Methods of Functional Analysis and Topology
Vol. 24 (2018), no. 2, pp. 178–186

ON THE INVERSE EIGENVALUE PROBLEMS FOR A JACOBI MATRIX WITH MIXED GIVEN DATA

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To the memory of M.L. Gorbachuk

Abstract. We give necessary and sufficient conditions for existence and uniqueness of a solution to inverse eigenvalues problems for Jacobi matrix with given mixed initial data. We also propose effective algorithms for solving these problems.

1. Introduction

Denote by J_n the Jacobi matrix

$$J_n = \begin{pmatrix} b_1 & a_1 & & & & & \\ a_1 & b_2 & a_2 & & & & \\ & a_2 & b_3 & a_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_{n-2} & b_{n-1} & a_{n-1} \\ & & & & & a_{n-1} & b_n \end{pmatrix}, \tag{1}$$

where $a_1,\ a_2,\ \dots,\ a_{n-1}$ and $b_1,\ b_2,\ \dots,\ b_n$ are parameters of the Jacobi matrix, and $a_j > 0$, b_j are real numbers.

We will denote by J_r the submatrix of the matrix J_n, consisting of the first r rows and columns of the matrix J_n. The characteristic polynomial of the matrix J_r will be denoted by $D_r(\lambda)$,

$$D_r(\lambda) = \det(\lambda I - J_r), \qquad r = 1, \dots, n. \tag{2}$$

All zeros of the polynomial $D_r(\lambda)$, where J_r is a Jacobi matrix of the form (1), are real and simple. Zeros $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ of the characteristic polynomial $D_n(\lambda)$ of the Jacobi matrix J_n given by (1) are eigenvalues of the matrix J_n. The zeros $\mu_1 < \mu_2 < \cdots < \mu_{n-1}$ of the characteristic polynomial $D_{n-1}(\lambda)$ are eigenvalues of the submatrix J_{n-1} that is obtained from the matrix J_n by removing the last row and the last column from the matrix. Eigenvalues of the matrices J_n and J_{n-1} alternate,

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n. \tag{3}$$

Property (3) can be written in one of the following equivalent forms:

$$\begin{aligned} &a) \quad (-1)^{n-j} D_{n-1}(\lambda_j) > 0, \qquad j = 1, \dots, n, \\ &b) \quad (-1)^{n-j} D_n(\mu_j) > 0, \qquad j = 1, \dots, n-1. \end{aligned} \tag{4}$$

The common sequence of eigenvalues of the Jacobi matrices J_n and J_{n-1} will be called an extended spectrum of the matrix J_n. It is well known that a Jacobi matrix is uniquely determined by its extended spectrum, and condition (3) is necessary and sufficient for

2010 *Mathematics Subject Classification.* Primary 47A10, 15A29, 65F18.
Key words and phrases. Jacobi matrix, inverse eigenvalue problems, mixed initial data.

the numbers $\{\lambda_j\}_{j=1}^n$, together with $\{\mu_j\}_{j=1}^{n-1}$, to form an extended spectrum of the Jacobi matrix J_n. Expansion into a continuous fraction,

$$\frac{D_n(\lambda)}{D_{n-1}(\lambda)} = \lambda - b_n - \cfrac{A_{n-1}}{\lambda - b_{n-1} - \cfrac{A_{n-2}}{\lambda - b_{n-2} - \ldots - \cfrac{A_1}{\lambda - b_1}}}, \tag{5}$$

gives one of algorithms that allow to solve an inverse problem, — to recover a Jacobi matrix from its extended spectrum.

Inverse eigenvalue problems with given mixed initial data (IEPMD) arise in the case where a part of elements of the Jacobi matrix is known. Then it is natural to consider only a part of the extended spectrum of the Jacobi matrix as known.

Uniqueness of solution of the inverse problem to a great extend depends on how all elements of the Jacobi matrix (1) are partitioned into known or unknown ones, see [2, Introduction]). For example, the Jacobi matrix J_4 with $b_1 = b_4 = 0$, $a_1 = a_3 = 1$, $b_2 = \frac{3}{4} - x$, $b_3 = \frac{3}{4} + x$, $a_2^2 = \frac{9}{16} - x^2$, where $|x| < \frac{3}{4}$, has the spectrum consisting of the four eigenvalues $(-1, -\frac{1}{2}, 1, 2)$ that does not depend on x. Hence it is impossible to recover the matrix with three unknown elements b_2, b_3, a_2 from the spectrum of the matrix J_4. However, the matrix can be uniquely recovered from a part of the extended spectrum of the matrix J_4. It is sufficient to consider two eigenvalues of the matrix J_4, λ_1 and λ_2 subject to the condition $\lambda_1\lambda_2 \neq -1$, and one eigenvalue μ_1 of the matrix J_3.

