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SELF-CONSISTENT TRANSLATIONAL MOTION OF REFERENCE FRAMES AND SIGN-DEFINITENESS OF TIME IN UNIVERSAL KINEMATICS

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This paper is dedicated to the memory of professor Myroslav Gorbachuk

Abstract. Universal kinematics as mathematical objects may be interesting for astrophysics, because there exists a hypothesis that, in the large scale of the Universe, physical laws (in particular, the laws of kinematics) may be different from the laws acting in a neighborhood of our solar System. The present paper is devoted to investigation of self-consistent translational motion of reference frames in abstract universal kinematics. In the case of self-consistent translational motion we can give a clear and unambiguous definition of displacement as well as the average and the instantaneous speed of the reference frame. Hence the uniform rectilinear motion is a particular case of self-consistent translational motion. So, the investigation of self-consistently translational motion is technically necessary for definition of classes of inertially-related reference frames (being in the state of uniform rectilinear mutual motion) in universal kinematics. In the paper we investigate the correlations between self-consistent translational motion and definiteness of time direction for reference frames in universal kinematics.

1. Introduction

The concept of inertial reference frame plays a key role in the classical mechanics and special relativity theory, because the basic fundamental laws of physics have the simplest formulation in inertial reference frames. Usually it is supposed that inertial reference frames belong to a single equivalence class. Namely, any two inertial reference frames are moving rectilinearly with a constant speed one relatively to another.

In the papers [6, 8–10, 13], we had constructed a new class of abstract mathematical objects, namely universal kinematics, which are intended for mathematical modeling of the evolution of physical systems in the framework of various laws of kinematics. Also in these papers it had been shown that universal kinematics can be applied for mathematically strict foundation of the kinematics of the special theory of relativity and its tachyon extensions. Investigation of universal kinematics may be interesting for Astrophysics, because there exists a hypothesis that, in the large scale of the Universe, physical laws (in particular, the laws of kinematics) may be different from the laws, acting in a neighborhood of our solar System (that is, different from those based on Lorentz-Poincare or Galilean coordinate transformations for inertial reference frames). That is why, in connection with the statement in the first paragraph, there naturally arises a problem to study the uniform rectilinear motion of reference frames at the level of abstract universal kinematics. But, since the uniform rectilinear motion of reference frames is the motion with constant speed, at first it is reasonable to investigate at an abstract level a more general case, where the reference frames are moving in a such way that it is possible

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to unambiguously introduce the concept of displacement and, therefore, the medium or instantaneous speed of one reference frame relatively to another. In the paper [15] this kind of motion of the reference frames was called self-consistent translational or sc-translational, where the mathematically strict definition of self-consistent translational (ie sc-translational) reference frames in universal kinematics was given.

In the present paper we proceed the investigations begun in [15]. In particular, we investigate correlations between self-consistent translational motion and definiteness of time direction for reference frames in universal kinematics.

2. Vector universal kinematics and their properties

For further understanding of this paper, main concepts and a notation system for theories of changeable sets, kinematic sets, and universal kinematics, are needed. These theories were developed in [4–10]. Some of these papers were published in Ukrainian. That is why, for the convenience of the reader, main results of these papers were "converted" into English and collected in the preprint [13], where one can find the most complete and detailed explanation of these theories. Hence, we refer to [13] the reader who is not familiar with the essential concepts. So, during citation of needed main results we sometimes will give the dual reference of these results (in one of the papers [4–10] as well as in [13]).

Definition 1. **(a):** A kinematic set $\mathfrak{C}$ is called *vector* if and only if

$$\forall \mathfrak{l} \in \mathcal{L}k(\mathfrak{C}) \;\; \mathbb{L}s(\mathfrak{l}) \neq \emptyset.$$

(b): A universal kinematics $\mathcal{F} = \left(\mathfrak{C}, \overleftarrow{\mathcal{Q}}\right)$ is called *vector* if and only if $\mathfrak{C}$ is a vector kinematic set.

Using the system of notations accepted in [8–10, 13], we deduce the following corollary of Definition 1.

Corollary 1. *A universal kinematics $\mathcal{F}$ is vector if and only if $\forall \mathfrak{l} \in \mathcal{L}k(\mathcal{F}) \;\; \mathbb{L}s(\mathfrak{l}) \neq \emptyset$.*

Let $\mathfrak{C}$ be an arbitrary vector kinematic set or universal kinematics. Then, according to Definition 1 and Corollary 1, for every reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ the relation $\mathbb{L}s(\mathfrak{l};\mathfrak{C}) \neq \emptyset$ holds true. Hence, in accordance with the system of notations accepted in [8–10, 13], we have

$$\forall \mathfrak{l} \in \mathcal{L}k(\mathfrak{C}) \quad \mathfrak{Ps}(\mathfrak{l};\mathfrak{C}) \neq \emptyset.$$

Moreover, for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ and arbitrary elements $a_1, \dots, a_n \in \mathbf{Zk}(\mathfrak{l}, \mathfrak{C})$, $\lambda_1, \dots, \lambda_n \in \mathfrak{Ps}(\mathfrak{l};\mathfrak{C})$ the following element is defined:

$$(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l},\mathfrak{C}} \quad (n \in \mathbb{N}).$$

Taking into account the abbreviated variants of notations introduced in [8–10, 13], in the case where the kinematic set or universal kinematics $\mathfrak{C}$ is known in advance, we use the notations $\mathbb{L}s(\mathfrak{l})$, $\mathfrak{Ps}(\mathfrak{l})$, $\mathbf{Zk}(\mathfrak{l})$, $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l}}$ instead of $\mathbb{L}s(\mathfrak{l};\mathfrak{C})$, $\mathfrak{Ps}(\mathfrak{l};\mathfrak{C})$, $\mathbf{Zk}(\mathfrak{l},\mathfrak{C})$, $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l},\mathfrak{C}}$ (correspondingly). Moreover in the cases where the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ is known from the context, we use the abbreviated notation $\lambda_1 a_1 + \cdots + \lambda_n a_n$ instead of $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l}}$.

3. Coordinate transforms in universal kinematics and coordinate transform operators

Definition 2. Let $\mathfrak{Q}_1, \mathfrak{Q}_2$ be any *coordinate spaces*[1], and $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$, $\mathbb{T}_2 = (\mathbf{T}_1, \leq_2)$ ($\mathbf{T}_1, \mathbf{T}_2 \neq \emptyset$) be any linearly ordered sets. Any bijection $\mathscr{U}$ from $\mathbf{T}_1 \times \mathbf{Zk}(\mathfrak{Q}_1)$ to

[1] Definition of coordinate space can be found in [6, Definition 3] or [13, Definition 2.14.2].

$\mathbf{T}_2 \times \mathbf{Zk}(\mathfrak{Q}_2)$ ($\mathscr{U} : \mathbf{T}_1 \times \mathbf{Zk}(\mathfrak{Q}_1) \longleftrightarrow \mathbf{T}_2 \times \mathbf{Zk}(\mathfrak{Q}_2)$) is called a coordinate transform operator from $(\mathbb{T}_1, \mathfrak{Q}_1)$ to $(\mathbb{T}_2, \mathfrak{Q}_2)$. The set of all coordinate transform operators from $(\mathbb{T}_1, \mathfrak{Q}_1)$ to $(\mathbb{T}_2, \mathfrak{Q}_2)$ is denoted by

$$\mathbf{Pk}(\mathbb{T}_1, \mathfrak{Q}_1; \mathbb{T}_2, \mathfrak{Q}_2).$$

Directly from Definition 2, using definition of the universal kinematics as well as system of notations for the universal kinematics (see. [9, 13]), we deduce the following proposition.

