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ON EXTENSIONS OF LINEAR FUNCTIONALS WITH APPLICATIONS TO NON-SYMMETRICALLY SINGULAR PERTURBATIONS

MYKOLA DUDKIN AND TETIANA VDOVENKO

This paper is dedicated to the 75th anniversary of V. D. Koshmanenko

Abstract. The article is devoted to extensions of linear functionals, generated by scalar products, in a scale of Hilbert spaces. Such extensions are used to consider non-symmetrically singular rank one perturbations of $\mathcal{H}_{-2}$-class. For comparison, we give main definitions and descriptions of singular non-symmetric perturbations of $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$-classes.

Together with outstanding mathematicians S .Albeverio [2], P. Kurasov [3], W. Karwowski [11], L. Nizhnik [16], V. D. Koshmanenko is one of the founders of the singular perturbation theory of self-adjoint operators in general cases (for an abstract self-adjoint operator and an abstract Hilbert space).

In our article we continue investigations of singularly perturbed operators in a case of non-symmetric perturbation. In this connection, we would like to show one simple, but useful in applications, expansion method of liner functionals generated by scalar products in rigged Hilbert spaces.

1. An extension of functionals for an equipped Hilbert space

Let $\mathcal{H}$ be a separable Hilbert space with a norm $\|\cdot\|$ and a scalar product $(\cdot,\cdot)$. Consider an unbounded positive self-adjoint operator $A \geq cI$, $c > 1$ with a domain $\mathfrak{D}(A)$ in $\mathcal{H}$.

Via the operator A, we introduce an A-scale of Hilbert spaces [1, 13]: $\mathcal{H}_k = \mathcal{H}_k(A)$, $k \in \mathbb{Z}$ (briefly $A > 1$), where $\mathcal{H}_\alpha = \mathfrak{D}(A^{\alpha/2})$, $\alpha \in \mathbb{N}$ and the norm is generated by the scalar product

$$(\varphi,\psi)_\alpha = (A^{\alpha/2}\varphi, A^{\alpha/2}\psi), \quad \varphi,\psi \in \mathfrak{D}(A^{\alpha/2});$$

$\mathcal{H}_0 = \mathcal{H}$; the space $\mathcal{H}_{-\alpha}$, $\alpha \in \mathbb{N}$ is constructed as a completion of $\mathcal{H}$ with respect to the norm

$$(f,g)_{-\alpha} = (A^{-\alpha/2}f, A^{-\alpha/2}g), \quad f,g \in \mathcal{H}.$$

By $\langle \omega, \varphi\rangle_{-\alpha,\alpha}$, $\alpha \in \mathbb{N}$ we denote the dual scalar product for $\omega \in \mathcal{H}_{-\alpha}$ and $\varphi \in \mathcal{H}_\alpha$.

The operator A has the following properties in the scale $\mathcal{H}_k(A)$ [1, 13]:

$$\begin{aligned} A^{\alpha/2} &: \mathcal{H}_\alpha \longrightarrow \mathcal{H}_0, \\ A^{-\alpha/2} &: \mathcal{H}_0 \longrightarrow \mathcal{H}_\alpha, \\ (A^{\alpha/2})^{cl} &: \mathcal{H}_0 \longrightarrow \mathcal{H}_{-\alpha}, \\ (A^{-\alpha/2})^{cl} &: \mathcal{H}_{-\alpha} \longrightarrow \mathcal{H}_0, \quad \alpha \in \mathbb{N}, \end{aligned} \tag{1}$$

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where "*cl*" denotes the closure of an operator in the corresponding space. The scalar products have the following properties in the scale $\mathcal{H}_k(A)$ [1, 13]:

$$(2)\quad \begin{aligned} &(f,\varphi)_0 = \langle f,\varphi\rangle_{-\alpha,\alpha}, \quad f\in\mathcal{H}_0, \quad \varphi\in\mathcal{H}_\alpha;\\ &\langle\omega,\varphi\rangle_{-\alpha,\alpha} = (\omega,(A^\alpha)^{cl}\varphi)_{-\alpha} = ((A^{-\alpha})^{cl}\omega,\varphi)_\alpha = ((A^{-\alpha/2})^{cl}\omega,(A^{\alpha/2})^{cl}\varphi)_0,\\ &\omega\in\mathcal{H}_{-\alpha}, \quad \varphi\in\mathcal{H}_\alpha, \quad \alpha\in\mathbb{N};\\ &\langle\omega,\varphi\rangle_{-\alpha,\alpha} = \langle\omega,\varphi\rangle_{-\beta,\beta}, \quad \alpha<\beta, \quad \alpha,\beta\in\mathbb{N}, \quad \omega\in\mathcal{H}_{-\alpha}, \quad \varphi\in\mathcal{H}_\beta.\end{aligned}$$

