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A PROBABILISTIC PROOF OF THE VITALI COVERING LEMMA

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Abstract. The classical Vitali Covering Lemma on $\mathbb{R}$ states that there exists a constant $c > 0$ such that, given a finite collection of intervals $\{I_j\}$ in $\mathbb{R}$, there exists a disjoint subcollection $\{\tilde{I}_j\} \subseteq \{I_j\}$ such that $|\cup \tilde{I}_j| \geq c|\cup I_j|$. We provide a new proof of this covering lemma using probabilistic techniques and Padovan numbers.

1. Introduction

The *Vitali Covering Lemma* is one of the fundamental tools of modern analysis and geometric measure theory. Its d-dimensional version may be stated as follows.

Theorem 1. (Vitali Covering Lemma on $\mathbb{R}^d$). *Let $\{B_j\}$ be a finite collection of open balls in $\mathbb{R}^d$. Then there exists a disjoint subcollection $\{\tilde{B}_j\} \subseteq \{B_j\}$ such that*

$$|\cup \tilde{B}_j| \geq 3^{-d}\,|\cup B_j|\,.$$

This theorem is very well known and has many minor variations; see, e.g., [6, 9]. The proof on $\mathbb{R}^d$ relies on both an initial ordering of the balls so that they are nonincreasing in size as well as the triangle inequality, taking advantage of the observation that if two open balls B_1 and B_2 intersect and $|B_1| \geq |B_2|$, then B_2 is necessarily contained in the 3-fold concentric dilate of B_1. (This accounts for the term 3^{-d} above.) In the one-dimensional case, the ordering of the real line may be exploited to provide a better constant:

Theorem 2. (Vitali Covering Lemma on $\mathbb{R}$). *Let $\{I_j\}$ be a finite collection of open intervals in $\mathbb{R}$. Then there exists a disjoint subcollection $\{\tilde{I}_j\} \subseteq \{I_j\}$ such that*

$$|\cup \tilde{I}_j| \geq \frac{1}{2}\,|\cup I_j|\,.$$

The reader may consult [7] as well as [8] for a straightforward proof, attributed to W. H. Young, of the above.

Covering lemmas have been proven for collections of objects other than balls. For instance, in [4] A. Córdoba and R. Fefferman proved a covering lemma for rectangles with sides parallel to the axes that implies that these rectangles differentiate functions in $L^p(\mathbb{R}^2)$ for $p > 1$ and, moreover, functions in the Orlicz class $L(\log L)(\mathbb{R}^2)_{loc}$. Strömberg, Córdoba, and Fefferman used similar ideas in [5, 14] to prove that rectangles in $\mathbb{R}^2$ oriented in a direction of the form 2^{-j} also differentiate functions in $L^p(\mathbb{R}^2)$ for $p \geq 2$ (subsequent work of Nagel, Stein, and Wainger [10] and independently by Bateman [2] shows that these rectangles also differentiate $L^p(\mathbb{R}^2)$ for $p > 1$.)

Unfortunately, over the past few decades there has been relatively little progress in developing covering lemmas to yield improved differentiation results. (This is not to say

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there have not been some fantastic results in the theory of differentiation of integrals, see, for instance, the work of Bateman and Katz in [2, 3].) For example, we still do not know if rectangles in $\mathbb{R}^2$ oriented in lacunary directions must differentiate functions locally in an Orlicz class of the form $L(\log L)^k(\mathbb{R}^2)_{loc}$ for some k.

This paper considers a different approach to covering lemmas, one inspired by probabilistic reasoning and "Erdös-type" methods. As these methods have been highly successful in the field of combinatorics (many examples may be found in the excellent text [1] by Alon and Spencer), it is natural to ask whether they may be of use in the area of differentiation of integrals. To answer, we start at the very beginning: an attempt to reprove the Vitali Covering Lemma on $\mathbb{R}$ via probabilistic techniques. We are pleased to present a new proof in the next section. Of particular interest is the proof's natural but unexpected encounter with the so-called *Padovan sequence*. This sequence, named after Richard Padovan, is considered to have been originally discovered by the Dutch architect Dom Hans Van Der Laan [11], and the interested reader may find an introduction to them by Ian Stewart in [13]. In the last section of the paper we provide topics of research that we believe are suitable for further exploration.

2. A Probabilistic proof of the Vitali Covering Lemma on $\mathbb{R}$

The goal of this section is to provide a probabilistic proof of the following:

Theorem 3. *Let $\{I_j\}_{j=1}^N$ be a finite collection of intervals in $\mathbb{R}$. Then there exists a disjoint subcollection $\{\tilde{I}_j\} \subseteq \{I_j\}$ such that*

$$|\cup \tilde{I}_j| \geq \frac{1}{4}|\cup I_j|.$$

Proof. We begin by making some useful reductions. Without loss of generality we may assume that each $I_j = [x_j, y_j]$ is closed. Moreover, we may assume without loss of generality that the collection is minimal in the sense that no given I_j is covered by the union of the remaining I_k's. In addition, we may label the intervals $\{I_j\}$ such that $x_j \leq x_k$ for all $j < k$.

