Methods of Functional Analysis and Topology
Vol. 24 (2018), no. 1, pp. 41–52

ADJUNCTION FORMULA, POINCARÉ RESIDUE AND HOLOMORPHIC DIFFERENTIALS ON RIEMANN SURFACES

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Abstract. There is still a big gap between knowing that a Riemann surface of genus g has g holomorphic differential forms and being able to find them explicitly. The aim of this paper is to show how to construct holomorphic differential forms on compact Riemann surfaces. As known, the dimension of the space $H^1(\mathcal{D},\mathbb{C})$ of holomorphic differentials of a compact Riemann surface $\mathcal{D}$ is equal to its genus, $\dim H^1(\mathcal{D},\mathbb{C}) = g(\mathcal{D}) = g$. When the Riemann surface is concretely described, we show that one can usually present a basis of holomorphic differentials explicitly. We apply the method to the case of relatively complicated Riemann surfaces.

1. Adjunction formula and Poincaré residue

Let M be a compact complex manifold of dimension n and let $\mathcal{D} \subset M$ be a smooth hypersurface. Let $N_{\mathcal{D}}$ be the normal bundle on $\mathcal{D}$ which can be defined as the quotient $N_{\mathcal{D}} = \dfrac{T'_M|_{\mathcal{D}}}{T'_{\mathcal{D}}}$, where T'_M and $T'_{\mathcal{D}}$ are the tangent bundles on M and $\mathcal{D}$ respectively (The sign $'$ is added to recall the adjective holomorphic). The dual of $N_{\mathcal{D}}$ is called the conormal bundle and we denote it by $N^*_{\mathcal{D}}$. This is a sub-bundle of the cotangent bundle $T'^*_M|_{\mathcal{D}}$; i.e., the set of cotangent vectors on M vanishing on $T'_{\mathcal{D}} \subset T'_M|_{\mathcal{D}}$. We suppose that $\mathcal{D}$ is locally defined by functions $f_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)$. Therefore, the line bundle $[\mathcal{D}]$ on M is defined by the transition functions $g_{\alpha\beta} = \dfrac{f_\alpha}{f_\beta}$. Since $f_\alpha \equiv 0$ on $\mathcal{D} \cap \mathcal{U}_\alpha$, df_α is a section of the conormal bundle $N^*_{\mathcal{D}}$ on $\mathcal{D} \cap \mathcal{U}_\alpha$. Moreover, by assumption, $\mathcal{D}$ is smooth so df_α does not vanish on $\mathcal{D} \cap \mathcal{U}_\alpha$. Moreover, we have on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{D}$, that $df_\alpha = d(g_{\alpha\beta} f_\beta) = g_{\alpha\beta} df_\beta$, and since $\mathcal{D}$ is smooth, df_α is non-zero on $\mathcal{D}$. So this shows that the sections $df_\alpha \in H^1(\mathcal{U}_\alpha \cap \mathcal{D}, \mathcal{O}_{\mathcal{D}}(N^*_{\mathcal{D}}))$ determine a global section, which does not vanish on $\mathcal{D}$, of the line bundle $N^*_{\mathcal{D}} \otimes [\mathcal{D}]|_{\mathcal{D}}$. Therefore, the latter is trivial (a bundle that has its holomorphic sections is trivial) and we have the adjunction formula

$$N^*_{\mathcal{D}} = [-\mathcal{D}]|_{\mathcal{D}}. \tag{1}$$

Now let $K_M = \Lambda^n TM$ be the canonical bundle of M where n is the dimension of M. This is a line bundle contrary to tangent and cotangent bundles. Recall that the sections of K_M are the holomorphic n-forms, i.e., $\mathcal{O}(K_M) = \Omega^n_M$. To determine the canonical bundle $K_{\mathcal{D}}$ of $\mathcal{D}$ as a function of K_M, one proceeds as follows: consider the exact sequence

$$0 \longrightarrow N^*_{\mathcal{D}} \longrightarrow T'^*_M|_{\mathcal{D}} \longrightarrow T'^*_{\mathcal{D}} \longrightarrow 0.$$

We deduce that

$$T'^*_M|_{\mathcal{D}} = N^*_{\mathcal{D}} \oplus T'^*_{\mathcal{D}}, \quad \left(\Lambda^n T'^*_M\right)\Big|_{\mathcal{D}} \simeq \Lambda^{n-1} T'^*_{\mathcal{D}} \otimes N^*_{\mathcal{D}}, \quad K_M|_{\mathcal{D}} = K_{\mathcal{D}} \otimes N^*_{\mathcal{D}},$$

2010 *Mathematics Subject Classification.* 34M45, 34M05, 11G10, 14H40, 14K02, 14H70.

Key words and phrases. Adjunction formula, Poincaré residue, Riemann surfaces, holomorphic differentials.

and taking account of formula (1), one obtains

$$K_{\mathcal{D}} = K_M|_{\mathcal{D}} \otimes N_{\mathcal{D}} = K_M|_{\mathcal{D}} \otimes [\mathcal{D}]|_{\mathcal{D}} = (K_M \otimes [\mathcal{D}])|_{\mathcal{D}}.$$

Therefore, we get the following.

Proposition 1. *Let $\mathcal{D} \subset M$ be a smooth hypersurface, $N^*\mathcal{D}$ be the dual bundle of the normal bundle $N_{\mathcal{D}}$ on $\mathcal{D}$ and $K_{\mathcal{D}}$ (resp. K_M) the canonical bundle on $\mathcal{D}$ (respectively M). Then we have the following adjunction formulas:*

$$N^*_{\mathcal{D}} = [-\mathcal{D}]|_{\mathcal{D}}, \quad K_{\mathcal{D}} = (K_M \otimes [\mathcal{D}])|_{\mathcal{D}}.$$

Example 1. Let $Z_0, \dots, Z_n$ be homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$ and

$$x_1 = \frac{Z_1}{Z_0}, \quad x_2 = \frac{Z_2}{Z_0}, \quad \dots, \quad x_n = \frac{Z_n}{Z_0},$$

the affine coordinates in $\mathbb{P}^n(\mathbb{C})\backslash\{Z_0 = 0\}$. Let us define the divisor of the n-meromorphic form

$$\omega = \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

To do this, consider on $\mathbb{P}^n(\mathbb{C})\backslash\{Z_n = 0\}$ the following affine coordinates:

$$u_0 = \frac{Z_0}{Z_n}, \quad u_1 = \frac{Z_1}{Z_n}, \quad \dots \quad, u_{n-1} = \frac{Z_{n-1}}{Z_n}.$$

