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STRONG CONVERGENCE IN TOPOLOGICAL SPACES

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ABSTRACT. Study of summability theory in an arbitrary topological space is not always an easy issue as many of the convergence methods need linear structure in the space. The concept of statistical convergence is one of the exceptional concepts of summability theory that can be considered in a topological space. There is a strong relationship between this convergence method and strong convergence which is another interesting concept of summability theory. However, dependence of the strong convergence to the metric, studying similar relationship directly in arbitrary Hausdorff spaces is not possible. In this paper we introduce a convergence method which extends the notion of strong convergence to topological spaces. This new definition not only helps us to investigate a similar relationship in a topological space but also leads to study a new type of convergence in topological spaces. We also give a characterization of statistical convergence.

1. INTRODUCTION

The classical summability theory, whose main aim is to make a non-convergent sequence converge, has been studied mostly in a linear space or in a space that has a group structure. Since the concept of convergence is one of the main concepts of topology, study of the summability theory in a topological space is an important subject of functional analysis. Some authors have studied summability theory in a topological space by assuming either topological space to have a group structure or a linear structure [1, 2, 3, 4, 5, 6, 7]. However, this assumptions restrict the scope. Therefore introducing summability methods in a topological space without any linear or group structure has became a popular problem. One of these methods is statistical convergence that can be studied in Hausdorff topological spaces without restricting the scope [8, 9, 10, 11]. More general method in a topological space that is introduced with the help of a probability distribution defined on the Borel sets of a topological space is distributional convergence [11, 12].

As the structure of statistical convergence is compatible with topological structure, i.e., it can be characterized considering the elements of topological base it can be studied in arbitrary Hausdorff topological spaces whereas similar idea vanishes when the issue is to study strong convergence whose definition strongly relies on the metric function. In this paper we study this concept in a topological space by considering a class of pre-metrics on the space that have some certain properties which are closely related to elements of topological base.

Let X be a Hausdorff topological space and let $A = (a_{nk})$ be a non-negative regular summability matrix. Then a sequence $x = (x_k)$ in X is said to be A-statistically convergent to $\alpha \in X$ [8, 9, 10] if for any open set U that contains α

$$\lim_n \sum_{k: x_k \notin U} a_{nk} = 0.$$

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It is not difficult to see that the definition of A-statistical convergence can be given with the elements of the base of the topology instead of open sets. Note here that we call a summability matrix regular when it preserves the limits of the convergent sequences [13]. Considering the base of the usual topology of reals we can easily get the following well-known definition of the statistical convergence of a real valued sequence [14, 15, 16, 17, 18]. Let $x = (x_k)$ be a real sequence and let $A = (a_{nk})$ be a non-negative regular summability matrix. Then x is said to be A-statistically convergent to the real number L if for any $\varepsilon > 0$

$$\lim_n \sum_{k:|x_k - L| > \varepsilon} a_{nk} = 0.$$

Let (X, d) be a metric space and let $A = (a_{nk})$ be non-negative regular summability matrix. Then a sequence $x = (x_k)$ in X is said to be A-strongly convergent to $\alpha \in X$ [19, 20, 21, 22] if

$$\lim_n \sum_k d(x_k, \alpha) a_{nk} = 0.$$

2. $A_{\mathcal{T}}$-Strong convergence

The dependence of the definition of strong convergence on the metric functions prevents the concept from being studied in arbitrary topological spaces. Therefore the well-known relationship [14, 19, 20] between strong convergence and statistical convergence can not be carried to topological spaces directly. However, this relationship is not basically coming from the usual properties of the metric functions. The motivation behind the following definition is to consider the necessary properties of the metric functions to define the concept of strong convergence in topological Lspaces and to get a similar relationship between this new concept and statistical convergence. Thus, the study of statistical convergence and more generally the study of summability will be easy in topological spaces.