Proposition 1. *Let $\mathcal{F}$ be an arbitrary universal kinematics and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ be any reference frames of $\mathcal{F}$. Then we have*

$$[\mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{F}] \in \mathbf{Pk}(\mathbb{T}\mathbf{m}(\mathfrak{l}), \mathsf{BG}(\mathfrak{l}); \mathbb{T}\mathbf{m}(\mathfrak{m}), \mathsf{BG}(\mathfrak{m})).$$

Thus, every universal coordinate transform between reference frames of universal kinematics is a coordinate transform operator. Conversely, it turns out that for an arbitrary coordinate transform operator $\mathscr{U}$ there exists a universal kinematics $\mathcal{F}$ such that $\mathscr{U}$ is a universal coordinate transform between some reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$.

Proposition 2 ([15]). *For any coordinate transform operator $\mathscr{U} \in \mathbf{Pk}(\mathbb{T}_1, \mathfrak{Q}_1; \mathbb{T}_2, \mathfrak{Q}_2)$ there exist a universal kinematics $\mathcal{F}$ and reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ such that*

$$\begin{array}{ll} \mathbb{T}\mathbf{m}(\mathfrak{l}) = \mathbb{T}_1; & \mathsf{BG}(\mathfrak{l}, \mathcal{F}) = \mathfrak{Q}_1; \\ \mathbb{T}\mathbf{m}(\mathfrak{m}) = \mathbb{T}_2; & \mathsf{BG}(\mathfrak{m}, \mathcal{F}) = \mathfrak{Q}_2; \\ [\mathfrak{m} \leftarrow \mathfrak{l}] = \mathscr{U}. & \end{array}$$

Moreover in the case where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are vector coordinate spaces (that is $\mathbb{L}s(\mathfrak{Q}_1) \neq \emptyset$, $\mathbb{L}s(\mathfrak{Q}_2) \neq \emptyset$), the universal kinematics $\mathcal{F}$ also is vector.

4. Trajectories generated by the motion of reference frames in the universal kinematics

Definition 3. Let $\mathcal{F}$ be any universal kinematics and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ be any reference frames of $\mathcal{F}$. The trajectory of a point $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$ (under the motion of the reference frame $\mathfrak{m}$ relatively the frame $\mathfrak{l}$ in the kinematics $\mathcal{F}$) is defined as the following set:

$$\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}, \mathcal{F}]}(\mathrm{x}) = \{[\mathfrak{l} \leftarrow \mathfrak{m}](t, \mathrm{x}) \mid t \in \mathbf{Tm}(\mathfrak{m})\} \subseteq \mathbb{M}k(\mathfrak{l}).$$

Remark 1 (*on physical content of the trajectory $\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}, \mathcal{F}]}(\mathrm{x})$).* We can think that any universal kinematics $\mathcal{F}$ is some abstract "world", which not necessarily coincides with ours. Imagine that in the "world" $\mathcal{F}$ there exists a material point with constant coordinates x relatively to some reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$. Then $\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}, \mathcal{F}]}(\mathrm{x})$ is the trajectory of the motion of the point x relatively the (other) reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$.

In the case where the universal kinematics $\mathcal{F}$ is known in advance, we use the abbreviated notation

$$\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$$

instead of $\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}, \mathcal{F}]}(\mathrm{x})$.

The next propositions 3 and 4 describe some properties of the trajectories introduced above. In these propositions, $\mathcal{F}$ is an arbitrary universal kinematics and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ are any reference frames of $\mathcal{F}$.

Proposition 3 ([15]). *For arbitrary $\mathrm{x}, \mathrm{y} \in \mathbf{Zk}(\mathfrak{m})$, the condition $\mathrm{x} \neq \mathrm{y}$ implies the relation*

$$\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x}) \cap \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{y}) = \emptyset.$$

Proposition 4 ([15]). *For any element $\mathrm{w} \in \mathbb{M}k(\mathfrak{l})$ there exists a unique element $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$ such that*

$$\mathrm{w} \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x}).$$

5. Self-consistent translational reference frames in universal kinematics

In this section we formulate a strict definition of self-consistent translational reference frames in universal kinematics. First we present some technical definitions and propositions, necessary for this aim.

Definition 4. Let $\mathcal{F}$ be any vector universal kinematics and $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ be any reference frame of $\mathcal{F}$.

(1) The set
$$A^{\langle +\mathrm{x};\, \mathfrak{l}\rangle} := \{(\mathsf{tm}(\mathrm{w}),\, \mathrm{x} + \mathsf{bs}(\mathrm{w})) \mid \mathrm{w} \in A\}$$
is called a *parallel shift* of the set $A \subseteq \mathbb{M}k(\mathfrak{l})$ by the vector $\mathrm{x} \in \mathbf{Zk}(\mathfrak{l})$ (in the frame $\mathfrak{l}$). In the cases where it does not lead to misunderstanding we use the abbreviated notation
$$A^{\langle +\mathrm{x}\rangle}$$
instead of $A^{\langle +\mathrm{x};\, \mathfrak{l}\rangle}$.

(2) We say that the set $A \subseteq \mathbb{M}k(\mathfrak{l})$ is *parallel* to the set $B \subseteq \mathbb{M}k(\mathfrak{l})$ *relatively* the reference frame $\mathfrak{l}$ (denoted by $A \parallel_{\mathfrak{l}}^{\mathcal{F}} B$) if and only if there exists an element $\mathrm{x} \in \mathbf{Zk}(\mathfrak{l})$ such that $B = A^{\langle +\mathrm{x}\rangle}$, that is,
$$B = \{(\mathsf{tm}(\mathrm{w}),\, \mathrm{x} + \mathsf{bs}(\mathrm{w})) \mid \mathrm{w} \in A\}.$$
In the case where the universal kinematics $\mathcal{F}$ is known in advance, we use the abbreviated notation
$$A \parallel_{\mathfrak{l}} B$$
instead of $A \parallel_{\mathfrak{l}}^{\mathcal{F}} B$.

Proposition 5 ([15]). *Let $\mathcal{F}$ be any vector universal kinematics and $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ be any reference frame of $\mathcal{F}$. Then the following statements hold:*

(1) $A^{\langle +\mathbf{0}\rangle} = A$ *(for an arbitrary set $A \subseteq \mathbb{M}k(\mathfrak{l})$), where $\mathbf{0} = \mathbf{0}_{\mathfrak{l}} = \mathbf{0}_{\mathfrak{l},\mathcal{F}}$ is the zero vector of the linear space generated by the linear structure $\mathbb{L}s(\mathfrak{l})$;*

(2) $\left(A^{\langle +\mathrm{x}\rangle}\right)^{\langle +\mathrm{y}\rangle} = A^{\langle +(\mathrm{x}+\mathrm{y})\rangle}$ *(for every $A \subseteq \mathbb{M}k(\mathfrak{l})$ and $\mathrm{x}, \mathrm{y} \in \mathbf{Zk}(\mathfrak{l})$), in particular,* $\left(A^{\langle +\mathrm{x}\rangle}\right)^{\langle +(-\mathrm{x})\rangle} = A^{\langle +\mathbf{0}\rangle} = A$;

(3) *The binary relation $\parallel_{\mathfrak{l}}$ is an equivalence relation on the set $2^{\mathbb{M}k(\mathfrak{l})} = \{A \mid A \subseteq \mathbb{M}k(\mathfrak{l})\}$ (i.e., $\parallel_{\mathfrak{l}}$ is a reflexive, symmetric, and transitive relation on $2^{\mathbb{M}k(\mathfrak{l})}$).*

Definition 5. Let $\mathcal{F}$ be any universal kinematics.