We have

$$u_0 = \frac{1}{x_n}, \quad u_1 = \frac{x_1}{x_n}, \quad \dots \quad, u_{n-1} = \frac{x_{n-1}}{x_n},$$

and

$$\omega = \frac{1}{u_0^{n+1}} du_0 \wedge du_1 \wedge \dots \wedge du_{n-1}.$$

Hence the divisor of ω is $(\omega) = -(n+1)H_\infty$, where H_∞ is a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and

$$K_{\mathbb{P}^n(\mathbb{C})} = [(\omega)] = [-(n+1)H_\infty].$$

Let $\omega \in \Omega^n_M(\mathcal{D})$ be a meromorphic n-form on $\mathcal{D}$ with at worst a simple pole along $\mathcal{D}$. Let $(z_1, \dots, z_n)$ be complex local coordinates on an open subset of M and assume that $\mathcal{D}$ is defined locally by functions $f(z_1, \dots, z_n)$. Therefore,

$$\omega = \frac{g(z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n}{f(z_1, \dots, z_n)} \in K_M,$$

where $g(z_1, \dots, z_n)$ is a holomorphic function. Taking into account the second adjunction formula (Proposition 1), one can write ω in the form

$$\omega = \frac{df}{f} \wedge \widetilde{\omega} \in K_M,$$

where $\widetilde{\omega}$ is a holomorphic $(n-1)$-form. Therefore,

$$\widetilde{\omega} = (-1)^{j-1} \frac{g(z_1, \dots, z_n) dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n}{\frac{\partial f}{\partial z_j}},$$

where j is such that $\dfrac{\partial f}{\partial z_j} \neq 0$. It is therefore possible to define an application (corresponding to the second adjunction formula) by setting

$$PR : \Omega^n_M(\mathcal{D}) \longrightarrow \Omega^{n-1}_V, \quad \omega \longmapsto PR(\omega) = \widetilde{\omega}|_{f=0},$$

and we call $PR(\omega)$ the Poincaré residue of ω. It is a useful formula as we shall see later that under certain conditions it allows explicitly to find holomorphic differentials on submanifolds.

From the exact sequence of sheaves

$$0 \longrightarrow \Omega^n_M \longrightarrow \Omega^n_M(\mathcal{D}) \overset{PR}{\longrightarrow} \Omega^{n-1}_{\mathcal{D}} \longrightarrow 0,$$

we obtain the long exact cohomology sequence

$$0 \longrightarrow H^0(M,\Omega^n_M) \longrightarrow H^0(M,\Omega^n_M(\mathcal{D})) \overset{PR}{\longrightarrow} H^0(\mathcal{D},\Omega^{n-1}_{\mathcal{D}})$$
$$\longrightarrow H^1(M,\Omega^n_M) \longrightarrow H^1(M,\Omega^n_M(\mathcal{D})) \longrightarrow \cdots$$

It should be noted that the kernel of the application of the Poincaré residue consists simply of an n-holomorphic form on M and this application is surjective if $H^1(M,\Omega^n_M) = 0$.

2. How to construct holomorphic differential forms on some compact Riemann surfaces

Let now M be an Abelian surface (a complex algebraic torus) and let $\mathcal{D}$ be a positive divisor on M. Define $\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on} M : (f) \geq -\mathcal{D}\}$, i.e., for $\mathcal{D} = \sum k_j \mathcal{D}_j$ a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a k_j-fold pole along $\mathcal{D}_j$. The divisor $\mathcal{D}$ is called ample when a basis $(f_0,\ldots,f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds M smoothly into $\mathbb{P}^N(\mathbb{C})$ for some k via the map

$$M \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad p \longmapsto [1 : f_1(p) : \ldots : f_N(p)],$$

then $k\mathcal{D}$ is called very ample. If we assume $\mathcal{D}$ to be very ample, then M is embedded into $\mathbb{P}^N(\mathbb{C})$ using the functions of $\mathcal{L}(\mathcal{D})$ where $N = \dim \mathcal{L}(\mathcal{D}) - 1$ and in particular $\mathcal{D} \subseteq \mathbb{P}^N(\mathbb{C})$, $g = N + 2$, (the geometric genus of $\mathcal{D}$).

Let $\dot{z} = f(z)$, $z \in \mathbb{C}^n$, be a system of differential equations governing a flow X_1 and let X_2 be any other flow on M commuting with the first (X_1 and X_2 are linearly independent vector fields in a neighborhood of a point p in M). Let the vector field have Laurent solutions

$$z_j(t) = t^{-1}\left(z_j^{(0)} + z_j^{(1)}t + +z_j^{(2)}t^2 + \cdots\right), \quad 1 \leq j \leq n \tag{2}$$

whose coefficients are rational functions on an $(n-1)$-dimensional affine variety. Consider the complex affine invariant manifold (thought of as lying in $\mathbb{C}^n$) defined by the intersection of the constants of the motion of this system and let $\mathcal{D}$ be the pole solutions (2) restricted to this invariant manifold, i.e., the set of Laurent solution which remain confined to a fixed affine invariant manifold, related to specific values of the constants (generic constants).