Next, for the simplicity, we consider only some part of the A-scale,

$$(3)\qquad \mathcal{H}_-\supset\mathcal{H}_0\supset\mathcal{H}_+,$$

where $\mathcal{H}_- = \mathcal{H}_{-1}$, $\mathcal{H}_+ = \mathcal{H}_1$, i.e., we have the equipped Hilbert space $\mathcal{H}_0$ with a positive and a negative spaces [7]. in this connection, we denote $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{-1,1}$. As it is well known [7], $\mathcal{H}_-$ is a space of linear continuous functionals on $\mathcal{H}_+$. Hence, each element $\omega\in\mathcal{H}_-$ generates a linear continuous functional $\mathcal{I}_\omega(\varphi)$, $\varphi\in\mathcal{H}_+$ of the form $\mathcal{I}_\omega(\varphi)=\langle\omega,\varphi\rangle$ (but this functional is an unbounded and a densely defined one in $\mathcal{H}_0$).

Let us consider an extension by linearity of the functional $\mathcal{I}_\omega(\varphi)$ on some elements of $\mathcal{H}_0$ (or in general on $\mathcal{H}_-$). An example further in the first section illustrates a possibility of such a situation. Following the article [4] we denote the extension $\mathcal{I}_{\omega^{ex}} = \langle\omega^{ex},\varphi\rangle$, where $\varphi\in\mathcal{H}_+\cup\Phi$, $\Phi\subset\mathcal{H}_0$ (or in general $\Phi\subset\mathcal{H}_-$). We extend $\mathcal{I}_\omega(\varphi)$ by assigning an arbitrary convenient value $c_\varphi := \mathcal{I}_{\omega^{ex}}(\varphi)\in\mathbb{C}$. Under an extension by linearity we understand the equality

$$\mathcal{I}_{\omega^{ex}}(a\varphi+b\psi) = \bar{a}\mathcal{I}_{\omega^{ex}}(\varphi)+\bar{b}\mathcal{I}_{\omega^{ex}}(\psi),$$
$$\forall a,b\in\mathbb{C}, \quad \varphi,\psi\in\Phi\cup\mathcal{H}_+, \quad \Phi\subset\mathcal{H}_0(\Phi\subset\mathcal{H}_-).$$

Analogously we consider:

- the functional $\mathcal{I}_\psi(\varphi) = (\psi,\varphi)_+$, $\varphi,\psi\in\mathcal{H}_+$, and its extension
$$\mathcal{I}_{\psi^{ex}}(\varphi) = (\psi^{ex},\varphi)_+, \quad \psi\in\mathcal{H}_+, \quad \varphi\in\mathcal{H}_+\cup\Phi_+, \quad \Phi_+\subset\mathcal{H}_0;$$
- the functional $\mathcal{I}_\omega(\phi) = (\omega,\phi)_-$, $\omega,\phi\in\mathcal{H}_-$, and its extension
$$\mathcal{I}_{\omega^{ex}}(\phi) = (\omega^{ex},\phi)_-, \quad \omega\in\mathcal{H}_-, \quad \phi\in\mathcal{H}_-\cup\Phi_-, \quad \Phi_-\subset\mathcal{H}_{-2};$$
- the functional $\mathcal{I}_f(g) = (f,g)_0$, $f,g\in\mathcal{H}_0$, and its extension
$$\mathcal{I}_{f^{ex}}(g) = (f^{ex},g)_0, \quad f\in\mathcal{H}_0, \quad g\in\mathcal{H}_0\cup\Phi_0, \quad \Phi_0\subset\mathcal{H}_-.$$

For simplicity we suppose that the subsets Φ, Φ_i, $i=\{0,+,-\}$ are one dimensional.

Proposition 1. *If we put $\psi=(A^{-1})^{cl}\omega$, $f=(A^{-1/2})^{cl}\omega$, and suppose that $A^{1/2}\Phi_+ = \Phi_0$, $(A^{1/2})^{cl}\Phi_0 = \Phi_-$ and $\Phi_+=\Phi$, then we can choose extensions of the functionals $\mathcal{I}_\omega(\varphi)$, $\mathcal{I}_\omega(\phi)$, $\mathcal{I}_\psi(\varphi)$ and $\mathcal{I}_f(g)$ so that*

$$(4)\qquad \mathcal{I}_{\omega^{ex}}(\varphi) = \mathcal{I}_{\omega^{ex}}(\phi) = \mathcal{I}_{\psi^{ex}}(\varphi) = \mathcal{I}_{f^{ex}}(g).$$

Proof. The chain (4) has a form of the second line in (2),

$$(5)\quad \langle\omega,\varphi\rangle = (\omega,(A)^{cl}\varphi)_- = ((A^{-1})^{cl}\omega,\varphi)_+ = ((A^{-1/2})^{cl}\omega,(A^{1/2})^{cl}\varphi)_0, \quad \omega\in\mathcal{H}_-,$$

if $\varphi\in\mathcal{H}_+$. For $\varphi\in\Phi$ we can put a joint convenient constant. □

Proposition 1 has a generalization for an arbitrary scale $\mathcal{H}_k$, $k\in\mathbb{Z}$. Namely, let us choose $k_1,k_2,k_3\in\mathbb{Z}$, such that $k_1<k_2<k_3$, $|k_2-k_1|=|k_3-k_2|$, and consider the part of the scale $\mathcal{H}_k$, $k\in\mathbb{Z}$,

$$(6)\qquad \mathcal{H}_{k_1}\supset\mathcal{H}_{k_2}\supset\mathcal{H}_{k_3}.$$

The triplet (6) is associated uniquely with a positive self-adjoint operator $\mathcal{A}$ (see [1, 13]).