(1) We say that a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is *trajectory-regular* relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (in the kinematics $\mathcal{F}$) if and only if the following condition is fulfilled:

(a): for each $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$ the trajectory $\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$ is an abstract trajectory from $\mathbb{T}\mathbf{m}(\mathfrak{l})$ to $\mathbf{Zk}(\mathfrak{l})$ (that is, $\forall\, \mathrm{w}_1, \mathrm{w}_2 \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$ the equality $\mathsf{tm}(\mathrm{w}_1) = \mathsf{tm}(\mathrm{w}_2)$ leads to the equality $\mathsf{bs}(\mathrm{w}_1) = \mathsf{bs}(\mathrm{w}_2)$ (and therefore to the equality $\mathrm{w}_1 = \mathrm{w}_2$).

In the next two items $\mathcal{F}$ is any *vector* universal kinematics.

(2) We say that a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is *self-consistent quasitranslational* (the abbreviated name for the term is *sc-quasitranslational*) relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (in the kinematics $\mathcal{F}$), if and only if the following condition is satisfied:

(b): for every $\mathrm{x}, \mathrm{y} \in \mathbf{Zk}(\mathfrak{m})$ the relation $\mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x}) \parallel_{\mathfrak{l}} \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{y})$ is true.

(3) A reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is called *self-consistent translational* (the shortened name of the term is *sc-translational*) relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (in kinematics $\mathcal{F}$) if and only if $\mathfrak{m}$ is sc-quasitranslational and trajectory-regular

relatively $\mathfrak{l}$ in the kinematics $\mathcal{F}$, that is, if and only if the both above conditions (a) and (b) are satisfied.

Note that further we will use the abbreviated variants of the terms, introduced in the items 2 and 3 of Definition 5 (that is we will use the terms "*sc-quasitranslational*" and "*sc-translational*" instead of "self-consistent quasitranslational" and "self-consistent translational" correspondingly).

Remark 2. In the paper [15] it is explained that on a physical level *Definition 5 can describe reference frames connected with solid bodies being in the state of translational motion* in the framework of the laws of the classical mechanics. Also in [15] it is shown that in the framework of the relativity theory the highlighted above conclusion is not true in the general case. This effect is stipulated by the fact that the Lorentz length contraction can not be uniform in the accelerated reference frames, and so a rigid non-inertial reference frame does not look like rigid in a "fixed" inertial frame [20, 21].

6. Sign-definiteness of time leads to trajectory-regularity

In the papers [11, 12, 14], the notion of time direction between reference frames of universal kinematics had been introduced. Below we recall and complement the definition of this notion.

Definition 6. Let $\mathcal{F}$ be any universal kinematics.

(1) We say that a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is *time-positive* in $\mathcal{F}$ relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (notated by $\mathfrak{m} \Uparrow^{+}_{\mathcal{F}} \mathfrak{l}$) if and only if for an arbitrary $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{M}k(\mathfrak{l})$ such that $\mathsf{bs}(\mathrm{w}_1) = \mathsf{bs}(\mathrm{w}_2)$ and $\mathsf{tm}(\mathrm{w}_1) <_{\mathfrak{l}} \mathsf{tm}(\mathrm{w}_2)$ it is true that $\mathsf{tm}([\mathfrak{m} \leftarrow \mathfrak{l}]\, \mathrm{w}_1) <_{\mathfrak{m}} \mathsf{tm}([\mathfrak{m} \leftarrow \mathfrak{l}]\, \mathrm{w}_2)$.

(2) We say that a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is *time-negative* in $\mathcal{F}$ relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (denoted by $\mathfrak{m} \Downarrow^{-}_{\mathcal{F}} \mathfrak{l}$) if and only if for arbitrary $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{M}k(\mathfrak{l})$ such that $\mathsf{bs}(\mathrm{w}_1) = \mathsf{bs}(\mathrm{w}_2)$ and $\mathsf{tm}(\mathrm{w}_1) <_{\mathfrak{l}} \mathsf{tm}(\mathrm{w}_2)$ we have $\mathsf{tm}([\mathfrak{m} \leftarrow \mathfrak{l}]\, \mathrm{w}_1) >_{\mathfrak{m}} \mathsf{tm}([\mathfrak{m} \leftarrow \mathfrak{l}]\, \mathrm{w}_2)$.

(3) We say that a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is *time-sign-defined* in $\mathcal{F}$ relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ (denoted by $\mathfrak{m} \Updownarrow^{\pm}_{\mathcal{F}} \mathfrak{l}$) if and only if at least one of the relations $\mathfrak{m} \Uparrow^{+}_{\mathcal{F}} \mathfrak{l}$ or $\mathfrak{m} \Downarrow^{-}_{\mathcal{F}} \mathfrak{l}$ is fulfilled.

Remark 3 (*on physical content of Definition 6).* According to Remark 1, we can imagine that any universal kinematics $\mathcal{F}$ is some abstract "world". In every such a "world" $\mathcal{F}$ there exists a fixed for this "world" set of reference frames $\mathcal{L}k(\mathcal{F})$. We will agree that for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the arrows of a clock, fixed in the frame $\mathfrak{l}$ are rotating clockwise relatively to the frame $\mathfrak{l}$. We say that the reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is time-positive (time-negative) relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ if and only if the observer in the reference frame $\mathfrak{m}$ (fixed relatively to $\mathfrak{m}$) observes that the arrows of the clock, fixed in the frame $\mathfrak{l}$, are rotating clockwise (counterclockwise) in the frame $\mathfrak{m}$, correspondingly.

Proposition 6. *If a reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ is time-sign-defined relatively to the reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ in the kinematics $\mathcal{F}$, then $\mathfrak{m}$ is trajectory-regular relatively to the frame $\mathfrak{l}$ in $\mathcal{F}$.*

Proof. Suppose that $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ and $\mathfrak{l} \Uparrow^{+}_{\mathcal{F}} \mathfrak{m}$. Consider any elements $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$ and $\mathrm{w}_1, \mathrm{w}_2 \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$ such that

$$\mathsf{tm}(\mathrm{w}_1) = \mathsf{tm}(\mathrm{w}_2)\,. \tag{1}$$

Let us prove that $\mathrm{w}_1 = \mathrm{w}_2$.

Assume the contrary, $\mathrm{w}_1 \neq \mathrm{w}_2$. Since $\mathrm{w}_1, \mathrm{w}_2 \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$, then, by Definition 3, the time points $t_1, t_2 \in \mathbf{Tm}(\mathfrak{m})$ exist such that

$$\mathrm{w}_1 = [\mathfrak{l} \leftarrow \mathfrak{m}](t_1, \mathrm{x}); \quad \mathrm{w}_2 = [\mathfrak{l} \leftarrow \mathfrak{m}](t_2, \mathrm{x}). \tag{2}$$

According to [9, equalities (6), (7)] or [13, equalities (3.3), (3.4)], the mapping $[\mathfrak{l} \leftarrow \mathfrak{m}]$ is a bijection between $\mathbb{M}k(\mathfrak{m})$ and $\mathbb{M}k(\mathfrak{l})$. So, since $\mathrm{w}_1 \neq \mathrm{w}_2$, using (2), we deduce $t_1 \neq t_2$. Hence only the following two cases are possible: $t_1 <_{\mathfrak{m}} t_2$ or $t_1 >_{\mathfrak{m}} t_2$. But, taking into account the relation $\mathfrak{l} \Uparrow^{+}_{\mathcal{F}} \mathfrak{m}$ and equalities (2), by Definition 6 (item 1) in the case $t_1 <_{\mathfrak{m}} t_2$ we obtain

$$\mathsf{tm}(\mathrm{w}_1) = \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}](t_1, \mathrm{x})) <_{\mathfrak{l}} [\mathfrak{l} \leftarrow \mathfrak{m}](t_2, \mathrm{x}) = \mathsf{tm}(\mathrm{w}_2).$$

Similarly in the case $t_1 >_{\mathfrak{m}} t_2$ we deduce

$$\mathsf{tm}(\mathrm{w}_1) >_{\mathfrak{l}} \mathsf{tm}(\mathrm{w}_2).$$

Thus, in the both cases we have $\mathsf{tm}(\mathrm{w}_1) \neq \mathsf{tm}(\mathrm{w}_2)$, which contradicts the equality (1). Hence, our assumption is wrong. That is why, we have $\mathrm{w}_1 = \mathrm{w}_2$.