Let us index the elements of the Jacobi matrix J_n in (1) as follows:

$$c_1 = b_1, \quad c_2 = a_1, \quad c_3 = b_2, \quad \ldots, \quad c_{2n-1} = b_n \tag{6}$$

assuming that $c_{2k-1} = b_k$, $c_{2k} = a_k$.

Inverse eigenvalue problem for the Jacobi matrix J_n with mixed data (IEPMD) can be formulated in the following way.

Problem (IEPMD)**.** *Let, in a Jacobi matrix J_n of the form* (1)*, the first l elements, considered with respect to the sequential indexing* (6)*, be unknown, and the remaining $2n-1-l$ elements $\{c_j\}_{j=l+1}^{2n-1}$ be given. Suppose that $l_1 \leqslant l$, and the numbers $\lambda_1, \ldots, \lambda_{l_1}$, $l_2 = l - l_1$, and the numbers $\mu_1, \ldots, \mu_{l_2}$ are given. We need to recover the matrix J_n in such a way that the numbers $\{\lambda_j\}_{j=1}^{l_1}$ would be eigenvalues of the matrix J_n and that numbers $\{\mu_j\}_{j=1}^{l_2}$ would be eigenvalue of the matrix J_{n-1}.*

A particular case of the IEPMD for $l = n$, $l_1 = n$, and $l_2 = 0$ is the well-known Hochstadt inverse eigenvalue problem (HIEP), for which a theorem on uniqueness of solution of the inverse problem was obtained in [4, 5] under the condition that a solution exists. For $l \leqslant n$, $l_1 = l$, a theorem on uniqueness of a solution of the inverse problem was obtained in [3] in the case where l arbitrary eigenvalues of the matrix J_n are given with $l < n$. Solution uniqueness theorems were also obtained in [6] for the inverse problem in the case where $l > n$, $l_1 = n$, and $l_2 = l - n$.

Theorem 1 of this paper gives uniqueness of a solution of the IEPMD problem in a general case. We also propose an algorithm for solving such an inverse problem. In this paper, we treat the cases $l > n$ and $l_1 = n$ (Theorem 3), or $l_2 = n-1$ (Theorem 4) in more details. For these cases, together with an algorithm for solving the inverse problems, we formulate and prove necessary and sufficient conditions for existence and uniqueness of a solution to the IEPMD problem in terms of initial data.

2. Auxiliary results

2.1. **Determinant identity.** Let us partition all elements of the Jacobi matrix J_n of the form (1) into two groups using the sequential indexing (6); the first group contains l elements, and the second one $2n - 1 - l$ elements. Let $r = \left[\frac{l+1}{2}\right]$ be the integer part of the number $\frac{l+1}{2}$, that is, the number $r = \frac{l}{2}$ if l is an even number, and $r = \frac{l+1}{2}$ if l is odd. Denote by $Q_m(\lambda)$ the characteristic polynomial of the submatrix formed by the last m rows and m columns of the Jacobi matrix J_n, and by $\check{Q}_k(\lambda)$ the characteristic poynomial of the $k \times k$-submatrix formed by the last k rows and columns of the Jacobi matrix J_{n-1}. Using known determinant identities (see, e.g. [2]), we have the following:

$$\begin{aligned} D_n(\lambda) &= D_r(\lambda)Q_{n-r}(\lambda) - a_r^2 D_{r-1}(\lambda)Q_{n-r-1}(\lambda), \\ D_{n-1}(\lambda) &= D_r(\lambda)\check{Q}_{n-r-1}(\lambda) - a_r^2 D_{r-1}(\lambda)\check{Q}_{n-r-2}(\lambda). \end{aligned} \tag{7}$$

Considering the two equations in (7) as a system of linear equations with respect to $D_r(\lambda)$ and $a_r^2 D_{r-1}(\lambda)$ and using that $Q_{n-r}(\lambda)\check{Q}_{n-r-2}(\lambda) - Q_{n-r-1}(\lambda)\check{Q}_{n-r-1}(\lambda) = -\prod_{j=r+1}^{n-1} a_j^2$ (see [2, Lemma 1]) we get two more identities,

$$\begin{aligned} \prod_{j=r+1}^{n-1} a_j^2 D_r(\lambda) &= D_{n-1}(\lambda)Q_{n-r-1}(\lambda) - D_n(\lambda)\check{Q}_{n-r-2}(\lambda), \\ \prod_{j=r}^{n-1} a_j^2 D_{r-1}(\lambda) &= D_{n-1}(\lambda)Q_{n-r}(\lambda) - D_n(\lambda)\check{Q}_{n-r-1}(\lambda). \end{aligned} \tag{8}$$

