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SYMMETRIC EXTENSIONS OF SYMMETRIC LINEAR RELATIONS (OPERATORS) PRESERVING THE MULTIVALUED PART

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Dedicated to the memory of Professor M. L. Gorbachuk

Abstract. Let $\mathfrak{H}$ be a Hilbert space and let A be a symmetric linear relation (in particular, a nondensely defined operator) in $\mathfrak{H}$. By using the concept of a boundary triplet for A^* we characterize symmetric extensions $\widetilde{A} \supset A$ preserving the multivalued part of A. Such a characterization is given in terms of an abstract boundary parameter and the Weyl function of the boundary triplet. Application of these results to the Hamiltonian system $Jy' - B(t)y = \lambda\Delta(t)y$ enabled us to describe its matrix solutions generating the generalized Fourier transform with the nonempty set of respective spectral functions.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space and let T be a closed linear operator in $\mathfrak{H}$. Identifying T with its graph leads to the concept of a linear relation in $\mathfrak{H}$ (i.e., a subspace in $\mathfrak{H} \oplus \mathfrak{H}$), which turns out to be useful in the spectral theory of linear operators and its applications (see e.g. [6, 15, 19, 21, 27]). For the linear relation T in $\mathfrak{H}$ the subspace $\operatorname{mul} T \subset \mathfrak{H}$ given by $\operatorname{mul} T = \{h \in \mathfrak{H} : \{0, h\} \in T\}$ is called the multivalued part of T. Clearly in the case $T \subset \widetilde{T}$ one has $\operatorname{mul} T \subset \operatorname{mul} \widetilde{T}$. Note also that T is an operator if and only if $\operatorname{mul} T = \{0\}$.

Let A be a symmetric linear relation (in particular, nondensely defined operator) in $\mathfrak{H}$ with deficiency indices $n_\pm(A)$, let A^* be its adjoint and let $\mathfrak{N}_\lambda = \ker(A^* - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, be the defect subspace of A. If the operator part of A is densely defined (in particular if A is a densely defined operator), then $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ for any symmetric extension $\widetilde{A}$ of A; otherwise it is not true. The paper is devoted to the problem of characterization of symmetric extensions $\widetilde{A} \supset A$ such that $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$. For self-adjoint extensions $\widetilde{A} = \widetilde{A}^*$ this problem has been studied in a number of papers. Namely, let $\operatorname{Self}_0(A)$ be the set of all extensions $\widetilde{A} = \widetilde{A}^*$ of A with $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ (in the case of an operator A the later means that $\widetilde{A}$ is an operator). In the paper by A. V. Štraus [30] the class $\operatorname{Self}_0(A)$ for an operator A is parameterized by means of unitary operators $U : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ with special properties. In the case $n_+(A) = n_-(A)$ another description of the set $\operatorname{Self}_0(A)$ is given by the Krein-Naimark formula for resolvents

$$(1.1) \qquad (\widetilde{A}_\theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\theta - Q(\lambda))^{-1}\gamma^*(\overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $A_0 = A_0^*$ is a fixed extension of A, $\gamma(\lambda)$ is a so called γ-field and $Q(\lambda)$ is a Q-function of the pair (A, A_0). It was shown by H. Langer and B. Textorius in [21] that formula (1.1) gives a bijective correspondence $\widetilde{A} = \widetilde{A}_\theta$ between all extensions $\widetilde{A} \in \operatorname{Self}_0(A)$ and all linear relations $\theta = \theta^*$ in the auxiliary Hilbert space $\mathcal{H}$ satisfying the admissibility

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condition

$$s-\lim_{y\to\infty}\tfrac{1}{y}[Q(iy)+(Q(iy)-Q^*(i))(\theta-Q(iy))^{-1}(Q(iy)-Q(i))]=0. \tag{1.2}$$

In particular $A_0\in\mathrm{Self}_0(A)$ if and only if

$$s-\lim_{y\to\infty}\tfrac{1}{y}Q(iy)=0. \tag{1.3}$$

During the last period an approach to the extension theory of symmetric relations A based on the concept of a boundary triplet for A^* has been extensively developed (see [9, 14, 13, 18, 20] and references therein). Such a triplet is of the form $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and Γ_j are linear mappings from A^* to $\mathcal{H}$ such that the mapping $\Gamma=(\Gamma_0,\Gamma_1)^\top$ is surjective and the abstract Green identity

$$(f',g)-(f,g')=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})-(\Gamma_0\widehat{f},\Gamma_1\widehat{g}),\quad \widehat{f}=\{f,f'\},\ \widehat{g}=\{g,g'\}\in A^*$$

holds. With each boundary triplet Π one associates the Weyl function $M(\cdot)$ of Π defined by $\Gamma_1\upharpoonright\mathfrak{N}_\lambda=M(\lambda)\Gamma_0\upharpoonright\mathfrak{N}_\lambda,\ \lambda\in\mathbb{C}\setminus\mathbb{R}$ [14, 22]. It turns out that $M(\cdot)$ belongs to the class $R[\mathcal{H}]$ of Nevanlinna functions with values in the set $\boldsymbol{B}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$. A connection between Krein-Naimark formula (1.1) and boundary triplets has been discovered in [14, 22, 11, 12]. Namely, it is shown in these papers that each boundary triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* gives rise to formula (1.1) with $A_0=\ker\Gamma_0,\ \gamma(\lambda)=(\Gamma_0\upharpoonright\mathfrak{N}_\lambda)^{-1},\ \theta=\Gamma\widetilde{A}_\theta,\ Q(\lambda)=M(\lambda)$ and the admissibility condition (1.2) is equivalent to the following two simpler conditions:

$$s-\lim_{y\to\infty}\tfrac{1}{y}(\theta-M(iy))^{-1}=0\quad\text{and}\quad s-\lim_{y\to\infty}\tfrac{1}{y}(\theta^{-1}-M^{-1}(iy))^{-1}=0. \tag{1.4}$$

In the case $\theta=B\in\boldsymbol{B}(\mathcal{H})$ conditions (1.4) are equivalent to a unique condition

$$s-\lim_{y\to\infty}\tfrac{1}{y}(B-M(iy))^{-1}=0. \tag{1.5}$$

In the present paper we extend the above results to symmetric extension $\widetilde{A}$ of a symmetric linear relation A in $\mathfrak{H}$. It turns out that for symmetric extensions $\widetilde{A}$ of A the criterions for $\mathrm{mul}\,\widetilde{A}=\mathrm{mul}\,A$ differ essentially from those for self-adjoint extensions.

For a Hilbert space $\mathcal{H}$ let $R_u[\mathcal{H}]$ be the set of operator-functions $M(\cdot)\in R[\mathcal{H}]$ such that $\mathrm{Im}M(\lambda)\geq\alpha_\lambda I_{\mathcal{H}}$ with $\alpha_\lambda>0,\ \lambda\in\mathbb{C}_+$, and let $R_\Pi[\mathcal{H}]$ be the set of all $M(\cdot)\in R_u[\mathcal{H}]$ such that

$$s-\lim_{y\to+\infty}\tfrac{1}{iy}M(iy)=0\quad\text{and}\quad\lim_{y\to+\infty}y\,\mathrm{Im}(M(iy)h,h)=+\infty,\quad 0\neq h\in\mathcal{H}. \tag{1.6}$$

Assume that $\mathcal{H}$ is decomposed as $\mathcal{H}=\mathcal{H}'\oplus\dot{\mathcal{H}}$ and let

$$M(\lambda)=\begin{pmatrix}M_1(\lambda)&M_2(\lambda)\\M_3(\lambda)&M_4(\lambda)\end{pmatrix}:\mathcal{H}'\oplus\dot{\mathcal{H}}\to\mathcal{H}'\oplus\dot{\mathcal{H}},\quad\lambda\in\mathbb{C}_+ \tag{1.7}$$

be the block representation of an operator-function $M(\cdot)\in Ru[\mathcal{H}]$. Then for each operator function $K(\cdot)\in R[\dot{\mathcal{H}}]$ the equality (the Redheffer transform)

$$m_K(\lambda)=M_1(\lambda)-M_2(\lambda)(K(\lambda)+M_4(\lambda))^{-1}M_3(\lambda),\quad\lambda\in\mathbb{C}_+ \tag{1.8}$$

defines the operator-function $m_K(\cdot)\in R_u[\mathcal{H}']$. Moreover, we show in the paper that if $m_K(\cdot)$ satisfies

$$s-\lim_{y\to+\infty}\tfrac{1}{y}m_K(iy)=0 \tag{1.9}$$

for some $K\in R_\Pi[\mathcal{H}]$, then it satisfies (1.9) for any $K\in R_\Pi[\mathcal{H}]$. This fact enables us to introduce the following definition.

Definition 1.1. Let a Hilbert space $\mathcal{H}$ be decomposed as $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$. An operator-function $M(\cdot) \in R_u[\mathcal{H}]$ with the block representation (1.7) is referred to the class $R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ if for some (and hence for all) $K(\cdot) \in R_{\Pi}[\dot{\mathcal{H}}]$ the operator-function $m_K(\cdot) \in R_u[\mathcal{H}']$ of the form (1.8) satisfies (1.9).

Next assume that a symmetric relation A in $\mathfrak{H}$ has equal deficiency indices and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then according to [18, 22] the abstract boundary conditions

$$\widetilde{A}_\theta := \{\hat{f} \in A^* : \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \theta\}$$

give a parametrization $\widetilde{A} = \widetilde{A}_\theta$ of all symmetric extensions $\widetilde{A}$ of A by means of symmetric linear relations $\theta = \Gamma \widetilde{A}$ in $\mathcal{H}$. If in particular $\mathcal{H}'$ is a closed subspace in $\mathcal{H}$ and $\theta = \{0\} \oplus \mathcal{H}'$, then $\dot{A} := \widetilde{A}_\theta$ is given by

$$\dot{A} = \{\widehat{f} \in A^* : \Gamma_0 \widehat{f} = 0,\ \Gamma_1 \widehat{f} \in \mathcal{H}'\}. \tag{1.10}$$

Moreover, if $B(=\theta)$ is a bounded symmetric operator from $\mathcal{H}'$ to $\mathcal{H}$, then

$$\widetilde{A}_B = \{\widehat{f} \in A^* : \Gamma_0 \widehat{f} \in \mathcal{H}',\ \Gamma_1 \widehat{f} - B\Gamma_0 \widehat{f} = 0\}. \tag{1.11}$$

In terms of the γ-field $\gamma(\lambda)$ and Q-function $Q(\lambda) = M(\lambda)$ of the pair (A, A_0) the extension $\dot{A}$ of the form (1.10) can be defined as follows. Let $\mathfrak{N}_i' = \gamma(i)\mathcal{H}' (\subset \mathfrak{N}_i)$ and let $U : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ be a unitary operator, which defines A_0 by means of the Von Neumann formula. Then $\dot{A}$ is defined by the same formula with the isometry $V = U \restriction \mathfrak{N}_i'$.

The criterions for $\operatorname{mul} \dot{A} = \operatorname{mul} A$ and $\operatorname{mul} \widetilde{A}_B = \operatorname{mul} A$ are given in the following two theorems proved in the paper.

Theorem 1.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, let $M(\cdot) \in R_u[\mathcal{H}]$ be the Weyl function of Π, let $\mathcal{H}$ be decomposed as $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$ and let $\dot{A}$ by the symmetric extension* (1.10) *of A. Then $\operatorname{mul} \dot{A} = \operatorname{mul} A$ if and only if $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$.*

Theorem 1.3. *Let Π and $M(\cdot)$ be the same as in Theorem 1.2. Moreover, let $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$, let B be a bounded symmetric operator from $\mathcal{H}'$ to $\mathcal{H}$ with the block representation $B = (B_1, B_2)^\top : \mathcal{H}' \to \mathcal{H}' \oplus \dot{\mathcal{H}}$, let $M(\lambda)$ has the block representation* (1.7) *and let*

$$N(\lambda) = \begin{pmatrix} (B_1 - M_1(\lambda))^{-1} & (B_1 - M_1(\lambda))^{-1}(B_2^* - M_2(\lambda)) \\ (B_2 - M_3(\lambda))(B_1 - M_1(\lambda))^{-1} & M_4(\lambda) + (B_2 - M_3(\lambda))(B_1 - M_1(\lambda))^{-1}(B_2^* - M_2(\lambda)) \end{pmatrix}. \tag{1.12}$$

Then $N(\cdot) \in R_u[\mathcal{H}]$ and the symmetric extension $\widetilde{A}_B$ of A given by (1.11) *satisfies $\operatorname{mul} \widetilde{A}_B = \operatorname{mul} A$ if and only if $N(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$.*

In the case $\mathcal{H}' = \mathcal{H}$ one has $\dot{A} = A_0 (= \ker \Gamma_0)$ and the condition $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ in Theorem 1.2 turns into condition (1.3) (with $Q(\lambda) = M(\lambda)$). Moreover, in this case the operator B in Theorem 1.3 is a bounded self-adjoint operator in $\mathcal{H}$, $\widetilde{A}_B = \widetilde{A}_B^*$, $N(\lambda) = (B - M(\lambda))^{-1}$ and the condition $N(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ in this theorem takes the form (1.5).

We also specify criterions for $\operatorname{mul} \widetilde{A}_\theta = \operatorname{mul} A$ in the case of a general symmetric θ and for θ with the closed domain (see Theorems 3.9 and 3.11).

Actually the above results with certain modifications are obtained for symmetric relations A with arbitrary (possibly unequal) deficiency indices. To this end we use a boundary triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* with possibly unequal Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ (see [24, 25]).

In the last section the obtained results are applied to Hamiltonian systems. Namely, we characterise matrix solutions of such systems generating the generalized Fourier transform with the nonempty set of respective spectral functions.

2. Linear relations and boundary triplets

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote separable Hilbert spaces; $\boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $\boldsymbol{B}(\mathcal{H}) := \boldsymbol{B}(\mathcal{H}, \mathcal{H})$; $A \upharpoonright \mathcal{L}$ is a restriction of the operator $A \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ onto the linear manifold $\mathcal{L} \subset \mathcal{H}_1$; $\mathbb{C}_+$ $(\mathbb{C}_-)$ is the open upper (lower) half-plane of the complex plane. If $\mathcal{H}$ is a subspace in $\widetilde{\mathcal{H}}$, then $P_{\mathcal{H}} (\in \boldsymbol{B}(\widetilde{\mathcal{H}}))$ denote the orthoprojection in $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ and $P_{\widetilde{\mathcal{H}},\mathcal{H}} (\in \boldsymbol{B}(\widetilde{\mathcal{H}}, \mathcal{H}))$ is the same orthoprojection considered as an operator from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$. Moreover, $I_{\mathcal{H},\widetilde{\mathcal{H}}} \in \boldsymbol{B}(\mathcal{H}, \widetilde{\mathcal{H}})$ denote the operator embedding $\mathcal{H}$ into $\widetilde{\mathcal{H}}$.

Recall that a linear relation $T : \mathcal{H}_0 \to \mathcal{H}_1$ from a Hilbert space $\mathcal{H}_0$ to a Hilbert space $\mathcal{H}_1$ is a linear manifold in the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$. If $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ one speaks of a linear relation T in $\mathcal{H}$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ $(\widetilde{\mathcal{C}}(\mathcal{H}))$. A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ and $\operatorname{mul} T$ the domain, range, kernel and the multivalued part of T respectively; moreover, we denote by $T^{-1} (\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ and $T^* (\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ the inverse and adjoint linear relations of T.

2.2. **Linear relations from a Hilbert space to its subspace.** In the following $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, $P_1 := P_{\mathcal{H}_0,\mathcal{H}_1}$ and $P_2 = P_{\mathcal{H}_2}$. For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we let

$$g_\theta(\widehat{h}) = 2\operatorname{Im}(h_1, h_0)_{\mathcal{H}_0} + ||P_2 h_0||^2, \quad \widehat{h} = \{h_0, h_1\} \in \theta.$$

Definition 2.1. [23]. A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ if $g_\theta(\widehat{h}) \geq 0$, $g_\theta(\widehat{h}) \leq 0$ or $g_\theta(\widehat{h}) = 0$, $\widehat{h} \in \theta$, respectively. A relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ if it belongs to the class $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ respectively and there is not an extension $\widetilde{\theta} \supset \theta$, $\widetilde{\theta} \neq \theta$ in the corresponding class. We also let $\operatorname{Self}(\mathcal{H}_0, \mathcal{H}_1) = \operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1) \cap \operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$.