In the case $\mathfrak{l} \Downarrow^{-}_{\mathcal{F}} \mathfrak{m}$ the proof is similar. □

Combining Definition 5 (item 3) and Proposition 6, we deduce the following theorem.

Theorem 1. *Let $\mathcal{F}$ be any vector universal kinematics. If a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is sc-quasitranslational relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ and the reference frame $\mathfrak{l}$ is time-sign-defined relatively to $\mathfrak{m}$, then the reference frame $\mathfrak{m}$ is sc-translational relatively to $\mathfrak{l}$ in the kinematics $\mathcal{F}$.*

In general, a statement converse to Proposition 6 is not true. And the next example is designed to affirm this fact. First we introduce one notation needed for a presentation of this example.

Notation 1. Let $d \in \mathbb{N}$ be any natural number. We further denote by $\widehat{\mathbb{R}^d}$ the (vector) coordinate space generated by the space $\mathbb{R}^d$, that is, a coordinate space for which the following conditions are satisfied:

1) $\mathbf{Zk}\left(\widehat{\mathbb{R}^d}\right) = \mathbb{R}^d$;

2) $\mathfrak{Ps}\left(\widehat{\mathbb{R}^d}\right) = \mathbb{R}$;

3) $(\mathrm{x}+\mathrm{y})_{\widehat{\mathbb{R}^d}} = (\mathrm{x}_1+\mathrm{y}_1, \ldots, \mathrm{x}_d+\mathrm{y}_d)$, where $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_d) \in \mathbb{R}^d$, $\mathrm{y} = (\mathrm{y}_1, \ldots, \mathrm{y}_d) \in \mathbb{R}^d$;

4) $(\lambda \mathrm{x})_{\widehat{\mathbb{R}^d}} = (\lambda \mathrm{x}_1, \ldots, \lambda \mathrm{x}_d)$, $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_d) \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$;

5) $\langle x, y \rangle_{\widehat{\mathbb{R}^d}} = \mathrm{x}_1\mathrm{y}_1 + \cdots + \mathrm{x}_d\mathrm{y}_d$, $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_d) \in \mathbb{R}^d$, $\mathrm{y} = (\mathrm{y}_1, \ldots, \mathrm{y}_d) \in \mathbb{R}^d$, in particular, $\|\mathrm{x}\|_{\widehat{\mathbb{R}^d}} = \sqrt{\langle \mathrm{x}, \mathrm{x} \rangle_{\widehat{\mathbb{R}^d}}} = \sqrt{\mathrm{x}_1^2 + \cdots + \mathrm{x}_d^2}$ (where $\langle x, y \rangle_{\widehat{\mathbb{R}^d}}$ and $\|\mathrm{x}\|_{\widehat{\mathbb{R}^d}}$ are the inner product and the norm in the coordinate space $\widehat{\mathbb{R}^d}$, correspondingly).

In particular for the case $d = 1$ we denote

$$\widehat{\mathbb{R}} := \widehat{\mathbb{R}^1}.$$

Example 1. Let $\mathscr{T}(t) : \mathbb{R} \mapsto \mathbb{R}$ be any bijective and non-monotonous mapping from $\mathbb{R}$ onto $\mathbb{R}$. For example, we may put

$$\mathscr{T}(t) := \begin{cases} \frac{1}{t}, & t > 0 \\ t, & t \leq 0 \end{cases} \qquad (t \in \mathbb{R}).$$

The non-monotonicity of the mapping $\mathscr{T}$ ensures the existence of $t_1, t_2, t_3, t_4 \in \mathbb{R}$, for which the following conditions holds true:

$$t_1 < t_2, \quad t_3 < t_4, \quad \mathscr{T}(t_1) < \mathscr{T}(t_2), \quad \mathscr{T}(t_3) > \mathscr{T}(t_4). \tag{3}$$

Since the mapping $\mathscr{T}$ is a bijection of $\mathbb{R}$ onto itself, the mapping

$$\mathscr{U}(t,\mathrm{x}) := (\mathscr{T}(t),\mathrm{x}) \quad \big((t,\mathrm{x}) \in \mathbb{R}^2\big) \tag{4}$$

is a bijection between $\mathbb{R}^2$. Hence,

$$\mathscr{U} \in \mathbf{Pk}\left(\mathbb{R}_{ord}, \widehat{\mathbb{R}}; \mathbb{R}_{ord}, \widehat{\mathbb{R}}\right),$$

where $\mathbb{R}_{ord} = (\mathbb{R}, \leq)$ is the linearly ordered set generated by the standard linear order relation $\leq$ on $\mathbb{R}$. So, according to Proposition 2, there exist a vector universal kinematics $\mathcal{F}$ and reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ such that

$$\mathbb{T}\mathbf{m}(\mathfrak{l}) = \mathbb{T}\mathbf{m}(\mathfrak{m}) = \mathbb{R}_{ord}; \tag{5}$$

$$\mathsf{BG}(\mathfrak{l},\mathcal{F}) = \mathsf{BG}(\mathfrak{m},\mathcal{F}) = \widehat{\mathbb{R}}; \tag{6}$$

$$[\mathfrak{l} \leftarrow \mathfrak{m}, \mathcal{F}] = \mathscr{U}. \tag{7}$$

Taking into account the denoted system for the universal kinematics (see [9, 13]) and formula (6), we get

$$\mathbf{Zk}(\mathfrak{l},\mathcal{F}) = \mathbf{Zk}(\mathfrak{m},\mathcal{F}) = \mathbf{Zk}\left(\widehat{\mathbb{R}}\right) = \mathbb{R}.$$

Thus, according to (5), we have

$$\mathbf{Tm}(\mathfrak{l}) = \mathbf{Tm}(\mathfrak{m}) = \mathbb{R};$$
$$\mathbb{M}k(\mathfrak{l}) = \mathbf{Tm}(\mathfrak{l}) \times \mathbf{Zk}(\mathfrak{l}) = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2; \quad \mathbb{M}k(\mathfrak{m}) = \mathbb{R}^2.$$

Consider any element $\mathrm{x} \in \mathbb{R} = \mathbf{Zk}(\mathfrak{m})$. According to (7) and Definition 3, we get

$$\begin{aligned} \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x}) &= \{[\mathfrak{l} \leftarrow \mathfrak{m}](t,\mathrm{x}) \mid t \in \mathbf{Tm}(\mathfrak{m})\} \\ &= \{[\mathfrak{l} \leftarrow \mathfrak{m}](t,\mathrm{x}) \mid t \in \mathbb{R}\} = \{\mathscr{U}(t,\mathrm{x}) \mid t \in \mathbb{R}\} \\ &= \{(\mathscr{T}(t),\mathrm{x}) \mid t \in \mathbb{R}\} = \{(\tau,\mathrm{x}) \mid \tau \in \mathbb{R}\}. \end{aligned} \tag{8}$$

Using (8) it is easy to verify that for arbitrary $\mathrm{w}_1, \mathrm{w}_2 \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$ we have $\mathsf{bs}(\mathrm{w}_1) = \mathsf{bs}(\mathrm{w}_2)$. So, condition (a) of Definition 5 is readily fulfilled, and, according to item 1 of this definition, the reference frame $\mathfrak{m}$ is trajectory-regular relatively to the reference frame $\mathfrak{l}$ (in the kinematics $\mathcal{F}$).