Identities (7) permit to construct an algorithm for solving the IEPMD problem formulated in Introduction. Indeed, if l is the number of unknown elements of the Jacobi matrix J_n, with the sequential indexing (6), and $r = \left[\frac{l+1}{2}\right]$ is the integer part of the number $\frac{l+1}{2}$, then all elements of the matrix J_n with indices greater than l are known. Hence, all the polynomials Q_α and $\check{Q}_\beta$ that enter identities (7) are known. Moreover, if l is an odd number, then a_r is also known, since it has the index $2r = l+1$ with respect to indexing (6). Since $\lambda_1, \dots, \lambda_{l_1}$ in the IEPMD problem are eigenvalues of the matrix J_n, we have that $D_n(\lambda_j) = 0$ for $j = 1, \dots, l_1$. Similarly, the numbers $\mu_1, \dots, \mu_{l_2}$ are eigenvalues of the matrix J_{n-1}, hence $D_{n-1}(\mu_k) = 0$, $k = 1, \dots, l_2$. It follows from (7) that

$$\begin{aligned} &D_r(\lambda_j)Q_{n-r}(\lambda_j) - a_r^2 D_{r-1}(\lambda_j)Q_{n-r-1}(\lambda_j) = 0, \qquad j = 1, \dots, l_1, \\ &D_r(\mu_k)\check{Q}_{n-r-1}(\mu_k) - a_r^2 D_{r-1}(\mu_k)\check{Q}_{n-r-2}(\mu_k) = 0, \qquad k = 1, \dots, l_2. \end{aligned} \tag{9}$$

System (9) uniquely defines the polynomials $D_r(\lambda)$, $D_{r-1}(\lambda)$, and the number a_r for even l. We will deal with this problem in Section 2.2. Since we know $D_r(\lambda)$ and $D_{r-1}(\lambda)$, the algorithm in (5) can be continued, thus permitting to recover all unknown elements of the matrix J_n.

2.2. **Some facts from the theory of fractional-rational function interpolation.** For the reader's convenience, we give some simple facts with proofs from the theory of fractional-rational interpolation [1] in a form suitable for proofs of the main results in the paper.

Lemma 1. *Let $p(x)$ and $q(x)$ be two relatively prime polynomials of degrees n and m, correspondingly. Let the coefficient of the highest power of the argument in the polynomial $p(x)$ be equal to* 1. *Consider a sequence $x_1, x_2, \dots, x_{n+m+1}$ of $n+m+1$ distinct values of the argument, interpolation nodes, and two sequences α_j, β_j, $j = 1, \dots, n+m+1$, of interpolation values that are not zero for each j, $|\alpha_j| + |\beta_j| \neq 0$, and such that*

$$p(x_j)\alpha_j + q(x_j)\beta_j = 0, \qquad j = 1, \dots, n+m+1. \tag{10}$$

Then system (10) *uniquely defines the pair of polynomials p, q.*

Proof. Let $\alpha_j \neq 0$ and $\beta_j \neq 0$ for all j, and polynomials p and q satisfy system (10), and assume that there is another pair of polynomials, $\widetilde{p}(x)$ and $\widetilde{q}(x)$, such that they have degrees n and m, correspondingly, the coefficient of the highest power of the polynomial $\widetilde{p}(x)$ is 1, and the pair satisfies system (10). Then it follows from (10) that

$$\frac{p(x_j)}{q(x_j)} = \frac{\widetilde{p}(x_j)}{\widetilde{q}(x_j)} = -\frac{\beta_j}{\alpha_j}, \qquad j = 1, \ldots, n+m+1. \tag{11}$$

This means that the polynomial $p(x)\widetilde{q}(x) - \widetilde{p}(x)q(x)$ has degree $n+m$ and $n+m+1$ distinct zeros $x_1, x_2, \ldots, x_{n+m+1}$. But then it is a zero polynomial. Since the polynomials p and q are relatively prime and $p(x)\widetilde{q}(x) - \widetilde{p}(x)q(x) \equiv 0$, we have that $p(x) \equiv \widetilde{p}(x)$ and $q(x) \equiv \widetilde{q}(x)$.