Let $J_{\mathcal{H}_0,\mathcal{H}_1} \in \boldsymbol{B}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ be an operator defined by

$$J_{\mathcal{H}_0,\mathcal{H}_1} = \begin{pmatrix} 0 & 0 & -I_{\mathcal{H}_1} \\ 0 & -iI_{\mathcal{H}_2} & 0 \\ I_{\mathcal{H}_1} & 0 & 0 \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1 \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1. \tag{2.1}$$

Then $J^*_{\mathcal{H}_0,\mathcal{H}_1} = J^{-1}_{\mathcal{H}_0,\mathcal{H}_1} = -J_{\mathcal{H}_0,\mathcal{H}_1}$ and hence the equality

$$[\widehat{h}, \widehat{k}] = (iJ_{\mathcal{H}_0,\mathcal{H}_1} \widehat{h}, \widehat{k}), \quad \widehat{h}, \widehat{k} \in \mathcal{H}_0 \oplus \mathcal{H}_1 \tag{2.2}$$

defines the indefinite inner product $[\cdot, \cdot]$ in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Since

$$[\widehat{h}, \widehat{h}] = g_\theta(\widehat{h}), \quad \widehat{h} = \{h_0, h_1\} \in \theta, \tag{2.3}$$

it follows that the relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\operatorname{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, $\operatorname{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ if and only if it is a nonnegative, nonpositive or neutral subspace in the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ with the product (2.2) respectively [5]. Moreover, θ belongs to the class $\operatorname{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ or $\operatorname{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if it is a maximal nonnegative or maximal nonpositive subspace respectively.

In the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ the classes $\operatorname{Dis}_0(\mathcal{H}, \mathcal{H}) =: \operatorname{Dis}_0(\mathcal{H})$, $\operatorname{Ac}_0(\mathcal{H}, \mathcal{H}) =: \operatorname{Ac}_0(\mathcal{H})$ and $\operatorname{Sym}_0(\mathcal{H}, \mathcal{H}) =: \operatorname{Sym}_0(\mathcal{H})$ coincide with the well-known classes of dissipative, accumulative and symmetric linear relations in $\mathcal{H}$ respectively; moreover, $\operatorname{Dis}(\mathcal{H}, \mathcal{H}) =: \operatorname{Dis}(\mathcal{H})$, $\operatorname{Ac}(\mathcal{H}, \mathcal{H}) =: \operatorname{Ac}(\mathcal{H})$ and $\operatorname{Self}(\mathcal{H}, \mathcal{H}) =: \operatorname{Self}(\mathcal{H})$ are the classes of maximal dissipative, maximal accumulative and self-adjoint linear relations in $\mathcal{H}$ respectively. Observe also that in this case the operator $J_{\mathcal{H},\mathcal{H}} =: J_{\mathcal{H}}$ is

$$J_{\mathcal{H}} = \begin{pmatrix} 0 & -I_{\mathcal{H}} \\ I_{\mathcal{H}} & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}.$$

With each $\lambda \in \mathbb{C}_-$ one associates the operator $X_\lambda \in \boldsymbol{B}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ given by

(2.4)
$$X_\lambda = \frac{1}{\sqrt{-2\mathrm{Im}\lambda}} \begin{pmatrix} -\lambda I_{\mathcal{H}_1} & 0 & I_{\mathcal{H}_1} \\ 0 & \sqrt{-2\mathrm{Im}\lambda}\, I_{\mathcal{H}_2} & 0 \\ -\overline{\lambda} I_{\mathcal{H}_1} & 0 & I_{\mathcal{H}_1} \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1 \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1.$$

One can easily verify that X_λ is invertible and hence the (Cayley) transform

$$\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1) \ni \theta \to \eta = \eta_\lambda(\theta) := X_\lambda \theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1), \quad \lambda \in \mathbb{C}_-$$

is an automorphism of $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. It is clear that

$$\eta_\lambda(\theta) = \{\{(h_1 - \lambda P_1 h_0) \oplus \sqrt{-2\mathrm{Im}\lambda} P_2 h_0, h_1 - \overline{\lambda} P_1 h_0\} : \{h_0, h_1\} \in \theta\}. \tag{2.5}$$

Recall that an operator $B \in \boldsymbol{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is called a contraction (isometry) if $||Bf|| \le ||f||$ (resp. $||Bf|| = ||f||$), $f \in \mathfrak{H}_1$.

Lemma 2.2. *The equality $\eta = \eta_\lambda(\theta)(= X_\lambda \theta)$ establishes a bijective correspondence between*

(1) *all $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ and all contractions $\eta \in \boldsymbol{B}(\mathrm{dom}\,\eta, \mathcal{H}_1)$ with the closed domain $\mathrm{dom}\,\eta \subset \mathcal{H}_0$;*
(2) *all $\theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$ ($\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$) and all contractions (resp. isometries) $\eta^{-1} \in \boldsymbol{B}(\mathrm{ran}\,\eta, \mathcal{H}_0)$ with the closed domain $\mathrm{ran}\,\eta \subset \mathcal{H}_1$;*
(3) *all $\theta \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ and all contractions $\eta \in \boldsymbol{B}(\mathcal{H}_0, \mathcal{H}_1)$;*
(4) *all $\theta \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ and all contractions $\eta^{-1} \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_0)$.*

Hence $\dim \theta \le \dim \mathcal{H}_0$, $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$, and $\dim \theta \le \dim \mathcal{H}_1$, $\theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1) \cup \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$.

Proof. Let $\widehat{J} \in \boldsymbol{B}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ be an operator given by

$$\widehat{J} = \begin{pmatrix} I_{\mathcal{H}_1} & 0 & 0 \\ 0 & I_{\mathcal{H}_2} & 0 \\ 0 & 0 & -I_{\mathcal{H}_1} \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1 \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \oplus \mathcal{H}_1.$$

Then the immediate checking shows that

$$X_\lambda^* \widehat{J} X_\lambda = i J_{\mathcal{H}_0, \mathcal{H}_1}. \tag{2.6}$$

Let $\widehat{h} = \{h_0, h_1\} \in \theta$ and let $\widehat{f} = \{f_0, f_1\} := X_\lambda \widehat{h}$. Then by (2.2), (2.3) and (2.6)

$$||f_0||^2 - ||f_1||^2 = (\widehat{J}\widehat{f}, \widehat{f}) = (iJ_{\mathcal{H}_0,\mathcal{H}_1}\widehat{h}, \widehat{h}) = g_\theta(\widehat{h}),$$

which yields statements (1)–(4). □

For a relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we let

$$\mathfrak{M}_\lambda(\theta) = \mathrm{ran}\,(\theta - \lambda I_{\mathcal{H}_0}) = \{(h_1 - \lambda P_1 h_0) \oplus (-\lambda P_2 h_0) : \{h_0, h_1\} \in \theta\}, \quad \lambda \in \mathbb{C}_-, \tag{2.7}$$

$$\mathfrak{M}'_\lambda(\theta) = \{(h_1 - \lambda P_1 h_0) \oplus \sqrt{-2\mathrm{Im}\lambda} P_2 h_0) : \{h_0, h_1\} \in \theta\}, \quad \lambda \in \mathbb{C}_-, \tag{2.8}$$

$$\mathfrak{M}_\lambda(\theta) = \mathrm{ran}\,(\theta - \lambda P_1) = \{h_1 - \lambda P_1 h_0 : \{h_0, h_1\} \in \theta\}, \quad \lambda \in \mathbb{C}_+ \tag{2.9}$$

(in (2.7) θ is considered as a relation in $\mathcal{H}_0(\supset \mathcal{H}_1)$). Clearly, the equalities (2.7)–(2.9) define linear manifolds $\mathfrak{M}_\lambda(\theta)$ and $\mathfrak{M}'_\lambda(\theta)$ in $\mathcal{H}_0$ for $\lambda \in \mathbb{C}_-$ and $\mathfrak{M}_\lambda(\theta)$ in $\mathcal{H}_1$ for $\lambda \in \mathbb{C}_+$. Moreover,

$$\mathfrak{M}'_\lambda(\theta) = \operatorname{dom} \eta_\lambda(\theta), \quad \lambda \in \mathbb{C}_-, \quad \mathfrak{M}_\lambda(\theta) = \operatorname{ran} \eta_{\overline{\lambda}}(\theta), \quad \lambda \in \mathbb{C}_+, \tag{2.10}$$

$$\mathfrak{M}_\lambda(\theta) = T_\lambda \mathfrak{M}'_\lambda(\theta), \quad \lambda \in \mathbb{C}_-, \tag{2.11}$$

where $T_\lambda \in \boldsymbol{B}(\mathcal{H}_0)$ is an isomorphism defined by the block representation

$$T_\lambda = \operatorname{diag}(I_{\mathcal{H}_1}, -\tfrac{\lambda}{\sqrt{-2\mathrm{Im}\lambda}} I_{\mathcal{H}_2}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Definition 2.3. The subspaces $\mathfrak{N}_\lambda(\theta) \subset \mathcal{H}_0$, $\lambda \in \mathbb{C}_+$, and $\mathfrak{N}_\lambda(\theta) \subset \mathcal{H}_1$, $\lambda \in \mathbb{C}_-$, defined by

$$\mathfrak{N}_\lambda(\theta) = \mathcal{H}_0 \ominus \mathfrak{M}_{\overline{\lambda}}(\theta), \quad \lambda \in \mathbb{C}_+, \quad \mathfrak{N}_\lambda(\theta) = \mathcal{H}_1 \ominus \mathfrak{M}_{\overline{\lambda}}(\theta), \quad \lambda \in \mathbb{C}_-$$

are called the defect subspaces of $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

Theorem 2.4. (1) *Let* $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$. *Then :* (a) $\mathfrak{M}_\lambda(\theta)$ *is a closed subspace in* $\mathcal{H}_0$, $\lambda \in \mathbb{C}_-$*;* (b) *the set of extensions* $\widetilde{\theta} \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ *of* θ *is not empty and for each such an extension* $\widetilde{\theta}$ *the equality*

$$\dim(\widetilde{\theta}/\theta) = \dim \mathfrak{N}_\lambda(\theta), \quad \lambda \in \mathbb{C}_+ \tag{2.12}$$

is valid. Hence the dimension of $\mathfrak{N}_\lambda(\theta)$ *does not depend on* $\lambda \in \mathbb{C}_+$.

(2) *Let* $\theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$. *Then :* (a) $\mathfrak{M}_\lambda(\theta)$ *is a closed subspace in* $\mathcal{H}_1$, $\lambda \in \mathbb{C}_+$*;* (b) *the set of extensions* $\widetilde{\theta} \in \mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ *of* θ *is not empty and for each such an extension* $\widetilde{\theta}$ *the equality*

$$\dim(\widetilde{\theta}/\theta) = \dim \mathfrak{N}_\lambda(\theta), \quad \lambda \in \mathbb{C}_- \tag{2.13}$$

is valid. Therefore the dimension of $\mathfrak{N}_\lambda(\theta)$ *does not depend on* $\lambda \in \mathbb{C}_-$.

Proof. (1) Let $\lambda \in \mathbb{C}_-$, let $\mathfrak{M}'_\lambda(\theta)$ be given by (2.8) and let $\mathfrak{N}'_{\overline{\lambda}}(\theta) := \mathcal{H}_0 \ominus \mathfrak{M}'_\lambda(\theta)$. Then by (2.10) and Lemma 2.2 $\mathfrak{M}'_\lambda(\theta)$ is a closed subspace in $\mathcal{H}_0$, which in view of (2.11) yields statement (a). Next, by Lemma 2.2 $\eta := \eta_\lambda(\theta) \in \boldsymbol{B}(\mathfrak{M}'_\lambda(\theta), \mathcal{H}_1)$, $||\eta|| \leq 1$ and the equality $\widetilde{\eta} = \eta_\lambda(\widetilde{\theta})$ gives a bijective correspondence between all extensions $\widetilde{\theta} \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ of θ and all contractive extensions $\widetilde{\eta} \in \boldsymbol{B}(\mathcal{H}_0, \mathcal{H}_1)$ of η. Since the set of such $\widetilde{\eta}$ is not empty (for instance, one can put $\widetilde{\eta} = \eta P_{\mathcal{H}_0, \mathfrak{M}'_\lambda(\theta)}$), the set of extensions $\widetilde{\theta} \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ of θ is not empty as well. Moreover, each $\widetilde{\eta}$ has the block representation

$$\widetilde{\eta} = (\eta, \eta') : \mathfrak{M}'_\lambda(\theta) \oplus \mathfrak{N}'_{\overline{\lambda}}(\theta) \to \mathcal{H}_1$$

with some $\eta' \in \boldsymbol{B}(\mathfrak{N}'_{\overline{\lambda}}(\theta), \mathcal{H}_1)$. Therefore $\widetilde{\eta} = \eta \dotplus \eta'$ and, consequently,

$$\dim \widetilde{\theta}/\theta = \dim \widetilde{\eta}/\eta = \dim \eta' = \dim \mathfrak{N}'_{\overline{\lambda}}(\theta).$$

Since by (2.11)

$$\dim \mathfrak{N}'_{\overline{\lambda}}(\theta) = \dim \mathfrak{N}_{\overline{\lambda}}(\theta), \quad \lambda \in \mathbb{C}_-, \tag{2.14}$$

the equality (2.12) is valid.

Similarly by using Lemma 2.2 one proves statement (2). □

Theorem 2.4 enables one to give the following definition.

Definition 2.5. If $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$ ($\theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1)$), then the cardinal number $n_+(\theta) = \dim \mathfrak{N}_\lambda(\theta)$, $\lambda \in \mathbb{C}_+$ (resp. $n_-(\theta) = \dim \mathfrak{N}_\lambda(\theta)$, $\lambda \in \mathbb{C}_-$) is called the deficiency index of θ.

If $\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)(\Leftrightarrow \theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1) \cap \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1))$, then the cardinal numbers $n_+(\theta) = \dim \mathfrak{N}_\lambda(\theta)$, $\lambda \in \mathbb{C}_+$, and $n_-(\theta) = \dim \mathfrak{N}_\lambda(\theta)$, $\lambda \in \mathbb{C}_-$, are called the deficiency indices of θ.

Remark 2.6. In the case $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ and $\theta\in \mathrm{Sym}_0(\mathcal{H})$ the equalities (2.7) and (2.9) take the form $\mathfrak{M}_\lambda(\theta)=\mathrm{ran}\,(\theta-\lambda I_{\mathcal{H}})$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Therefore in this case $\mathfrak{N}_\lambda(\theta)$ is the defect subspace and $n_\pm(\theta)$ are deficiency indices of the symmetric relation θ in the sense of the well-known definition (see e.g. [3, 10]).

In the following proposition we parameterize the class $\mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ in terms of isometries $V\in \boldsymbol{B}(\mathcal{H}',\mathcal{H}_0)$, where $\mathcal{H}'\subset\mathcal{H}_1$.

Proposition 2.7. *The equalities*

$$K_0=\begin{pmatrix} i(I_{\mathcal{H}',\mathcal{H}_1}-V_1)\\ \sqrt{2}V_2\end{pmatrix}:\mathcal{H}'\to\underbrace{\mathcal{H}_1\oplus\mathcal{H}_2}_{\mathcal{H}_0},\quad K_1=V_1+I_{\mathcal{H}',\mathcal{H}_1}(\in\boldsymbol{B}(\mathcal{H}',\mathcal{H}_1)),\tag{2.15}$$

$$\theta=\{\{K_0h,K_1h\}:h\in\mathcal{H}'\}\tag{2.16}$$

establish a bijective correspondence between all pairs $\{\mathcal{H}',V\}$ consisting of a subspace $\mathcal{H}'\subset\mathcal{H}_1$ and an isometry $V\in\boldsymbol{B}(\mathcal{H}',\mathcal{H}_0)$ with the block representation

$$V=\begin{pmatrix}V_1\\ V_2\end{pmatrix}:\mathcal{H}'\to\underbrace{\mathcal{H}_1\oplus\mathcal{H}_2}_{\mathcal{H}_0}\tag{2.17}$$

and all relations $\theta\in\mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$. Moreover, $\mathrm{gr}V=\eta_{-i}^{-1}(\theta)$, where $\eta_{-i}(\theta)$ is the Cayley transform (2.5) *of θ with $\lambda=-i$, and the following equalities hold:*

$$\dim(\mathcal{H}_1\ominus\mathrm{dom}\,V)=n_-(\theta),\quad \dim(\mathcal{H}_0\ominus\mathrm{ran}\,V)=n_+(\theta).\tag{2.18}$$

Proof. According to Lemma 2.2 the equality

$$\mathrm{gr}V=\eta_{-i}^{-1}(\theta)=\{\{h_1-iP_1h_0,(h_1+iP_1h_0)\oplus\sqrt{2}P_2h_0\}:\{h_0,h_1\}\in\theta\}\tag{2.19}$$

gives a bijective correspondence between all $\theta\in\mathrm{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$ and all isometries $V\in\boldsymbol{B}(\mathcal{H}',\mathcal{H}_0)$ with $\mathcal{H}'\subset\mathcal{H}_1$. Moreover, by (2.10) one has

$$\mathcal{H}'=\mathrm{dom}\,V=\mathrm{ran}\,\eta_{-i}(\theta)=\mathfrak{M}_i(\theta),\quad \mathrm{ran}\,V=\mathrm{dom}\,\eta_{-i}(\theta)=\mathfrak{M}'_{-i}(\theta).\tag{2.20}$$

Let V has the block representation (2.17). Then $\mathrm{gr}V=\{\{h,V_1h\oplus V_2h\}:h\in\mathcal{H}'\}$ and (2.19) is equivalent to

$$\theta=\{\{i(h-V_1h)\oplus\sqrt{2}V_2h,V_1h+h\}:h\in\mathcal{H}'\},$$

which in turn is equivalent to (2.15), (2.16). Moreover, in view of (2.20) $\mathcal{H}_1\ominus\mathrm{dom}\,V=\mathfrak{N}_{-i}(\theta)$ and $\mathcal{H}_0\ominus\mathrm{ran}\,V=\mathfrak{N}'_i(\theta)$. This and (2.14) yield the equalities (2.18). □

Proposition 2.7 implies the following well-known result (see e.g. [18]).