Suppose there is an interval different from I_k in $\{I_j\}$ that contains y_k as an interior point. Without loss of generality, let I_l be the interval of this type that extends furthest to the right. Now let $y_k \in I \neq I_k, I_l$. Suppose towards a contradiction that $I = [x, y] \in \{I_j\}$. If $x \leq x_k$, then $I_k \subseteq I$, and by the minimality of $\{I_j\}$, we have that $I_k \notin \{I_j\}$, a contradiction. On the other hand, if $x > x_k$, then $I \subseteq I_k \cup I_l$, and by the minimality of $\{I_j\}$, we have that $I \notin \{I_j\}$, another contradiction. Thus, y_k is only contained in I_k and I_l, and $l = k + 1$ follows. Accordingly, without loss of generality, we may assume the intervals $\{I_j\}_{j=1}^N$ are such that $y_j = x_{j+1}$ for all $j < N$.

We now begin to reason in a probabilistic manner. We call $\{\tilde{I}_j\}$ a *maximally disjoint* subcollection of $\{I_j\}_{j=1}^N$ if the intervals in $\{\tilde{I}_j\}$ are pairwise disjoint and, moreover, every interval in $\{I_j\}$ intersects the union of the intervals in $\{\tilde{I}_j\}$. Our strategy is the following:

(i) Count the number of maximally disjoint subcollections of $\{I_j\}_{j=1}^N$.

(ii) Given an interval $I \in \{I_j\}_{j=1}^N$, show that it must lie in $\frac{1}{4}$ of the maximally disjoint subcollections. In particular, show that, given a maximally disjoint subcollection, the probability that a given interval $I \in \{I_j\}$ is a member of that subcollection is greater than or equal to $\frac{1}{4}$.

Observe that once (i) and (ii) are proven, Theorem 3 immediately follows via linearity of expectation.

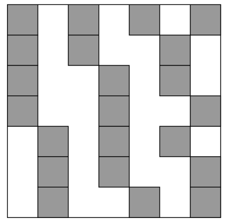

FIGURE 1. Maximally Disjoint Subcollections of Seven Intervals

Denote the number of maximally disjoint subcollections of $\{I_j\}_{j=1}^N$ by Q_N. By inspection, we see that $Q_1 = 1$, $Q_2 = Q_3 = 2$, $Q_4 = 3$, $Q_5 = 4$, $Q_6 = 5$, and $Q_7 = 7$. Figure 1 illustrates the seven different maximally disjoint subcollections associated to seven intervals. Observe that if $N \geq 4$, the number of maximally disjoint subcollections that contain I_1 is exactly Q_{N-2} and the number of maximally disjoint subcollections that contain I_2 is Q_{N-3}. By symmetry, the number of maximally disjoint subcollections that contain I_N is Q_{N-2} and the number that contain I_{N-1} is Q_{N-3}. More generally, if $3 \leq k \leq N-2$, then the number of maximally disjoint subcollections that contain I_k is the number of maximally disjoint subcollections of $\{I_1, \ldots, I_{k-2}\}$ times the number of maximally disjoint subcollections of $\{I_{k+2}, \ldots, I_N\}$, or $Q_{k-2} \times Q_{N-(k+1)}$.

If $N \geq 3$, any maximally disjoint subcollection of $\{I_j\}_{j=1}^N$ must contain exactly one of I_1 or I_2, but not both. From the above argument we know how many maximally disjoint subcollections contain I_1 and how many contain I_2, and we find the recursive relation

$$Q_N = Q_{N-2} + Q_{N-3}.$$

This establishes (i). To show (ii), we see it suffices to show that the inequalities

$$\frac{Q_{N-2}}{Q_N} \geq \frac{1}{4},$$

$$\frac{Q_{N-3}}{Q_N} \geq \frac{1}{4},$$

and

$$\frac{Q_{k-2}Q_{N-(k+1)}}{Q_N} \geq \frac{1}{4}, \quad k = 3, \ldots, N-2$$

hold for an arbitrary integer $N \geq 4$. (The cases $N = 1, 2, 3$ follow readily by inspection.)