But, from the other hand, the reference frame $\mathfrak{l}$ is not time-sign-defined relatively to the reference frame $\mathfrak{m}$ in $\mathcal{F}$. And now our aim is to verify this fact. Choose any fixed number $\mathrm{x} \in \mathbb{R}$. Denote

$$\mathrm{w}_i := (t_i, \mathrm{x}) \quad \left(i \in \overline{1,4}\right),$$

where $\overline{m,n} = \{m, \ldots, n\}$ ($\forall m, n \in \mathbb{N}$, $m \leq n$). Then, applying (3), (4), (7), we deduce that

$$\begin{aligned} \mathsf{tm}(\mathrm{w}_1) = t_1 < t_2 = \mathsf{tm}(\mathrm{w}_2); &\quad \mathsf{tm}(\mathrm{w}_3) < \mathsf{tm}(\mathrm{w}_4); \\ \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\mathrm{w}_1) &= \mathsf{tm}(\mathscr{U}(t_1,\mathrm{x})) = \mathsf{tm}((\mathscr{T}(t_1),\mathrm{x})) \\ &= \mathscr{T}(t_1) < \mathscr{T}(t_2) = \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\mathrm{w}_2); \\ \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\mathrm{w}_3) &> \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\mathrm{w}_4). \end{aligned}$$

So, according to Definition 6, the reference frame $\mathfrak{l}$ is not time-positive or time-negative relatively to the reference frame $\mathfrak{m}$ in $\mathcal{F}$. Thus, $\mathfrak{l}$ is not time-sign-defined relatively $\mathfrak{m}$ in $\mathcal{F}$, despite the fact that $\mathfrak{m}$ is trajectory-regular relatively $\mathfrak{l}$ in $\mathcal{F}$.

Now we have seen that in the general case Proposition 6 can not be inverted. But, despite this, it turns out that under some additional conditions of topological type, a statement converse to this proposition becomes true. To formulate these topological conditions as well as to prove the declared result we need to introduce some auxiliary concepts and to prove some auxiliary, technical results. This will be done in the next section.

7. Auxiliary topological concepts and results

7.1. **Auxiliary concepts and facts from the theory of linearly ordered topological spaces.** Let $\mathbb{T} = (\mathbf{T}, \leq)$ be any linearly ordered set. Recall [2] that every such linearly ordered set can be equipped with the natural "internal" topology $\mathfrak{Tpi}[\mathbb{T}]$ generated by the base consisting of open sets of the kind

(i): $(-\infty, \infty) = \mathbf{T}$;
(ii): $(\tau, \infty) = \{t \in \mathbf{T} \mid t > \tau\}$, where $\tau \in \mathbf{T}$;
(iii): $(-\infty, \tau) = \{t \in \mathbf{T} \mid t < \tau\}$, where $\tau \in \mathbf{T}$;
(iv): $(\tau_1, \tau_2) = \{t \in \mathbf{T} \mid \tau_1 < t < \tau_2\}$, where $\tau_1, \tau_2 \in \mathbf{T}$, $\tau_1 < \tau_2$.

Let $\mathcal{F}$ be an arbitrary universal kinematics and $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ be any reference frame of $\mathcal{F}$. Further we denote by $\mathfrak{Tpi}(\mathfrak{l})$ the internal topology of the linearly ordered set $\mathbb{T}\mathbf{m}(\mathfrak{l}) = (\mathbf{Tm}(\mathfrak{l}), \leq_{\mathfrak{l}})$, i.e.,

$$\mathfrak{Tpi}(\mathfrak{l}) := \mathfrak{Tpi}[\mathbb{T}\mathbf{m}(\mathfrak{l})].$$

We say that a topological space $(\mathscr{X}, \mathfrak{S}_{\mathscr{X}})$ is connected (where $\mathfrak{S}_{\mathscr{X}} \subseteq 2^{\mathscr{X}}$ is the topology on the space $\mathscr{X}$) if and only if $\mathscr{X}$ can not be represented as a union $\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2$ of two disjoint non-empty open subsets $\mathscr{X}_1, \mathscr{X}_2 \in \mathfrak{S}_{\mathscr{X}}$ (cf [3, Corollary 6.1.2 and remark after it]). Also we denote by $\mathbf{card}(\mathbf{M})$ the cardinality of any set $\mathbf{M}$.

Lemma 1. *Suppose that the following holds true::*

1) $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$; $\mathbb{T}_2 = (\mathbf{T}_2, \leq_2)$ *are linearly ordered sets, and* $\mathbf{card}(\mathbf{T}_1) > 1$.

2) $\mathbf{T}_1$ *is a connected topological space in the topology* $\mathfrak{Tpi}[\mathbb{T}_1]$.

3) $f : \mathbf{T}_1 \mapsto \mathbf{T}_2$ *is a continuous injective mapping (i.e., a continuous invertible function) from the topological space* $(\mathbf{T}_1, \mathfrak{Tpi}[\mathbb{T}_1])$ *to the topological space* $(\mathbf{T}_2, \mathfrak{Tpi}[\mathbb{T}_2])$.

Then the function f is strictly monotone (that is, strictly increasing or strictly decreasing on $\mathbf{T}_1$*).*

Proof. Since, by the conditions of the lemma, $\mathbf{card}(\mathbf{T}_1) > 1$, the linearly ordered set $\mathbb{T}_1$ contains at least one non-maximal element.

Let $\tau \in \mathbf{T}_1$ be an arbitrary non-maximal element. Since $\mathbf{T}_1$ is a connected topological space, according to [16, Lemma 3.1, item (d)], the interval $(\tau, +\infty)$ is a connected set in $\mathbf{T}_1$. The function f is continuous. So, according to [3, Theorem 6.1.3], it maps the set $(\tau, +\infty)$, connected in $\mathbf{T}_1$, into the set $f((\tau, +\infty))$, connected in the space $\mathbf{T}_2$. Since the function f is an injective mapping, we have that $f((\tau, +\infty)) \not\ni f(\tau)$. Hence, the set $f((\tau, +\infty))$ is contained in the union of two open sets,

$$f((\tau, +\infty)) \subseteq (f(\tau), +\infty) \cup (-\infty, f(\tau)). \tag{9}$$

Taking into account the established above fact that the set $f((\tau, +\infty))$ is connected, we see that (9) is possible only in the following two cases:

(a): $f((\tau, +\infty)) \subseteq (f(\tau), +\infty)$ or
(b): $f((\tau, +\infty)) \subseteq (-\infty, f(\tau))$.

Consider case (a). Let us prove that

$$\forall \tau_1 >_1 \tau \quad f((-\infty, \tau_1)) \subseteq (-\infty, f(\tau_1)), \tag{10}$$

where $\tau_1 >_1 \tau$ means that $\tau <_1 \tau_1$ and $\tau <_1 \tau_1$ means that $\tau \leq_1 \tau_1$ and $\tau \neq \tau_1$. Consider any $\tau_1 >_1 \tau$. Similarly to the cases *(a)–(b)* we can prove that for element τ_1 the following alternative take place:

(a1): $f((-\infty, \tau_1)) \subseteq (-\infty, f(\tau_1))$ or
(b1): $f((-\infty, \tau_1)) \subseteq (f(\tau_1), +\infty)$.

