

ERGOREGIONS BETWEEN TWO ERGOSPHERES

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In blessed memory of Miroslav L'vovich Gorbachuk

ABSTRACT. For a stationary spacetime metric, black holes are spatial regions out of which disturbances do not propagate. In our previous work an existence and regularity theorem was proven for black holes in two space dimensions, in the case where the boundary of the ergoregion is a simple closed curve surrounding a singularity. In this paper we study the case of an annular ergoregion, whose boundary has two components.

1. INTRODUCTION

On $\mathbb{R}^{1+2} \cong \mathbb{R}_t^1 \times \mathbb{R}_x^2$, let g be a stationary pseudo-Riemannian (Lorentzian) metric with signature $(+1, -1, -1)$, and consider the associated wave equation $\square_g u = 0$.

As we take the general point of view of *analogue spacetimes*, we largely ignore issues of coordinate invariance, working in the global system of coordinates (t, x) . For convenience we sometimes write $t = x^0$, $x = (x^1, x^2)$, and denote the corresponding components of the metric $g_{ij} = g_{ij}(x)$, $0 \leq i, j \leq 2$. Here, the assumption that g is stationary means these depend only on x . Inverting the matrix of the g_{ij} gives the components of the inverse metric, $g^{ij} = g^{ij}(x)$. We assume $g^{00}(x) > 0$ for all x , which corresponds to the natural time orientation.

The wave equation is then

$$(1.1) \quad \square_g u = \sum_{i,j=0}^2 \frac{1}{\sqrt{|g(x)|}} \partial_{x^i} [\sqrt{|g(x)|} g^{ij}(x) \partial_{x^j} u] = 0,$$

where $|g(x)| = |\det[g_{ij}]_{i,j=0}^2|$.

Denote by Ω the *ergoregion*, which is the set of $x \in \mathbb{R}^2$ where $g_{00}(x) < 0$, i.e. ∂_t is not timelike. In other words, the ergoregion is the region where the spatial part of the wave operator is not elliptic. Note that by Cramer's rule and our sign conventions we have $g_{00}(x) = g^{00}(x)\Delta(x)$, where $\Delta = g^{11}g^{22} - (g^{12})^2$ and $g^{00}(x) > 0$, so $\Omega = \{\Delta(x) < 0\}$.

In [4] we discussed the case of an ergosphere surrounding a singularity. In this paper we consider the case of an annular domain Ω , i.e. one whose boundary $\partial\Omega = \{\Delta(x) = 0\}$ consists of two nested Jordan curves, each of which is smooth in the sense that $\partial\Delta/\partial x \neq 0$ when $\Delta = 0$. Informally we say that there are “two ergospheres”. Assume the components of the metric g are defined in a fixed spatial neighborhood of $\bar{\Omega}$.

1.1. Examples of spacetimes with two ergospheres. An example of a metric in three space dimensions having two ergospheres is the celebrated Kerr metric, which we will write in Kerr-Schild coordinates [7], [12], [13].

The Hamiltonian in cylindrical coordinates (ρ, φ, z) is given by

$$(1.2) \quad H = \tau^2 - \xi_\rho^2 - \left(\frac{1}{\rho}\xi_\varphi\right)^2 - \xi_z^2 + K(-\tau + \hat{b} \cdot \hat{\xi})^2,$$

where $\hat{\xi} = (\xi_\rho, \frac{1}{\rho}\xi_\varphi, \xi_z)$, $\hat{b} = (b_\rho, b_\varphi, b_z)$, with

$$(1.3) \quad K = K(\rho, z) = \frac{2mr^3}{r^4 + a^2z^2}, \quad b_\rho = \frac{\rho r}{r^2 + a^2}, \quad b_\varphi = \frac{a\rho}{r^2 + a^2}, \quad b_z = \frac{z}{r}.$$

Here r is defined by the relation

$$(1.4) \quad \frac{\rho^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

Therefore note that $b_\rho^2 + b_\varphi^2 + b_z^2 = 1$.

For the Kerr metric it is well known that there are two ergospheres called the *outer ergosphere* and *inner ergosphere*, which occur where $K = 1$. There are also outer and inner horizons given by the equations $r = r_\pm = m \pm \sqrt{m^2 - a^2}$, assuming $0 < a < m$.