Consider a basis $f_0 = 1$, f_1,…,f_N of $\mathcal{L}(\mathcal{D})$; these functions will be polynomials in $z_1,\ldots,z_n$ and if the Laurent series z_j (see (2)) are substituted into these polynomials f_j, they have the following property

$$f_j(z_1(t),\ldots,z_n(t)) = t^{-1}\left(f_j^{(0)} + f_j^{(1)}t + +f_j^{(2)}t^2 + \cdots\right). \tag{3}$$

Let $W = \{f_1^{(0)},\ldots,f_N^{(0)}\}$ be the span of the residues of the functions $f_0 = 1$, f_1,…,f_N in $\mathcal{L}(\mathcal{D})$, the residues being meromorphic functions on $\mathcal{D}$. Let dt_1, dt_2 be the holomorphic 1-forms on M defined by $dt_i(X_j) = \delta_{ij}$. Giving the vector fields X_1 and X_2 above and associated times t_1 and t_2 defines on $\mathcal{D}$ the following differentials $\omega_i = dt_i|_{\mathcal{D}}$, restrictions of dt_1 and dt_2 to $\mathcal{D}$:

$$\omega_1 = dt_1|_{\mathcal{D}} = \frac{1}{\Delta f_i^{(0)}} d\left(\frac{f_j^{(0)}}{f_i^{(0)}}\right), \quad \omega_2 = dt_2|_{\mathcal{D}} = -f^{(0)}X_2\left\{\frac{1}{f_i}\right\}\omega_1$$

with

$$\Delta = \lim_{t\to 0} \det \left(\begin{array}{cc} X_1\left\{\frac{1}{f_i}\right\} & X_2\left\{\frac{1}{f_i}\right\} \\ X_1\left\{\frac{f_j}{f_i}\right\} & X_2\left\{\frac{f_j}{f_i}\right\} \end{array} \right)\Bigg|_{\text{Laurent solutions } z(t,p,\mathcal{D})}.$$

Taking into account the adjunction formula for the Poincaré residue applied to the 2-form $\omega = f_j dt_1 \wedge dt_2$, with $f_j \in \mathcal{L}(\mathcal{D})$, we have

$$\begin{aligned} \omega = \frac{dt_1 \wedge dt_2}{\frac{1}{f_j}} \longrightarrow \text{Rés } \omega|_{\mathcal{D}} &= -\left.\frac{dt_1}{\frac{\partial}{\partial t_2}\left(\frac{1}{f_j}\right)}\right|_{\mathcal{D}}, \\ &= \left.\frac{dt_2}{\frac{\partial}{\partial t_1}\left(\frac{1}{f_j}\right)}\right|_{\mathcal{D}}, \\ &= \left.\frac{dt_2}{\frac{\partial}{\partial t_1}\left(\frac{t_1}{f_j^{(0)}} + \circ(t_1^2)\right)}\right|_{\mathcal{D}}, \\ &= f_j^{(0)} dt_2\Big|_{\mathcal{D}}, \\ &= f_j^{(0)}\omega_2, \end{aligned}$$

where $f_j^{(0)}$ is the residue appearing in (3) and $\dfrac{\partial}{\partial t_j}$ is the derivative according to the vector field X_j. Similarly, using the second vector field X_2, we find $\omega = \widetilde{f}_j^{(0)}\omega_1$, with $\widetilde{f}_j^{(0)}$ the residue.

Since M is an Abelian surface and $\mathcal{D}$ a positive divisor, we deduce from the Kodaira-Nakano vanishing theorem [5, 11] that $H^1(M, \Omega_M^2(\mathcal{D}) = 0$ and again from the above exact sequence of sheaves

$$0 \longrightarrow \Omega_M^2 \longrightarrow \Omega_M^2(\mathcal{D}) \stackrel{PR}{\longrightarrow} \Omega_{\mathcal{D}}^1 \longrightarrow 0,$$

we get a long exact sequence

$$0 \longrightarrow H^0(M,\Omega_M^2) \longrightarrow H^0(M,\Omega_M^2(\mathcal{D})) \stackrel{PR}{\longrightarrow} H^0(\mathcal{D},\Omega_{\mathcal{D}}^1) \longrightarrow H^1(M,\Omega_M^2) \longrightarrow 0.$$

Note that $dt_1 \wedge dt_2$ is the only holomorphic 2-form on the Abelian surface M. Then $\dim H^0(\mathcal{D},\Omega_{\mathcal{D}}^1) = g$ (genus of $\mathcal{D}$), $H^1(M,\Omega_M^2) = H_{\overline{\partial}}^{1,2}(M) = H_{\overline{\partial}}^{1,0}(M)$, i.e., the space of holomorphic 1-forms on M, which is 2-dimensional by the Dolbeault theorem [5, 11] and we have $H^0(M,\Omega_M^2) = \mathbb{C}$. It follows that except for a 2-dimensional space, every holomorphic 1-form on $\mathcal{D}$ is the Poincaré Residue of a meromorphic 2-form on M with pole along $\mathcal{D}$. Therefore, we have the following proposition :

Proposition 2. *The differentials ω_1, ω_2, $f_j^{(0)}\omega_2, 1 \leq j \leq g-2$, form a basis for the space of holomorphic differential forms on $\mathcal{D}$, i.e.,*

$$H^0(\mathcal{D},\Omega_{\mathcal{D}}^1) = \{\omega_1,\omega_2\} \oplus \left\{f_j^{(0)}\omega_2, 1 \leq j \leq g-2\right\} = \{\omega_1,\omega_2\} \oplus W\omega_2.$$

The space W consists of all the residues of the functions of $\mathcal{L}(\mathcal{D})$.

As an immediate consequence of this proposition, we have

Corollary 1. *The embedding of $\mathcal{D}$ into $\mathbb{P}^N(\mathbb{C})$ is the canonical embedding,*

$$\mathcal{D} \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad p \longmapsto (\omega_1,\omega_2,f_1^{(0)}\omega_2,\ldots,f_{g-2}^{(0)}\omega_2).$$

From the equation

$$\frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial w}dw = 0$$

on the Riemann surface $\mathcal{D}$ whose affine equation is $F(w,z) = 0$, we deduce that the rational differentials are written on $\mathcal{D}$

$$\omega = \frac{P(w,z)}{\frac{\partial F}{\partial w}}dz = -\frac{P(w,z)}{\frac{\partial F}{\partial z}}dw,$$

where $P(w,z)$ is an arbitrary rational function. In fact, the formula of the Poincaré residue states that all holomorphic differentials on the Riemann surface $\mathcal{D}$ are written in the form above with $P(w,z)$ a polynomial of degree $\leq$ ded $F-3$.

Let's start with a relatively simple cases. We shall first consider the well-known case of elliptic and hyperelliptic curves, leaving the case of complicated Riemann surfaces to be studied in the last section.