If some $\beta_k = 0$ in system (10), then $p(x_k) = 0$, since $\alpha_k \neq 0$ by the condition of the lemma. Similarly, the condition $\alpha_k = 0$ yields that $q(x_k) = 0$. This permits to eliminate all arguments x_k in system (10) for which α_k or β_k is zero. Indeed, let $\alpha_k = 0$ for only $k = 1, \ldots, n_1$ and $\beta_k = 0$ for only $k = n_1+1, \ldots, n_1+n_2$. Then $p(x) = \prod_{k=n_1+1}^{n_1+n_2}(x - x_k)\widehat{p}(x)$, $q(x) = \prod_{k=1}^{n_1}(x - x_k)\widehat{q}(x)$, where the degree of the polynomial $\widehat{p}$ equals $n - n_1$ and the degree of the polynomial $\widehat{q}$ is $m - n_1$. System (10), in this case, is equivalent to the same system for the polynomials $\widehat{p}$, $\widehat{q}$,

$$\widehat{p}(x_j)\widehat{\alpha}_j + \widehat{q}(x_j)\widehat{\beta}_j = 0, \qquad j = n_1+n_2+1, \ldots, n+m+1, \tag{12}$$

where $\widehat{\alpha}_j = \prod_{k=n_1}^{n_1+n_2}(x_j - x_k)\alpha_j$, $\widehat{\beta}_j = \prod_{k=1}^{n_1}(x_j - x_k)\beta_j$. We have $\widehat{\alpha}_j \neq 0$ and $\widehat{\beta}_j \neq 0$ for all j in (12), and, according to what has been proved above, the polynomials $\widehat{p}$, $\widehat{q}$ are uniquely defined by system (12). Consequently, system (10) uniquely defines a pair of polynomials p and q as well. □

Remark 1. *We can also assume that the uniqueness given by Lemma 1 is preserved with the assumption that the leading coefficient of the polynomial $q(x)$ equals 1. To this end, it is sufficient to consider the interpolation system (10) with only $n+m$ interpolation nodes.*

By writing the polynomials p and q in Lemma 1 in terms of their coefficients, $p(x) = x^n + a_1x^{n-1} + \cdots + a_n$ and $q(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$, system (10) becomes a linear algebraic system of order $n+m+1$ with respect to the coefficients $a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$. We will call this linear system a defining interpolation system for a pair of polynomials with respect to their coefficients. One can also use the Lagrange formula to find the polynomials p and q in terms of their values $p(x_1), \ldots, p(x_n), q(x_{n+1}), \ldots, q(x_{n+m+1})$. Then, for $j > n$, $p(x_j)$ is given as a linear sum of $p(x_1), \ldots, p(x_n)$, and, for $j \leqslant n$, the values $q(x_j)$ are expressed as a linear sum of $q(x_{n+1}), \ldots, q(x_{n+m+1})$. In such a case, the interpolation system (10) turns into a linear algebraic system with respect to $n+m+1$ values of the polynomials in interpolation nods, $p(x_1), \ldots, p(x_n), q(x_{n+1}), \ldots, q(x_{n+m+1})$. This system will be called a defining interpolation system for the pair of polynomials p and q with respect to their values in the interpolation nodes. Of course, the defining systems with respect to the coefficients and values in the interpolation nodes are equivalent.

Lemma 2. *If a defining interpolation system for a pair of polynomials p and q with respect to their coefficients (or values in the interpolation nodes) has a pair of relatively prime polynomials as a solution, then the determinant of this system is not zero.*

Proof. By Lemma 1, a pair of relatively prime polynomials p and q gives rise to the interpolation system (10) that has a unique solution. This is possible only if the determinant of this linear system is not zero. □

3. Main results

Let us consider the IEPMD problem that has been formulated in the Introduction.

Theorem 1. *Suppose, that the IEPMD problem has a solution for a Jacobi matrix J_n in* (1) *with the first l unknowns, according to the sequential indexing* (6)*, and given numbers $\lambda_1, \dots, \lambda_{l_1}, \mu_1, \dots, \mu_{l_2}$, $l_1 + l_2 = l$. Then the solution of such an inverse problem is unique.*

Proof. Let a solution of the IEPMD problem exist. Set $r = \left[\frac{l+1}{2}\right]$. Then system (9) is satisfied. We can apply the results of Lemmas 1 and 2 and Remark 1 to system (9). Indeed, the given numbers $\lambda_1, \dots, \lambda_{n_1}, \mu_1, \dots, \mu_{n_2}$ can be considered as interpolation nodes and system (9) as a particular case of system (10). An interpolation pair of polynomials for system (9) are the polynomials D_r and D_{r-1} for odd l, if a_r is a given parameter. If l is even, then the polynomials $D_r(\lambda)$ and $a_r^2 D_{r-1}(\lambda)$ make an interpolation pair. Here the numbers α_j and β_j in system (10) are taken to be the values $Q_{n-r}(\lambda_j)$ and $-Q_{n-r-1}(\lambda_j)$, if $j \leqslant n_1$, or $\check{Q}_{n-r-1}(\mu_k)$ and $-\check{Q}_{n-r-2}(\mu_k)$, if $k = 1, \dots, n_2$.