Two examples of 2+1 spacetimes with two ergospheres may be obtained by reduction from the 3+1 dimensional Kerr metric:

- Set $z = 0$ and $\xi_z = 0$ to obtain the Hamiltonian

$$(1.5) \quad H_1 = \tau^2 - \xi_\rho^2 - (\frac{1}{\rho}\xi_\varphi)^2 + K(-\tau + b_\rho\xi_\rho + b_\varphi(\frac{1}{\rho}\xi_\varphi))^2,$$

where $K = \frac{2m}{r} = \frac{2m}{\sqrt{\rho^2 - a^2}}$. This corresponds to the equatorial plane of the Kerr metric.

- Since the Kerr Hamiltonian H is independent of φ , we have that ξ_φ is constant. If we set $\xi_\varphi = 0$, then we obtain the Hamiltonian

$$(1.6) \quad H_2 = \tau^2 - \xi_\rho^2 - \xi_z^2 + K(-\tau + b_\rho\xi_\rho + b_z\xi_z)^2$$

This corresponds to a reduction by rotational symmetry.

Related to the first situation we consider so-called *acoustic metrics* with Hamiltonians

$$(1.7) \quad H = \left(\tau + A\xi_\rho + B(\frac{1}{\rho}\xi_\varphi)\right)^2 - \xi_\rho^2 - (\frac{1}{\rho}\xi_\varphi)^2$$

with $A = A(\rho)$, $B = B(\rho)$ (cf [11], [4]). Here the ergosphere is where $A^2 + B^2 = 1$, and we consider the case where the ergoregion is an annular domain between $\rho = \rho_-$ and $\rho = \rho_+$.

Another example is the optical (Gordon) metric [8], [5], whose inverse metric tensor has components

$$g^{ij} = \eta^{ij} + (n(x)^2 - 1)u^i u^j,$$

where η^{ij} is the inverse of the Minkowski metric tensor with signature $(+1, -1, -1)$; $n(x)$ is the index of refraction, which describes the propagation of light in a moving dielectric; $(u^0, u^1, u^2, u^3) = (1 - \frac{|w|^2}{c^2})(1, \frac{w}{c})$ is the 4-velocity of the medium flow, with $w(x) = (w_1, w_2, w_3)$ the velocity of the dielectric; and c is the speed of light in a vacuum.

The Gordon Hamiltonian is given by

$$(1.8) \quad H = \tau^2 - \sum_{j=1}^3 \xi_j^2 + (n(x)^2 - 1) \left(\sum_{j=1}^3 u^j \xi_j\right)^2.$$

The plan of the paper is as follows. In Section 2 we review the role of so-called ‘zero-energy’ null geodesics. In Section 3 we analyze the first example above of a reduction of the Kerr metric. In Section 4 we generalize the results of Section 3 to the case of acoustic metrics.

In Section 5, we study acoustic metrics where we allow double roots and more than two horizons. In Section 6, we prove some results for a general metric on a 2+1 dimensional spacetime with two ergospheres, and we describe the behavior of the zero energy null geodesics. In Section 7 we study the case where both ergospheres are also horizons.

2. ZERO ENERGY NULL GEODESICS IN THE ERGOREGION

On the cotangent space $T^*\mathbb{R}^{1+2}$ we use global coordinates (t, x, τ, ξ) , often denoting $t = x^0$, $x = (x^1, x^2)$, $\tau = \xi_0$, and $\xi = (\xi_1, \xi_2)$.

As in the analysis of [2], [4], the main idea will be to analyze the dynamics of the *zero energy null geodesics*. Recall that for a point $x \in \Omega$, the forward light-cone at x consists of null-bicharacteristics with increasing t . Its spatial projection is a cone based at $x \in \mathbb{R}^2$, whose edges correspond to null-bicharacteristics with $\tau = 0$ (thus ‘zero energy’). These edges may be described by pair of smooth, autonomous vector fields $X^\pm = X^\pm(x)$, $x \in \Omega$, which give the corresponding null-geodesic flow parameterized by t .