The recursive relation for Q_j coincides with that of the *Padovan numbers* P_j, defined by

$$P_0 = P_1 = P_2 = 1, \quad P_j = P_{j-2} + P_{j-3} \quad \text{for} \quad j \geq 3\,. \tag{2.1}$$

In particular, we have $P_{j+1} = Q_j$. In terms of the Padovan numbers, it suffices to show that

$$\frac{P_{N-1}}{P_{N+1}} \geq \frac{1}{4}, \tag{2.2}$$

$$\frac{P_{N-2}}{P_{N+1}} \geq \frac{1}{4}, \quad \text{and} \tag{2.3}$$

$$\frac{P_{k-1}P_{N-k}}{P_{N+1}} \geq \frac{1}{4}, \quad k = 1, 2, \ldots, N \tag{2.4}$$

hold for an arbitrary integer $N \geq 4$. As $P_0 = P_1 = 1$, the first two inequalities are implied by the third one, so it suffices to show (2.4).

We now associate the Padovan numbers with the generating function

$$P(x) = \sum_{n=0}^{\infty} P_n x^n.$$

The recursive relation satisfied by the Padovan numbers yields

$$\begin{aligned} P(x) &= 1 + x + x^2 + \sum_{n=3}^{\infty} (P_{n-2} + P_{n-3}) x^n \\ &= 1 + x + x^2 P(x) + x^3 P(x). \end{aligned}$$

Solving for $P(x)$ yields

$$P(x) = \frac{1+x}{1-x^2-x^3}.$$

Standard partial fraction decomposition permits us to express $P(x)$ as

$$P(x) = -\left(\frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} \right),$$

where

$$\begin{aligned} \alpha &= \sqrt[3]{\frac{25}{54} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{25}{54} - \sqrt{\frac{23}{108}}} - \frac{1}{3} \sim 0.75488 \quad \text{and} \\ \beta = \bar{\gamma} &= -\frac{1 + \alpha + i\sqrt{4\alpha(1+\alpha) - (1+\alpha)^2}}{2} \sim -0.8774 - 0.7449i, \end{aligned}$$

are the roots of the cubic equation $1 - x^2 - x^3 = 0$,

$$\begin{aligned} A &= \frac{1+\alpha}{(\alpha-\beta)(\alpha-\gamma)} \sim 0.5451, \text{ and} \\ B = \bar{C} &= \frac{1+\beta}{(\beta-\alpha)(\beta-\gamma)} \sim -0.2726 + 0.0740i. \end{aligned}$$

Expressing $\frac{1}{x-\alpha}$, $\frac{1}{x-\beta}$, $\frac{1}{x-\gamma}$ in the standard way as geometric series, we find

$$P(x) = \sum_{n=0}^{\infty} \left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}} + \frac{C}{\gamma^{n+1}} \right) x^n.$$

This enables us to express the numbers P_n as

$$P_n = \frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}} + \frac{C}{\gamma^{n+1}}.$$

Now, from (2.1) an elementary induction gives the estimates

$$\frac{2\alpha^5}{\alpha^{n+1}} \leq P_n \leq \frac{4\alpha^7}{\alpha^{n+1}} \quad \text{for} \quad n \geq 4. \tag{2.5}$$

Accordingly, we estimate

$$\frac{P_{k-1}P_{N-k}}{P_{N+1}} \geq \frac{\frac{2\alpha^5}{\alpha^k}\frac{2\alpha^5}{\alpha^{N-k+1}}}{\frac{4\alpha^7}{\alpha^{N+2}}} = \alpha^4 > 0.32 \quad \text{for} \quad k-1, N-k, \quad \text{and} \quad N+1 \geq 4,$$

and it remains to verify (2.4) for $k = 1, 2, 3, 4$. If $N \geq 8$ we may handle the cases $k = 1, 2, 3, 4$ by estimating P_{N+1} and P_{N-k} with (2.5) while using the exact value for P_{k-1}. This yields

$$\frac{P_{k-1}P_{N-k}}{P_{N+1}} \geq \frac{\alpha^2}{2} > 0.28,$$

the smallest value occurring for the $k = 3$ case. This leaves us to deal with the fewer than 28 cases associated to $1 \leq k \leq 4$ and $1 \leq N \leq 7$. The associated values for $\frac{P_{k-1}P_{N-k}}{P_{N+1}}$ are provided in the following table.

$\frac{P_{k-1}P_{N-k}}{P_{N+1}}$	k = 1	k = 2	k = 3	k = 4
N = 1	1			
N = 2	1/2	1/2		
N = 3	1/2	1/2	1/2	
N = 4	2/3	1/3	1/3	2/3
N = 5	1/2	1/2	1/4	1/2
N = 6	3/5	2/5	2/5	2/5
N = 7	4/7	3/7	2/7	4/7
N = 8	5/9	4/9	1/3	4/9

TABLE 1. $\frac{P_{k-1}P_{N-k}}{P_{N+1}}$ for small N, k

We see then that a minimum value of $\frac{1}{4}$ occurs for $\frac{P_{k-1}P_{N-k}}{P_{N+1}}$ when $k = 3$ and $N = 5$. The following figure illustrates this scenario. Observe that the third of five intervals occurs only once in the four possible maximally disjoint subcollections of $\{[0, 1/5], [1/5, 2/5], \ldots, [4/5, 1]\}$. This shows that the probability that a given interval $I \in \{I_j\}_{j=1}^N$ lies in a maximally disjoint subcollection is greater than or equal to $\frac{1}{4}$, proving (ii) and completing the proof of the theorem.