Conditions of Lemma 1 are satisfied, since the polynomials $D_r(\lambda)$ and $D_{r-1}(\lambda)$ have alternating zeros; the same is true for the pair $Q_{n-r}(\lambda)$ and $Q_{n-r-1}(\lambda)$, and the pair $\check{Q}_{n-r-1}(\lambda)$ and $\check{Q}_{n-r-2}(\lambda)$.

Hence, if the IEPMD problem has a solution, then the polynomials $D_r(\lambda)$ and $D_{r-1}(\lambda)$ are defined uniquely, and, if the number l is odd, the number a_r is defined uniquely as well. This implies that a solution of the IEPMD problem is unique. □

Example 1. *Let us recover a Jacobi matrix J_4 with $l = 4$ unknown b_1, b_2, a_1, a_2 and the known elements $b_3 = \frac{6}{7}$, $b_4 = 0$, $a_3 = \sqrt{7}$, if two eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$ of the matrix J_4 and two eigenvalues $\mu_1 = 0$, $\mu_2 = 3$ of the matrix J_3 are given.*

In this case, system (9) *is the following:*

$$\begin{aligned} D_2(-1)Q_2(-1) - a_2^2 D_1(-1)Q_1(-1) &= 0, \\ D_2(1)Q_2(1) - a_2^2 D_1(1)Q_1(1) &= 0, \\ D_2(0)\check{Q}_1(0) - a_2^2 D_1(0) &= 0, \\ D_2(3)\check{Q}_1(3) - a_2^2 D_1(3) &= 0. \end{aligned} \tag{13}$$

Since $Q_2(\lambda) = \lambda(\lambda - \frac{6}{7}) - 7$, $Q_1(\lambda) = \lambda$, $\check{Q}_1(\lambda) = \lambda - \frac{6}{7}$, setting $D_2(\lambda) = \lambda^2 + \alpha\lambda + \beta$, $D_1(\lambda) = \lambda + \gamma$, we get from system (12) *that $\alpha = -\frac{1}{7}$, $\beta = -\frac{12}{7}$, $\gamma = \frac{1}{3}$, $a_2 = \frac{6\sqrt{6}}{7}$. Hence, $D_2(\lambda) = \lambda^2 - \frac{1}{7}\lambda - \frac{12}{7}$, $D_1(\lambda) = \lambda + \frac{1}{3}$. According to algorithm* (5)*, we have $\frac{D_2(\lambda)}{D_1(\lambda)} = \lambda - b_2 - \frac{a_1^2}{\lambda - b_1}$. Thus $b_1 = -\frac{1}{3}$, $b_2 = \frac{10}{21}$, $a_1 = \frac{\sqrt{14}}{3}$.*

The matrix J_4 has the form

$$\begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{14}}{3} & 0 & 0 \\ \frac{\sqrt{14}}{3} & \frac{10}{21} & \frac{6\sqrt{6}}{7} & 0 \\ 0 & \frac{6\sqrt{6}}{7} & \frac{6}{7} & \sqrt{7} \\ 0 & 0 & \sqrt{7} & 0 \end{pmatrix}. \tag{14}$$

Remark 2. *In Example 1, we have that the number $l = 4$ of unknown elements of the Jacobi matrix J_4 is equal to its order. Hence, in this case, we can consider the Hochstadt inverse eigenvalue problem with all eigenvalues $\{-3, -1, 1, 4\}$ of the matrix J_4 being known. Of course, a solution of such an inverse problem is given by matrix* (14).