Assume that Condition (b1) is satisfied. Since $\tau_1 >_1 \tau$, we have $\tau \in (-\infty, \tau_1)$. Therefore, by condition (b1), we have

$$f(\tau) >_2 f(\tau_1). \tag{11}$$

From the other hand, we have assumed the case (a). So, we get

$$f(\tau) <_2 f(\tau_1). \tag{12}$$

But inequalities (11) and (12) can not be fulfilled together. The last contradiction shows, that Condition (b1) can not hold. Hence Condition (a1) holds (for each value $\tau_1 \in \mathbf{T}_1$ such that $\tau_1 >_1 \tau$). Thus, (10) is proved.

From relation (10) and Condition (a) it follows that the function f is strictly increasing on the set $[\tau, +\infty) = \{t \in \mathbf{T}_1 \mid t \geq_1 \tau\}$. Similarly we can verify that in the case (b) the function f is strictly decreasing on $[\tau, +\infty)$.

Thus, the function f is strictly monotone on $[\tau, +\infty)$. Since τ is an arbitrary non-maximal element of $\mathbf{T}_1$, the function f is strictly monotone on every interval $[\tau_2, +\infty)$ such that $\tau_2 \leq \tau$. Since τ is a non-maximal element of $\mathbf{T}_1$, the interval $[\tau, +\infty)$ contains at least two elements. Therefore, if the function f is strictly increasing on $[\tau, +\infty)$, it will be strictly increasing on every interval $[\tau_2, +\infty)$ such that $\tau_2 \leq \tau$. That is, it will be strictly increasing on $\mathbf{T}_1$. Similarly, if the function f is strictly decreasing on $[\tau, +\infty)$, then it is strictly decreasing on $\mathbf{T}_1$. □

7.2. **Some information from the theory of separately continuous mappings.** Let $(\mathscr{X}, \mathfrak{S}_{\mathscr{X}})$, $(\mathscr{Y}, \mathfrak{S}_{\mathscr{Y}})$ and $(\mathscr{Z}, \mathfrak{S}_{\mathscr{Z}})$ be topological spaces, where $\mathfrak{S}_{\mathscr{S}} \subseteq 2^{\mathscr{S}}$ is a topology on the topological space $\mathscr{S}$ ($\mathscr{S} \in \{\mathscr{X}, \mathscr{Y}, \mathscr{Z}\}$). By $\mathbf{C}(\mathscr{X}, \mathscr{Y})$ we denote the collection of all continuous mappings from $\mathscr{X}$ to $\mathscr{Y}$. For a mapping $\boldsymbol{f} : \mathscr{X} \times \mathscr{Y} \mapsto \mathscr{Z}$ and a point $(x, y) \in \mathscr{X} \times \mathscr{Y}$ we write

$$\boldsymbol{f}^x(y) := \boldsymbol{f}_y(x) := \boldsymbol{f}(x, y).$$

Recall [17] that a mapping $\boldsymbol{f} : \mathscr{X} \times \mathscr{Y} \mapsto \mathscr{Z}$ is reffered to as separately continuous if and only if $\boldsymbol{f}^x \in \mathbf{C}(\mathscr{Y}, \mathscr{Z})$ and $\boldsymbol{f}_y \in \mathbf{C}(\mathscr{X}, \mathscr{Z})$ for every point $(x, y) \in \mathscr{X} \times \mathscr{Y}$ (see also [19, 22]). The set of all separately continuous mappings $\boldsymbol{f} : \mathscr{X} \times \mathscr{Y} \mapsto \mathscr{Z}$ is denoted by $\mathbf{CC}(\mathscr{X} \times \mathscr{Y}, \mathscr{Z})$ [17, 19, 22].

For next considerations we need the following lemma.

Lemma 2. *Let $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$, $\mathbb{T}_2 = (\mathbf{T}_2, \leq_2)$ be linearly ordered sets and $(\mathscr{X}, \mathfrak{S}_{\mathscr{X}})$ be a topological space.* Suppose the following conditions*:*

- **(a):** *topological spaces $\mathbf{T}_1$ and $\mathscr{X}$ are connected (where $\mathbf{T}_i$ are considered as topological spaces with the topology $\mathfrak{Tpi}[\mathbb{T}_i]$ $(i \in \overline{1, 2})$);*
- **(b):** *the mapping $\boldsymbol{f} : \mathbf{T}_1 \times \mathscr{X} \mapsto \mathbf{T}_2$ is separately continuous;*
- **(c):** *for every $x \in \mathscr{X}$ the mapping $\boldsymbol{f}_x(t) = \boldsymbol{f}(t, x)$ $(t \in \mathbf{T}_1)$ is injective on $\mathbf{T}_1$.*

Then one and only one of the following alternative assertions is true:

1. *the function $\boldsymbol{f}_x(t)$ is strictly increasing on $\mathbf{T}_1$ for each $x \in \mathscr{X}$;*
2. *the function $\boldsymbol{f}_x(t)$ is strictly decreasing on $\mathbf{T}_1$ for each $x \in \mathscr{X}$.*

Proof. We do not consider the case where $\mathbf{card}(\mathbf{T}_1) = 1$, because in this case the function $\boldsymbol{f}_x(t)$ is strictly increasing and strictly decreasing on $\mathbf{T}_1$ simultaneously, according to the rules of formal logic.

So, further we assume that

$$\mathbf{card}(\mathbf{T}_1) > 1. \tag{13}$$

Consider any fixed $x \in \mathscr{X}$. In accordance with conditions of the lemma, $\boldsymbol{f}_x(t)$ is an injective continuous mapping from $\mathbf{T}_1$ to $\mathbf{T}_2$. Hense from Lemma 1 we deduce the following:

(1^0): $\boldsymbol{f}_x$ is a strictly monotone mapping from the linearly ordered set $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ into the linearly ordered set $\mathbb{T}_2$ (for every element $x \in \mathscr{X}$).

So, it remains to prove that the direction of monotonicity for the function $\boldsymbol{f}_x$ does not depend on the element $x \in \mathscr{X}$.

Further we consider $\mathbf{T}_2 \times \mathbf{T}_2$ as a topologicas space with the Tychonoff topology on the square of the topological space $(\mathbf{T}_2, \mathfrak{Tpi}[\mathbb{T}_2])$. According to Condition (13), the elements $t_0, t_1 \in \mathbf{T}_1$ exist such that $t_0 <_1 t_1$. Consider a function $\Phi : \mathscr{X} \mapsto \mathbf{T}_2 \times \mathbf{T}_2$, defined by

$$\Phi(x) := (\boldsymbol{f}_x(t_0), \boldsymbol{f}_x(t_1)) \in \mathbf{T}_2 \times \mathbf{T}_2 \quad (x \in \mathscr{X}).$$

Using conditions (b) and (a) of the lemma as well as [18, Theorem 3.3], we see that the function Φ is a continuous mapping from the connected topological space $\mathscr{X}$ into the topological space $\mathbf{T}_2 \times \mathbf{T}_2$. Hence, according to [3, Theorem 6.1.3], the range $\mathfrak{R}(\Phi) = \Phi(\mathscr{X})$ is connected set in $\mathbf{T}_2 \times \mathbf{T}_2$. Since $t_0 \neq t_1$, according to Statement *(1^0)*, we have $\boldsymbol{f}_x(t_0) \neq \boldsymbol{f}_x(t_1)$ ($\forall x \in \mathscr{X}$). Therefore, the set $\mathfrak{R}(\Phi) = \Phi(\mathscr{X})$ (connected in $\mathbf{T}_2 \times \mathbf{T}_2$) is included into the union of two disjoint sets,

$$\Phi(\mathscr{X}) \subseteq \mathbf{U}_1 \cup \mathbf{U}_2, \quad \text{where} \tag{14}$$
$$\mathbf{U}_1 = \{(\tau_1, \tau_2) \in \mathbf{T}_2 \times \mathbf{T}_2 \mid \tau_1 <_2 \tau_2\};$$
$$\mathbf{U}_2 = \{(\tau_1, \tau_2) \in \mathbf{T}_2 \times \mathbf{T}_2 \mid \tau_1 >_2 \tau_2\}.$$