Let (X, τ) be a Hausdorff space, let $\mathcal{B}_\alpha$ be the family of elements of the base of τ that contains $\alpha \in X$ and let $B_T(\alpha, \varepsilon) := \{\beta \in X \mid T(\beta, \alpha) < \varepsilon\}$. We denote the set of functions $T : X \times X \to [0, \infty)$ that satisfy the following condition with $\mathcal{L}(X)$: “For any $\varepsilon > 0$ and for any $\alpha \in X$ there exists $U_\varepsilon \in \mathcal{B}_\alpha$ such that $U_\varepsilon \subset B_T(\alpha, \varepsilon)$”. Although any function from $\mathcal{L}(X)$ is a pre-metric (see e.g. [23]) on X, the topology τ need not be pre-metrizable. Here the property that is satisfied by these pre-metrics makes them somewhat *compatible* with the topology τ.

Definition 1. Let X be a Hausdorff topological space, let $x = (x_k)$ be a sequence in X, let $A = (a_{nk})$ be a non-negative regular summability matrix and let $\mathcal{T} \subset \mathcal{L}(X)$. Then x is said to be $A_{\mathcal{T}}$-strongly convergent to point $\alpha \in X$ if for any $T \in \mathcal{T}$

$$\lim_n \sum_k T(x_k, \alpha) a_{nk} = 0,$$

where we assume that the series is convergent for each positive integer n.

The classical strong convergence method is regular due to the regularity of the matrix. As $\mathcal{T} \subset \mathcal{L}(X)$ and the matrix is regular it is not difficult to see that $A_{\mathcal{T}}$-strong convergence is regular. Besides, the metric on a metrizable topological space X belongs $\mathcal{L}(X)$.

The following example illustrates a $C_{\mathcal{T}}$-strongly convergent sequence in a *non-metrizable* Hausdorff space where $C = (c_{nk})$ is the well known Cesàro matrix defined by $c_{nk} = \frac{1}{n}$ if $k \le n$ and zero otherwise.

Example 1. Let τ_L be the lower limit topology on the set of all real numbers $\mathbb{R}$. It is known that $(\mathbb{R}, \tau_L)$ is not metrizable but it is a Hausdorff space [24]. Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \text{ is a perfect square}, \\ 1, & \text{otherwise} \end{cases}$$

and consider the family of functions $\mathcal{T} = \{T_r\}_{r \geq 0}$ defined for any $r \geq 0$ that

$$T_r(x, y) = \begin{cases} x - y, & x \geq y, \\ r, & x < y. \end{cases}$$

Let $T_r \in \mathcal{T}$. For any $\varepsilon > 0$ and for any $\alpha \in X$ if we consider $U_\varepsilon = [\alpha, \alpha + \varepsilon) \in \mathcal{B}_\alpha$ then for all $\beta \in B(\varepsilon)$ we have $T(\beta, \alpha) = \beta - \alpha < \varepsilon$. Thus $\mathcal{T} \subset \mathcal{L}(X)$. On the other hand it is easy to see that x is not convergent in the corresponding topology whereas for any $r \geq 0$ we get that

$$\lim_n \sum_k T(x_k, 1) c_{nk} = \lim_n \frac{1}{n} \sum_{\substack{1 \leq k \leq n \\ k = j^2}} r = 0,$$

which proves that x is $C_{\mathcal{T}}$-strongly convergent to 1.

It is known that in metric spaces if a sequence is A-strongly convergent then it is A-statistically convergent and if it is bounded and A-statistically convergent then it is A-strongly convergent [14]. Thus, these two concepts are equivalent over the space of all bounded sequences. Moreover Khan and Orhan [25] have characterized the A-strong convergence of a sequence by proving "a real sequence is A-strongly convergent to zero if and only if it is A-statistically convergent to zero and A-uniformly integrable". Note that a real sequence is said to be A-uniformly integrable if

$$\lim_{c \to \infty} \sup_n \sum_{|x_k| > c} |x_k| \, a_{nk} = 0,$$

where $A = (a_{nk})$ is a non-negative regular summability matrix [25].