Remark 3. *Even in the case where the number l of unknown elements of the Jacobi matrix J_n does not exceed its order, $l \leqslant n$, together with the inverse problem for l eigenvalues of the matrix J_n, it is also expedient to consider the inverse problem for $l_1 < l$ eigenvalues of the matrix J_n and $l_2 = l - l_1$ eigenvalues of the matrix J_{n-1}. Such a general setting for the IEPMD problem can be useful for applications, since it can experimentally be easier to determine l_1 and l_2 eigenvalues in two experiments with the matrices J_n and J_{n-1} as opposed to determining $l = l_1 + l_2$ eigenvalues of the only matrix J_n in a single experiment.*

Uniqueness of solution of an IEPMD problem, based on Theorem 1, together with an effective algorithm for finding a solution allow to formulate necessary and sufficient conditions on the initial data in the IEPMD problem, which would yield existence of a solution. This conditions come from a possibility to apply, to the initial data, the algorithm that would give a Jacobi matrix.

Theorem 2. *For an IEPMD problem to have a solution for an odd number l of unknown elements of the Jacobi matrix J_n, it is necessary and sufficient that the linear algebraic system* (9) *constructed from the initial data for the coefficients of the polynomials $D_r(\lambda)$ and $D_{r-1}(\lambda)$ have a unique solution and all zeros of the obtained polynomials $D_r(\lambda)$ and $D_{r-1}(\lambda)$ be real, simple, and mutually alternating.*

Proof. Necessity follows from Theorem 1. If conditions of Theorem 2 are fulfilled, then algorithm (5) can be applied to $\frac{D_r(\lambda)}{D_{r-1}(\lambda)}$. As a result, we obtain a Jacobi matrix J_r. Supplementing the matrix J_r with elements of the matrix J_n, given by the initial data, we get a solution of the IEPMD problem. □

More effective existence conditions for solutions of an IEPMD problem can be obtained in terms of the initial data in a special case of the IEPMD problem if, for the number of unknown elements of the Jacobi matrix J_n, we have $l > n$, all eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ of the matrix J_n are given, together with some of $l - n$ eigenvalues $\mu_1, \dots, \mu_{l-n}$ of the matrix J_{n-1}. In this case, $D_n(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)$, $D_{n-1}(\lambda) = p(\lambda) \prod_{k=1}^{l-n} (\lambda - \mu_k)$, where the polynomial $p(\lambda)$ has degree $2n - 1 - l$. By substituting these expressions into the second identity in (8) we get

$$\prod_{j=r}^{n-1} a_j^2 D_{r-1}(\lambda) = p(\lambda) \prod_{k=1}^{l-n} (\lambda - \mu_k) Q_{n-r}(\lambda) - \prod_{j=1}^{n} (\lambda - \lambda_j) \check{Q}_{n-r-1}(\lambda). \tag{15}$$

Since $r = \left[\frac{l+1}{2}\right]$, the degree of the polynomial $D_{r-1}(\lambda)$ in (15) is less than $l - r$ that is the degree of the polynomial $A(\lambda) = \prod_{j=1}^{l-n} (\lambda - \mu_j) Q_{n-r}(\lambda)$. Hence the polynomial $p(\lambda)$ equals the quotient of the polynomial $\prod_{j=1}^{n} (\lambda - \lambda_j) \check{Q}_{n-r-1}(\lambda)$ divided by the polynomial $A(\lambda)$. If l is odd, then the degrees of the polynomials $D_{r-1}(\lambda)$ and $A(\lambda)$ are the same. Hence, it follows from (15) that

$$p(\lambda) = \left\{ \frac{\prod_{j=1}^{n} (\lambda - \lambda_j) \check{Q}_{n-r-1}(\lambda)}{\prod_{j=1}^{l-n} (\lambda - \mu_j) Q_{n-r}(\lambda)} \right\}_{\text{quit}} + \frac{1 - (-1)^l}{2} \prod_{j=r}^{n-1} a_j^2. \tag{16}$$

The symbol $\left\{\frac{B(\lambda)}{A(\lambda)}\right\}_{\text{quit}}$ in (16) denotes a polynomial obtained by dividing the polynomial $B(\lambda)$ by the polynomial $A(\lambda)$ using the Euclidean algorithm. Identity (16) permits to obtain the polynomial $D_{n-1}(\lambda) = p(\lambda)\prod_{j=1}^{l-n}(\lambda - \mu_j)$ in terms of the initial data in the IEPMD problem. This, together with the polynomial $D_n(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j)$ and algorithm (5), gives an effective algorithm for solving the IEPMD problem in the case where $l > n$, $l_1 = n$, $l_2 = l - n$. Moreover, we have the following theorem.