Proposition 2.8. *Let $\mathcal{H}$ be a Hilbert space. Then the equality*

$$\theta=\{\{i(I_{\mathcal{H}',\mathcal{H}}-V)h,(V+I_{\mathcal{H}',\mathcal{H}})h\}:h\in\mathcal{H}'\}\tag{2.21}$$

gives a bijective correspondence between all pairs $\{\mathcal{H}',V\}$ formed by a subspace $\mathcal{H}'\subset\mathcal{H}$ and an isometry $V\in\boldsymbol{B}(\mathcal{H}',\mathcal{H})$ and all relations $\theta\in\mathrm{Sym}_0(\mathcal{H})$. Moreover, $\theta=\theta^*$ if and only if $\mathcal{H}'=\mathcal{H}$ and $V\in\boldsymbol{B}(\mathcal{H})$ is a unitary operator.

As is known for each relation $\theta\in\mathrm{Sym}_0(\mathcal{H})$ the decompositions

$$\mathcal{H}=\widehat{\mathcal{H}}\oplus\mathrm{mul}\,\theta,\quad \theta=\mathrm{gr}B\oplus(\{0\}\oplus\mathrm{mul}\,\theta)\tag{2.22}$$

hold with a closed symmetric operator B in $\widehat{\mathcal{H}}$, which is called an operator part of θ. Clearly, $\mathrm{dom}\,B=\mathrm{dom}\,\theta(\subset\widehat{\mathcal{H}})$,

$$\mathrm{gr}B=\theta\cap\widehat{\mathcal{H}}^2\tag{2.23}$$

and $\theta \in \mathrm{Self}(\mathcal{H})$ if and only if $B = B^*$. Moreover, B is a bounded operator if and only if $\operatorname{dom}\theta$ is closed, in which case $B \in \boldsymbol{B}(\operatorname{dom}\theta, \widehat{\mathcal{H}})$ (if $\theta \in \mathrm{Self}(\mathcal{H})$, then $B \in \boldsymbol{B}(\widehat{\mathcal{H}})$). These arguments yield the following lemma.

Lemma 2.9. *Let $\mathcal{H}_1$ be a Hilbert space. Then for any relation $\theta \in \mathrm{Sym}_0(\mathcal{H}_1)$ with the closed domain there are a decomposition*

$$\mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \dot{\mathcal{H}}_1 \tag{2.24}$$

and a pair of operators $B_1 = B_1^* \in \boldsymbol{B}(\mathcal{H}_1')$ and $B_2 \in \boldsymbol{B}(\mathcal{H}_1', \dot{\mathcal{H}}_1)$ such that

$$\theta = \{\{h_1, B_1 h_1 \oplus h_2 \oplus B_2 h_1\} : h_1 \in \mathcal{H}_1',\ h_2 \in \mathcal{H}_2'\}. \tag{2.25}$$

Moreover, $\mathcal{H}_1' = \operatorname{dom}\theta$, $\mathcal{H}_2' = \operatorname{mul}\theta$ and $B = (B_1, B_2)^\top$ is the operator part of θ (hence decomposition (2.24) *and operators B_j in* (2.25) *are uniquely defined by θ).*

Definition 2.10. The representation of a linear relation $\theta \in \mathrm{Sym}_0(\mathcal{H}_1)$ with the closed domain by means of (2.24) and (2.25) will be cold canonical. Such a representation will be written as $\theta = \{\mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \dot{\mathcal{H}}_1, B_1, B_2\}$.

Definition 2.11. A pair (C_0, C_1) of operators $C_j \in \boldsymbol{B}(\mathcal{H})$, $j \in \{0,1\}$, is said to be a self-adjoint operator pair if $\mathrm{Im}(C_1 C_0^*) = 0$ and $(C_0 \pm iC_1)^{-1} \in \boldsymbol{B}(\mathcal{H})$. In the case $\dim\mathcal{H} < \infty$ the last condition can be replaced with $\operatorname{ran}(C_0, C_1) = \mathcal{H}$.

The set of all self-adjoint operator pairs (C_0, C_1) with $C_j \in \boldsymbol{B}(\mathcal{H})$ will be denoted by $SP(\mathcal{H})$.

As is known, for each pair $(C_0, C_1) \in SP(\mathcal{H})$ the equality

$$\theta = \{\{h, h'\} \in \mathcal{H}^2 : C_0 h + C_1 h' = 0\} \tag{2.26}$$

defines a relation $\theta \in \mathrm{Self}(\mathcal{H})$ and, conversely, for each $\theta \in \mathrm{Self}(\mathcal{H})$ there is a pair $(C_0, C_1) \in SP(\mathcal{H})$ such that (2.26) holds.

Proposition 2.12. *Let $(C_0, C_1) \in SP(\mathcal{H})$ and let $\theta \in \mathrm{Self}(\mathcal{H})$ be given by* (2.26). *Then*

(1) $\operatorname{dom}\theta$ *is closed if and only if* $\operatorname{ran} C_1$ *is closed.*

(2) *If $\mathcal{K}_1 := \operatorname{ran} C_1$ is closed, $\mathcal{H}_2' := \ker C_1$, $\mathcal{H}_1' := \mathcal{H} \ominus \mathcal{H}_2'$ and $\mathcal{K}_2 := \mathcal{H} \ominus \mathcal{K}_1$, then $\operatorname{dom}\theta = \mathcal{H}_1'$, $\operatorname{mul}\theta = \mathcal{H}_2'$ and the block representations of C_0 and C_1 are*

$$C_0 = \begin{pmatrix} C_{01} & C_{02} \\ 0 & C_{03} \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_2' \to \mathcal{K}_1 \oplus \mathcal{K}_2, \quad C_1 = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_2' \to \mathcal{K}_1 \oplus \mathcal{K}_2. \tag{2.27}$$

Moreover, $C_{11}^{-1} \in \mathcal{B}(\mathcal{K}_1, \mathcal{H}_1')$ and the operator part $B'(\in \boldsymbol{B}(\mathcal{H}_1'))$ of θ is $B' = -C_{11}^{-1} C_{01}$.

Proof. (1) Since $\operatorname{ran}(C_0 + iC_1) = \mathcal{H}$, it follows that $\operatorname{ran}(C_0, C_1) = \mathcal{H}$. Therefore (see e.g. [23, Proposition 3.1]) $\theta^* = \{\{-C_1^* h, C_0^* h\} : h \in \mathcal{H}\}$ and in view of the equality $\theta = \theta^*$ one has $\operatorname{dom}\theta = \operatorname{ran} C_1^*$. Hence $\operatorname{dom}\theta$ is closed if and only if $\operatorname{ran} C_1^*$ is closed or, equivalently, if and only if $\operatorname{ran} C_1$ is closed.

(2) Since $C_1 h = 0 \Leftrightarrow C_0 \cdot 0 + C_1 h = 0 \Leftrightarrow h \in \operatorname{mul}\theta$, it follows that $\operatorname{mul}\theta = \ker C_1 = \mathcal{H}_2'$. Since $\operatorname{ran} C_1$ is closed, by statement (1) $\operatorname{dom}\theta$ is closed. Therefore $\operatorname{dom}\theta = \mathcal{H} \ominus \mathcal{H}_2' = \mathcal{H}_1'$.

It follows from (2.26) and (2.23) that $\mathrm{gr}B' = \{\{h, h'\} \in \mathcal{H}_1'^2 : C_0 h + C_1 h' = 0\}$. Hence

$$C_0 \upharpoonright \mathcal{H}_1' = -C_1 \upharpoonright \mathcal{H}_1' \cdot B' \tag{2.28}$$

and, consequently, $\operatorname{ran}(C_0 \upharpoonright \mathcal{H}_1') \subset \operatorname{ran} C_1 = \mathcal{K}_1$. Thus equalities (2.27) are valid.

Next, the second equality in (2.27) yields $\ker C_{11} = \ker(C_1 \upharpoonright \mathcal{H}_1') = \{0\}$ and $\operatorname{ran} C_{11} = \operatorname{ran} C_1 = \mathcal{K}_1$. Therefore the operator C_{11} is invertible. Moreover, by (2.28) and (2.27) $C_{01} = -C_{11} B'$, which implies the required equality for B'. □

The following lemma will be useful in the sequel.

Lemma 2.13. *Let $\mathcal{H}'$ be a subspace in $\mathcal{H}_1$, let $V \in \boldsymbol{B}(\mathcal{H}', \mathcal{H}_0)$ be an isometry with the block representation* (2.17) *and let $\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ be respective relation* (2.15), (2.16). *Assume also that $n_-(\theta) \le n_+(\theta)$. Then*

(1) *There exist a Hilbert space $\widetilde{\mathcal{H}}_0 \supset \mathcal{H}_1$ and a unitary operator $U \in \boldsymbol{B}(\mathcal{H}_0, \widetilde{\mathcal{H}}_0)$ such that $V^{-1} \subset U$.*

(2) *If $U \in \boldsymbol{B}(\mathcal{H}_0, \widetilde{\mathcal{H}}_0)$ is a unitary operator from statement* (1) *and*

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}_1 \oplus \widetilde{\mathcal{H}}_2}_{\widetilde{\mathcal{H}}_0} \tag{2.29}$$

is the block representation of U, then the equality

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \widetilde{\mathcal{H}}_0 \oplus \mathcal{H}_1, \tag{2.30}$$

where

$$z_1 = \begin{pmatrix} \frac{1}{2}(u_1 + I_{\mathcal{H}_1}) & -\frac{i}{\sqrt{2}} u_2 \\ \frac{i}{\sqrt{2}} u_3 & u_4 \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}_1 \oplus \widetilde{\mathcal{H}}_2}_{\widetilde{\mathcal{H}}_0}, \tag{2.31}$$

$$z_2 = \begin{pmatrix} -\frac{i}{2}(u_1 - I_{\mathcal{H}_1}) \\ \frac{1}{\sqrt{2}} u_3 \end{pmatrix} : \mathcal{H}_1 \to \underbrace{\mathcal{H}_1 \oplus \widetilde{\mathcal{H}}_2}_{\widetilde{\mathcal{H}}_0}, \tag{2.32}$$

$$z_3 = \left(\tfrac{i}{2}(u_1 - I_{\mathcal{H}_1}), \tfrac{1}{\sqrt{2}} u_2\right) : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \mathcal{H}_1, \quad z_4 = \tfrac{1}{2}(u_1 + I_{\mathcal{H}_1}) \tag{2.33}$$

define the operator $Z \in \boldsymbol{B}(\mathcal{H}_0 \oplus \mathcal{H}_1, \widetilde{\mathcal{H}}_0 \oplus \mathcal{H}_1)$ such that

$$Z^* J_{\widetilde{\mathcal{H}}_0, \mathcal{H}_1} Z = J_{\mathcal{H}_0, \mathcal{H}_1}, \quad Z J_{\mathcal{H}_0, \mathcal{H}_1} Z^* = J_{\widetilde{\mathcal{H}}_0, \mathcal{H}_1}, \tag{2.34}$$

$$Z\theta = \{0\} \oplus \mathcal{H}' = \{\{0, h\} : h \in \mathcal{H}'\}. \tag{2.35}$$

If in addition $\dim \theta < \infty$ (in particular, if $\dim \mathcal{H}_1 < \infty$), then there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{H}_0)$ such that $V^{-1} \subset U$ and for this operator U statement (2) *holds with $\widetilde{\mathcal{H}}_0 = \mathcal{H}_0$ and $\widetilde{\mathcal{H}}_2 = \mathcal{H}_2$.*

Proof. (1) Since $n_-(\theta) \le n_+(\theta)$, it follows from (2.18) that $\dim(\mathcal{H}_1 \ominus \mathrm{ran}\, V^{-1}) \le \dim(\mathcal{H}_0 \ominus \mathrm{dom}\, V^{-1})$. Therefore there exist a Hilbert space $\widetilde{\mathcal{H}}_0 \supset \mathcal{H}_1$ with

$$\dim(\widetilde{\mathcal{H}}_0 \ominus \mathrm{ran}\, V^{-1}) = \dim(\mathcal{H}_0 \ominus \mathrm{dom}\, V^{-1})$$

and a unitary operator $U \in \boldsymbol{B}(\mathcal{H}_0, \widetilde{\mathcal{H}}_0)$ such that $V^{-1} \subset U$.

(2) Let $U \in \boldsymbol{B}(\mathcal{H}_0, \widetilde{\mathcal{H}}_0)$ be an operator from statement (1) with the block representation (2.29) and let Z be given by (2.30) - (2.33). Then the equalities $U^*U = I_{\mathcal{H}_0}$ and $UU^* = I_{\widetilde{\mathcal{H}}_0}$ yield

$$u_1^* u_1 + u_3^* u_3 = I_{\mathcal{H}_1}, \quad u_2^* u_1 + u_4^* u_3 = 0, \quad u_2^* u_2 + u_4^* u_4 = I_{\mathcal{H}_2}, \tag{2.36}$$

$$u_1 u_1^* + u_2 u_2^* = I_{\mathcal{H}_1}, \quad u_3 u_1^* + u_4 u_2^* = 0, \quad u_3 u_3^* + u_4 u_4^* = I_{\widetilde{\mathcal{H}}_2} \tag{2.37}$$

and the immediate calculations with taking (2.36) and (2.37) into account give the equalities (2.34). Moreover, since $UV = I_{\mathcal{H}', \widetilde{\mathcal{H}}_0} = \begin{pmatrix} I_{\mathcal{H}', \mathcal{H}_1} \\ 0 \end{pmatrix}$, it follows from (2.29) and (2.17) that

$$u_1 V_1 + u_2 V_2 = I_{\mathcal{H}', \mathcal{H}_1}, \quad u_3 V_1 + u_4 V_2 = 0.$$

These equalities together with (2.31)–(2.33) and (2.15) yield

$$z_1 K_0 + z_2 K_1 = 0, \quad z_3 K_0 + z_4 K_1 = 2 I_{\mathcal{H}', \mathcal{H}_1}. \tag{2.38}$$

It follows from (2.16) that $\{h_0,h_1\}\in\theta$ if and only if $h_0=K_0h$ and $h_1=K_1h$ with some $h\in\mathcal{H}'$. Moreover, in view of (2.38) one has

$$Z\{h_0,h_1\}=\{(z_1K_0+z_2K_1)h,(z_3K_0+z_4K_1)h\}=\{0,2h\}.$$

This proves (2.35).

Assume now that $\dim\theta<\infty$. Since by Proposition 2.7 $\mathrm{gr}V^{-1}=\eta_{-i}(\theta)=X_{-i}\theta$, it follows that $\dim(\mathrm{dom}\,V^{-1})=\dim(\mathrm{ran}\,V^{-1})<\infty$. Therefore $\dim(\mathcal{H}_0\ominus\mathrm{dom}\,V^{-1})=\dim(\mathcal{H}_0\ominus\mathrm{ran}\,V^{-1})$, which implies the last statement of the lemma. □

The following corollary is immediate from Lemma 2.13.