In this section we extend the mentioned strong relation of the concepts of A-statistical convergence and A-strong convergence to the Hausdorff topological spaces. Let (X, τ) be a Hausdorff space. We denote the set of functions $T : X \times X \to [0, \infty)$ that satisfy the following condition with $\mathcal{Q}(X)$: "For all $\alpha \in X$ and for all $B \in \mathcal{B}_\alpha$ there exists $M > 0$ such that for all $\beta \notin B$, $T(\beta, \alpha) > M$".

Theorem 1. *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix and let $\mathcal{T} \subset \mathcal{L}(X)$. Then*
i) *If x is an $A_{\mathcal{T}}$-strongly convergent sequence to α in X and if $\mathcal{T} \cap \mathcal{Q}(X) \neq \varnothing$ then x is A-statistically convergent to α.*
ii) *If $x = (x_k)$ is an A-statistically convergent to α in X and if*

$$\sup_{T \in \mathcal{T}} \sup_k T(x_k, \alpha) < \infty, \tag{2.1}$$

then x is $A_{\mathcal{T}}$-strongly convergent to α.

Proof. **i)** Let x be a sequence in X that is $A_{\mathcal{T}}$-strongly convergent to α, let $B \in \mathcal{B}_\alpha$ and let $T_0 \in \mathcal{T} \cap \mathcal{Q}(X)$. Then there exists $M > 0$ such that for all $\beta \notin B$, $T_0(\beta, \alpha) > M$.

Now as T_0 is non-negative we have

$$\begin{aligned}\sum_k T_0(x_k,\alpha)a_{nk} &= \sum_{k:x_k\notin B} T_0(x_k,\alpha)a_{nk} + \sum_{k:x_k\in B} T_0(x_k,\alpha)a_{nk}\\ &\geq \sum_{k:x_k\notin B} T_0(x_k,\alpha)a_{nk}\\ &\geq M\sum_{k:x_k\notin B} a_{nk}.\end{aligned}$$

Thus we get

$$0\leq \sum_{k:x_k\notin B} a_{nk} \leq \frac{1}{M}\sum_k T_0(x_k,\alpha)a_{nk}.$$

Since B is an arbitrary element of the topological base the last inequality implies x is A-statistically convergent to α.

ii) Suppose that x is A-statistically convergent to α and (2.1) holds. Let T be an arbitrary element of $\mathcal{T}$. Then for any $\varepsilon>0$ there exists $B\in\mathcal{B}_\alpha$ such that for all $\beta\in B$, $T(\beta,\alpha)<\varepsilon$. Thus one can get

$$\begin{aligned}0 &\leq \sum_k T(x_k,\alpha)a_{nk}\\ &= \sum_{k:x_k\notin B} T(x_k,\alpha)a_{nk} + \sum_{k:x_k\in B} T(x_k,\alpha)a_{nk}\\ &\leq \sup_{T\in\mathcal{T}}\sup_k T(x_k,\alpha)\sum_{k:x_k\notin B} a_{nk} + \varepsilon\sum_k a_{nk}.\end{aligned} \tag{2.2}$$

Since A is regular and T is arbitrary, (2.2) implies that x is $A_{\mathcal{T}}$-strongly convergent to α. □

Remark 1. Note that the family $\mathcal{T}$ in Example 1 is a subfamily of $\mathcal{Q}(X)$. To see that let $T_r\in\mathcal{T}$, $\alpha\in\mathbb{R}$ and let $B\in\mathcal{B}_\alpha$. Then the most general form of B is like $[\alpha,\gamma)$. Thus we have for any $\beta\notin B$ that $T_r(\beta,\alpha)>\min\{r/2, T(\gamma,\alpha)/2\}$. Therefore, considering Theorem 1 one can have that the sequence $x=(x_k)$ in Example 1 is $C_{\mathcal{T}}$-statistically convergent to one.

