

4.3. **Concluding remarks.** Let us illustrate the sharpness and α-dependence of Theorem 1.1 in the finite dimensional case. Consider the matrix

$$A = \begin{pmatrix} 5 & 1/6 \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

First let $\nu = 0$ and consider the sharpness. Take
(4.5)
$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1/24 \\ 0 & 1 \end{pmatrix}$. It is simple to check that $T^{-1} = \begin{pmatrix} 1 & -1/24 \\ 0 & 1 \end{pmatrix}$

and $TAT^{-1} = S$. In addition, $q_0 = \|A - S\| = 1/6$ and $\kappa_T = \|T^{-1}\|\|T\| \approx 1.08$. In the considered case $\lambda_1(S) = 1, \lambda_2(S) = 5, d_1 = 2$. So $d_1 > 2q_0$,

$$\zeta_0(S) = \frac{1}{(d_1 - 2q_0)^2} = 9/25, \eta_0(S) = \frac{d_1}{d_1 - q_0} = 12/11.$$

Theorem 1.1 implies

$$\kappa_T \leq \eta_0(S)(1 + 2q_0\sqrt{\zeta_0(S)})^2 \approx 1.57.$$

We can see that in the considered trivial case Theorem 1.1 gives us a rather rough bound.

Now let $\nu = 1/2$, and consider the α-dependence. Take $A_\alpha = A+\alpha I$ with A defined by (4.4) and $S_\alpha = S+\alpha I$ with S defined as in (4.5). Then $\lambda_1(S_\alpha) = 1+\alpha, \lambda_2(S_\alpha) = 5+\alpha$,

$$S^{-1/2} = \begin{pmatrix} 1/\sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } (A-S)S^{-1/2} = \begin{pmatrix} 0 & 1/6 \\ 0 & 0 \end{pmatrix}.$$

Consequently, $q_{1/2} = 1/6$ and condition (4.3) is provided by the inequality $2q_{1/2}(\lambda_2(S)+\alpha)^{1/2} < d_1$ for $\alpha < 31$. We can write

$$\eta_{1/2}(S_\alpha) = \frac{d_1}{d_1 - q_{1/2}(\alpha + \lambda_2(S))^{1/2}} = \frac{2}{2 - \frac{1}{6}(\alpha+5)^{1/2}}$$

and

$$\zeta_{1/2}(S_\alpha) = \frac{\lambda_2(S)+\alpha}{(d_1 - 2q_{1/2}(\alpha+\lambda_2(S))^{1/2})^2} = \frac{5+\alpha}{(2 - \frac{1}{3}\sqrt{5+\alpha})^2}.$$

Theorem 1.1 implies

$$\kappa_T \le \eta_{1/2}(S_\alpha)(1+2q_{1/2}\sqrt{\zeta_{1/2}(S_\alpha)})^2 = \frac{12}{12-\sqrt{\alpha+5}}\left(1+\frac{3\sqrt{5+\alpha}}{6-\sqrt{5+\alpha}}\right)^2$$

$(0 < \alpha < 31)$. This inequality shows that the dependence of the conditions of Theorem 1.1 on α is rather complicated.

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References

1. N. E. Benamara and N. K. Nikolskii, *Resolvent tests for similarity to a normal operator*, Proc. London Math. Soc. **78** (1999), 585–626.
2. T. Betcke, S. N. Chandler-Wilde, I. G. Graham, S. Langdon, M. Lindner, *Condition number estimates for combined potential integral operators in acoustics and their boundary element discretisation*, Numer. Methods Partial Differential Equ. **27** (2011), 31–69.
3. J A. van Casteren, *Operators similar to unitary or selfadjoint ones*, Pacific J. Math. **104** (1983), no. 1, 241–255.
4. S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and M. Lindner, *Condition number estimates for combined potential boundary integral operators in acoustic scattering*, J. Int. Eqn. Appl. **21** (2009), 229–279.
5. N. Dunford and J. T. Schwartz, *Linear Operators,* Part 3: *Spectral Operators*, Wiley-Interscience Publishers, Inc., New York, 1971.
6. M. M. Faddeev and R. G. Shterenberg, *On similarity of differential operators to a selfadjoint one*, Math. Notes **72** (2002), 292–303.
7. M. I. Gil', *A bound for similarity condition numbers of unbounded operators with Hilbert–Schmidt hermitian components*, J. Aust. Math. Soc. **97** (2014), no. 3, 1–12.
8. M. I. Gil', *On condition numbers of spectral operators in a Hilbert space*, Analysis and Mathematical Physics **5** (2015), 363–372.
9. M. I. Gil', *An inequality for similarity condition numbers of unbounded operators with Schatten–von Neumann Hermitian components*, Filomat **30** (2016), no. 13, 3415–3425.
10. M. I. Gil', *Rotations of eigenvectors under unbounded perturbations*, Journal of Spectral Theory **7** (2017), no. 1, 191–199.
11. Guoliang Chen, Yimin Wei and Yifeng Xue, *The generalized condition numbers of bounded linear operators in Banach spaces*, J. Aust. Math. Soc. **76** (2004), 281–290.
12. I. M. Karabash, *J-selfadjoint ordinary differential operators similar to selfadjoint operators*, Methods Funct. Anal. Topology **6** (2000), no. 2, 22–49.
13. I. M. Karabash, A. S. Kostenko, and M. M. Malamud, *The similarity problem for J-nonnegative Sturm–Liouville operators*, J. Differential Equations **246** (2009), 964–997.
14. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
15. A. Kostenko, *The similarity problem for indefinite Sturm–Liouville operators with periodic coefficients*, Oper. Matrices **5** (2011), no. 4, 707–722.
16. A. Kostenko, *The similarity problem for indefinite Sturm–Liouville operators and the help inequality*, Advances in Mathematics **246** (2013), 368–413.
17. M. M. Malamud, *Similarity of a triangular operator to a diagonal operator*, Journal of Mathematical Sciences **115** (2003), no. 2, 2199–2222.
18. S. V. Parter and Sze-Ping Wong, *Preconditioning second-order elliptic operators: condition numbers and the distribution of the singular values*, Journal of Scientific Computing **6** (1991), no. 2, 129–157.
19. B. Pruvost, *Analytic equivalence and similarity of operators*, Integr. Equ. Oper. Theory **44** (2002), 480–493.
20. M. Seidel and B. Silbermann, *Finite sections of band-dominated operators, norms, condition numbers and pseudospectra*, Operator Theory: Advances and Applications, Vol. 228, 375–390. Springer, Basel, 2013.

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