We will prove that the sets $\mathbf{U}_1$ and $\mathbf{U}_2$ are open. Suppose, that $(\tau_1, \tau_2) \in \mathbf{U}_1$. Then we have $\tau_1 <_2 \tau_2$. If there exists an element $\widetilde{\tau} \in \mathbf{T}_2$ such that $\tau_1 <_2 \widetilde{\tau} <_2 \tau_2$, then the set $\widetilde{\mathbf{U}} = (-\infty, \widetilde{\tau}) \times (\widetilde{\tau}, \infty)$ is open in $\mathbf{T}_2 \times \mathbf{T}_2$ (according to definition of the Tychonoff product topology, see [18]). Moreover, it contains the point (τ_1, τ_2) and satisfies the condition $\widetilde{\mathbf{U}} \subseteq \mathbf{U}_1$. In the case where the point τ, satisfying $\tau_1 <_2 \widetilde{\tau} <_2 \tau_2$, does not exist, the sets $(-\infty, \tau_1] = (-\infty, \tau_2)$ and $[\tau_2, \infty) = (\tau_1, \infty)$ are open in $\mathbf{T}_2$. Hence in this case the set $\widetilde{\mathbf{U}} = (-\infty, \tau_1] \times [\tau_2, \infty)$ is open in $\mathbf{T}_2 \times \mathbf{T}_2$. Moreover, we have $(\tau_1, \tau_2) \in \widetilde{\mathbf{U}} \subseteq \mathbf{U}_1$. Thus, we have seen that in the both cases for every point $(\tau_1, \tau_2) \in \mathbf{U}_1$ the set $\widetilde{\mathbf{U}}$ exists such that $\widetilde{\mathbf{U}}$ is open in $\mathbf{T}_2 \times \mathbf{T}_2$ and $(\tau_1, \tau_2) \in \widetilde{\mathbf{U}} \subseteq \mathbf{U}_1$. This means that the set $\mathbf{U}_1$ is open in $\mathbf{T}_2 \times \mathbf{T}_2$. Similarly it can be proved that the set $\mathbf{U}_2$ is open in $\mathbf{T}_2 \times \mathbf{T}_2$. Above we have proved that the set $\Phi(\mathscr{X})$ is connected. So, since the sets $\mathbf{U}_1$, $\mathbf{U}_2$ are open, the inclusion (14) is possible only in the following two cases:

(a): $\Phi(\mathscr{X}) \subseteq \mathbf{U}_1$;
(b): $\Phi(\mathscr{X}) \subseteq \mathbf{U}_2$.

In the case (a) we have

$$\boldsymbol{f}_x(t_0) <_2 \boldsymbol{f}_x(t_1) \quad (\forall x \in \mathscr{X}).$$

Therefore, according to Statement *(1^0)*, we see that the function $\boldsymbol{f}_x$ is strictly increasing on $\mathbf{T}_1$ for each $x \in \mathscr{X}$. Similarly in the case (b) $\boldsymbol{f}_x$ is strictly decreasing on $\mathbf{T}_1$ for each $x \in \mathscr{X}$. □

7.3. **Some classes of universal kinematics.** In the next definition we introduce some classes of universal kinematics, needed for presentation of the main results of the paper.

Definition 7. Let $\mathcal{F}$ be an arbitrary universal kinematics. We say that $\mathcal{F}$ belongs to the class:

(V) if and only if $\mathcal{F}$ is a vector universal kinematics in the sense of Definition 1, that is, if and only if for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the condition $\mathbb{L}s(\mathfrak{l}) \neq \emptyset$ holds;

(T) if and only if for each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the condition $\mathcal{T}p(\mathfrak{l}) \neq \emptyset$ is satisfied (the universal kinematics of class (T) is also called *topological*);

(C_0) if and only if for each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the ordered pair $(\mathbf{Tm}(\mathfrak{l}), \mathfrak{Tpi}(\mathfrak{l}))$ is a connected topological space, (the universal kinematics of class (C_0) is also called *time-connected*);

(C_1) if and only if for each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the ordered pair $(\mathbf{Zk}(\mathfrak{l}), \mathcal{T}p(\mathfrak{l}))$ is a connected topological space (the universal kinematics of class (C_1) is also called by *space-connected*).

We say that a universal kinematics $\mathcal{F}$ belongs to the *combined class* $(\alpha_1, \ldots, \alpha_n)$ *(where* $n \in \mathbb{N}$, $\alpha_i \in \{$"V", "T", "C_0", "C_1"$\}$ $(i \in \overline{1,n})$*)*, if and only if $\mathcal{F}$ belongs all classes $(\alpha_1), \ldots, (\alpha_n)$ simultaneously. In particular:

(TV) We say that a universal kinematics $\mathcal{F}$ is *topological-vector*, if and only if $\mathcal{F}$ belongs to the class (TV), that is, if and only if for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the conditions $\mathbb{L}s(\mathfrak{l}) \neq \emptyset$ and $\mathcal{T}p(\mathfrak{l}) \neq \emptyset$ are fulfilled.

8. Conditions under which trajectory-regularity leads to sign-definiteness of time

Let $\mathcal{F}$ be an arbitrary topological universal kinematics, $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ any reference frame of $\mathcal{F}$ and $(\mathscr{X}, \mathfrak{S}_{\mathscr{X}})$ any topological space. We say that a mapping $\boldsymbol{f} : \mathbb{M}k(\mathfrak{l}) \mapsto \mathfrak{X}$ is *separately continuous relatively space and time variables* if and only if $\boldsymbol{f} \in \mathbf{CC}(\mathbf{Tm}(\mathfrak{l}) \times \mathbf{Zk}(\mathfrak{l}), \mathscr{X})$, where $\mathbf{Tm}(\mathfrak{l})$ and $\mathbf{Zk}(\mathfrak{l})$ are considered as topological spaces with the topologies $\mathfrak{Tpi}(\mathfrak{l})$ and $\mathcal{T}p(\mathfrak{l})$, correspondingly.

Proposition 7. *Let a reference frame* $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ *be trajectory-regular relatively to the frame* $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ *in a topological universal kinematics* $\mathcal{F}$*. Impose also the following additional conditions:*

- **(a):** *The topological spaces* $(\mathbf{Tm}(\mathfrak{m}), \mathfrak{Tpi}(\mathfrak{m}))$ *and* $(\mathbf{Zk}(\mathfrak{m}), \mathcal{T}p(\mathfrak{m}))$ *are connected.*
- **(b):** *The mapping* $\mathbb{M}k(\mathfrak{m}) \ni \mathrm{w} \longmapsto \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\,\mathrm{w}) \in \mathbf{Tm}(\mathfrak{l})$ *is separately continuous relatively to the space and time variables (where* $\mathbf{Tm}(\mathfrak{l})$ *is considered as a topological space with the topology* $\mathfrak{Tpi}(\mathfrak{l})$*).*

Then the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ *is time-sign-defined relatively to the reference frame* $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ *in the kinematics* $\mathcal{F}$ *(ie* $\mathfrak{l} \Updownarrow^{\pm}_{\mathcal{F}} \mathfrak{m}$*).*

Proof. Denote

$$\boldsymbol{f}(t, \mathrm{x}) := \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}](t, \mathrm{x})), \quad t \in \mathbf{Tm}(\mathfrak{m}), \quad \mathbf{Zk}(\mathfrak{m}).$$