Example 2. Consider the elliptic curve $\mathcal{E}$ whose affine equation is

$$F(w,z) = w^2 - z(z-1)(z-\lambda) = 0, \quad \lambda \neq 0, 1.$$

Let us show that

$$\omega = \frac{dz}{\sqrt{z\,(z-1)\,(z-\lambda)}}$$

is the unique holomorphic differential on $\mathcal{E}$. Indeed, the branch points of $\mathcal{E}$ are: $0, 1, \lambda, \infty$. Since $g(\mathcal{E}) = 1$, then there is a single holomorphic differential on $\mathcal{E}$. The Poincaré residue formula is written in this case

$$\omega = \frac{z^k w^j}{2\sqrt{z(z-1)(z-\lambda)}}dz.$$

Consider $j = 0$, from where

$$\omega = \frac{z^k}{2\sqrt{z(z-1)(z-\lambda)}}dz.$$

In the neighborhood of the point $z = 0$, one chooses as local parameter $t = \sqrt{z}$. Therefore

$$\omega = \frac{t^{2k}}{\sqrt{(t^2-1)(t^2-\lambda)}}dt = at^{2k}(1 + o(t^2))dt, \quad a \equiv \text{constant},$$

which shows that ω is holomorphic for $k \geq 0$. In the neighborhood of $z = 1$ and $z = \lambda$, the same method is used and the same conclusion is obtained. In the neighborhood of $z = \infty$, one chooses as local parameter $t = \frac{1}{\sqrt{z}}$ from which follows

$$\omega = -\frac{dt}{\sqrt{t^{2k}(1-t^2)(1-\lambda t^2)}}dt = bt^{-2k}(1 + o(t^2))dt, \quad b \equiv \text{constant},$$

which shows that ω is holomorphic for $k \leq 0$. So in the neighborhood of $0, 1, \lambda, \infty$, the differential form ω is holomorphic if $k = 0$, i.e., if

$$\omega = \frac{dz}{\sqrt{z\,(z-1)\,(z-\lambda)}}.$$

It is also outside these points because if $c \in \mathbb{C}$, $z_0 \neq 0, 1, \lambda, \infty$, we choose as local parameter $t = z - z_0$, from where

$$\omega = \frac{dt}{2\sqrt{(t+z_0)(t+z_0-1)(t+z_0-\lambda)}}dt = c(1 + o(t^3))dt, \quad c \equiv \text{constant},$$

is holomorphic. In conclusion,

$$\omega = \frac{dz}{\sqrt{z\,(z-1)\,(z-\lambda)}}$$

is the only holomorphic differential on the elliptic curve $\mathcal{E}$.

Example 3. Let us show that the following differentials

$$\omega_k = \frac{z^{k-1}dz}{\sqrt{\prod_{j=1}^{2g+1}\,(z-z_j)}}, \quad k=1,2,\ldots,g,$$

form a basis of holomorphic differentials on the hyperelliptic curve $\mathcal{H}$ of genus g whose affine equation is

$$F(w,z) = w^2 - \prod_{j=1}^{2g+1}\,(z-z_j) = 0.$$

Indeed, the branch points of $\mathcal{H}$ are : $z_1,\ldots,z_{2g+1},\infty$. The number of holomorphic differentials on $\mathcal{H}$ is equal to the genus g of $\mathcal{H}$ and it suffices to use a reasoning similar to that of the previous example. In the neighborhood of $z=z_i$, $1\leq i\leq 2g+1$, one chooses a local parameter $t=\sqrt{z-z_i}$ and one obtains

$$\frac{2\left(z_i+t^2\right)^{k-1}tdt}{\sqrt{\prod_{j=1}^{2g+1}\,(t^2+z_i-z_j)}},$$

which are holomorphic for $k\geq 1$. Similarly, in the neighborhood of $z=\infty$, the local parameter $t=\dfrac{1}{\sqrt{z}}$ is chosen and we obtain

$$\omega = -\frac{2t^{2g+1}dt}{t^{2(k-g)}\sqrt{\prod_{j=1}^{2g+1}\,(1-z_jt^2)}},$$

which are holomorphic for $k\leq g$. Outside the points $z_1,\ldots,z_{2g+1},\infty$ the differentials in questions are obviously holomorphic.

Example 4. Let $\mathcal{D}$ be a Riemann surface defined by the equation

$$F(w,z) \equiv w^4 + 2P_2(z)w^2 + Q_4(z) = 0,$$

where $P_2(z)$ and $Q_4(z)$ are two polynomials of degree 2 and 4 respectively. Note that the quotient $\mathcal{E}=\mathcal{D}/\tau$ of $\mathcal{D}$ by the involution $\tau:\mathcal{D}\longrightarrow\mathcal{D}$, $(w,z)\longmapsto(-w,z)$, is an elliptic curve $\mathcal{E}$ defined by

$$y^2 = P_2^2(z) - Q_4(z).$$

The Riemann surface $\mathcal{D}$ is a double ramified covering of $\mathcal{E}$,

$$\mathcal{C}:\left\{\begin{array}{l} w^2 = -P_2(z)+y, \\ y^2 = P_2^2(z)-Q_4(z) \end{array}\right.$$

with four simple branch points $p_i\equiv(w=0, y=P_2(z), 4 \text{ roots of } Q_4(z)=0)$, $1\leq i\leq 4$, and four points at infinity q_i, $1\leq i\leq 4$. The divisors of w and z are

$$(w) = -\sum_{i=1}^4 q_i + \sum_{i=1}^4 p_i, \quad (w) = -\sum_{i=1}^4 q_i + 4 \text{ roots}.$$

According to the Riemann-Hurwitz formula [5, 11], the genus of $\mathcal{D}$ is

$$g(\mathcal{D} = n(g(\mathcal{E}-1)+1+\frac{\sharp\{\text{branch points}\}}{2} + 0 + 1 + \frac{4}{2} = 3.$$