Corollary 2.14. *Assume that $\mathcal{H}$ is a Hilbert space, $\mathcal{H}'$ is a subspace in $\mathcal{H}$, $V\in\boldsymbol{B}(\mathcal{H}',\mathcal{H})$ is an isometry and θ is a symmetric relation* (2.21) *in $\mathcal{H}$. Assume also that $n_-(\theta)=n_+(\theta)$. Then*

(1) *There exists a unitary operator $U\in\boldsymbol{B}(\mathcal{H})$ such that $V^{-1}\subset U$.*

(2) *If U is an operator from statement* (1), *then the equality*

$$Z=\begin{pmatrix}U+I & -i(U-I)\\ i(U-I) & U+I\end{pmatrix}:\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}$$

defines the operator $Z\in\boldsymbol{B}(\mathcal{H}\oplus\mathcal{H})$ such that $Z^*J_{\mathcal{H}}Z=ZJ_{\mathcal{H}}Z^*=J_{\mathcal{H}}$ and $Z\theta=\{0\}\oplus\mathcal{H}'$.

2.3. **The classes $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$, $\widetilde{R}(\mathcal{H})$ and $R[\mathcal{H}]$.** By using Lemma 2.2 one can easily verify that $(\theta-iP_1)^{-1}\in\boldsymbol{B}(\mathcal{H}_1,\mathcal{H}_0)$ for each $\theta\in\mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1)$.

Definition 2.15. [23, 25]. A function $\tau(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ is referred to the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ if $-\tau(\lambda)\in\mathrm{Ac}(\mathcal{H}_0,\mathcal{H}_1)$, $\lambda\in\mathbb{C}_+$, and the operator-function $(\tau(\lambda)+iP_1)^{-1}$ is holomorphic on $\mathbb{C}_+$.

A function $\tau(\cdot)\in\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ belongs to the class $\widetilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$ if $-\tau(\lambda)=\theta$, $\lambda\in\mathbb{C}_+$, with some $\theta\in\mathrm{Self}(\mathcal{H}_0,\mathcal{H}_1)$.

It turns out that the equality

$$\tau(\lambda)=\{C_0(\lambda),C_1(\lambda)\}:=\{\{h_0,h_1\}\in\mathcal{H}_0\oplus\mathcal{H}_1:C_0(\lambda)h_0+C_1(\lambda)h_1=0\}\tag{2.39}$$

with $\lambda\in\mathbb{C}_+$ enables one to identify a function $\tau(\cdot)\in\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ and a pair of holomorphic operator-functions $C_j(\cdot):\mathbb{C}_+\to[\mathcal{H}_j,\mathcal{H}_0]$, $j\in\{0,1\}$, satisfying

$$2\,\mathrm{Im}(C_1(\lambda)P_1C_0^*(\lambda))+C_0(\lambda)P_2C_0^*(\lambda)\geq 0,\quad (C_0(\lambda)-iC_1(\lambda)P_1)^{-1}\in[\mathcal{H}_0]\tag{2.40}$$

for all $\lambda\in\mathbb{C}_+$. Moreover, for a function $\tau(\cdot)\in\widetilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$ one has $\tau(\lambda)=\{C_0,C_1\}$, $\lambda\in\mathbb{C}_+$, that is $C_j(\lambda)$ does not depend on λ (for more details see [23, 25]).

In the case $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ the class $\widetilde{R}(\mathcal{H},\mathcal{H})$ coincides with the well-known class $\widetilde{R}(\mathcal{H})$ of Nevanlinna $\widetilde{\mathcal{C}}(\mathcal{H})$-valued functions (Nevanlinna operator pairs) $\tau(\lambda)=\{C_0(\lambda),C_1(\lambda)\}$, while the class $\widetilde{R}^0(\mathcal{H},\mathcal{H})$ turns into the class $\widetilde{R}^0(\mathcal{H})$ of all $\tau(\cdot)\in\widetilde{R}(\mathcal{H})$ such that $\tau(\lambda)=\theta$, $\lambda\in\mathbb{C}_+$, with some $\theta=\theta^*\in\widetilde{\mathcal{C}}(\mathcal{H})$. Recall also the following definition.

Definition 2.16. A holomorphic operator-function $M(\cdot):\mathbb{C}_+\to\boldsymbol{B}(\mathcal{H})$ is referred to the class $R[\mathcal{H}]$ if $\mathrm{Im}M(\lambda)\geq 0$, $\lambda\in\mathbb{C}_+$. A function $M(\cdot)\in R[\mathcal{H}]$ is referred to the class $R_u[\mathcal{H}]$ if $\mathrm{Im}M(\lambda)\geq\alpha_\lambda I_{\mathcal{H}}$ with some $\alpha_\lambda>0$, $\lambda\in\mathbb{C}_+$.

Clearly, $R[\mathcal{H}]\subset\widetilde{R}(\mathcal{H})$.

As is known (see e.g. [8]) a function $M(\cdot)\in R[\mathcal{H}]$ admits the integral representation

$$M(\lambda)=\mathcal{A}_M+\lambda\mathcal{B}_M+\int_{\mathbb{R}}\left(\frac{1}{t-\lambda}-\frac{1}{1+t^2}\right)d\Sigma_M(t),$$

where $\mathcal{A}_M = \mathcal{A}_M^* \in \boldsymbol{B}(\mathcal{H})$, $\mathcal{B}_M \in \boldsymbol{B}(\mathcal{H})$, $\mathcal{B}_M \geq 0$ and $\Sigma_M(\cdot) : \mathbb{R} \to \boldsymbol{B}(\mathcal{H})$ is a nondecreasing strongly left-continuous operator-function such that $\Sigma_M(0) = 0$ and

$$\int_{\mathbb{R}} (t^2+1)^{-1} (d\Sigma_M(t)h, h) < \infty, \quad h \in \mathcal{H}.$$

The operators $\mathcal{A}_M$, $\mathcal{B}_M$ and the function $\Sigma_M(\cdot)$ are uniquely defined by $M(\cdot)$. In particular,

$$\mathcal{B}_M = s - \lim_{y \to +\infty} \tfrac{1}{iy} M(iy). \tag{2.41}$$

Definition 2.17. An operator-function $M(\cdot) \in R_u[\mathcal{H}]$ is referred to the class $R_\Pi[\mathcal{H}]$ if it satisfies (1.6) or, equivalently, if

$$\mathcal{B}_M = 0 \quad \text{and} \quad \int_{\mathbb{R}} (d\Sigma_M(t)h, h) = +\infty, \quad 0 \neq h \in \mathcal{H}.$$

The following proposition is well known (see e.g. [25, Proposition 2.2]).

Proposition 2.18. *Assume that the Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$ and*

$$M(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}}_{\mathcal{H}} \to \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}}_{\mathcal{H}}, \quad \lambda \in \mathbb{C}_+ \tag{2.42}$$

is the block representation of an operator-function $M(\cdot) \in R[\mathcal{H}]$. Then: (i) $M_1(\cdot) \in R[\mathcal{H}']$ *and* $M_4(\cdot) \in R[\dot{\mathcal{H}}]$*;* (ii) $\mathcal{B}_M = 0$ *if and only if* $\mathcal{B}_{M_1} = 0$ *and* $\mathcal{B}_{M_4} = 0$.

Proposition 2.19. *Let the conditions of Proposition 2.18 be satisfied and let $M(\cdot) \in R_u[\mathcal{H}]$. Then the equality*

$$m_K(\lambda) = M_1(\lambda) - M_2(\lambda)(K(\lambda) + M_4(\lambda))^{-1} M_3(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{2.43}$$

defines a mapping $R[\dot{\mathcal{H}}] \ni K(\cdot) \to m_K(\cdot) \in R_u[\mathcal{H}']$ of the set $R[\dot{\mathcal{H}}]$ into $R_u[\mathcal{H}']$.

Proof. Let $K(\cdot) \in R[\dot{\mathcal{H}}]$ and let $\widetilde{K}(\lambda) = \operatorname{diag}(0, K(\lambda)) \in \boldsymbol{B}(\mathcal{H}' \oplus \dot{\mathcal{H}})$. Then

$$\widetilde{K}(\lambda) + M(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ M_3(\lambda) & K(\lambda) + M_4(\lambda) \end{pmatrix} : \mathcal{H}' \oplus \dot{\mathcal{H}} \to \mathcal{H}' \oplus \dot{\mathcal{H}}, \quad \lambda \in \mathbb{C}_+.$$

Since $\widetilde{K}(\cdot) \in R[\mathcal{H}]$ and $M(\cdot) \in R_u[\mathcal{H}]$, it follows that $(\widetilde{K} + M)(\cdot) \in R_u[\mathcal{H}]$ and hence $(K + M_4)(\cdot) \in R_u[\mathcal{H}]$. Therefore the operators $\widetilde{K}(\lambda) + M(\lambda)$ and $K(\lambda) + M_4(\lambda)$ are invertible and by the Frobenius formula one has

$$P_{\mathcal{H},\mathcal{H}'} \left(-(\widetilde{K}(\lambda) + M(\lambda))^{-1} \right) \upharpoonright \mathcal{H}' = -m_K^{-1}(\lambda). \tag{2.44}$$

Since $(\widetilde{K} + M)(\cdot) \in R_u[\mathcal{H}]$, it follows that $-(\widetilde{K} + M)^{-1}(\cdot) \in R_u[\mathcal{H}]$ and by (2.44) $-m_K^{-1}(\cdot) \in R_u[\mathcal{H}']$. Therefore $m_K(\cdot) \in R_u[\mathcal{H}']$. □

Remark 2.20. The transform (2.43) is an analog of the Redheffer transform for contractive operator-functions (see e.g. [2]).

2.4. **Boundary triplets and self-adjoint extensions.** Let $\mathfrak{H}$ be a Hilbert space, let A be a closed symmetric linear relation in $\mathfrak{H}$ with deficiency indices $n_\pm(A)$, let $\mathfrak{N}_\lambda(A)$ be the defect subspace of A and let $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Denote also by Ext_A the set of all proper extensions of A, i.e., the set of all relations $\widetilde{A} \in \widetilde{\mathcal{C}}(\mathfrak{H})$ such that $A \subset \widetilde{A} \subset A^*$. As is known (see e.g [22, 30]) each dissipative, accumulative or symmetric extension $\widetilde{A} \supset A$ is proper (that is, $\widetilde{A} \subset A^*$).

As before we assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, $P_1 := P_{\mathcal{H}_0,\mathcal{H}_1}$ and $P_2 = P_{\mathcal{H}_2}$.

Definition 2.21. [24]. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0,1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$, $\widehat{f} \in A^*$, from A^* into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$:

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \widehat{f}, P_2 \Gamma_0 \widehat{g})_{\mathcal{H}_2}.$$

According to [24] a boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* exists if and only if $n_-(A) \le n_+(A)$, in which case $\dim \mathcal{H}_1 = n_-(A)$ and $\dim \mathcal{H}_0 = n_+(A)$.

Proposition 2.22. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then*

(1) $\ker \Gamma = A$ *and* Γ *is a bounded operator from* A^* *into* $\mathcal{H}_0 \oplus \mathcal{H}_1$*;*

(2) *The set of all proper extensions $\widetilde{A} \in \mathrm{Ext}_A$ is parameterized by linear relations $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. More precisely, the mapping*

$$\theta \to \widetilde{A}_\theta := \{\hat{f} \in A^* : \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \theta\} \tag{2.45}$$

establishes a bijective correspondence $\widetilde{A} = \widetilde{A}_\theta$ between all relations $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and all extensions $\widetilde{A} \in \mathrm{Ext}_A$. The equality $\widetilde{A} = \widetilde{A}_\theta$ means that $\theta = \Gamma \widetilde{A} = \{\{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\} : \widehat{f} \in \widetilde{A}\}$.

(3) *The following equivalences hold:*

$$\widetilde{A}_\theta \in \mathrm{Dis}_0(\mathfrak{H}) \iff \theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1), \quad \widetilde{A}_\theta \in \mathrm{Ac}_0(\mathfrak{H}) \iff \theta \in \mathrm{Ac}_0(\mathcal{H}_0, \mathcal{H}_1), \tag{2.46}$$

$$\widetilde{A}_\theta \in \mathrm{Sym}_0(\mathfrak{H}) \iff \theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1), \quad \widetilde{A}_\theta \in \mathrm{Self}(\mathfrak{H}) \iff \theta \in \mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1). \tag{2.47}$$

Moreover, the deficiency indices of $\widetilde{A}_\theta$ and θ are connected by

$$n_+(\widetilde{A}_\theta) = n_+(\theta), \quad \text{if} \quad \widetilde{A}_\theta \in \mathrm{Dis}_0(\mathfrak{H}); \quad n_-(\widetilde{A}_\theta) = n_-(\theta), \quad \text{if} \quad \widetilde{A}_\theta \in \mathrm{Ac}_0(\mathfrak{H}); \tag{2.48}$$

$$n_+(\widetilde{A}_\theta) = n_+(\theta) \quad \text{and} \quad n_-(\widetilde{A}_\theta) = n_-(\theta), \quad \text{if} \quad \widetilde{A}_\theta \in \mathrm{Sym}_0(\mathfrak{H}). \tag{2.49}$$

Proof. Statements (1), (2) and equivalences (2.46), (2.47) are proved in [24]. Next, assume that $\widetilde{A}_\theta \in \mathrm{Dis}_0(\mathfrak{H})$ and hence $\theta \in \mathrm{Dis}_0(\mathcal{H}_0, \mathcal{H}_1)$. Let $\widetilde{\theta} \in \mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ be an extension of θ. Then $\widetilde{A}_\theta \subset \widetilde{A}_{\widetilde{\theta}}$ and by the first equivalence in (2.46) $\widetilde{A}_{\widetilde{\theta}} \in \mathrm{Dis}(\mathfrak{H})$. It follows from Theorem 2.4, (1) that $n_+(\theta) = \dim \widetilde{\theta}/\theta$ and $n_+(\widetilde{A}_\theta) = \dim \widetilde{A}_{\widetilde{\theta}}/\widetilde{A}_\theta$. Moreover, by statement (1) $\dim \widetilde{A}_{\widetilde{\theta}}/\widetilde{A}_\theta = \dim \widetilde{\theta}/\theta$. This proves the first equality in (2.48). The second equality in (2.48) can be proved similarly. Finally, (2.49) is a consequence of (2.48) □

According to [24] for each boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* the operator $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A)$, $\lambda \in \mathbb{C}_+$, isomorphically maps $\widehat{\mathfrak{N}}_\lambda(A)$ onto $\mathcal{H}_0$. Therefore the equality

$$\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A) = M_+(\lambda) \Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A), \quad \lambda \in \mathbb{C}_+ \tag{2.50}$$

correctly defines the operator function $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$, which is called the Weyl function of the triplet Π. This function is holomorphic on $\mathbb{C}_+$ and the block representation

$$M_+(\lambda) = (M(\lambda), N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+ \tag{2.51}$$

defines the operator function $M(\cdot) \in R_u[\mathcal{H}_1]$.

As is known a linear relation $\widetilde{A} = \widetilde{A}^*$ in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ satisfying $A \subset \widetilde{A}$ is called an exit space self-adjoint extension of A. Such an extension $\widetilde{A}$ is called minimal if there is no a nontrivial subspace $\mathfrak{H}' \subset \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$ reducing $\widetilde{A}$. In the following we denote by $\widetilde{\mathrm{Self}}(A)$ the set of all minimal exit space self-adjoint extensions of $A \in \mathrm{Sym}_0(\mathfrak{H})$. Moreover, we denote by $\mathrm{Self}(A)$ the set of all extensions $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$ of A (such an extension is called canonical). As is known, for each $A \in \mathrm{Sym}_0(\mathfrak{H})$ one has $\widetilde{\mathrm{Self}}(A) \neq \emptyset$. Moreover, $\mathrm{Self}(A) \neq \emptyset$ if and only if $n_+(A) = n_-(A)$, in which case $\mathrm{Self}(A) \subset \widetilde{\mathrm{Self}}(A)$.

Parametrization of all extensions $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ and all $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ with $\mathrm{mul}\, \widetilde{A} = \mathrm{mul}\, A$ in terms of a boundary triplet for A^* is specified in the following theorem.