Explicitly, let $\sigma(x, \xi) = \sum_{i,j=0}^2 g^{ij}(x) \xi_i \xi_j$ be the symbol of \square_g , and consider a general bicharacteristic curve $(x, \xi) = (x(t), \xi(t)) \in T^*\mathbb{R}^{1+2}$, parameterized by t , i.e.

$$(2.1) \quad \begin{aligned} \frac{dx^i}{dt} &= \frac{\partial_{\xi_i} \sigma(x, \xi)}{\partial_\tau \sigma(x, \xi)} = \frac{\sum_{j=0}^2 2g^{ij}(x) \xi_j}{\sum_{j=0}^2 g^{0j}(x) \xi_j}, \\ \frac{d\xi_i}{dt} &= \frac{\partial_{x^i} \sigma(x, \xi)}{\partial_\tau \sigma(x, \xi)} = \frac{\sum_{i,j=0}^2 \partial_{x^i} g^{ij}(x) \xi_i \xi_j}{\sum_{j=0}^2 g^{0j}(x) \xi_j}. \end{aligned}$$

Note that as the metric is stationary, $\xi_0 = \tau$ is constant. Then for a null-bicharacteristic with $\tau = 0$ we have the characteristic equation

$$(2.2) \quad \sum_{i,j=1}^2 g^{ij}(x) \xi_i \xi_j = 0.$$

We may solve for ξ_1 to obtain $\xi_1 = \frac{-g^{12} \pm \sqrt{-\Delta}}{g^{11}} \xi_2$, and substituting this relation and $\tau = 0$ into the above, we obtain

$$(2.3) \quad \frac{dx^i}{dt} = 2 \frac{g^{i1} \frac{-g^{12} \pm \sqrt{-\Delta}}{g^{11}} + g^{i2}}{g^{01} \frac{-g^{12} \pm \sqrt{-\Delta}}{g^{11}} + g^{02}} = 2 \frac{g^{11} g^{i2} - g^{12} g^{i1} \pm g^{i1} \sqrt{-\Delta}}{g^{11} g^{02} - g^{12} g^{01} \pm g^{01} \sqrt{-\Delta}} =: X^{\pm, i}(x).$$

Note that the choice of sign is arbitrary, but we may make a consistent choice throughout the ergoregion. To analyze the examples introduced in Section 1, we will study the dynamics of the vector fields X^\pm .

We recall from the analysis of [4] the following fact:

Lemma 2.1. *If γ is a limit cycle for one of the vector fields X^\pm , then γ is a horizon.*

3. THE CASE OF THE EQUATORIAL PLANE FOR THE KERR METRIC

We consider our first example of a 2+1 spacetime obtained by reduction from the Kerr metric. Recall that we set $z = 0$ and $\xi_z = 0$ to obtain the Hamiltonian

$$(3.1) \quad H = \tau^2 - \xi_\rho^2 - \left(\frac{1}{\rho} \xi_\varphi\right)^2 + K(-\tau + b_\rho \xi_\rho + b_\varphi \left(\frac{1}{\rho} \xi_\varphi\right))^2,$$

where $K = \frac{2m}{r} = \frac{2m}{\sqrt{\rho^2 - a^2}}$. We consider the region $a < \rho < \sqrt{4m^2 + a^2}$.

We calculate

$$\begin{aligned} \frac{1}{2} H_\tau &= \tau - K(-\tau + b_\rho \xi_\rho + b_\varphi \left(\frac{1}{\rho} \xi_\varphi\right)), \\ \frac{1}{2} H_{\xi_\rho} &= -\xi_\rho + K(-\tau + b_\rho \xi_\rho + b_\varphi \left(\frac{1}{\rho} \xi_\varphi\right)) b_\rho, \\ \frac{1}{2} H_{\xi_\varphi} &= -\frac{1}{\rho} \left(\frac{1}{\rho} \xi_\varphi\right) + K(-\tau + b_\rho \xi_\rho + b_\varphi \left(\frac{1}{\rho} \xi_\varphi\right)) \left(\frac{1}{\rho} b_\varphi\right) \end{aligned}$$

and therefore the zero energy null geodesic flow is given by

$$(3.2) \quad \begin{aligned} \frac{d\rho}{dt} &= \frac{-\xi_\rho + K(b_\rho \xi_\rho + b_\varphi(\frac{1}{\rho} \xi_\varphi))b_\rho}{-K(b_\rho \xi_\rho + b_\varphi(\frac{1}{\rho} \xi_\varphi))}, \\ \frac{d\varphi}{dt} &= \frac{-\frac{1}{\rho}(\frac{1}{\rho} \xi_\varphi) + K(b_\rho \xi_\rho + b_\varphi(\frac{1}{\rho} \xi_\varphi))(\frac{1}{\rho} b_\varphi)}{-K(b_\rho \xi_\rho + b_\varphi(\frac{1}{\rho} \xi_\varphi))}. \end{aligned}$$