On the other hand the family of functions $\mathcal{T}=\{T_r\}_{r\geq 0}$ defined for any $r\geq 0$ by

$$T_r(x,y)=\begin{cases} x-y, & x\geq y,\\ \frac{1}{r+1}, & x<y\end{cases}$$

is a subfamily of $\mathcal{L}(X)\cap\mathcal{Q}(X)$ that satisfies (2.1). In fact, $\sup_{r\geq 0}\sup_k T_r(x_k,1)=1$.

Consider a metric space (X,d). As $d\in\mathcal{L}(X)\cap\mathcal{Q}(X)$, considering $\mathcal{T}=\{d\}$ we see that any bounded sequence satisfies (2.1). Thus we obtain the following corollary immediately:

Corollary 1. *Let (X,d) be a metric space and let $A=(a_{nk})$ be a non-negative regular summability matrix. Then*
i) *If a sequence in X is A-strongly convergent to point $\alpha\in X$ then it is A-statistically convergent to α.*
ii) *If a bounded sequence in X is A-statistically convergent to point $\alpha\in X$ then it is A-strongly convergent to α.*

We remark again here that Corollary 1 is a well-known result of summability theory (see e.g [14]).

As mentioned before the concept of strong convergence is characterized with the help of the concept of uniform integrability. However, in arbitrary topological spaces the idea can not be conducted in a similar way. Following theorem generalizes this idea in topological spaces. In the theorem we use a condition which is similar to the concept of $(a_{n,k})$-compactly uniform integrability of [26].

Theorem 2. *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix and let $\mathcal{T} \subset \mathcal{L}(X) \cap \mathcal{Q}(X)$.*
i) *If a sequence $x = (x_k)$ in X is $A_{\mathcal{T}}$-strongly convergent to $\alpha \in X$ then for any $T \in \mathcal{T}$ and for any $\varepsilon > 0$ there exists a compact subset K_T of X such that*

$$\sup_n \sum_{x_k \notin K_T} T(x_k, L) a_{nk} < \varepsilon$$

holds.
ii) *If a sequence $x = (x_k)$ in X is A-statistically convergent to $\alpha \in X$ and if for any $\varepsilon > 0$ and for any $T \in \mathcal{T}$ there exists a compact subset K of X such that*

$$\sup_n \sum_{x_k \notin K} T(x_k, L) a_{nk} < \varepsilon$$

holds and for any compact subset E of X and $T \in \mathcal{T}$ there exists $M > 0$ such that $\sup_{x_k \in E} T(x_k, \alpha) < M$ then x is $A_{\mathcal{T}}$-strongly convergent to α.

Proof. **i)** Assume that x is $A_{\mathcal{T}}$-strongly convergent to α. Then for any given $T \in T$ and for any given $\varepsilon > 0$ there exists $n_0(T, \varepsilon)$ such that

$$\sum_k T(x_k, \alpha) a_{nk} < \varepsilon, \tag{2.3}$$

whenever $n > n_0$. As the series are convergent, for $n = 1, 2, \ldots, n_0$ there exists $k_0(T)$ such that

$$\sum_{k > k_0} T(x_k, \alpha) a_{nk} < \varepsilon.$$

Now consider the compact subset $K_T = \{x_1, x_2, \ldots, x_{k_0}\}$. Then it is obvious that for any $n = 1, 2, \ldots, n_0$

$$0 \leq \sum_{x_k \notin K_T} T(x_k, \alpha) a_{nk} \leq \sum_{k > k_0} T(x_k, \alpha) a_{nk} < \varepsilon. \tag{2.4}$$