Theorem 2.23. [24, 26]. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $M_+(\cdot)$ be the Weyl function of Π. Then*

(1) *The equalities*

$$\widetilde{A}(\lambda) = \widetilde{A}_{-\tau(\lambda)} = \{\widehat{f} \in A^* : C_0(\lambda)\Gamma_0\widehat{f} - C_1(\lambda)\Gamma_1\widehat{f} = 0\}(\in \mathrm{Ext}_A), \tag{2.52}$$

$$P_{\widetilde{\mathfrak{H}},\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (\widetilde{A}(\lambda) - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \tag{2.53}$$

give a bijective correspondence $\widetilde{A} = \widetilde{A}_\tau$ between all functions (operator pairs) $\tau = \tau(\lambda) = \{C_0(\lambda), C_1(\lambda)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ defined by (2.39) and all extensions $\widetilde{A} \in \widetilde{\mathrm{Self}}(A)$ Moreover, $\widetilde{A}_\tau \in \mathrm{Self}(A)$ if and only if $\tau \in \widetilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

(2) *If $\tau = \{C_0(\lambda), C_1(\lambda)\} \in \widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, then the equalities*

$$\Phi_\tau(\lambda) = P_1(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)(= -P_1(\tau(\lambda) + M_+(\lambda))^{-1}), \tag{2.54}$$

$$\widehat{\Phi}_\tau(\lambda) = M_+(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_0(\lambda) \upharpoonright \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+ \tag{2.55}$$

define the functions $\Phi_\tau(\cdot)$, $\widehat{\Phi}_\tau(\cdot) \in R[\mathcal{H}_1]$ and the equality $\mathrm{mul}\,\widetilde{A}_\tau = \mathrm{mul}\,A$ holds if and only if

$$s - \lim_{y\to+\infty} \tfrac{1}{iy}\Phi_\tau(iy) = s - \lim_{y\to+\infty} \tfrac{1}{iy}\widehat{\Phi}_\tau(iy) = 0. \tag{2.56}$$

Remark 2.24. If $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$, then the triplet Π turns into the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* in the sense of [18, 9]. In this case $n_+(A) = n_-(A) = \dim \mathcal{H}$ and the function $M(\lambda)(= M_+(\lambda)) \in R[\mathcal{H}]$ coincides with the Weyl function of Π introduced in [14, 22]. Observe also that for the triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ the results of this subsection coincides with those from [11, 12, 14, 22].

3. Symmetric extensions preserving the multivalued part

In what follows A is a closed symmetric linear relation in $\mathfrak{H}$ with $n_-(A) \le n_+(A)$. Clearly, for each symmetric extension $\dot{A}$ of A one has $\widetilde{\mathrm{Self}}(\dot{A}) \subset \widetilde{\mathrm{Self}}(A)$.

Proposition 3.1. *Assume that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $M_+(\cdot)$ is the Weyl function of Π. Let $\mathcal{H}'$ be a subspace in $\mathcal{H}_1$, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be decomposed as*

$$\mathcal{H}_1 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1, \quad \mathcal{H}_0 = \mathcal{H}' \oplus \dot{\mathcal{H}}_0 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1 \oplus \mathcal{H}_2 \tag{3.1}$$

(here $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1 = \dot{\mathcal{H}}_0 \ominus \dot{\mathcal{H}}_1$), let

$$\Gamma_0 = (\Gamma_0', \dot{\Gamma}_0)^\top : A^* \to \mathcal{H}' \oplus \dot{\mathcal{H}}_0, \quad \Gamma_1 = (\Gamma_1', \dot{\Gamma}_1)^\top : A^* \to \mathcal{H}' \oplus \dot{\mathcal{H}}_1 \tag{3.2}$$

be the block representations of Γ_0 and Γ_1 and let

$$M_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_{2+}(\lambda) \\ M_3(\lambda) & M_{4+}(\lambda) \end{pmatrix} : \mathcal{H}' \oplus \dot{\mathcal{H}}_0 \to \mathcal{H}' \oplus \dot{\mathcal{H}}_1, \quad \lambda \in \mathbb{C}_+ \tag{3.3}$$

be the block representation of $M_+(\lambda)$. Then

(1) *The equalities*

$$\dot{A} = \widetilde{A}_{\{0\}\oplus\mathcal{H}'} = \{\widehat{f} \in A^* : \Gamma_0'\widehat{f} = 0, \dot{\Gamma}_0\widehat{f} = \dot{\Gamma}_1\widehat{f} = 0\}, \tag{3.4}$$

$$\dot{A}^* = \{\widehat{f} \in A^* : \Gamma_0'\widehat{f} = 0\} \tag{3.5}$$

define a closed symmetric extension $\dot{A}$ of A and its adjoint $\dot{A}^*$. Moreover, $n_-(\dot{A}) \le n_+(\dot{A})$.

(2) *The collection $\dot{\Pi} = \{\dot{\mathcal{H}}_0 \oplus \dot{\mathcal{H}}_1, \dot{\Gamma}_0 \upharpoonright \dot{A}^*, \dot{\Gamma}_1 \upharpoonright \dot{A}^*\}$ is a boundary triplet for $\dot{A}^*$ and the Weyl function $\dot{M}_+(\cdot)$ of this triplet is $\dot{M}_+(\lambda) = M_{4+}(\lambda)$, $\lambda \in \mathbb{C}_+$.*

(3) *If $\dot{\tau} = \{\dot{C}_0(\lambda), \dot{C}_1(\lambda)\} \in \widetilde{R}(\dot{\mathcal{H}}_0, \dot{\mathcal{H}}_1)$ and $\widetilde{A} = \widetilde{A}_{\dot{\tau}} \in \widetilde{\mathrm{Self}}(\dot{A})$, then*

(i) *the equality* $\operatorname{mul}\widetilde{A} = \operatorname{mul}\dot{A}$ *holds if and only if the following two conditions are satisfied:*

$$s - \lim_{y\to+\infty} \tfrac{1}{iy} P_{\dot{\mathcal{H}}_0,\dot{\mathcal{H}}_1}(\dot{C}_0(iy) - \dot{C}_1(iy)M_{4+}(iy))^{-1}\dot{C}_1(iy) = 0, \tag{3.6}$$

$$s - \lim_{y\to+\infty} \tfrac{1}{iy} M_{4+}(iy)(\dot{C}_0(iy) - \dot{C}_1(iy)M_{4+}(iy))^{-1}\dot{C}_0(iy) \upharpoonright \dot{\mathcal{H}}_1 = 0; \tag{3.7}$$

(ii) *the operator-function*

$$m_{\dot{\tau}}(\lambda) = M_1(\lambda) + M_{2+}(\lambda)(\dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda))^{-1}\dot{C}_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{3.8}$$

belongs to $R[\mathcal{H}']$ *and the equality* $\operatorname{mul}\widetilde{A} = \operatorname{mul} A$ *holds if and only if in addition to* (3.6) *and* (3.7) *the following condition is satisfied:*

$$s - \lim_{y\to+\infty} \tfrac{1}{iy} m_{\dot{\tau}}(iy) = 0. \tag{3.9}$$

Proof. For a triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (i.e., in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$) statements (1) and (2) are proved in [11, Proposition 4.1]. In the case of the triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ the proof is similar.

Let us prove statement (3). The equivalence of the equality $\operatorname{mul}\widetilde{A} = \operatorname{mul}\dot{A}$ to (3.6) and (3.7) directly follows from Theorem 2.23, (2) applied to the triplet $\dot{\Pi}$. Next, in view of (2.52) and (2.53) the extension $\widetilde{A}$ is defined by (2.53) with

$$\widetilde{A}(\lambda) = \widetilde{A}_{-\dot{\tau}}(\lambda) = \{\widehat{f} \in \dot{A}^* : \dot{C}_0(\lambda)\dot{\Gamma}_0\widehat{f} - \dot{C}_1(\lambda)\dot{\Gamma}_1\widehat{f} = 0\}(\in \operatorname{Ext}_{\dot{A}}), \quad \lambda \in \mathbb{C}_+. \tag{3.10}$$

It follows from (3.5) that (3.10) can be represented as

$$\widetilde{A}(\lambda) = \{\widehat{f} \in A^* : \Gamma_0'\widehat{f} = 0,\ \dot{C}_0(\lambda)\dot{\Gamma}_0\widehat{f} - \dot{C}_1(\lambda)\dot{\Gamma}_1\widehat{f} = 0\}(\in \operatorname{Ext}_A), \quad \lambda \in \mathbb{C}_+. \tag{3.11}$$

Letting

$$C_0(\lambda) = \begin{pmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \dot{C}_0(\lambda) \end{pmatrix} : \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}_0}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}_0}_{\mathcal{H}_0}, \tag{3.12}$$

$$C_1(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \dot{C}_1(\lambda) \end{pmatrix} : \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}_1}_{\mathcal{H}_1} \to \underbrace{\mathcal{H}' \oplus \dot{\mathcal{H}}_0}_{\mathcal{H}_0} \tag{3.13}$$

and taking (3.2) into account one rewrites (3.11) as (2.52). Thus the extension $\widetilde{A}$ belongs to $\widetilde{\operatorname{Self}}(A)$ and in the triplet Π for A^* it is given by (2.52) and (2.53) with $C_0(\lambda)$ and $C_1(\lambda)$ of the form (3.12) and (3.13). Therefore by Theorem 2.23 the equality $\operatorname{mul}\widetilde{A} = \operatorname{mul} A$ holds if and only if the operator-functions $\Phi_\tau(\cdot)$ and $\widehat{\Phi}_\tau(\cdot)$ of the form (2.54) and (2.55) satisfy (2.56).

It follows from (3.3) and (3.12), (3.13) that

$$(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} = \begin{pmatrix} I & 0 \\ -\dot{C}_1(\lambda)M_3(\lambda) & \dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda) \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} I & 0 \\ (\dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda))^{-1}\dot{C}_1(\lambda)M_3(\lambda) & (\dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda))^{-1} \end{pmatrix}.$$

Moreover, $P_1 = \begin{pmatrix} I_{\mathcal{H}'} & 0 \\ 0 & P_{\dot{\mathcal{H}}_0,\dot{\mathcal{H}}_1} \end{pmatrix}$, $C_0(\lambda) \upharpoonright \mathcal{H}_1 = \begin{pmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \dot{C}_0(\lambda) \upharpoonright \dot{\mathcal{H}}_1 \end{pmatrix}$ and the immediate calculation shows that

$$\Phi_\tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & P_{\dot{\mathcal{H}}_0,\dot{\mathcal{H}}_1}(\dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda))^{-1}\dot{C}_1(\lambda) \end{pmatrix} : \mathcal{H}' \oplus \dot{\mathcal{H}}_1 \to \mathcal{H}' \oplus \dot{\mathcal{H}}_1,$$

$$\widehat{\Phi}_\tau(\lambda) = \begin{pmatrix} m_{\dot{\tau}}(\lambda) & * \\ * & M_{4+}(\lambda)(\dot{C}_0(\lambda) - \dot{C}_1(\lambda)M_{4+}(\lambda))^{-1}\dot{C}_0(\lambda) \upharpoonright \dot{\mathcal{H}}_1 \end{pmatrix} : \mathcal{H}' \oplus \dot{\mathcal{H}}_1 \to \mathcal{H}' \oplus \dot{\mathcal{H}}_1$$

for all $\lambda\in\mathbb{C}_+$ (the entries $*$ do not matter). Since by Theorem 2.23, (2) $\Phi_\tau(\cdot)$, $\widehat{\Phi}_\tau(\cdot)\in R[\mathcal{H}_1]$, it follows from Proposition 2.18 that $m_{\dot\tau}(\cdot)\in R[\mathcal{H}']$ and conditions (2.56) are equivalent to (3.6), (3.7) and (3.9). Hence $\operatorname{mul}\widetilde{A}=\operatorname{mul}A$ if and only if conditions (3.6)–(3.9) are satisfied. □

Lemma 3.2. *Let an operator-function $K(\cdot)$ belongs to the class $R_\Pi[\mathcal{H}]$. Then*

(1) $-K^{-1}(\cdot)\in R_\Pi[\mathcal{H}]$.

(2) *For each function $\tau(\cdot)\in\widetilde{R}(\mathcal{H})$ one has $(\tau(\lambda)+K(\lambda))^{-1}\in\boldsymbol{B}(\mathcal{H})$, $(\tau^{-1}(\lambda)+K^{-1}(\lambda))^{-1}\in\boldsymbol{B}(\mathcal{H})$ and*

$$s-\lim_{y\to+\infty}\tfrac{1}{y}(\tau(iy)+K(iy))^{-1}=s-\lim_{y\to+\infty}\tfrac{1}{y}(\tau^{-1}(iy)+K^{-1}(iy))^{-1}=0.$$

Proof. According to [14, Corollary 2] there exist a Hilbert space $\mathfrak{H}$, a densely defined symmetric operator A in $\mathfrak{H}$ and a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* such that $K(\cdot)$ is the Weyl function of Π. Moreover, according to [22] $\widehat{\Pi}=\{\mathcal{H},\Gamma_1,-\Gamma_0\}$ is a boundary triplet for A^* as well and the Weyl function of $\widehat{\Pi}$ is $-K^{-1}(\cdot)$. Therefore by [14, Corollary 2] $-K^{-1}(\cdot)\in R_\Pi[\mathcal{H}]$, which proves statement (1).

Next assume that $\tau=\tau(\cdot)\in\widetilde{R}(\mathcal{H})$ and let $\widetilde{A}=\widetilde{A}_\tau\in\widetilde{\operatorname{Self}}(A)$ be the respective extension of A in the triplet Π (see Theorem 2.23, (1)). Since A is densely defined, it follows that $\operatorname{mul}\widetilde{A}=\{0\}(=\operatorname{mul}A)$. Moreover, $\widetilde{A}=\widetilde{A}_{-\tau^{-1}}$ in the triplet $\widehat{\Pi}$ and application of Theorem 2.23, (2) yields statement (2). □

Theorem 3.3. *Assume that $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ is a boundary triplet for A^*, $M_+(\cdot)$ is the Weyl function of Π and $M(\cdot)\in R_u[\mathcal{H}_1]$ is the operator function defined by* (2.51) *(in the case $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ $M(\cdot)$ is the Weyl function of Π). Let $\mathcal{H}'$ be a subspace in $\mathcal{H}_1$, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be decomposed as in* (3.1)*, let Γ_0 and Γ_1 have the block representations* (3.2) *and let $\dot A\in\operatorname{Ext}_A$ be a symmetric extension* (3.4) *of A. Assume also that*

$$M(\lambda)=\begin{pmatrix}M_1(\lambda) & M_2(\lambda)\\ M_3(\lambda) & M_4(\lambda)\end{pmatrix}:\mathcal{H}'\oplus\dot{\mathcal{H}}_1\to\mathcal{H}'\oplus\dot{\mathcal{H}}_1,\quad \lambda\in\mathbb{C}_+ \tag{3.14}$$

is the block representation of $M(\cdot)$ and let $K(\cdot)\in R_\Pi[\dot{\mathcal{H}}_1]$ (see Definition 2.17). Then the equality $\operatorname{mul}\dot A=\operatorname{mul}A$ holds if and only if the operator-function $m_K(\cdot)\in R_u[\mathcal{H}']$ of the form (2.43) *satisfies*

$$s-\lim_{y\to+\infty}\tfrac{1}{iy}m_K(iy)=0. \tag{3.15}$$

Proof. It follows from (2.51) and (3.14) that $M_+(\lambda)$ has the block representation (3.3), where

$$M_{2+}(\lambda)=(M_2(\lambda),N_2(\lambda)):\dot{\mathcal{H}}_1\oplus\mathcal{H}_2\to\mathcal{H}', \tag{3.16}$$

$$M_{4+}(\lambda)=(M_4(\lambda),N_4(\lambda)):\dot{\mathcal{H}}_1\oplus\mathcal{H}_2\to\dot{\mathcal{H}}_1 \tag{3.17}$$

and the entries $M_j(\lambda)$, $j\in\{1,\dots,4\}$, are taken from (3.14). Let

$$\dot C_0(\lambda)=\begin{pmatrix}-K(\lambda) & 0\\ 0 & I_{\mathcal{H}_2}\end{pmatrix}:\underbrace{\dot{\mathcal{H}}_1\oplus\mathcal{H}_2}_{\dot{\mathcal{H}}_0}\to\underbrace{\dot{\mathcal{H}}_1\oplus\mathcal{H}_2}_{\dot{\mathcal{H}}_0}, \tag{3.18}$$

$$\dot C_1(\lambda)=(I_{\dot{\mathcal{H}}_1},0)^\top:\dot{\mathcal{H}}_1\to\underbrace{\dot{\mathcal{H}}_1\oplus\mathcal{H}_2}_{\dot{\mathcal{H}}_0},\quad\lambda\in\mathbb{C}_+. \tag{3.19}$$

Then

$$2\mathrm{Im}(\dot C_1(\lambda)P_{\dot{\mathcal H}_0,\dot{\mathcal H}_1}\dot C_0^*(\lambda))+\dot C_0(\lambda)P_{\mathcal H_2}\dot C_0^*(\lambda)=2\mathrm{Im}\left[\begin{pmatrix} I_{\dot{\mathcal H}_1}\\ 0\end{pmatrix}(-K^*(\lambda),0)\right]$$
$$+\begin{pmatrix}-K(\lambda) & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}\begin{pmatrix}-K^*(\lambda) & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}=\begin{pmatrix}2\mathrm{Im}K(\lambda) & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}\geq 0.$$

Moreover,

$$\dot C_0(\lambda)-i\dot C_1(\lambda)P_{\dot{\mathcal H}_0,\dot{\mathcal H}_1}=\begin{pmatrix}-K(\lambda) & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}-i\begin{pmatrix}I_{\dot{\mathcal H}_1} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-(K(\lambda)+iI_{\dot{\mathcal H}_1}) & 0\\ 0 & I_{\mathcal H_2}\end{pmatrix}$$

and hence the operator $\dot C_0(\lambda)-i\dot C_1(\lambda)P_{\dot{\mathcal H}_0,\dot{\mathcal H}_1}$ is invertible. Thus, conditions (2.40) for $\dot C_0(\lambda)$ and $\dot C_1(\lambda)$ are satisfied and, consequently, the pair $\dot\tau_0:=\{\dot C_0(\lambda),\dot C_1(\lambda)\}$ belongs to $\widetilde R(\dot{\mathcal H}_0,\dot{\mathcal H}_1)$.