Since $\dim H^1(\mathcal{D},\mathbb{C}) = 3$, then there exist 3 holomorphic differential forms on $\mathcal{D}$. We have

$$\frac{\partial F}{\partial w}(w,z) = 4(w^2 + P_2(z))w = 4yw,$$

and from the Poincaré residue formula, we know that the 3 holomorphic differentials on $\mathcal{D}_\varepsilon$ are of the form

$$\frac{g(w,z)dz}{\frac{\partial F}{\partial w}(w,z)} = \frac{g(\alpha,\beta,\varepsilon)dz}{4yw},$$

where $g(w,z)$ is a polynomial with $\deg g \leq \deg F - 3 = 1$. Therefore, the 3 holomorphic differentials $\omega_1, \omega_2, \omega_3$ on $\mathcal{D}$ can be written in the form

$$\omega_1 = \frac{dz}{y}, \quad \omega_2 = \frac{zdz}{yw}, \quad \omega_3 = \frac{dz}{yw}.$$

Example 5. Consider the following Riemann surface $\mathcal{C}$ of genus 3 :

$$F(\theta,\zeta) \equiv \left[\theta^2 + a\beta^2\gamma^2 + \left(b\gamma^2 + c\beta^2\right)\zeta\right]^2 - d^2\zeta\beta^2\gamma^2 = 0,$$

where θ is an arbitrary parameter, a, b, c, d are generic constants and where α, β, γ parametrizes the elliptic curve $\mathcal{E}$,

$$\alpha^2 = \gamma^2 - d_2^2, \quad \beta^2 = \gamma^2 + d_1^2.$$

(d_1, d_2 are two constants $\neq 0$). The map $\tau : \mathcal{C} \longrightarrow \mathcal{C}$, $(\theta,\zeta) \longmapsto (-\theta,\zeta)$, is an involution on $\mathcal{C}$ and the quotient $\mathcal{C}_0 = \mathcal{C}/\tau$ is an elliptic curve $\mathcal{C}_0$ defined by

$$\eta^2 = d^2\zeta\left(\zeta + d_1^2 + d_2^2\right)\left(\zeta + d_2^2\right).$$

The curve $\mathcal{C}$ is a double ramified covering of $\mathcal{C}_0$,

$$\mathcal{C} \longrightarrow \mathcal{C}_0, \quad (\theta,\eta,\zeta) \longmapsto (\eta,\zeta),$$

$$\mathcal{C} : \begin{cases} \theta^2 = -a\beta^2\gamma^2 - \left(b\gamma^2 + c\beta^2\right)\zeta + \eta, \\ \eta^2 = d^2\zeta\left(\zeta + d_1^2 + d_2^2\right)\left(\zeta + d_2^2\right). \end{cases}$$

From the Poincaré residue formula, the 3 holomorphic differentials $\omega_1, \omega_2, \omega_3$ on $\mathcal{C}$ are of the form

$$P(\theta,\zeta)\frac{d\zeta}{\frac{\partial F}{\partial \theta}(\theta,\zeta)}\bigg|_{F(\theta,\zeta)=0} = P(\theta,\zeta)\frac{d\zeta}{4\theta\eta},$$

where P is a polynomial of degree $\leq \deg F - 3 = 1$. Therefore, we have $\frac{d\zeta}{\eta}$, $\frac{\zeta d\zeta}{\theta\eta}$ and $\frac{d\zeta}{\theta\eta}$, forming a basis for the space of holomorphic differentials on $\mathcal{C}$.

3. Applications to more complicated Riemann surfaces

We now show how to obtain the holomorphic differential forms on more complicated Riemann surfaces.

3.1. **A Riemann surface related to the problem of the rotation of a solid body around a fixed point.** Consider two Riemann surfaces $\mathcal{D}_\varepsilon$, $\epsilon \equiv \pm i = \pm\sqrt{-1}$, defined by

$$P(\alpha,\beta,\epsilon) \equiv (\alpha^2 - 1)((\alpha^2 - 1)\beta^4 - P(\beta) + c = 0,$$

where $P(\beta) \equiv a\beta^2 - 2\epsilon b\beta - 1$, $\epsilon \equiv \pm i = \pm\sqrt{-1}$ and a, b, c are generic constants. These Riemann surfaces was obtained by the author [9] during the study of the complete algebraic integrability of the Kowalewski's top. The latter being a solid body rotating about a fixed point, whose equations of motion take on the form

$$\begin{aligned} \dot{m}_1 &= m_2m_3, & \dot{\gamma}_1 &= 2m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &= -m_1m_3 + 2\gamma_3, & \dot{\gamma}_2 &= m_1\gamma_3 - 2m_3\gamma_1, \\ \dot{m}_3 &= -2\gamma_2, & \dot{\gamma}_3 &= m_2\gamma_1 - m_1\gamma_2. \end{aligned} \tag{4}$$

The second flow commuting with the first is regulated by the equations

$$
\begin{array}{lrcl}
(5) & \dot{m}_1 &=& Dm_3 + 2B\gamma_3, \qquad \dot{\gamma}_1 = D\gamma_3, \\
& \dot{m}_2 &=& Cm_3 - 2A\gamma_3, \qquad \dot{\gamma}_2 = C\gamma_3, \\
& \dot{m}_3 &=& -Dm_1 - Cm_2 + 2\left(A\gamma_2 - B\gamma_1\right), \qquad \dot{\gamma}_3 = -D\gamma_1 - C\gamma_2,
\end{array}
$$

where $A = m_1^2 - m_2^2 - 4\gamma_1$, $B = 2\left(m_1m_2 - 2\gamma_2\right)$, $C = Am_1 + Bm_2$ and $D = Am_2 - Bm_1$. The differential equations (4) admit two distinct families of Laurent solutions depending on five free parameters. The first and second meromorphic solutions, with the Laurent series, are

$$
\begin{array}{rcll}
m_1(t) &=& \frac{\alpha}{t}\varepsilon\left(\alpha^2 - 2\right)\beta + o(t), & \gamma_1(t) = \frac{1}{2t^2} + o(t), \\
m_2(t) &=& \frac{\varepsilon\alpha}{t} - \alpha^2\beta + o(t), & \gamma_2(t) = \frac{\varepsilon}{2t^2} + o(t), \\
m_3(t) &=& \frac{\epsilon}{t} + \alpha\beta + o(t), & \gamma_3(t) = \frac{\beta}{t} + o(t).
\end{array}
$$

Proposition 3. *The differentials*

$$
(6) \qquad \omega_0 = \frac{d\beta}{u}, \quad \omega_1 = \frac{(\alpha^2 - 1)\beta^2 d\beta}{\alpha u}, \quad \omega_2 = \frac{d\beta}{\alpha u}
$$

form a basis for the space of holomorphic differential forms on $\mathcal{D}_\varepsilon$.