Thus from (2.3) and (2.4) we get that

$$\sup_n \sum_{x_k \notin K_T} T(x_k, L) a_{nk} < \varepsilon.$$

ii) Let $x = (x_k)$ be a sequence in X that is A-statistically convergent to α, let $\varepsilon > 0$ and let $T \in \mathcal{T}$. From the hypothesis there exists a compact subset K of X such that

$$\sup_n \sum_{x_k \notin K} T(x_k, L) a_{nk} < \varepsilon / 2$$

and there exists $M > 0$ such that $\sup_{x_k \in K} T(x_k, \alpha) < M$. On the other hand as $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\alpha$ such that $U_\varepsilon \subset B_T(\alpha, \varepsilon/2)$. Now one can write for any positive integer n that

$$\begin{aligned}\sum_{x_k\in K} T(x_k,\alpha)a_{nk} &\le \sum_{x_k\in K\cap V_\varepsilon} T(x_k,\alpha)a_{nk} + \sum_{x_k\in K\setminus U_\varepsilon} T(x_k,\alpha)a_{nk}\\ &\le \frac{\varepsilon}{2}\sum_k a_{nk} + \sup_{x_k\in K} T(x_k,\alpha)\sum_{x_k\notin U_\varepsilon} a_{nk}\\ &< \frac{\varepsilon}{2}\sum_k a_{nk} + M\sum_{x_k\notin U_\varepsilon} a_{nk}.\end{aligned}$$

Clearly this implies

$$\begin{aligned}\sum_k T(x_k,\alpha)a_{nk} &\le \sum_{x_k\in K} T(x_k,\alpha)a_{nk} + \sum_{x_k\notin K} T(x_k,\alpha)a_{nk}\\ &\le \frac{\varepsilon}{2}\sum_k a_{nk} + M\sum_{x_k\notin V_\varepsilon} a_{nk} + \sup_n\sum_{x_k\notin K} T(x_k,\alpha)a_{nk}\\ &< \frac{\varepsilon}{2}\sum_k a_{nk} + M\sum_{x_k\notin U_\varepsilon} a_{nk} + \varepsilon/2.\end{aligned}$$

Now from the A-statistical convergence of x and the last inequality we have

$$0 \le \limsup_n \sum_k T(x_k,\alpha)a_{nk} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we get

$$\lim_n \sum_k T(x_k,\alpha)a_{nk} = 0.$$

Therefore, x is A_T-strongly convergent to α. □

Remark 2. Let $(X,\|\|)$ be a finite dimensional normed space and let $A = (a_{nk})$ be a non-negative regular summability matrix. If $x = (x_k)$ is a bounded sequence in X then it is obvious that

$$\sup_n \sum_{x_k\notin K} \|x_k-\alpha\|\, a_{nk} = 0,$$

where $K = \left\{x\in X : \|x\| \le \sup_k \|x_k\|\right\}$. Note that K is compact. On the other hand, we get for any compact subset E of X that

$$\sup_{x_k\in E} \|x_k-\alpha\| \le \|\alpha\| + \sup_k \|x_k\|.$$

Considering $\mathcal{T} = \{\|.\|\}$ all these imply that Corollary 1 is a consequence of Theorem 2 whenever the space is finite dimensional normed space. Here opposite to the classical case Theorem 1 can not be obtained from Theorem 2 in arbitrary Hausdorff spaces as for given $T\in\mathcal{T}$ the set $\left\{x\in X : T(x,\alpha) \le \sup_{T\in\mathcal{T}}\sup_k T(x_k,\alpha)\right\}$ does not have to be even compact. Note also that last theorem is a generalization of Theorem 2.1 of [25] for sequences in the topological spaces. For the sake of completeness we give the sketch of proof of that result with the help of Theorem 2.

Corollary 2. *A real sequence is A-strongly convergent to zero if and only if it is A-statistically convergent to zero and A-uniformly integrable.*

Proof. Note that we study in the usual topology of real numbers and so we consider the absolute value metric. If x is A-strongly convergent to zero then it is A-statistically convergent to zero. Considering $\mathcal{T} = \{|.|\}$, from Theorem 2 for any $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}$ such that

$$\sup_n \sum_{x_k \notin K} |x_k| \, a_{nk} < \varepsilon.$$

As K is bounded there exists $c_0 > 0$ such that $K \subset [-c, c]$ whenever $c > c_0$. Thus we can write

$$\sup_n \sum_{|x_k| > c} |x_k| \, a_{nk} \leq \sup_n \sum_{x_k \notin K} |x_k| \, a_{nk} < \varepsilon.$$