Then for every $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$ we obtain a mapping $\boldsymbol{f}_{\mathrm{x}} : \mathbf{Tm}(\mathfrak{m}) \mapsto \mathbf{Tm}(\mathfrak{l})$, acting by the formula, $\boldsymbol{f}_{\mathrm{x}}(t) := \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}](t, \mathrm{x}))$ $(t \in \mathbf{Tm}(\mathfrak{m}))$. We will prove that the mapping $\boldsymbol{f}_{\mathrm{x}}$ is injective on $\mathbf{Tm}(\mathfrak{m})$. Assume that $\boldsymbol{f}_{\mathrm{x}}(t_1) = \boldsymbol{f}_{\mathrm{x}}(t_2)$ for some $t_1, t_2 \in \mathbf{Tm}(\mathfrak{m})$. Then we obtain

$$\mathsf{tm}(\mathrm{w}_1) = \mathsf{tm}(\mathrm{w}_2), \tag{15}$$

where $\mathrm{w}_i = [\mathfrak{l} \leftarrow \mathfrak{m}](t_i, \mathrm{x})$ $(i \in \overline{1,2})$. According to Definition 3, we have, $\mathrm{w}_1, \mathrm{w}_2 \in \mathbf{trj}_{[\mathfrak{l} \leftarrow \mathfrak{m}]}(\mathrm{x})$. So, by Definition 5 (item 1), from (15) we obtain the equality $\mathrm{w}_1 = \mathrm{w}_2$. Therefore,

$$[\mathfrak{l} \leftarrow \mathfrak{m}](t_1, \mathrm{x}) = [\mathfrak{l} \leftarrow \mathfrak{m}](t_2, \mathrm{x}).$$

Thence, using [9, equalities (6), (7)] or [13, equalities (3.3), (3.4)], we receive that $(t_1, \mathrm{x}) = (t_2, \mathrm{x})$, i.e., $t_1 = t_2$. Thus for every $t_1, t_2 \in \mathbf{Tm}(\mathfrak{m})$ the equality $\boldsymbol{f}_{\mathrm{x}}(t_1) = \boldsymbol{f}_{\mathrm{x}}(t_2)$ implies $t_1 = t_2$, and so $\boldsymbol{f}_{\mathrm{x}}$ is an injective mapping from $\mathbf{Tm}(\mathfrak{m})$ to $\mathbf{Tm}(\mathfrak{l})$. Moreover, according to condition (b) of the proposition, the mapping $\boldsymbol{f} : \mathbf{Tm}(\mathfrak{m}) \times \mathbf{Zk}(\mathfrak{m}) \mapsto \mathbf{Tm}(\mathfrak{l})$ is separately continuous. Hence using condition (a) of the proposition and Lemma 2, we see that one and only one of the following alternative statements is true:

(1) the function $\boldsymbol{f}_{\mathrm{x}}(t)$ is strictly increasing on $\mathbf{Tm}(\mathfrak{m})$ for each $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$;
(2) the function $\boldsymbol{f}_{\mathrm{x}}(t)$ is strictly decreasing on $\mathbf{Tm}(\mathfrak{m})$ for each $\mathrm{x} \in \mathbf{Zk}(\mathfrak{m})$.

In the case of statement 1, we have $\mathfrak{l} \Uparrow^{+}_{\mathcal{F}} \mathfrak{m}$ (according to Definition 6, item 1) and, similarly, in the case of statement 2 we obtain $\mathfrak{l} \Downarrow^{-}_{\mathcal{F}} \mathfrak{m}$. So, in the both cases we deduce, $\mathfrak{l} \Updownarrow^{\pm}_{\mathcal{F}} \mathfrak{m}$. □

Let $\mathcal{F}$ be a topological-vector universal kinematicsis. Taking into account the definition of a coordinate space (see. [6, Definition 3] or [13, Definition 2.14.2]) as well as the system of notations in the theory of universal kinematicsis (see [13, Subsection 22.2]) we assure that for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the set $\mathbf{Zk}(\mathfrak{l})$ forms a topological vector space with the topology $\mathcal{T}p(\mathfrak{l})$ and linaear algebraic operations generated by the linear structure $\mathbb{L}s(\mathfrak{l})$. It is known that every topological vector space is connected [1, Theorem 3.1]. Hence, applying Proposition 7 as well as Definition 5 (item 3), we deduce the following theorem, "converse' to Theorem 1.

Theorem 2. *Let $\mathcal{F}$ be an arbitrary topological-vector universal kinematics.*

If a reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is sc-translational relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ and the following additional conditions are fulfilled:

(a): *the topological space $(\mathbf{Tm}(\mathfrak{m}), \mathfrak{Tpi}(\mathfrak{m}))$ is connected;*
(b): *the mapping $\mathbb{M}k(\mathfrak{m}) \ni \mathrm{w} \longmapsto \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\,\mathrm{w}) \in \mathbf{Tm}(\mathfrak{l})$ is separately continuous relatively to the space and time variables;*

then the reference frame $\mathfrak{l}$ is time-sign-defined relatively the reference frame $\mathfrak{m}$ in the kinematics $\mathcal{F}$ (ie $\mathfrak{l} \Updownarrow^{\pm}_{\mathcal{F}} \mathfrak{m}$).

Definition 8. We say that a topological universal kinematics $\mathcal{F}$ is C-*regular* (or of class (C^1_1)) if and only if:

(1) The kinematics $\mathcal{F}$ is of the class $(\mathrm{C}_0\mathrm{C}_1)$ (that is, for each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the ordered pairs $(\mathbf{Tm}(\mathfrak{l}), \mathfrak{Tpi}(\mathfrak{l}))$ and $(\mathbf{Zk}(\mathfrak{l}), \mathcal{T}p(\mathfrak{l}))$ are connected topological spaces.
(2) For any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ the mapping $\mathbb{M}k(\mathfrak{m}) \ni \mathrm{w} \longmapsto \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\,\mathrm{w}) \in \mathbf{Tm}(\mathfrak{l})$ is separately continuous relatively to the space and time variables.

As it was said before, every topological vector space is connected. So we get the following proposition.

Proposition 8. *A topological-vector universal kinematics $\mathcal{F}$ is* C*-regular if and only if $\mathcal{F}$ is of class (C_0) (that is, for each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ the ordered pair $(\mathbf{Tm}(\mathfrak{l}), \mathfrak{Tpi}(\mathfrak{l}))$ is a connected topological space) and for any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ the mapping $\mathbb{M}k(\mathfrak{m}) \ni \mathrm{w} \longmapsto \mathsf{tm}([\mathfrak{l} \leftarrow \mathfrak{m}]\,\mathrm{w}) \in \mathbf{Tm}(\mathfrak{l})$ is separately continuous relatively to the space and time variables.*

Using propositions 6, 7 and Definition 8, we obtain the following corollary.

Corollary 2. *A reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ of a* C*-regular topological universal kinematics $\mathcal{F}$ is trajectory-regular relatively to a reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ in the kinematics $\mathcal{F}$ if and only if $\mathfrak{l} \Updownarrow^{\pm}_{\mathcal{F}} \mathfrak{m}$ (that is, if and only if the frame $\mathfrak{l}$ is time-sign-defined relatively to the frame $\mathfrak{m}$ in $\mathcal{F}$).*

Using Theorem 1 and Theorem 2, we obtain the following corollary.

Corollary 3. *Let $\mathcal{F}$ be a* C*-regular topological-vector universal kinematics. A reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{F})$ is sc-translational relatively to the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{F})$ in kinematics $\mathcal{F}$ if and only if the following conditions are satisfied:*

(1) *the frame* $\mathfrak{m}$ *is sc-quasitranslational relatively* $\mathfrak{l}$ *in kinematics* $\mathcal{F}$*;*
(2) $\mathfrak{l} \Updownarrow_{\mathcal{F}}^{\pm} \mathfrak{m}$.

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