Proof. Note that the application

$$
\sigma_\epsilon : \mathcal{D}_\epsilon \longrightarrow \mathcal{D}_\epsilon, \quad (\alpha, \beta, \epsilon) \longmapsto (-\alpha, \beta, \epsilon)
$$

is an involution on $\mathcal{D}_\epsilon$ and the quotient $\mathcal{D}_\epsilon^0 = \mathcal{D}_\epsilon / \sigma_\epsilon$ by this involution is an elliptic curve defined by

$$
(7) \qquad u^2 = P^2(\beta) - 4c\beta^4.
$$

A direct calculation shows that each $\mathcal{D}_\epsilon$ can be seen as a 2-sheeted ramified covering of $\mathcal{D}_\epsilon^0$

$$
\mathcal{D}_\epsilon \longrightarrow \mathcal{D}_\epsilon^0, (\alpha, u, \beta, \epsilon) \longmapsto (u, \beta, \epsilon),
$$

$$
\mathcal{D}_\epsilon : \begin{cases} \alpha^2 = \dfrac{2\beta^4 + P(\beta) + u}{2\beta^4}, \\ u^2 = P^2(\beta) - 4c\beta^4. \end{cases}
$$

Each Riemann surface $\mathcal{D}_\epsilon$ has four points at infinity and four branch points on the elliptic curve $\mathcal{D}_\epsilon^0$ and according to the Riemann-Hurwitz formula, the genus of $\mathcal{D}_\epsilon$ is $g(\mathcal{D}_\epsilon) = 3$. Since $\dim H^1(\mathcal{D}_\varepsilon, \mathbb{C}) = 3$, then there exist 3 holomorphic differential forms on $\mathcal{D}_\varepsilon$. From the Poincaré residue formula, we know that the 3 holomorphic differentials on $\mathcal{D}_\varepsilon$ are of the form

$$
\frac{g(\alpha, \beta, \varepsilon) d\beta}{\frac{\partial P}{\partial \alpha}(\alpha, \beta, \varepsilon)} = \frac{g(\alpha, \beta, \varepsilon) d\beta}{\alpha u},
$$

where $g(\alpha, \beta, \varepsilon)$ is a polynomial of at most degree five in α and β. It is easy to verify that

$$
\omega_0 = \frac{d\beta}{u}, \quad \omega_1 = \frac{(\alpha^2 - 1)\beta^2 d\beta}{\alpha u}, \quad \omega_2 = \frac{d\beta}{\alpha u}
$$

form effectively a basis of holomorphic differentials on $\mathcal{D}_\varepsilon$. ∎

It was shown by the author [9] that the affine surface defined by the four constants of motion

$$\begin{aligned} Q_1 &= \frac{1}{2}\left(m_1^2+m_2^2\right)+m_3^2+2\gamma_1, \\ Q_2 &= m_1\gamma_1+m_2\gamma_2+m_3\gamma_3, \\ Q_3 &= \gamma_1^2+\gamma_2^2+\gamma_3^2, \\ Q_4 &= \frac{1}{16}\left(m_1^2+m_2^2\right)^2-\frac{1}{2}\left(m_1^2-m_2^2\right)\gamma_1-m_1m_2\gamma_2+\gamma_1^2+\gamma_2^2 \end{aligned}$$

completes into an Abelian surface M by adjoining a divisor $\mathcal{D}\equiv\mathcal{D}_{\epsilon=i}+\mathcal{D}_{\epsilon=-i}$.

Proposition 4. *Let dt_1, dt_2 be the two holomorphic 1-forms on M defined by $dt_i(X_j)=\delta_{ij}$ where t_1, t_2 are local coordinates on M, X_1 is the flow (4) and X_2 the other flow (5) commuting with the first. Then, the restrictions $\omega_i=dt_i|_{\mathcal{D}}$ to $\mathcal{D}$ are given by*

$$dt_{1|\mathcal{D}_\epsilon}=\omega_1=\frac{k_1\left(\alpha^2-1\right)\beta^2d\beta}{\alpha u},\quad dt_{2|\mathcal{D}_\epsilon}=\omega_2=\frac{k_2d\beta}{\alpha u},$$

where $k_1,k_2\in\mathbb{C}$ and ω_1,ω_2 are the holomorphic differential forms (6) on $\mathcal{D}_\varepsilon$. In addition, the 9 holomorphic differential forms on $\mathcal{D}$ are : ω_1, ω_2, $f_1^{(0)}\omega_2,\ldots,f_7^{(0)}\omega_2$, where $f_1^{(0)},\ldots,f_7^{(0)}$ are the residues of the functions $f_1,\ldots,f_7$ respectively.

Proof. In the neighborhood of a point $p\in\mathcal{D}_\epsilon\cap\{\alpha u\neq 0\}$ (where u is given in (7)), we consider two coordinates τ and x on M by setting

$$\tau=\frac{1}{m_3}=-\epsilon t+o(t),\quad x=\begin{cases} x_1= & -i\beta+o(t)\text{ sur }\mathcal{D}_{\epsilon=i},\\ x_2= & i\beta+o(t)\text{ sur }\mathcal{D}_{\epsilon=-i}.\end{cases}$$

We have

$$dt_1=\frac{\frac{\partial x}{\partial t_2}d\tau-\frac{\partial\tau}{\partial t_2}dx}{\frac{\partial\tau}{\partial t_1}\frac{\partial x}{\partial t_2}-\frac{\partial\tau}{\partial t_2}\frac{\partial x}{\partial t_1}},\quad dt_2=\frac{-\frac{\partial x}{\partial t_1}dt+\frac{\partial\tau}{\partial t_1}dx}{\frac{\partial\tau}{\partial t_1}\frac{\partial x}{\partial t_2}-\frac{\partial\tau}{\partial t_2}\frac{\partial x}{\partial t_1}},$$