The characteristic equation is

$$(3.3) \quad (Kb_\rho^2 - 1)\xi_\rho^2 + 2Kb_\rho b_\varphi \xi_\rho(\frac{1}{\rho} \xi_\varphi) + K(b_\varphi^2 - 1)(\frac{1}{\rho} \xi_\varphi)^2 = 0.$$

Solving for ξ_ρ in terms of ξ_φ yields

$$(3.4) \quad \begin{aligned} \xi_\rho &= \frac{-Kb_\rho b_\varphi \pm \sqrt{K^2 b_\rho^2 b_\varphi^2 - (Kb_\rho^2 - 1)(Kb_\varphi^2 - 1)}}{Kb_\rho^2 - 1} (\frac{1}{\rho} \xi_\varphi) \\ &= \frac{-Kb_\rho b_\varphi \pm \sqrt{K(b_\rho^2 + b_\varphi^2) - 1}}{Kb_\rho^2 - 1} (\frac{1}{\rho} \xi_\varphi). \end{aligned}$$

The equation for zero energy null bicharacteristics becomes

$$(3.5) \quad \frac{d\rho^\pm}{dt} = \frac{\pm(Kb_\rho^2 - 1)\sqrt{K(b_\rho^2 + b_\varphi^2) - 1}}{\mp Kb_\rho \sqrt{K(b_\rho^2 + b_\varphi^2) - 1} - Kb_\varphi}.$$

The (+) family are the solutions for the positive square root in (3.4), and similarly for the (-) family.

Theorem 3.1. *The (+) family produces two horizons, while the (-) family does not produce any.*

Proof. With the (+) family, the denominator of (3.5) does not vanish, while the numerator is zero when $Kb_\rho^2 - 1 = 0$. Since we have $K = \frac{2m}{r}$, $b_\rho = \frac{r}{\rho}$, this happens when $r = m \pm \sqrt{m^2 - a^2}$. These zeros are limiting cycles, and the limiting cycle is an horizon in each case.

With the (-) family, since the denominator vanishes we reorganize (3.5) to get

$$(3.6) \quad \frac{d\rho^-}{dt} = \frac{-\sqrt{K(b_\rho^2 + b_\varphi^2) - 1}(Kb_\rho \sqrt{K(b_\rho^2 + b_\varphi^2) - 1} + Kb_\varphi)}{K^2(b_\rho^2 + b_\varphi^2)},$$

which is negative everywhere in the ergoregion. Thus $\rho^-(t)$ is strictly decreasing, and there is no horizon generated by the (-) family. \square

4. ACOUSTIC METRICS I

We consider a metric of the form

$$(4.1) \quad H = (\tau + A\xi_\rho + B(\frac{1}{\rho} \xi_\varphi))^2 - \xi_\rho^2 - (\frac{1}{\rho} \xi_\varphi)^2$$

with $A = A(\rho)$, $B = B(\rho)$. The ergospheres are where $A^2 + B^2 = 1$. We assume that this happens at $\rho = \rho_-$ and $\rho = \rho_+$, and that $A^2 + B^2 > 1$ on the interval $\rho \in (\rho_-, \rho_+)$.

We calculate

$$(4.2) \quad \begin{aligned} \frac{1}{2}H_\tau &= (\tau + A\xi_\rho + B(\frac{1}{\rho} \xi_\varphi)), \\ \frac{1}{2}H_{\xi_\rho} &= (\tau + A\xi_\rho + B(\frac{1}{\rho} \xi_\varphi))A - \xi_\rho, \\ \frac{1}{2}H_{\xi_\varphi} &= (\tau + A\xi_\rho + B(\frac{1}{\rho} \xi_\varphi))\frac{1}{\rho}B - \frac{1}{\rho}(\frac{1}{\rho} \xi_\varphi). \end{aligned}$$