As $c_0 = c_0(\varepsilon)$ the last inequality implies that

$$\lim_{c \to \infty} \sup_n \sum_{|x_k| > c} |x_k| \, a_{nk} = 0.$$

Conversely if x is A-uniformly integrable then for any $\varepsilon > 0$ there exists $c_0 > 0$ such that

$$\sup_n \sum_{|x_k| > c} |x_k| \, a_{nk} < \varepsilon,$$

whenever $c > c_0$ which implies

$$\sup_n \sum_{x_k \notin K} |x_k| \, a_{nk} < \varepsilon,$$

where $K = [-c, c]$. On the other hand for any compact subset E of $\mathbb{R}$ it is obvious that $\sup_{x_k \in E} |x_k| < \infty$. Now if we consider this fact together with the compactness of K one can get from Theorem 2 that x is A-strongly convergent to zero. □

The final theorem of this chapter is a characterization of statistical convergence.

Theorem 3. *Let X be a Hausdorff topological space, let $A = (a_{nk})$ be a non-negative regular summability matrix, let $T \in \mathcal{L}(X) \cap \mathcal{Q}(X)$ and let $x = (x_k)$ be a sequence in X. Then x is A-statistically convergent to $\alpha \in X$ if only if*

$$\lim_n \sum_k \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk} = 0. \tag{2.5}$$

Proof. Assume that x is A-statistically convergent to α and let $\varepsilon > 0$. As $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\alpha$ such that $U_\varepsilon \subset B_T(\alpha, \varepsilon)$. Then we have

$$\begin{aligned} \sum_k \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk} &= \sum_{x_k \notin U_\varepsilon} \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk} + \sum_{x_k \in U_\varepsilon} \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk} \\ &\leq \sum_{x_k \notin U_\varepsilon} a_{nk} + \varepsilon \sum_{x_k \in U_\varepsilon} a_{nk} \\ &\leq \sum_{x_k \notin U_\varepsilon} a_{nk} + \varepsilon. \end{aligned}$$

Therefore (2.5) holds.

Conversely, assume that (2.5) holds and let $B \in \mathcal{B}_\alpha$. Since $T \in \mathcal{Q}(X)$ there exists $M > 0$ such that for all $\beta \notin B$, $T(\beta, \alpha) > M$. Thus one can get

$$\begin{aligned} \sum_{x_k \notin B} a_{nk} &\leq \frac{1 + M}{M} \sum_{x_k \notin B} \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk} \\ &\leq \frac{1 + M}{M} \sum_k \frac{T(x_k, \alpha)}{1 + T(x_k, \alpha)} a_{nk}, \end{aligned}$$

which finishes the proof. □

We obtain the following characterization of statistical convergence of real sequences immediately.

Corollary 3. *Let $x = (x_k)$ be a real sequence and let $A = (a_{nk})$ be a nonnegative regular summability matrix. Then x is A-statistically convergent to real number L if and only if*

$$\lim_n \sum_k \frac{|x_k - L|}{1 + |x_k - L|} a_{nk} = 0.$$

3. Conclusion

Study of summability theory in topological spaces is an interesting subject of functional analysis. However, it is not an easy issue to study summability in topological spaces due to the lack of linearity. Even so there are some summability methods that can be defined in topological spaces. One of these methods is statistical convergence whose definition mainly depends on the base of the topology. Considering strong convergence in metric spaces some well known necessary conditions and sufficient conditions for statistical convergence have been obtained by several authors. In this paper we define a convergence method, A_T-strong convergence, which is a generalization of strong convergence in topological spaces via a class of particular functions. We obtain some relationships between statistical convergence and A_T-strong convergence similar to the classical case. Finally, using a function of this particular class we obtain a new characterization of statistical convergence.

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