$\dfrac{\partial}{\partial t_1}$ (resp. $\dfrac{\partial}{\partial t_2}$) is the derivative according to the vector field $X_1(4)$ (resp. $X_2(5)$). Using the Laurent series, we obtain

$$\frac{\partial\tau}{\partial t_1}=-\epsilon+o(t),\quad \frac{\partial x}{\partial t_1}=-2\alpha\beta^2+o(t),$$

$$\frac{\partial\tau}{\partial t_2}=-4\epsilon(\alpha^2-1)\beta^2+o(t),\quad \frac{\partial x}{\partial t_2}=8\alpha\left((\alpha^2-1)\beta^4-P(\beta)\right)+o(t),$$

where $P(\beta)\equiv c_1\beta^2-2\epsilon c_2\beta-1$. Hence

$$\begin{aligned} \omega_1 &\equiv dt_1|_{\mathcal{D}_\epsilon}=k_1\frac{(\alpha^2-1)\beta^2}{\alpha u}d\beta,\quad k_1\in\mathbb{C},\\ \omega_2 &\equiv dt_2|_{\mathcal{D}_\epsilon}=k_2\frac{d\beta}{\alpha u},\quad k_2\in\mathbb{C}.\end{aligned}$$

The space $\mathcal{L}(\mathcal{D})$ is spanned by the following functions

$$f_0=1,\quad f_1=m_1,\quad f_2=m_2,\quad f_3=m_3,\quad f_4=\gamma_3,\quad f_5=f_1^2+f_2^2,$$
$$f_6=4f_1f_4-f_3f_5,\quad f_7=\left(f_2\gamma_1-f_1\gamma_2\right)f_3+2f_4\gamma_2.$$

In addition, $\mathcal{D}$ is embedded into $\mathbb{P}^7(\mathbb{C})$ according to

$$p=(\alpha,u,\beta)\longmapsto\lim_{t\to 0}t(1,f_1(p),\ldots,f_7(p))=\left(0,f_1^{(0)}(p),\ldots,f_7^{(0)}(p)\right).$$

In a way that $\mathcal{D}_{\epsilon=i}$ and $\mathcal{D}_{\epsilon=-i}$ intersect each other transversely in four points at infinity $\left(\alpha=\pm1,\ u=\pm\beta^2\sqrt{a^2-4c},\ \beta=\infty\right)$ and that the geometric genus of $\mathcal{D}$ is 9. Since $\dim H^1(\mathcal{D},\mathbb{C})=9$, then there exist 9 holomorphic differential forms on $\mathcal{D}$. To determine them explicitly, we proceed as before. Consider the functions $f_0=1$, $f_1,\ldots,f_7$ of $\mathcal{L}(\mathcal{D})$

(see above). Taking into account the adjunction formula for the Poincaré residue applied to the 2-form $\omega = f_j dt_1 \wedge dt_2$, with $f_j \in \mathcal{L}(\mathcal{D})$, we have

$$\omega = \frac{dt_1 \wedge dt_2}{\frac{1}{f_j}} \longrightarrow \text{Res } \omega|_{\mathcal{D}} = -\left.\frac{dt_1}{\frac{\partial}{\partial t_2}\left(\frac{1}{f_j}\right)}\right|_{\mathcal{D}} = \left.\frac{dt_2}{\frac{\partial}{\partial t_1}\left(\frac{1}{f_j}\right)}\right|_{\mathcal{D}}$$

and we obtain as before

$$\text{Res } \omega|_{\mathcal{D}} = \left.\frac{dt_2}{\frac{\partial}{\partial t_1}\left(\frac{t_1}{f_j^{(0)}} + o(t_1^2)\right)}\right|_{\mathcal{D}} = f_j^{(0)} dt_2\Big|_{\mathcal{D}} = f_j^{(0)}\omega_2.$$

The differentials $\omega_1,\ \omega_2,\ f_1^{(0)}\omega_2,\ \ldots,\ f_7^{(0)}\omega_2$, form a basis for the space of holomorphic differential forms on $\mathcal{D}$, i.e.,

$$H^0(\mathcal{D}, \Omega^1_{\mathcal{D}}) = \{\omega_1, \omega_2\} \oplus \left\{f_j^{(0)}\omega_2, 1 \le j \le 7\right\} = \{\omega_1, \omega_2\} \oplus W\omega_2,$$

where the space W consists of all the residues of the functions of $\mathcal{L}(\mathcal{D})$. □

Corollary 2. *The embedding into $\mathbb{P}^7(\mathbb{C})$ via the residues of f_j is the canonical embedding of the curve $\mathcal{D}$ via its holomorphic differentials, except for the two differentials ω_1 and ω_2,*

$$p = (\alpha, u, \beta) \in \mathcal{D} \longmapsto \left\{\omega_2, f_1^{(0)}\omega_2, f_2^{(0)}\omega_2, \ldots, f_7^{(0)}\omega_2\right\} \in \mathbb{P}^7(\mathbb{C}).$$

3.2. **A Riemann surface related to the geodesic flow on the group $SO(4)$ for Manakov metric.** Recall that the equations of motion of the geodesic flow on the group $SO(4)$ for Manakov metric take on the form

$$\begin{aligned} \dot z_1 &= d_1 z_5 z_6, & \dot z_4 &= -d_2 z_2 z_6 + d_3 m_3 z_5,\\ \dot z_2 &= d_2 z_4 z_6, & \dot z_5 &= -d_3 z_3 z_4 + d_1 z_1 z_6,\\ \dot z_3 &= d_3 z_4 z_5, & \dot z_6 &= -d_1 z_1 z_5 + d_2 z_2 z_4, \end{aligned} \tag{8}$$

where $d_1^2 + d_2^2 + d_3^2 = 0$ and $d_1 d_2 d_3 = 1$. The second flow commuting with the first is regulated by the equations

$$\begin{aligned} \dot z_1 &= d_1 z_2 z_3, & \dot z_4 &= -d_3 z_2 z_6 - d_2 m_3 z_5,\\ \dot z_2 &= d_3 z_4 z_6 + d_1 z_1 z_3, & \dot z_5 &= d_2 z_3 z_4,\\ \dot z_3 &= -d_2 z_4 z_5 + d_1 z_1 z_2, & \dot z_6 &= d_3 z_2 z_4. \end{aligned} \tag{9}$$