Thus for $\tau = 0$

$$(4.3) \quad \begin{aligned} \frac{d\rho}{dt} &= \frac{(A\xi_\rho + B\frac{\xi_\varphi}{\rho})A - \xi_\rho}{A\xi_\rho + B\frac{\xi_\varphi}{\rho}}, \\ \frac{d\varphi}{dt} &= \frac{(A\xi_\rho + B\frac{\xi_\varphi}{\rho})\frac{B}{\rho} - \frac{\xi_\varphi}{\rho^2}}{A\xi_\rho + B\frac{\xi_\varphi}{\rho}}. \end{aligned}$$

We get the characteristic equation

$$(A^2 - 1)\xi_\rho^2 + 2AB\xi_\rho(\frac{1}{\rho}\xi_\varphi) + (B^2 - 1)(\frac{1}{\rho}\xi_\varphi)^2 = 0,$$

which leads to

$$(4.4) \quad \xi_\rho = \frac{-AB \pm \sqrt{A^2 + B^2 - 1}}{A^2 - 1}(\frac{1}{\rho}\xi_\varphi).$$

We assume there are two values $\rho = \rho_i$, $i = 1, 2$, such that $\rho_- < \rho_1 < \rho_2 < \rho_+$ and $A(\rho_i) = -1$, and we then claim that these are the horizons.

Let start with the case when $A < 0$ and $B > 0$. Substituting (4.4) into (4.3), we get

$$(4.5) \quad \begin{aligned} \frac{d\rho^+}{dt} &= \frac{(A^2 - 1)\sqrt{A^2 + B^2 - 1}}{A\sqrt{A^2 + B^2 - 1} - B}, \\ \frac{d\varphi^+}{dt} &= \frac{1}{\rho} \cdot \frac{\sqrt{A^2 + B^2 - 1}(AB - \sqrt{A^2 + B^2 - 1})}{A\sqrt{A^2 + B^2 - 1} - B} \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} \frac{d\rho^-}{dt} &= \frac{A(A^2 + B^2 - 1) - B\sqrt{A^2 + B^2 - 1}}{A^2 + B^2}, \\ \frac{d\varphi^-}{dt} &= \frac{1}{\rho} \cdot \frac{\sqrt{A^2 + B^2 - 1}(-AB - \sqrt{A^2 + B^2 - 1})}{-A\sqrt{A^2 + B^2 - 1} - B} \\ &= \frac{1}{\rho} \cdot \frac{(A\sqrt{A^2 + B^2 - 1} - B)(B^2 - 1)}{(-AB + \sqrt{A^2 + B^2 - 1})(A^2 + B^2)}. \end{aligned}$$

To derive the above we use that

$$(4.7) \quad \begin{aligned} (A\sqrt{A^2 + B^2 - 1} - B)(-A\sqrt{A^2 + B^2 - 1} - B) &= -(A^2 + B^2)(A^2 - 1), \\ (-AB + \sqrt{A^2 + B^2 - 1})(-AB - \sqrt{A^2 + B^2 - 1}) &= (A^2 - 1)(B^2 - 1). \end{aligned}$$

Theorem 4.1. *Suppose $A < 0$ and $B > 0$. Then the (+) family generates two horizons at $\rho = \rho_1$ and $\rho = \rho_2$, while the (-) family generates no horizons.*

The proof is the same as the proof of Theorem 3.1, with the horizons $\rho = \rho_1$, $\rho = \rho_2$ generated by the (+) family when $A(\rho_1) = A(\rho_2) = -1$, and no horizons generated by the (-) family.

Consider now the case when $A < 0$ and $B < 0$. Then analogous to (4.6), (4.7) we obtain

$$(4.8) \quad \begin{aligned} \frac{d\rho^+}{dt} &= \frac{A(A^2 + B^2 - 1) + B\sqrt{A^2 + B^2 - 1}}{A^2 + B^2}, \\ \frac{d\varphi^+}{dt} &= \frac{1}{\rho} \cdot \frac{(A\sqrt{A^2 + B^2 - 1} + B)(B^2 - 1)}{(-AB - \sqrt{A^2 + B^2 - 1})(A^2 + B^2)} \end{aligned}$$

and

$$(4.9) \quad \begin{aligned} \frac{d\rho^-}{dt} &= \frac{-(A^2 - 1)\sqrt{A^2 + B^2 - 1}}{-A\sqrt{A^2 + B^2 - 1} - B}, \\ \frac{d\varphi^-}{dt} &= \frac{1}{\rho} \cdot \frac{-AB\sqrt{A^2 + B^2 - 1} - (A^2 + B^2 - 1)}{-A\sqrt{A^2 + B^2 - 1} - B}. \end{aligned}$$