Since $M(\cdot)\in R_u[\mathcal H_1]$, it follows from (3.14) that $M_4(\cdot)\in R_u[\dot{\mathcal H}_1]$. Therefore the operator $M_4(\lambda)$ is invertible and $-M_4^{-1}(\cdot)\in R[\dot{\mathcal H}_1]$. Let us show that

$$P_{\dot{\mathcal H}_0,\dot{\mathcal H}_1}(\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda))^{-1}\dot C_1(\lambda)=-(M_4(\lambda)+K(\lambda))^{-1},\tag{3.20}$$

$$M_{4+}(\lambda)(\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda))^{-1}\dot C_0(\lambda)\restriction\dot{\mathcal H}_1=(M_4^{-1}(\lambda))+K^{-1}(\lambda))^{-1}.\tag{3.21}$$

Indeed, by (3.18), (3.19) and (3.17) one has

$$\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda)=\begin{pmatrix}-K(\lambda) & 0\\ 0 & I\end{pmatrix}-\begin{pmatrix}I\\ 0\end{pmatrix}(M_4(\lambda),N_4(\lambda))$$
$$=\begin{pmatrix}-(M_4(\lambda)+K(\lambda)) & -N_4(\lambda)\\ 0 & I\end{pmatrix}.$$

Therefore

$$(\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda))^{-1}$$
$$=\begin{pmatrix}-(M_4(\lambda)+K(\lambda))^{-1} & -(M_4(\lambda)+K(\lambda))^{-1}N_4(\lambda)\\ 0 & I\end{pmatrix}:\underbrace{\dot{\mathcal H}_1\oplus\mathcal H_2}_{\dot{\mathcal H}_0}\to\underbrace{\dot{\mathcal H}_1\oplus\mathcal H_2}_{\dot{\mathcal H}_0}$$

and, consequently,

$$(\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda))^{-1}\dot C_1(\lambda)=\begin{pmatrix}-(M_4(\lambda)+K(\lambda))^{-1}\\ 0\end{pmatrix}:\dot{\mathcal H}_1\to\dot{\mathcal H}_1\oplus\mathcal H_2.\tag{3.22}$$

This equality yields (3.20). Moreover, $\dot C_0(\lambda)\restriction\dot{\mathcal H}_1=\begin{pmatrix}-K(\lambda)\\ 0\end{pmatrix}$ and

$$M_{4+}(\lambda)(\dot C_0(\lambda)-\dot C_1(\lambda)M_{4+}(\lambda))^{-1}\dot C_0(\lambda)\restriction\dot{\mathcal H}_1=(M_4(\lambda),\,N_4(\lambda))$$
$$\times\begin{pmatrix}-(M_4(\lambda)+K(\lambda))^{-1} & -(M_4(\lambda)+K(\lambda))^{-1}N_4(\lambda)\\ 0 & I\end{pmatrix}\begin{pmatrix}-K(\lambda)\\ 0\end{pmatrix}$$
$$=M_4(\lambda)(M_4(\lambda)+K(\lambda))^{-1}K(\lambda),$$

which implies (3.21).

Next assume that $\widetilde A=\widetilde A_{\dot\tau_0}\in\widetilde{\mathrm{Self}}(\dot A)$ is the extension of $\dot A$ corresponding to the pair $\dot\tau_0$ in the triplet $\dot\Pi$ (see Proposition 3.1, (2)). It follows from (3.20), (3.21) and Lemma 3.2 that the pair $\dot\tau_0$ satisfies conditions (3.6) and (3.7). Therefore by Proposition 3.1, (3) one has $\mathrm{mul}\,\widetilde A=\mathrm{mul}\,\dot A$. On the other hand, the respective operator-function $m_{\dot\tau_0}(\cdot)$ of the form (3.8) is

$$m_{\dot\tau_0}(\lambda)=M_1(\lambda)+(M_2(\lambda),N_2(\lambda))\begin{pmatrix}-(M_4(\lambda)+K(\lambda))^{-1}\\ 0\end{pmatrix}M_3(\lambda)=m_K(\lambda)$$

(here we made use of (3.16) and (3.22)). Therefore by Proposition 3.1, (3) the equality $\operatorname{mul}\widetilde{A} = \operatorname{mul} A$ is equivalent to (3.15). This yields the statement of the theorem. □

Proposition 3.4. *Let under the conditions of Proposition 2.19 $K(\cdot) \to m_K(\cdot)$ be the mapping of the set $R[\dot{\mathcal{H}}]$ into $R_u[\mathcal{H}']$ defined by* (2.43)*. If $m_K(\cdot)$ satisfies* (3.15) *for some $K(\cdot) \in R_\Pi[\dot{\mathcal{H}}]$, then the same holds for any $K(\cdot) \in R_\Pi[\dot{\mathcal{H}}]$.*

Proof. According to [15] there exist a Hilbert space $\mathfrak{H}$, a symmetric (possibly non-densely defined) operator A in $\mathfrak{H}$ and a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $M(\cdot)$ is the Weyl function of Π. Let $\dot{A} = \widetilde{A}_{\{0\}\oplus\mathcal{H}'}$ be a symmetric extension of A (in the triplet Π). Then according to Theorem 3.3 for each $K(\cdot) \in R_\Pi[\dot{\mathcal{H}}]$ the equality (3.15) is equivalent to $\operatorname{mul}\dot{A} = \{0\}(= \operatorname{mul} A)$, which implies the required statement. □

Proposition 3.4 enables us to introduce Definition 1.1 of the class $R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$.

Remark 3.5. (1) Assume that $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$. Since $iI_{\dot{\mathcal{H}}} \in R_\Pi[\dot{\mathcal{H}}]$, it follows that a holomorphic operator-function $M(\cdot) : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H})$ with the block representation (2.42) belongs to $R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ if and only if $M(\cdot) \in R_u[\mathcal{H}]$ and the operator-function

$$m(\lambda) = M_1(\lambda) - M_2(\lambda)(M_4(\lambda) + iI_{\dot{\mathcal{H}}})^{-1} M_3(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{3.23}$$

satisfies (3.15).

(2) Statement of Theorem 3.3 can be reformulated as follows: the equality $\operatorname{mul}\dot{A} = \operatorname{mul} A$ holds if and only $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$.

In the following proposition we give a sufficient condition for an operator-function $M(\cdot)$ to belong to the class $R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$.

Proposition 3.6. *Let $M(\cdot) \in R_u[\mathcal{H}]$ and let $\mathcal{B}_M = 0$ (for $\mathcal{B}_M$ see* (2.41)*). Then $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ for each decomposition $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$.*

Proof. As was mentioned in the proof of Proposition 3.4 $M(\cdot)$ is the Weyl function of some boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* (A is a symmetric operator in $\mathfrak{H}$) and according to [22] $A_0 := \widetilde{A}_{\{0\}\oplus\mathcal{H}} = \{\widehat{f} \in A^* : \Gamma_0 \widehat{f} = 0\}$ is a self-adjoint extension of A. Moreover, since $\mathcal{B}_M = 0$, it follows that $\operatorname{mul} A_0 = \{0\}(= \operatorname{mul} A)$ (see [21, 15]). Let $\mathcal{H} = \mathcal{H}' \oplus \dot{\mathcal{H}}$ and let $\dot{A} = \widetilde{A}_{\{0\}\oplus\mathcal{H}'}$ (in the triplet Π). Then $\dot{A} \subset A_0$ and, consequently, $\operatorname{mul}\dot{A} \subset \operatorname{mul} A_0$. Therefore $\operatorname{mul}\dot{A} = \{0\}(= \operatorname{mul} A)$ and by Remark 3.5, (2) $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$. □

In the following theorem we provide a criterium, that guarantees the equality $\operatorname{mul}\widetilde{A} = \operatorname{mul} A$ for a symmetric extension $\widetilde{A}$ of a symmetric relation A with $n_-(A) \le n_+(A)$.

Theorem 3.7. *Assume that $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is a symmetric linear relation with $n_-(A) \le n_+(A)$, $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* and $M_+(\cdot)$ is the Weyl function of Π. Let $\widetilde{A} = \widetilde{A}_\theta \in \operatorname{Ext}_A$ be a symmetric extension of A corresponding to the boundary parameter $\theta \in \operatorname{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ (see Proposition 2.22) and let $n_-(\widetilde{A}) \le n_+(\widetilde{A})$. Then*

(1) *There exist a Hilbert space $\widetilde{\mathcal{H}}_0 \supset \mathcal{H}_1$ and an operator*

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \widetilde{\mathcal{H}}_0 \oplus \mathcal{H}_1, \tag{3.24}$$

satisfying (2.34) *and* (2.35) *with a certain subspace $\mathcal{H}' \subset \mathcal{H}_1$.*

(2) *Let $\widetilde{\mathcal{H}}_0$ be a Hilbert space, $\mathcal{H}'$ be a subspace in $\mathcal{H}_1$ and Z be an operator* (3.24) *from statement* (1)*. Moreover, let $\dot{\mathcal{H}}_1 = \mathcal{H}_1 \ominus \mathcal{H}'$, so that $\mathcal{H}_1 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1$. Then the equality*

$$\widetilde{M}_+(\lambda) = (z_3 + z_4 M_+(\lambda))(z_1 + z_2 M_+(\lambda))^{-1} (\in \boldsymbol{B}(\widetilde{\mathcal{H}}_0, \mathcal{H}_1)), \quad \lambda \in \mathbb{C}_+ \tag{3.25}$$

together with the block representation

$$\widetilde{M}_+(\lambda) = (\widetilde{M}(\lambda), \widetilde{N}_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+ \tag{3.26}$$

defines the operator-function $\widetilde{M}(\cdot) \in R_u[\mathcal{H}_1]$ *and the equality* $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ *holds if and only if* $\widetilde{M}(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$.

If in addition A *is an operator (that is,* $\operatorname{mul} A = \{0\}$*), then* $\widetilde{A}$ *is an operator if and only if* $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$.

Proof. Since $n_-(\widetilde{A}) \le n_+(\widetilde{A})$, it follows from (2.49) that $n_-(\theta) \le n_+(\theta)$. Therefore by Lemma 2.13 statement (1) is valid.

Let us prove statement (2). According to [25, Proposition 4.3] the equalities

$$\widetilde{\Gamma}_0 = z_1 \Gamma_0 + z_2 \Gamma_1, \quad \widetilde{\Gamma}_1 = z_3 \Gamma_0 + z_4 \Gamma_1$$

define a boundary triplet $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}_0 \oplus \mathcal{H}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* and the Weyl function $\widetilde{M}_+(\cdot)$ of $\widetilde{\Pi}$ is given by (3.25). Moreover, in this triplet $\widetilde{A} = \widetilde{A}_{\widetilde{\theta}}$ with $\widetilde{\theta} = Z\theta = \{0\} \oplus \mathcal{H}'$. This and Remark 3.5, (2) yield the required assertions. □

Remark 3.8. Lemma 2.13 provides an explicit construction of the coefficients z_j, $j \in \{1, \dots, 4\}$, in (3.25). Namely, assume that under the conditions of Theorem 3.7 the relation θ is represented in the form (2.15), (2.16) with an isometry $V \in \boldsymbol{B}(\mathcal{H}', \mathcal{H}_0)$ ($\mathcal{H}' \subset \mathcal{H}_1$) and $U \in \boldsymbol{B}(\mathcal{H}_0, \widetilde{\mathcal{H}}_0)$ is a unitary extension of V^{-1} with the block representation (2.29). Then by Lemma 2.13 one can take the operators z_j of the form (2.31)–(2.33) as coefficients in (3.25).

In the case $n_+(A) = n_-(A)$ and $n_+(\widetilde{A}) = n_-(\widetilde{A})$ statements of Theorem 3.7 can be rather simplified. Namely, the following theorem is valid.

Theorem 3.9. *Assume that* $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ *is a symmetric relation with* $n_+(A) = n_-(A)$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^* *and* $M(\cdot)$ *is the Weyl function of* Π. *Let* $\widetilde{A} = \widetilde{A}_\theta \in \operatorname{Ext}_A \cap \operatorname{Sym}_0(\mathfrak{H})$ *with the respective* $\theta \in \operatorname{Sym}_0(\mathcal{H})$ *and let* $n_+(\widetilde{A}) = n_-(\widetilde{A})$. *Assume also that* θ *is represented in the form* (2.21) *with an isometry* $V \in \boldsymbol{B}(\mathcal{H}', \mathcal{H})$ ($\mathcal{H}' \subset \mathcal{H}$). *Then*

(1) *There exist a unitary operator* $U \in \boldsymbol{B}(\mathcal{H})$ *such that* $V^{-1} \subset U$.

(2) *If* U *is an operator from statement* (1), *then the equality*

$$\widetilde{M}(\lambda) = (i(U - I) + (U + I)M(\lambda))((U + I) - i(U - I)M(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+ \tag{3.27}$$

correctly defines the operator-function $\widetilde{M}(\cdot) \in R_u[\mathcal{H}]$ *and the equality* $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ *holds if and only if* $\widetilde{M}(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}]$ *(here* $\dot{\mathcal{H}} = \mathcal{H} \ominus \mathcal{H}'$*).*

Proof. Since $n_+(\widetilde{A}) = n_-(\widetilde{A})$, it follows from (2.49) that $n_+(\theta) = n_-(\theta)$. Now the required statements are implied by Corollary 2.14 and Remark 3.5, (2). □

Lemma 3.10. *Assume that a Hilbert space* $\mathcal{H}$ *is decomposed as*

$$\mathcal{H} = \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \dot{\mathcal{H}}_1 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1,$$

where $\mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2'$,

(3.28)

$$M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} : \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \dot{\mathcal{H}}_1 \to \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \dot{\mathcal{H}}_1, \quad \lambda \in \mathbb{C}_+$$

is the block representation of an operator-function $M(\cdot) \in R_u[\mathcal{H}]$ *and* $N_1(\cdot) \in R_u[\mathcal{H}'_1 \oplus \dot{\mathcal{H}}_1]$ *and* $N_2(\cdot) \in R_u[\mathcal{H}'_2 \oplus \dot{\mathcal{H}}_1]$ *are the operator-functions given by*

$$N_1(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{13}(\lambda) \\ M_{31}(\lambda) & M_{33}(\lambda) \end{pmatrix}, \qquad N_2(\lambda) = \begin{pmatrix} M_{22}(\lambda) & M_{23}(\lambda) \\ M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix}. \tag{3.29}$$

Then $M(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$ if and only if $N_j(\cdot) \in R_{u0}[\mathcal{H}'_j \oplus \dot{\mathcal{H}}_1]$, $j \in \{1, 2\}$.