The following four quadrics are constants of motion for this system:

$$H_1 = z_2^2 - z_3^2 + z_4^2 = a, \qquad H_3 = z_1^2 - z_2^2 + z_6^2 = c,$$
$$H_2 = z_3^2 - z_1^2 + z_5^2 = b, \qquad H_4 = z_1 z_4 + a z_2 z_5 + b z_3 z_6 = d.$$

The system of differential equations (8) has one single 5-parameter family of Laurent series solution :

$$\begin{aligned} z_1 &= -\frac{\beta\gamma}{t}\left(1 + \frac{\alpha\theta}{\beta\gamma}t - \frac{1}{6}(a(4\alpha^2 + d_2^2 - d_3^2) + b(4\alpha^2 - d_1^2) + c(4\alpha^2 + d_1^2)\right.\\ &\qquad\qquad \left. + \frac{3d\alpha}{2\beta\gamma}(\beta^2 + \gamma^2))t^2 + o(t^3)\right),\\ z_4 &= \frac{d_1\alpha}{t}\left(1 + \frac{1}{6}(2a(\gamma^2 - d_3^2)) + b(2\gamma^2 - d_3^2) + c(2\beta^2 + d_2^2)\right.\\ &\qquad\qquad \left. + \frac{3d\beta\gamma}{2\alpha}t^2 + o(t^3)\right). \end{aligned}$$

while the others are obtained by the following cyclic permutation:
$z_1 \to z_2 \to z_3 \to z_1$, $a \to b \to c \to a$, $d \to d$, $\alpha \to \beta \to \gamma \to \alpha$,
$z_4 \to z_5 \to z_6 \to z_4$, $d_1 \to d_2 \to d_3 \to d_1$, $\theta \to \theta$.

During the study of the complete algebraic integrability of this problem, Haine [6] has shown that the affine surface defined by the four quadrics above completes into an Abelian surface M by adjoining a smooth Riemann surface $\mathcal{D}$ of the following affine equation:

$$\theta^2 + a\beta^2\gamma^2 + b\alpha^2\gamma^2 + c\alpha^2\beta^2 + d\alpha\beta\gamma = 0,$$

where θ is an arbitrary parameter, a, b, c, d are generic constants and where α, β, γ parametrizes the elliptic curve $\mathcal{E}$,

$$\alpha^2 = \gamma^2 - d_2^2, \quad \beta^2 = \gamma^2 + d_1^2.$$

Two questions arise : how many holomorphic differentials are there on $\mathcal{D}$ and how to determine them explicitly? The curve $\mathcal{D}$ is a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$. The branch points are defined by the 16 zeroes of $a\beta^2\gamma^2 + b\alpha^2\gamma^2 + c\alpha^2\beta^2 + d\alpha\beta\gamma$ on $\mathcal{E}$. The curve $\mathcal{D}$ is unramified at infinity and by Riemann-Hurwitz formula, the genus of $\mathcal{D}$ is 9. As before, let dt_1, dt_2 be the two holomorphic 1-forms on M defined by $dt_i(X_j) = \delta_{ij}$ where t_1, t_2 are local coordinates on M, X_1 is the flow (8) and X_2 the other flow (9) commuting with the first. Let us designate by $\frac{\partial}{\partial t_1}$ (resp. $\frac{\partial}{\partial t_2}$) the derivative according to the vector field $X_1(8)$ (resp. $X_2(9)$). Taking the differentials $\frac{\partial}{\partial t_1}\left(\frac{1}{z_6}\right)$, $\frac{\partial}{\partial t_2}\left(\frac{1}{z_6}\right)$, $\frac{\partial}{\partial t_1}\left(\frac{z_2}{z_6}\right)$, $\frac{\partial}{\partial t_2}\left(\frac{z_2}{z_6}\right)$ and using the Laurent series, we show (upon solving linearly for dt_1 and dt_2) that the restrictions $\omega_i = dt_i|_{\mathcal{D}}$ to $\mathcal{D}$ are given by

$$dt_{1|\mathcal{D}} = \omega_1 = \frac{k_1\alpha^3 d\alpha}{\alpha\beta\gamma\theta}, \quad dt_{2|\mathcal{D}} = \frac{k_2\alpha d\alpha}{\alpha\beta\gamma\theta},$$

where $k_1, k_2 \in \mathbb{C}$. In addition, the functions of $\mathcal{L}(\mathcal{D})$ are spanned by $f_0 = 1$, $f_1 = z_1$, $f_2 = z_2$, $f_3 = z_3$, $f_4 = z_4$, $f_5 = z_5$, $f_6 = z_6$ and one additional function $f_7 = d_2z_1z_4 - d_1z_2z_5$. The 9 holomorphic differential forms on $\mathcal{D}$ are : ω_1, ω_2, $f_1^{(0)}\omega_2, \ldots, f_7^{(0)}\omega_2$, where $f_1^{(0)}, \ldots, f_7^{(0)}$ are the residues of the functions $f_1, \ldots, f_7$ respectively. Therefore, we have the following proposition.

Proposition 5. *We have*

$$H^0(\mathcal{D}, \Omega^1_{\mathcal{D}}) = \{\omega_1, \omega_2\} \oplus \left\{f_j^{(0)}\omega_2, 1 \le j \le 7\right\} = \{\omega_1, \omega_2\} \oplus W\omega_2,$$

where the space W consists of all the residues of the functions of $\mathcal{L}(\mathcal{D})$. More specifically,

$$H^0(\mathcal{D}, \Omega^1_{\mathcal{D}}) = \{\omega_1, \omega_2\} \oplus \left(-\beta\gamma, -d_1\alpha\gamma, -d_2\alpha\beta, \alpha, \beta, \gamma, -\frac{\theta}{d_1}\right)\omega_2.$$

Remark 1. Upon putting $\zeta \equiv \alpha^2$, the curve $\mathcal{D}$ can also be seen as a 4-sheeted unramified covering of the Riemann surface $\mathcal{C}$ (see example 5) of genus 3 :

$$\mathcal{C} : F(\theta, \zeta) \equiv \left[\theta^2 + a\beta^2\gamma^2 + \left(b\gamma^2 + c\beta^2\right)\zeta\right]^2 - d^2\zeta\beta^2\gamma^2 = 0.$$

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Received 08/02/2017