Thus we have:

Theorem 4.2. *Suppose $A < 0$ and $B < 0$. Then the (+) family generates no horizons since $\frac{d\rho^+}{dt} < 0$ on (ρ_-, ρ_+) , while the (-) family generates two horizons at $\rho = \rho_i$, $i = 1, 3$ where $A(\rho_1) = A(\rho_2) = -1$.*

In the case when $A < 0$ and B changes sign between the two roots of $A = -1$, we have:

Theorem 4.3. *Suppose $A(\rho_1) = A(\rho_2) = -1$ and $\rho_0 \in (\rho_1, \rho_2)$ is such that $B(\rho_0) = 0$, with $B(\rho) > 0$ when $\rho < \rho_0$ and $B(\rho) < 0$ when $\rho > \rho_0$. Then the (+) family produces one horizon, and the (-) family also produces one horizon.*

Proof: Split into the intervals (ρ_-, ρ_0) and (ρ_0, ρ_+) . Each subinterval contains one root of $A = -1$. By the analysis in Theorem 4.1 the (+) family generates an horizon in the first subinterval while the (-) family does not, and by the analysis in Theorem 4.2 the (-) family generates an horizon in the second subinterval while the (+) family does not. This proves Theorem 4.3.

5. ACOUSTIC METRICS II

In this section we wish to consider cases when $A = -1$ has more than two roots, or when it has roots of higher multiplicity.

Example 5.1. The case when $a = m$ is called the extremal Kerr metric. In this case the outer and inner horizons coincide in a single horizon $r = m$.

More generally consider the case when $A = -1$ has a double root $A(\rho_1) = -1$, $A'(\rho_1) = 0$, with say $A''(\rho_1) > 0$. $A(\rho) + 1 = C(\rho)(\rho - \rho_1)^2$, $C(\rho) > 0$. Suppose $B > 0$.

We have that for equation (4.5), $\rho^+(t) \equiv \rho_1$ is a solution and therefore a horizon. Also $\frac{d\rho^+}{dt} > 0$ when $\rho \neq \rho_1$, so $\rho^+(t)$ will spiral toward ρ_1 from below as $t \rightarrow +\infty$, or from above when $t \rightarrow -\infty$. Note that when ρ approaches ρ_1 , the trajectory spirals as a power instead of logarithmically as in the case of a simple root.

Now consider the case when $A = -1$ has more than two simple roots. There must be an even number $2m$ roots $\rho_1, \rho_2, \dots, \rho_{2m-1}, \rho_{2m}$.

Take ρ_j, ρ_{j+1} . If $B > 0$ on $[\rho_j, \rho_{j+1}]$ then this is the same as the case of Theorem 4.1, so $\rho = \rho_j$, $\rho = \rho_{j+1}$ are horizons coming from the (+) family, while there are no horizons from the (-) family in this interval. If $B < 0$ then as in Theorem 4.2 $\rho = \rho_j$, $\rho = \rho_{j+1}$ are horizons from the (-) family while there are none for the (+) family. If B changes sign then one of $\rho = \rho_j$, $\rho = \rho_{j+1}$ is a horizon from the (+) family, and the other a horizon from the (-) family as in Theorem 4.3.

If A is always positive, we have the same picture when $A(\rho) = 1$, and the horizons are white.

Remark 5.2. Note that our definition of a horizon is local. We say that a horizon is a black hole horizon if null geodesics cannot move from inside to outside, and a white hole horizon if they cannot move from outside to inside.

6. THE GENERAL CASE

In this section we obtain several results in the general case. As in Section 2 we consider a general metric g and we assume there are two ergospheres Γ_1 and Γ_2 .

We assume the boundary is never characteristic, so that the (+) and (-) families are transversal to the boundary. It follows from transversality that as t increases one family starts on each Γ_i and the other ends, $i = 1, 2$.

We have all permutations of (+) and (-) starting and ending on Γ_1 and Γ_2 . For concreteness assume (+) starts on Γ_1 and (-) ends there. For Γ_2 , we can have that (+) starts and (-) ends, or the opposite.