Theorem 3. *There exists a matrix J_n solving an IEPMD problem for $l > n$ unknown elements with given $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\mu_1, \ldots, \mu_{l-n}$ such that the numbers $\{\lambda_j\}$ are eigenvalues of the matrix J_n and the numbers $\{\mu_j\}$ are eigenvalues of the matrix J_{n-1} if an only if*

$$(-1)^{n-1} D_{n-1}(\lambda_j) > 0, \qquad j = 1, \ldots, n, \tag{17}$$

where $D_{n-1}(\lambda) = p(\lambda)\prod_{j=1}^{n-1}(\lambda - \mu_j)$ and the polynomial $p(\lambda)$ is defined by (16) in terms of the initial data.

Proof. Condition (17) implies that using the pair of polynomials $D_n(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j)$ and $D_{n-1}(\lambda)$ defined in the theorem one can construct a Jacobi matrix $\widetilde{J}_n$ using algorithm (5). Here, all given numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ will be eigenvalues of the matrix $\widetilde{J}_n$ and the numbers $\mu_1, \ldots, \mu_{l-n}$ will be eigenvalues of the matrix $\widetilde{J}_{n-1}$ obtained from $\widetilde{J}_n$ by removing the last row and the last column. Hence, to prove the theorem, it is necessary to show that elements of the matrix $\widetilde{J}_n$with indices greater than l coincide with corresponding elements of the matrix J_n. Using identity (8) for $|\lambda| \to \infty$ we get

$$\frac{D_n(\lambda)}{D_{n-1}(\lambda)} - \frac{Q_{n-r}(\lambda)}{\breve{Q}_{n-r-1}(\lambda)} = -\prod_{j=r}^{n-1} a_j^2 |\lambda|^{-(2n-1-r)}(1 + o(1)). \tag{18}$$

Algorithm (5) shows that the expression $\frac{D_n(\lambda)}{D_{n-1}(\lambda)}$ can be expressed in terms of elements of the matrix $\widetilde{J}_n$ as a continued fraction,

$$\frac{D_n(\lambda)}{D_{n-1}(\lambda)} = \lambda - \widetilde{b}_n - \cfrac{\widetilde{a}_{n-1}^2}{\lambda - \widetilde{b}_{n-1} - \cdots - \frac{a_1^2}{\lambda - b_1}}. \tag{19}$$

According to algorithm (5), the expression $\frac{Q_{n-r}(\lambda)}{\breve{Q}_{n-r-1}(\lambda)}$ can be expressed in terms of given elements of the matrix J_n,

$$\frac{Q_{n-r}(\lambda)}{\breve{Q}_{n-r-1}(\lambda)} = \lambda - b_n - \cfrac{a_{n-1}^2}{\lambda - b_{n-1} - \cdots - \frac{a_r^2}{\lambda - b_{r+1}}}. \tag{20}$$

Substituting (19) and (20) into (18) yields the needed identities, $\widetilde{b}_n = b_n$, $\widetilde{a}_{n-1} = a_{n-1}$, $\ldots$, $\widetilde{a}_r = a_r$. Thus a construction of the Jacobi matrix $\widetilde{J}_n$ gives a solution of the IEPMD problem. □

Example 2. *Consider the case $n = 4$ where the matrix J_4 has $l = 5$ unknown parameters b_1, a_1, b_2, a_2, b_3, and $a_3 = \sqrt{7}$, $b_4 = 0$. Let $\{\lambda_j\} = \{-3, -1, 1, 4\}$ and $\mu_1 = 0$.*

Algorithm (16) *gives*

$$p(\lambda) = \left\{\frac{(\lambda+3)(\lambda+1)(\lambda-1)(\lambda-4)}{\lambda \cdot \lambda}\right\}_{\text{quit}} + 7 = \lambda^2 - \lambda - 6.$$

Hence, $D_3(\lambda) = p(\lambda)\cdot\lambda = \lambda^3 - \lambda^2 - 6\lambda$. *But* $D_4(\lambda) = (\lambda+3)(\lambda+1)(\lambda-1)(\lambda-4) = \lambda^4 - \lambda^3 - 13\lambda^2 + \lambda + 12$. *With such* $D_4(\lambda)$ *and* $D_3(\lambda)$*, algorithm* (5) *gives a matrix* J_4 *of form* (14).