$\square$

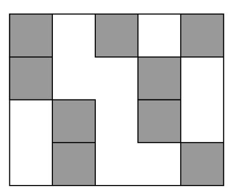

FIGURE 2. Maximally Disjoint Subcollections of Five Intervals

Remark : The factor of $\frac{1}{4}$ in the above proof could be improved upon very slightly, simply by recognizing that in the case of five intervals one could select either the first, third, and fifth intervals or the second, and fourth intervals and then look for the minimum value associated to higher values of N. Via this probabilistic technique, however, we cannot escape the asymptotic behavior as N tends to infinity for the fixed value $k = 3$ of $\frac{P_{k-1}P_{N-k}}{P_{N+1}}$, which is $\frac{\alpha^2}{2} \sim 0.28$. In this regard we are unable to achieve the sharp constant 1/2. However, we are pleased to have achieved the constant $\frac{1}{4}$ without having taken recourse to using ordering or Besicovitch-type arguments typical to proofs of covering lemmas .

3. FUTURE DIRECTIONS

Certainly this initial foray into proving covering theorems via probabilistic methods has been promising. Much more work in this area remains to be done. We believe that the following problems would be particularly suitable for further research in this area.

Problem 1: A natural next step would be to prove the Vitali Covering Lemma for balls or cubes in $\mathbb{R}^d$ via probabilistic methods. New ideas are clearly needed here, as the proof we have provided in the $\mathbb{R}^1$ context heavily relies on ordering properties of the real line that are unavailable in $\mathbb{R}^d$ for $d > 1$. In particular, we relied on being able to, given a finite collection of intervals $\{I_j\}$ in $\mathbb{R}$, find a subcollection $\{\tilde{I}_j\}$ such that $\cup I_j = \cup \tilde{I}_j$, where no more than two $\tilde{I}_j$ overlap at any given point. This type of reduction is unavailable in a higher dimensional scenario.

Problem 2: This problem involves a Vitali covering lemma on multiple scales simultaneously. Given an interval $I \subset \mathbb{R}$ and a constant $d > 0$, let dI denote the interval concentric with I and with length $d|I|$. Does there exist a constant $c > 0$ such that, given any finite collection of intervals $\{I_j\}$ and any positive integer k, there exists a subcollection $\{\tilde{I}_j\} \subseteq \{I_j\}$ satisfying

i) $|\cup \tilde{I}_j| \geq c|\cup I_j|$, and
ii) $\sum_j \chi_{2^k \tilde{I}_j} \leq 2^k$?

In seeking to prove this result, we have tried but at present failed to find an algorithm for finding the appropriate subcollection $\{\tilde{I}_j\}$. It is quite possible that the appropriate approach is to simply show the desired subcollection exists by probabilistic means.

Problem 3: Let $\mathcal{B}$ be a collection of open sets in $\mathbb{R}^d$ of finite measure. We may define the associated maximal operator $M_{\mathcal{B}}$ by

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| .$$

Suppose the maximal operator $M_{\mathcal{B}}$ is of *weak type* $(1,1)$, i.e., there exists a postive finite constant C such that

$$|\{x \in \mathbb{R}^d : M_{\mathcal{B}} f(x) > \alpha\}| \leq C \frac{\|f\|_1}{\alpha} \quad \text{for} \quad f \in L^1(\mathbb{R}^d) .$$

Must there be a covering lemma associated to $\mathcal{B}$? In particular, does there exist a constant $c > 0$ such that, given a finite collection $\{R_j\} \subseteq \mathcal{B}$, there exists a subcollection $\{\tilde{R}_j\} \subseteq \{R_j\}$ satisfying

i) $|\cup \tilde{R}_j| \geq c|\cup R_j|$, and
ii) $\sum_j \chi_{\tilde{R}_j} \leq c^{-1}$?

Note that, if $\mathcal{B}$ is the collection of balls in $\mathbb{R}^d$, $M_{\mathcal{B}}$ is the well-known Hardy-Littlewood maximal operator, and this operator is known to be of weak type $(1,1)$. (See, e.g., [12].) So the usual Vitali Covering Lemma corresponds to a special case of this problem. It is interesting that Córdoba and Fefferman have shown in [4] that covering lemmas do exist for maximal operators that are of weak type (p,p) for $1 < p < \infty$, but somewhat surprisingly the $p = 1$ case remains unresolved. The generality of this problem also suggests that an explicit covering algorithm may not be feasible, but possibly the existence of the desired $\{\tilde{R}_j\}$ may be proved by probabilistic methods.

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