Let (+) starts on both Γ_1 and Γ_2 . Then any (+) trajectory starting on Γ_1 cannot end on Γ_2 . Thus by the Poincaré-Bendixson theorem (cf. [6]), it approaches some limit cycle γ_1^+ , which must be a horizon by Lemma 2.1. Also any (+) trajectory starting on Γ_2 approaching some limit cycles γ_2^+ . Note γ_1^+ and γ_2^+ may coincide. Similarly if (-) family ends both on Γ_1 and Γ_2 there is at least one limiting cycle belonging to (-) family. Thus we get

Theorem 6.1. *Suppose one of the families starts on both Γ_1 and Γ_2 . Then to each of the (+) and (-) families corresponds at least one event horizon.*

In the case where (+) starts on Γ_1 and ends on Γ_2 one can not make a definite conclusion: it may be no event horizons produced by either family, or it may be some. The situation could be as in Section 4.

The total contribution is the sum of the contributions of the (+) and (-) family.

Thus, we have

Theorem 6.2. *If the (+) family starts on one ergosphere and ends on the other, then it may be that the (+) family does not generate any event horizon. In this case (-) family also starts on one ergosphere and ends on another. Thus it can be no event horizon produced by (-) family.*

Now we shall compare Theorems 6.1 and 6.2 with the results of Section 4. We start first with Theorem 4.3 where $A < 0$ and $B(\rho)$ changes sign: Say $B(\rho_0) = 0$, with $B(\rho) > 0$ when $\rho_- < \rho < \rho_0$, $B(\rho) < 0$ when $\rho_0 < \rho < \rho_+$. Consider the (+) family, for which we have (cf. (4.5))

$$(6.1) \quad \frac{d\rho^+}{dt} \approx \frac{-B^2\sqrt{A^2+B^2-1}}{A\sqrt{A^2+B^2-1}-B} \approx B\sqrt{A^2+B^2-1} > 0$$

near $\rho = \rho_-$ since $A^2 + B^2 - 1 \approx 0$, $B > 0$. Therefore $\rho_+(t)$ starts at $\rho = \rho_-$. Also $\frac{d\rho^+}{dt} < 0$ near $\rho = \rho_+$ since $B < 0$ near $\rho = \rho_+$.

Thus $\rho_+(t)$ starts on both ergospheres at $\rho = \rho_-$, $\rho = \rho_+$. Hence the conditions of Theorem 6.1 are satisfied and so there are event horizons corresponding to both the (+) and (-) families. This was directly argued in Theorem 4.3.

Consider now the case when in (4.5) we have $B > 0$ on $[\rho_-, \rho_+]$. Then it follows from (6.1) that $\frac{d\rho^+}{dt} > 0$ near $\rho = \rho_-$ and $\frac{d\rho^+}{dt} > 0$ near $\rho = \rho_+$. Therefore the (+) family starts at $\rho = \rho_-$ and ends at $\rho = \rho_+$. Also (-) family ends at $\rho = \rho_-$ and starts at $\rho = \rho_+$. Therefore the conditions of Theorem 6.2 are satisfied. The results of Theorem 6.2 are in agreement with the result of Theorem 4.1.

Now we shall describe, for definiteness, the phase portrait of (+) family in Ω .

Consider first the case of Theorem 4.3 or more generally of Theorem 6.1 with one (+) horizon only. Thus we have the event horizon $\rho = \rho_1$ where $\rho_- < \rho_1 < \rho_0$ and $A(\rho_1) = -1$, $B(\rho_1) > 0$. It follows from (4.5) at $\rho = \rho_1$ that

$$(6.2) \quad \frac{d\varphi^+}{dt} = \frac{1}{\rho^+} \frac{B(\rho_1)(-B(\rho_1) - B(\rho_1))}{-B(\rho_1) - B(\rho_1)} = \frac{1}{\rho_1} B(\rho_1).$$

Thus $\left. \frac{d\varphi^+}{dt} \right|_{\rho=\rho_1} > 0$. This means that the periodic (+) trajectory $\rho = \rho_1$ is traversed counterclockwise as t increases. Any (+) trajectory γ_1^+ starting at $\rho = \rho_-$ approaches the horizon $\rho = \rho_1$ spiraling counterclockwise. Analogously any (+) trajectory γ_2^+ starting at $\rho = \rho_+$ will also approach the horizon $\rho = \rho_1$ spiraling counterclockwise.