Proof. Clearly, (3.28) can be represented as (2.42) with $M_1(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}$, $M_2(\lambda) = (M_{13}(\lambda), M_{23}(\lambda))^\top$, $M_3(\lambda) = (M_{31}(\lambda), M_{32}(\lambda))$ and $M_4(\lambda) = M_{33}(\lambda)$. Let $m(\lambda)$ be given by (3.23). Then

$$m(\lambda) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} (M_{33} + iI_{\dot{\mathcal{H}}_1})^{-1} (M_{31}, M_{32}) = \begin{pmatrix} m_1(\lambda) & * \\ * & m_2(\lambda) \end{pmatrix},$$

where $M_{ij} = M_{ij}(\lambda)$ and $m_j(\lambda)$ is the operator-function (3.23) for $N_j(\cdot)$, $j \in \{1, 2\}$. This and Proposition 2.18 yield the required statement. □

In the following theorem we characterize explicitly symmetric extensions $\widetilde{A}_\theta \in \mathrm{Ext}_A$ with $\mathrm{mul}\, \widetilde{A}_\theta = \mathrm{mul}\, A$ under the additional condition that an abstract boundary parameter θ has the closed domain.

Theorem 3.11. *Assume that* A, Π *and* $M_+(\cdot)$ *are the same as in Theorem 3.7 and let* $M(\cdot) \in R_u[\mathcal{H}_1]$ *be the operator-function defined by the block representation* (2.51) *of* $M_+(\lambda)$*. Let* $\theta \in \mathrm{Sym}_0(\mathcal{H}_1)$ *(so that* $\theta \in \mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1)$ *as a subspace in* $\mathcal{H}_0 \oplus \mathcal{H}_1 \supset \mathcal{H}_1^2$*), let* $\mathrm{dom}\, \theta$ *be closed, let* $\widetilde{A} = \widetilde{A}_\theta \in Ext_A$ *be the respective symmetric extension of* A *and let* $\theta = \{\mathcal{H}'_1 \oplus \mathcal{H}'_2 \oplus \dot{\mathcal{H}}_1, B_1, B_2\}$ *be the canonical representation* (2.24), (2.25) *of* θ *with operators* $B_1 = B_1^* \in \boldsymbol{B}(\mathcal{H}'_1)$ *and* $B_2 \in \boldsymbol{B}(\mathcal{H}'_1, \dot{\mathcal{H}}_1)$ *(see Definition 2.10). Moreover, let* $M(\lambda)$ *has the block representation* (3.28)*, let* $\varphi_1(\lambda) = B_1 - M_{11}(\lambda)$*,* $\varphi_2(\lambda) = B_2 - M_{31}(\lambda)$*,* $\varphi_{2*}(\lambda) = B_2^* - M_{13}(\lambda)$ *and let*

$$N_1(\lambda) = \begin{pmatrix} \varphi_1^{-1}(\lambda) & \varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ \varphi_2(\lambda)\varphi_1^{-1}(\lambda) & M_{33}(\lambda) + \varphi_2(\lambda)\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \end{pmatrix} (\in \boldsymbol{B}(\mathcal{H}'_1 \oplus \dot{\mathcal{H}}_1)),$$

$$N_2(\lambda) = \begin{pmatrix} M_{22}(\lambda) + M_{21}(\lambda)\varphi_1^{-1}(\lambda)M_{12}(\lambda) & M_{23}(\lambda) - M_{21}(\lambda)\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ M_{32}(\lambda) - \varphi_2(\lambda)\varphi_1^{-1}(\lambda)M_{12}(\lambda) & M_{33}(\lambda) + \varphi_2(\lambda)\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \end{pmatrix}$$
$$(\in \boldsymbol{B}(\mathcal{H}'_2 \oplus \dot{\mathcal{H}}_1)),$$

where $\lambda \in \mathbb{C}_+$*. Then* $N_j(\cdot) \in R_u[\mathcal{H}'_j \oplus \dot{\mathcal{H}}_1]$ *and the equality* $\mathrm{mul}\, \widetilde{A} = \mathrm{mul}\, A$ *holds if and only if* $N_j(\cdot) \in R_{u0}[\mathcal{H}'_j \oplus \dot{\mathcal{H}}_1]$, $j \in \{1, 2\}$.

Proof. Let $\mathcal{H}' = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ and let $X_j \in \boldsymbol{B}(\mathcal{H}_1)$, $j \in \{1, \dots, 4\}$, be the operators given by the following block representations (with respect to decomposition (2.24) of $\mathcal{H}_1$):

$$X_1 = \begin{pmatrix} -B_1 & 0 & -B_2^* \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \; X_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} -I & 0 & 0 \\ 0 & 0 & 0 \\ -B_2 & 0 & 0, \end{pmatrix}, \; X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

The immediate checking shows that

$$X_1^* X_3 - X_3^* X_1 = 0, \quad X_1^* X_4 - X_3^* X_2 = I, \quad X_2^* X_4 - X_4^* X_2 = 0,$$
$$X_2 X_1^* - X_1 X_2^* = 0, \quad X_4 X_1^* - X_3 X_2^* = I, \quad X_4 X_3^* - X_3 X_4^* = 0.$$

Therefore the operator

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$$

with $Z_1 = \begin{pmatrix} X_1 & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0}$, $\quad Z_2 = \begin{pmatrix} X_2 \\ 0 \end{pmatrix} : \mathcal{H}_1 \to \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0}$, $Z_3 = (X_3, 0) : \underbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}_{\mathcal{H}_0} \to \mathcal{H}_1$ and $Z_4 = X_4$ satisfies $Z^* J_{\mathcal{H}_0,\mathcal{H}_1} Z = J_{\mathcal{H}_0,\mathcal{H}_1}$ and $Z J_{\mathcal{H}_0,\mathcal{H}_1} Z^* = J_{\mathcal{H}_0,\mathcal{H}_1}$. Moreover, by (2.25)

$$Z\theta = \{\{F_1(h_1,h_2) + F_2(h_1,h_2), F_3(h_1,h_2) + F_4(h_1,h_2) : h_1 \in \mathcal{H}_1', h_2 \in \mathcal{H}_2'\},$$

where

$$\begin{aligned}
F_1(h_1,h_2) &= Z_1 \restriction \mathcal{H}_1 \cdot h_1 = X_1(h_1 \oplus 0 \oplus 0) \oplus 0_{\mathcal{H}_2} = ((-B_1 h_1) \oplus 0 \oplus 0) \oplus 0_{\mathcal{H}_2}, \\
F_2(h_1,h_2) &= Z_2(B_1 h_1 \oplus h_2 \oplus B_2 h_1) = X_2(B_1 h_1 \oplus h_2 \oplus B_2 h_1) \oplus 0_{\mathcal{H}_2} \\
&= (B_1 h_1 \oplus 0 \oplus 0) \oplus 0_{\mathcal{H}_2}, \\
F_3(h_1,h_2) &= Z_3 \restriction \mathcal{H}_1 \cdot h_1 = X_3(h_1 \oplus 0 \oplus 0) = (-h_1) \oplus 0 \oplus (-B_2 h_1), \\
F_4(h_1,h_2) &= Z_4(B_1 h_1 \oplus h_2 \oplus B_2 h_1) = X_4(B_1 h_1 \oplus h_2 \oplus B_2 h_1) = 0 \oplus h_2 \oplus B_2 h_1.
\end{aligned}$$

Hence $Z\theta = \{\{0, (-h_1) \oplus h_2 \oplus 0 : h_1 \in \mathcal{H}_1', h_2 \in \mathcal{H}_2'\}\} = \{0\} \oplus (\mathcal{H}_1' \oplus \mathcal{H}_2') = \{0\} \oplus \mathcal{H}'$. Thus by Theorem 3.7 $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ if and only if the operator function $\widetilde{M}(\cdot) \in R_u[\mathcal{H}_1]$ defined by (3.25) and (3.26) belongs to $R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$. It follows from (2.51) that

$$\begin{aligned}
Z_1 + Z_2 M_+ &= \begin{pmatrix} X_1 & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} X_2 \\ 0 \end{pmatrix} (M(\lambda), N_+(\lambda)) = \begin{pmatrix} X_1 + X_2 M(\lambda) & X_2 N_+(\lambda) \\ 0 & I \end{pmatrix}, \\
(Z_1 + Z_2 M_+)^{-1} &= \begin{pmatrix} (X_1 + X_2 M(\lambda))^{-1} & * \\ 0 & I \end{pmatrix}, \quad Z_3 + Z_4 M_+ = (X_3 + X_4 M(\lambda), *),
\end{aligned}$$

where $M_+ = M_+(\lambda)$ and $*$ denotes the entries that do not matter. Therefore

$$\begin{aligned}
\widetilde{M}_+(\lambda) &= (X_3 + X_4 M(\lambda), *) \begin{pmatrix} (X_1 + X_2 M(\lambda))^{-1} & * \\ 0 & I \end{pmatrix} \\
&= ((X_3 + X_4 M(\lambda))(X_1 + X_2 M(\lambda))^{-1}, \widetilde{N}_+(\lambda))
\end{aligned}$$

with some $\widetilde{N}_+(\lambda)$ and in view of (3.26) $\widetilde{M}(\lambda)$ is

$$\widetilde{M}(\lambda) = (X_3 + X_4 M(\lambda))(X_1 + X_2 M(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{3.30}$$

Next,

$$\begin{aligned}
X_1 + X_2 M(\lambda) &= \begin{pmatrix} -B_1 & 0 & -B_2^* \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} -\varphi_1(\lambda) & M_{12}(\lambda) & -\varphi_{2*}(\lambda) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \\
(X_1 + X_2 M(\lambda))^{-1} &= \begin{pmatrix} -\varphi_1^{-1}(\lambda) & \varphi_1^{-1}(\lambda) M_{12}(\lambda) & -\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \\
X_3 + X_4 M(\lambda) &= \begin{pmatrix} -I & 0 & 0 \\ 0 & 0 & 0 \\ -B_2 & 0 & 0, \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} = \\
&= \begin{pmatrix} -I & 0 & 0 \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ -\varphi_2(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix}
\end{aligned}$$

and with respect to the decomposition (2.24) of $\mathcal{H}_1$ one has

$$\widetilde{M}(\lambda) = \begin{pmatrix} -I & 0 & 0 \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ -\varphi_2(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} \begin{pmatrix} -\varphi_1^{-1}(\lambda) & \varphi_1^{-1}(\lambda)M_{12}(\lambda) & -\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} \varphi_1^{-1}(\lambda) & * & \varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ * & M_{22}(\lambda) + M_{21}(\lambda)\varphi_1^{-1}(\lambda)M_{12}(\lambda) & M_{23}(\lambda) - M_{21}(\lambda)\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \\ \varphi_2(\lambda)\varphi_1^{-1}(\lambda) & M_{32}(\lambda) - \varphi_2(\lambda)\varphi_1^{-1}(\lambda)M_{12}(\lambda) & M_{33}(\lambda) + \varphi_2(\lambda)\varphi_1^{-1}(\lambda)\varphi_{2*}(\lambda) \end{pmatrix}.$$

This and Lemma 3.10 yield the statement of the theorem. □

Remark 3.12. In the case $n_+(A) = n_-(A) < \infty$ each boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* satisfies $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ (i.e., in fact $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$) and $\dim \mathcal{H} < \infty$. Therefore in this case $\mathrm{Sym}_0(\mathcal{H}_0, \mathcal{H}_1) = \mathrm{Sym}(\mathcal{H})$ and $\mathrm{dom}\,\theta$ is closed for each $\theta \in \mathrm{Sym}(\mathcal{H})$. Thus in the case $n_+(A) = n_-(A) < \infty$ Theorem 3.11 provides an explicit parametrization of all symmetric extensions $\widetilde{A} \in \mathrm{Ext}_A$ with $\mathrm{mul}\,\widetilde{A} = \mathrm{mul}\,A$.

The following corollary is immediate from Theorem 3.11.

Corollary 3.13. *Assume that under the conditions of Theorem 3.11* $\mathrm{mul}\,\theta = \{0\}$, *that is* $\theta = \mathrm{gr}B$, *where* $B \in \boldsymbol{B}(\mathcal{H}', \mathcal{H}_1)$ *is a symmetric operator in* $\mathcal{H}_1$ *defined on the closed subspace* $\mathcal{H}' \subset \mathcal{H}_1$. *Let* $\mathcal{H}_1$ *be decomposed as* $\mathcal{H}_1 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1$, *let* $B = (B_1, B_2)^\top : \mathcal{H}' \to \mathcal{H}' \oplus \dot{\mathcal{H}}_1$ *and* (3.14) *be block representations of* B *and* $M(\lambda)$ *respectively and let* $N(\cdot) : \mathbb{C}_+ \to \boldsymbol{B}(\mathcal{H}_1)$ *be the operator-function given for all* $\lambda \in \mathbb{C}_+$ *by the block representation* (1.12) *(with respect to the mentioned decomposition* $\mathcal{H}_1 = \mathcal{H}' \oplus \dot{\mathcal{H}}_1$*). Then*

(1) $\widetilde{A}$ *admits the representation in the form of abstract boundary conditions as*
$$\widetilde{A} = \{\widehat{f} \in A^* : \Gamma_0\widehat{f} \in \mathcal{H}',\ \Gamma_1\widehat{f} - B\Gamma_0\widehat{f} = 0\}.$$

(2) $N(\cdot) \in R_u[\mathcal{H}_1]$ *and the equivalence* $\mathrm{mul}\,\widetilde{A} = \mathrm{mul}\,A \iff N(\cdot) \in R_{u0}[\mathcal{H}' \oplus \dot{\mathcal{H}}_1]$ *is valid.*

Corollary 3.14. *Assume that* A, Π, $M_+(\cdot)$ *and* $M(\cdot)$ *are the same as in Theorem 3.11. Let* $\mathcal{H}'$ *be a subspace in* $\mathcal{H}_1$ *and let* $(C_0, C_1) \in SP(\mathcal{H}')$ *be a self-adjoint operator pair (see Definition 2.11). Assume that* $\mathcal{H}_2' := \ker C_1$ *and* $\mathcal{H}_1' = \mathcal{H}' \ominus \mathcal{H}_2'$, *so that* $\mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2'$ *and decomposition* (2.24) *of* $\mathcal{H}_1$ *holds with* $\dot{\mathcal{H}}_1 = \mathcal{H}_1 \ominus \mathcal{H}'$. *Moreover, let the subspace* $\mathcal{K}_1 := \mathrm{ran}\,C_1 \subset \mathcal{H}'$ *be closed and let* $C_{j1} := P_{\mathcal{H}', \mathcal{K}_1} C_j \upharpoonright \mathcal{H}_1' (\in \boldsymbol{B}(\mathcal{H}_1', \mathcal{K}_1))$, $j \in \{0, 1\}$ *(so that* C_{01} *and* C_{02} *are the left upper entries in matrices* (2.27)*). Assume also that* $M(\lambda)$ *has the block representation* (3.28) *and let*

$$\psi(\lambda) = -(C_{01} + C_{11}M_{11}(\lambda))^{-1}C_{11}, \tag{3.31}$$

$$N_1(\lambda) = \begin{pmatrix} \psi(\lambda) & -\psi(\lambda)M_{13}(\lambda) \\ -M_{31}(\lambda)\psi(\lambda) & M_{33}(\lambda) + M_{31}(\lambda)\psi(\lambda))M_{13}(\lambda) \end{pmatrix} (\in \boldsymbol{B}(\mathcal{H}_1' \oplus \dot{\mathcal{H}}_1)), \tag{3.32}$$

$$N_2(\lambda) = \begin{pmatrix} M_{22}(\lambda) + M_{21}(\lambda)\psi(\lambda)M_{12}(\lambda) & M_{23}(\lambda) + M_{21}(\lambda)\psi(\lambda)M_{13}(\lambda) \\ M_{32}(\lambda) + M_{31}(\lambda)\psi(\lambda)M_{12}(\lambda) & M_{33}(\lambda) + M_{31}(\lambda)\psi(\lambda)M_{13}(\lambda) \end{pmatrix} (\in \boldsymbol{B}(\mathcal{H}_2' \oplus \dot{\mathcal{H}}_1)), \tag{3.33}$$

where $\lambda \in \mathbb{C}_+$. *Then*

(1) *The equality (the abstract boundary conditions)*

$$\widetilde{A} = \{\widehat{f} \in A^* : \Gamma_0\widehat{f} \in \mathcal{H}',\ \Gamma_1\widehat{f} \in \mathcal{H}',\ C_0\Gamma_0\widehat{f} + C_1\Gamma_1\widehat{f} = 0\} \tag{3.34}$$

defines a symmetric extension $\widetilde{A} \in \mathrm{Ext}_A$.

(2) $N_j(\cdot)\in R_u[\mathcal{H}'_j\oplus\dot{\mathcal{H}}_1]$ *and the equality* $\operatorname{mul}\widetilde{A}=\operatorname{mul}A$ *holds if and only if* $N_j(\cdot)\in R_{u0}[\mathcal{H}'_j\oplus\dot{\mathcal{H}}_1],\ j\in\{1,2\}$.