Consider now one more case of an IEPMD problem where the number of unknown elements in the Jacobi matrix J_n is greater than the order n of the matrix J_n. Let the entire spectrum $\mu_1 < \mu_2 < \cdots < \mu_{n-1}$ of the matrix J_{n-1} and $l+1-n$ eigenvalues $\lambda_1, \ldots, \lambda_{l+1-n}$ of the matrix J_n be given. To solve the inverse problem, let us formulate an algorithm similar to the one in Theorem 3. The characteristic polynomial $D_{n-1}(\lambda)$ can be explicitly expressed in terms of the given spectrum, $D_{n-1}(\lambda) = \prod_{j=1}^{n-1}(\lambda - \mu_j)$. The characteristic polynomial $D_n(\lambda)$ can be written as a product, $\prod_{j=1}^{l+1-n}(\lambda - \lambda_j)\cdot q(\lambda)$, where the polynomial $q(\lambda)$ has degree $l+1-n$. Substituting these representations for $D_n(\lambda)$ and $D_{n-1}(\lambda)$ into identity (8) gives

$$\prod_{j=r}^{n-1} a_j^2 D_{r-1}(\lambda) = \prod_{j=1}^{n-1}(\lambda-\mu_j)Q_{n-r}(\lambda) - q(\lambda)\prod_{j=1}^{l+1-n}(\lambda-\lambda_j)\check{Q}_{n-r-1}(\lambda). \tag{21}$$

This identity, similarly to (15), gives

$$q(\lambda) = \left\{\frac{\prod_{j=1}^{n-1}(\lambda-\mu_j)Q_{n-r}(\lambda)}{\prod_{j=1}^{l+1-n}(\lambda-\lambda_j)\check{Q}_{n-r-1}(\lambda)}\right\}_{\text{quit}} - \frac{1-(-1)^l}{2}\prod_{j=r}^{n-1}a_j^2, \tag{22}$$

where $r = \left[\frac{l+1}{2}\right]$.

The polynomial $\{\ldots\}_{\text{quit}}$ can be effectively constructed from the initial data of the IEPMD problem. Hence, the representations $D_{n-1}(\lambda) = \prod_{j=1}^{n-1}(\lambda-\mu_j)$ and $D_n(\lambda) = q(\lambda)\prod_{j=1}^{l+1-n}(\lambda-\lambda_j)$, where the polynomial $q(\lambda)$ is given by (22) reduces the construction of J_n to algorithm (5).

Theorem 4. *An IEPMD problem with* $l \geqslant n$ *unknown parameters of the Jacobi matrix* J_n *has a solution for all eigenvalues* $\mu_1 < \mu_2 < \cdots < \mu_{n-1}$ *of the matrix* J_{n-1} *and* $\lambda_1, \ldots, \lambda_{l+1-n}$ *eigenvalues of the matrix* J_n *being given if and only if*

$$(-1)^{n-j}D_n(\mu_j) > 0, \qquad j = 1, \ldots, n-1, \tag{23}$$

where $D_n(\lambda) = q(\lambda)\prod_{j=1}^{l+1-n}(\lambda-\lambda_j)$ *and the polynomial* $q(\lambda)$ *is given by formula* (22).

Proof. Using the initial data we construct $D_{n-1}(\lambda) = \prod_{j=1}^{n-1}(\lambda-\mu_j)$ and $D_n(\lambda)$. Condition (23) is equivalent to that all zeros of the polynomials $D_n(\lambda)$ and $D_{n-1}(\lambda)$ alternate. Hence, using algorithm (5) applied to D_n and D_{n-1} we can construct a Jacobi matrix J_n and, as in Theorem 3, prove that it is a solution of the IEPMD problem. □

Example 3. *Let us construct a Jacobi matrix* J_4 *that has unknown elements* b_1*,* a_1*,* b_2*,* a_2*, and the elements* $b_3 = \frac{6}{7}$*,* $b_4 = 0$*,* $a_3 = \sqrt{7}$ *are known. Let the spectrum* $\{-2, 0, 3\}$ *of the matrix* J_3 *and one eigenvalue* $\lambda_1 = -1$ *of the matrix* J_4 *be given.*

Algorithm (22) *yields* $q(\lambda) = \left\{\frac{D_3(\lambda)Q_3(\lambda)}{(\lambda+1)(\lambda-\frac{6}{7})}\right\}_{\text{quit}} = \lambda^3 - 2\lambda^2 - 11\lambda + 12$. *Then* $D_4(\lambda) = (\lambda+1)q(\lambda) = \lambda^4 - \lambda^3 - 13\lambda^2 + \lambda + 12$. *Conditions* (23) *are satisfied. Algorithm* (5) *gives a matrix* J_4 *of the form* (14).

Acknowledgement. The research in this paper partially supported by the EU-financed project AMMODIT ("Approximation Methods for Molecular Modelling and Diagnosis Tasks", "MSCA-RISE-2014-645672-AMMODIT").

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Received 06/02/2018