Consider now the more involved case of (+) trajectories in the conditions of Theorem 4.1. We have two event horizons $\rho_- < \rho_1 < \rho_2 < \rho_+$. As in (4.1) we have $\frac{d\varphi^+}{dt} > 0$ when $\rho = \rho_1$, i.e. the periodic trajectory $\rho = \rho_1$ is traversed counterclockwise as t increases. At $\rho = \rho_2$ we have also $\frac{d\varphi^+}{dt} = \frac{1}{\rho_2} B(\rho_2) > 0$ since $B(\rho) > 0$ for all $\rho_- < \rho < \rho_+$. Therefore the horizon $\rho = \rho_2$ is also traversed counterclockwise. Let γ_0^+ be any (+) trajectory starting at $\rho = \rho_-$. It will approach $\rho = \rho_1$ spiraling counterclockwise as $t \rightarrow +\infty$. Consider any (+) trajectory γ_2^+ ending at $\rho = \rho_+$. It will approach $\rho = \rho_2$ as $t \rightarrow -\infty$, spiraling clockwise around the horizon $\rho = \rho_2$. Note that $\rho = \rho_2$ is traversed counterclockwise as $t \rightarrow +\infty$ and the direction is reversed when $t \rightarrow -\infty$. Consider any (+) trajectory between $\rho = \rho_1$ and $\rho = \rho_2$. It approaches $\rho = \rho_1$ when $t \rightarrow +\infty$ spiraling counter clockwise around $\rho = \rho_1$ and approaches $\rho = \rho_2$ when $t \rightarrow -\infty$ spiraling clockwise around $\rho = \rho_2$.

7. CHARACTERISTIC ERGOSPHERES

Consider a domain Ω between two ergospheres Γ_1 and Γ_2 that are also event horizons. Note that Γ_1 and Γ_2 are characteristic curves for the spatial part of \square_g , but there do not exist null geodesics which travel around them. It was shown in [4] that (+) and (-) families approach Γ_1 and Γ_2 when $t \rightarrow +\infty$ and $t \rightarrow -\infty$. Consider, for example, the (+) family. Suppose for definiteness that any (+) null-geodesic approaches Γ_1 as $t \rightarrow -\infty$ and approaches Γ_2 as $t \rightarrow +\infty$. For short, we will say that γ_+ “starts” on Γ_1 and γ_+ “ends” on Γ_2 . The situation here is similar to that of Section 6.

Suppose now that every null geodesic of the (+) family approaches Γ_1 when $t \rightarrow +\infty$ and also approaches Γ_2 when $t \rightarrow +\infty$, i.e. “ends” on Γ_1 and “ends” on Γ_2 . Then the null geodesics of the (-) family “start” on Γ_1 and Γ_2 . Then as in Section 6 (cf. Theorem 6.1) using the Poincaré-Bendixson theorem we obtain that there exists at least one horizon belonging to the (+) family and one horizon belonging to the (-) family.

In the case when null-geodesics of one family “start” on one ergosphere and “end” on the other, there may be no event horizon (cf. Theorem 6.2).

As an example of two characteristic ergospheres consider the Hamiltonian (1.6), i.e. the reduction of 3 + 1 dimensional Kerr metric when $\xi_\varphi = 0$ and φ is a constant. It was shown in [3] that $r = r_+$ and $r = r_-$ are two ergospheres where $r_\pm = m \pm \sqrt{m^2 - a^2}$, $0 < a < m$, and r is defined as in (1.4). Since $r = r_+$ and $r = r_-$ are event horizons in 3 + 1 Kerr metric they are also event horizons in 2 + 1 reduction. Since there are no event horizons between $r = r_-$ and $r = r_+$ the null-geodesics of one family travels from r_- to r_+ when t changed from $-\infty$ to $+\infty$ and for another family it travels from r_+ to r_- when $-\infty < t < +\infty$.

Remark 7.1. In this paper we consider the case when the ergosphere is either not characteristic at any point or totally characteristic. In this case, event horizon curves belong to either the (+) or (-) family. Therefore the total set of event horizons is a sum of the contributions of the (+) and (-) families. In the general case some points of each ergosphere can be characteristic (cf. [4]). Then as in [4] the event horizon consists of curves belonging to either the (+) or (-) family. Note that as in [4] it may happen that an event horizon has corners as the result of intersecting arcs from different families.

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