(3) *If in addition* $\ker C_1=\{0\}$, $M(\lambda)$ *has the block representation* (1.7) *and*

(3.35)
$$N(\lambda)=\begin{pmatrix}-(C_0+C_1M_1(\lambda))^{-1}C_1 & (C_0+C_1M_1(\lambda))^{-1}C_1M_2(\lambda)\\ M_3(\lambda)(C_0+C_1M_1(\lambda))^{-1}C_1 & M_4(\lambda)-M_3(\lambda)(C_0+C_1M_1(\lambda))^{-1}C_1M_2(\lambda)\end{pmatrix},\quad \lambda\in\mathbb{C}_+$$

(with respect to the decomposition $\mathcal{H}_1=\mathcal{H}'\oplus\dot{\mathcal{H}}_1$*), then* $N(\cdot)\in R_u[\mathcal{H}_1]$ *and the equality* $\operatorname{mul}\widetilde{A}=\operatorname{mul}A$ *holds if and only if* $N(\cdot)\in R_{u0}[\mathcal{H}'\oplus\dot{\mathcal{H}}_1]$.

Proof. (1) Let $\theta\in\operatorname{Self}(\mathcal{H}')$ be given by (2.26). Since $\mathcal{H}'^2\subset\mathcal{H}_1^2\subset\mathcal{H}_0\oplus\mathcal{H}_1$, it follows that $\theta\in\operatorname{Sym}_0(\mathcal{H}_1)\cap\operatorname{Sym}_0(\mathcal{H}_0,\mathcal{H}_1)$. Let $\widetilde{A}=\widetilde{A}_\theta\in\operatorname{Ext}_A$ be the respective extension of A given by (2.45). Then by Proposition 2.22 $\widetilde{A}\in\operatorname{Sym}_0(\mathfrak{H})$ and in view (2.26) $\widetilde{A}$ admits the representation (3.34).

(2) According to Proposition 2.12 $\operatorname{dom}\theta=\mathcal{H}'_1$, $\operatorname{mul}\theta=\mathcal{H}'_2$, C_0 and C_1 have the block representations (2.27) and the operator part $B'(\in\boldsymbol{B}(\mathcal{H}'_1))$ of θ is $B'=-C_{11}^{-1}C_{01}$. Next, consider θ as a symmetric relation in $\mathcal{H}_1$ and let B be the operator part of this relation. Clearly, the block representation of B is $B=(B',0)^\top$. Therefore by Lemma 2.9 the canonical representation of θ is $\theta=\{\mathcal{H}'_1\oplus\mathcal{H}'_2\oplus\dot{\mathcal{H}}_1,B_1,B_2\}$, where

$$B_1=-C_{11}^{-1}C_{01},\quad B_2=0. \tag{3.36}$$

Let $N_j(\cdot)\in R_u[\mathcal{H}'_j\oplus\dot{\mathcal{H}}_1],\ j\in\{1,2\}$, be the operator functions defined in Theorem 3.11. Then in view of (3.36)

$$\psi(\lambda):=\varphi_1^{-1}(\lambda)=-(C_{11}^{-1}C_{01}+M_{11}(\lambda))^{-1}=-(C_{01}+C_{11}M_{11}(\lambda))^{-1}C_{11},$$
$$\varphi_2(\lambda)=-M_{31}(\lambda),\quad \varphi_{2*}(\lambda)=-M_{13}(\lambda)$$

and hence $N_1(\lambda)$ and $N_2(\lambda)$ admit the representation (3.32), (3.33). Now statement (2) is implied by the equality $\widetilde{A}=\widetilde{A}_\theta$ and Theorem 3.11.

(3) If $\ker C_1=\{0\}$, then $\mathcal{H}'_2=\{0\}$, $\mathcal{H}'_1=\mathcal{H}'$ and by (2.27) $C_0=C_{01}$ and $C_1=C_{11}$. Therefore the required statements follow from statement (2). □

4. Applications to Hamiltonian systems

Let H be a finite-dimensional Hilbert space, let $\mathbb{H}=H\oplus H$ and let

$$J=\begin{pmatrix}0 & -I_H\\ I_H & 0\end{pmatrix}:H\oplus H\to H\oplus H.$$

As is known (see e.g. [4, 17]) a Hamiltonian differential system on an interval $\mathcal{I}=[a,b\rangle$, $-\infty<a<b\le\infty$, (with the regular endpoint a) is of the form

$$Jy'-B(t)y=\lambda\Delta(t)y,\quad t\in\mathcal{I},\quad \lambda\in\mathbb{C}, \tag{4.1}$$

where $B(t)=B^*(t)$ and $\Delta(t)\ge 0$ are $\boldsymbol{B}(\mathbb{H})$-valued functions on $\mathcal{I}$ integrable on each compact interval $[a,\beta]\subset\mathcal{I}$. Below we assume that system (4.1) is definite. The latter means that for some (and hence all) $\lambda\in\mathbb{C}$ there is only a trivial solution $y(t)\equiv 0$ of (4.1) such that $\Delta(t)y(t)=0$ (a.e. on $\mathcal{I}$).

Denote by $\mathfrak{H}(=L^2_\Delta(\mathcal{I}))$ the Hilbert space of functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ such that $\int_{\mathcal{I}}(\Delta(t)f(t),f(t))\,dt<\infty$. With system (4.1) one associates minimal and maximal linear relations $T_{\min}$ and $T_{\max}$ in $\mathfrak{H}$ (see e.g. [7]). It turns out that $T_{\min}$ is a closed symmetric relation with finite deficiency indices $n_\pm(T_{\min})\le\dim\mathbb{H}$, which coincide with the number of linearly independent solutions $y \in \mathfrak{H}$ of (4.1) for $\lambda \in \mathbb{C}_\pm$. Moreover, $T_{\max} = T_{\min}^*$ and for all $y, z \in \operatorname{dom} T_{\max}$ there exists the limit

$$[y,z]_b := \lim_{t\uparrow b}(Jy(t), z(t)).$$

Clearly each function $y \in \operatorname{dom} T_{\max}$ admits the representation

$$y(t) = \{y_0(t), y_1(t)\}(\in H \oplus H), \quad t \in \mathcal{I}. \tag{4.2}$$

Assume that $(C_0, C_1) \in SP(H)$ (see Definition 2.11). Then according to [27]

$$T := \{\{y,f\} \in T_{\max} : C_0 y_0(a) + C_1 y_1(a) = 0 \text{ and } [y,z]_b = 0,\, z \in \operatorname{dom} T_{\max}\} \tag{4.3}$$

is a closed symmetric extension of $T_{\min}$ (here $y_0(a)$ and $y_1(a)$ are taken from (4.2)).

Let $\varphi(\cdot,\lambda)(\in \boldsymbol{B}(H,\mathbb{H}))$ be the operator solution of (4.1) with the initial value $\varphi(a,\lambda) = (C_1^*, -C_0^*)^\top(\in \boldsymbol{B}(H, H\oplus H))$. Denote also by $\mathfrak{H}_b$ the set of all functions $f(\cdot) \in \mathfrak{H}$ with compact support. With each function $f(\cdot) \in \mathfrak{H}_b$ one associates the generalized Fourier transform $\widehat{f}(\cdot) : \mathbb{R} \to H$ given by

$$\widehat{f}(s) = \int_{\mathcal{I}} \varphi^*(t,s)\Delta(t) f(t)\, dt. \tag{4.4}$$

As is known a non-decreasing left-continuous function $\sigma(\cdot) : \mathbb{R} \to \boldsymbol{B}(H)$ with $\sigma(0) = 0$ is called a $\boldsymbol{B}(H)$-valued distribution function.

Definition 4.1. A $\boldsymbol{B}(H)$-valued distribution function $\sigma(\cdot)$ is called a q-pseudospectral function of the system (4.1) if the operator $(Vf)(s) = \widehat{f}(s)$, $f(\cdot) \in \mathfrak{H}_b$, admits a continuation to a partial isometry $V_\sigma \in [\mathfrak{H}, L^2(\sigma; H)]$ (for the Hilbert space $L^2(\sigma;H)$ see e.g. [16, Ch.13.5]).

According to [27] for each q-pseudospectral function $\sigma(\cdot)$ one has $\operatorname{mul} T \subset \ker V_\sigma$. Moreover, the inverse Fourier transform $f(t) = \int\limits_{\mathbb{R}} \varphi(t,s) d\sigma(s)\widehat{f}(s)$ exists only for $f(\cdot) \in \mathfrak{H} \ominus \ker V_\sigma$. These facts make natural the following definition.

Definition 4.2. [27]. A q-pseudospectral function $\sigma(\cdot)$ with the minimally possible kernel $\ker V_\sigma = \operatorname{mul} T$ of V_σ is called a pseudospectral function of the system (4.1) (with respect to the pair $(C_0, C_1) \in SP(H)$).

Definition 4.3. [1, 29]. A $\boldsymbol{B}(H)$-valued distribution function $\sigma(\cdot)$ is called a spectral function of the system (4.1) (with respect to the pair $(C_0, C_1) \in SP(H)$) if the following Parseval equality is valid:

$$\int_{\mathbb{R}} (d\sigma(s)\widehat{f}(s), \widehat{f}(s)) = \int_{\mathcal{I}} (\Delta(t) f(t), f(t))\, dt, \quad f(\cdot) \in \mathfrak{H}_b.$$

Assertion 4.4. [27]. (1) *A $\boldsymbol{B}(H)$-valued distribution function $\sigma(\cdot)$ is a spectral function if and only if it is a pseudospectral function such that V_σ is an isometry.*

(2) *Let $(C_0, C_1) \in SP(H)$ and let $T \in \operatorname{Ext}_{T_{\min}}$ be symmetric relation* (4.3). *Then the set of spectral functions (with respect to (C_0, C_1)) is not empty if and only if* $\operatorname{mul} T = \{0\}$, *in which case the sets of spectral and pseudospectral functions coincide.*

Definition 4.5. A self-adjoint operator pair $(C_0, C_1) \in SP(H)$ is referred to the class $SP_0(H)$ if the respective relation T of the form (4.3) satisfies $\operatorname{mul} T = \operatorname{mul} T_{\min}$ and to the class $SP_0'(H)$ if $\operatorname{mul} T = \{0\}(= \operatorname{mul} T_{\min})$.

Remark 4.6. (1) Clearly, $SP_0'(H) \subset SP_0(H)$ and by [28, Lemma 3.4] $SP_0(H) \neq \emptyset$.

(2) Since $\operatorname{mul} T_{\min} \subset \operatorname{mul} T$, it follows that for a pair $(C_0, C_1) \in SP_0(H)$ the partial isometry V_σ corresponding to the pseudospectral function $\sigma(\cdot)$ (with respect to (C_0, C_1)) has the minimal kernel in the following sense: for any pair $(C_0', C_1') \in SP(H)$ the partial isometry V_σ' corresponding to the pseudospectral function $\sigma'(\cdot)$ (with respect to (C_0', C_1'))

satisfies $\ker V_\sigma \subset \ker V'_\sigma$. Moreover, by Assertion 4.4 a pair $(C_0, C_1) \in SP(H)$ belongs to $SP'_0(H)$ if and only if the set of spectral functions (with respect to (C_0, C_1)) is not empty.

Our next goal is to characterise the classes $SP_0(H)$ and $SP'_0(H)$.

In the following we assume for simplicity that $n_+(T_{\min}) = n_-(T_{\min})$. Then according to [27] there exist a finite dimensional Hilbert space $\mathcal{H}_b$ and a surjective linear mapping

$$\Gamma_b = (\Gamma_{0b}, \Gamma_{1b})^\top : \operatorname{dom} T_{\max} \to \mathcal{H}_b \oplus \mathcal{H}_b \tag{4.5}$$

such that the following identity is valid

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z) - (\Gamma_{1b} y, \Gamma_{0b} z), \quad y, z \in \operatorname{dom} T_{\max} \tag{4.6}$$

(actually $\Gamma_b y$ is a singular boundary value of a function $y \in \operatorname{dom} T_{\max}$ at the endpoint b). Moreover, a collection $\Pi_d = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with

$$\mathcal{H} = H \oplus \mathcal{H}_b, \tag{4.7}$$

$$\Gamma_0\{y, f\} = y_0(a) \oplus \Gamma_{0b} y, \quad \Gamma_1\{y, f\} = y_1(a) \oplus (-\Gamma_{1b} y), \quad \{y, f\} \in T_{\max} \tag{4.8}$$

is a boundary triplet for $T_{\max}$.

The classes $SP_0(H)$ and $SP'_0(H)$ are characterized in the following two theorems.

Theorem 4.7. *Assume that system* (4.1) *is definite and* $n_+(T_{\min}) = n_-(T_{\min})$. *Let* $\Pi_d = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triplet* (4.7), (4.8) *for* $T_{\max}$ *and let* $M(\cdot)$ *be the Weyl function of* Π_d. *Let* $(C_0, C_1) \in SP(H)$, $\mathcal{K}_1 := \operatorname{ran} C_1 \subset H$, $\mathcal{H}'_2 := \ker C_1$ *and* $\mathcal{H}'_1 = H \ominus \mathcal{H}'_2$, *so that*

$$H = \mathcal{H}'_1 \oplus \mathcal{H}'_2, \quad \mathcal{H} = \mathcal{H}'_1 \oplus \mathcal{H}'_2 \oplus \mathcal{H}_b. \tag{4.9}$$

Moreover, let $C_{j1} := P_{H,\mathcal{K}_1} C_j \upharpoonright \mathcal{H}'_1 (\in \boldsymbol{B}(\mathcal{H}'_1, \mathcal{K}_1))$, $j \in \{0, 1\}$, *let* $M(\lambda)$ *has the block representation* (3.28) *(with respect to the second decomposition in* (4.9)*) and let* $N_j(\cdot) \in R_u[\mathcal{H}'_j \oplus \mathcal{H}_b]$, $j \in \{1, 2\}$, *be the operator functions defined by* (3.31) - (3.33) *(in* (3.28), (3.32) *and* (3.33) $\dot{\mathcal{H}}_1$ *should be replaced with* $\mathcal{H}_b$*). Then* $(C_0, C_1) \in SP_0(H)$ *if and only if* $N_j(\cdot) \in R_{u0}[\mathcal{H}'_j \oplus \mathcal{H}_b]$, $j \in \{1, 2\}$.

If in addition $\ker C_1 = \{0\}$ *and* $M(\lambda)$ *has the block representation* (1.7) *(with* $\mathcal{H}' = H$ *and* $\dot{\mathcal{H}}_1 = \mathcal{H}_b$*), then* $(C_0, C_1) \in SP_0(H)$ *if and only if* $N(\cdot) \in R_{u0}[H \oplus \mathcal{H}_b]$, *where* $N(\cdot) \in R_u[H \oplus \mathcal{H}_b]$ *is the operator function* (3.35).

Proof. Since $\dim H < \infty$, the subspace $\mathcal{K}_1$ is closed. Moreover, since Γ_b is surjective, it follows from (4.6) that (4.3) can be written as

$$T = \{\{y, f\} \in T_{\max} : C_0 y_0(a) + C_1 y_1(a) = 0,\ \Gamma_{0b} y = \Gamma_{1b} y = 0\},$$

that is, in the form (3.34). Now the required statements follow from Corollary 3.14 applied to the boundary triplet Π_d. □

Theorem 4.8. *Assume that system* (4.1) *is definite and* $n_+(T_{\min}) = n_-(T_{\min})$.*Then the set* $SP'_0(H)$ *is not empty if and only if* $\operatorname{mul} T_{\min} = \{0\}$. *Moreover, if* $\operatorname{mul} T_{\min} = \{0\}$, *then Theorem 4.7 holds with* $SP'_0(H)$ *instead of* $SP_0(H)$.

Proof. For each pair $(C_0, C_1) \in SP(H)$ one has $\operatorname{mul} T \supset \operatorname{mul} T_{\min}$. Therefore in the case $\operatorname{mul} T_{\min} \neq \{0\}$ the set $SP'_0(H)$ is empty. If $\operatorname{mul} T_{\min} = \{0\}$, then $SP'_0(H) = SP_0(H)$. Hence $SP'_0(H) \neq \emptyset$ and Theorem 4.7 holds with $SP'_0(H)$ instead of $SP_0(H)$. □

Remark 4.9. Note that the Weyl function $M(\lambda)$ in Theorems 4.7 and 4.8 is defined in terms of the boundary values of respective operator solutions of (4.1) at the endpoints a and b (see [27, Proposition 4.9]).

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