Let us consider the following:

- the functional $\mathcal{I}_\psi(\varphi) = (\psi,\varphi)_{k_3}$, $\varphi,\psi \in \mathcal{H}_{k_3}$, and its extension

$$\mathcal{I}_{\psi^{ex}}(\varphi) = (\psi^{ex},\varphi)_{k_3}, \quad \psi \in \mathcal{H}_{k_3}, \quad \varphi \in \mathcal{H}_{k_3} \cup \Phi_{k_3}, \quad \Phi_{k_3} \subset \mathcal{H}_{k_2};$$

- the functional $\mathcal{I}_\omega(\phi) = (\omega,\phi)_{k_1}$, $\omega,\phi \in \mathcal{H}_{k_1}$, and its extension

$$\mathcal{I}_{\omega^{ex}}(\phi) = (\omega^{ex},\phi)_{k_1}, \quad \omega \in \mathcal{H}_{k_1} \quad \phi \in \mathcal{H}_{k_1} \cup \Phi_{k_1}, \quad \Phi_{k_1} \subset \mathcal{H}_{k_1-|k_2-k_1|};$$

- the functional $\mathcal{I}_f(g) = (f,g)_{k_2}$, $f,g \in \mathcal{H}_{k_2}$, and its extension

$$\mathcal{I}_{f^{ex}}(g) = (f^{ex},g)_{k_2}, \quad f \in \mathcal{H}_{k_2}, \quad g \in \mathcal{H}_{k_2} \cup \Phi_{k_2}, \quad \Phi_{k_2} \subset \mathcal{H}_{k_1};$$

- the functional $\mathcal{I}_\omega(\varphi) = \langle\omega,\varphi\rangle_{k_1,k_3}$, $\omega \in \mathcal{H}_{k_1}$, $\varphi \in \mathcal{H}_{k_3}$ and its extension

$$\mathcal{I}_{\omega^{ex}}(\varphi) = \langle\omega^{ex},\varphi\rangle)_{k_1,k_3}, \quad \omega \in \mathcal{H}_{k_1}, \quad \varphi \in \mathcal{H}_{k_2} \cup \Phi, \quad \Phi \subset \mathcal{H}_{k_2}.$$

The next theorem generalizes Proposition 1.

Theorem 1. *If we put $\psi = (\mathcal{A}^{-1})^{cl}\omega$, $f = (\mathcal{A}^{-1/2})^{cl}\omega$, and suppose that $\mathcal{A}^{1/2}\Phi_{k_3} = \Phi_{k_2}$, $(\mathcal{A}^{1/2})^{cl}\Phi_{k_2} = \Phi_{k_1}$ and $\Phi_{k_3} = \Phi$, then we can choose extensions of functionals $\mathcal{I}_\omega(\varphi)$, $\mathcal{I}_\omega(\phi)$, $\mathcal{I}_\psi(\varphi)$ and $\mathcal{I}_f(g)$ so that*

$$\mathcal{I}_{\omega^{ex}}(\varphi) = \mathcal{I}_{\omega^{ex}}(\phi) = \mathcal{I}_{\psi^{ex}}(\varphi) = \mathcal{I}_{f^{ex}}(g). \tag{7}$$

Proof. The chain (6) has a form of the second line in (2) with the chains of indexes

$$\langle\omega,\varphi\rangle_{k_1,k_3}=(\omega,(\mathcal{A})^{cl}\varphi)_{k_1}=((\mathcal{A}^{-1})^{cl}\omega,\varphi)_{k_3}=((\mathcal{A}^{-1/2})^{cl}\omega,(\mathcal{A}^{1/2})^{cl}\varphi)_{k_2}, \quad \omega \in \mathcal{H}_{k_1}, \tag{8}$$

if $\varphi \in \mathcal{H}_{k_3}$. For $\varphi \in \Phi$ there we put a convenient constant. □

Example 1. Let $\mathcal{H}_0 = L_2([1,\infty), dx)$ be the space of square integrable functions on the half interval $[1,\infty)$ with respect to the Lebesgue measure, and A be an operator of a multiplication by the independent variable "x",

$$Af(x) = xf(x),\ \mathfrak{D}(A) = \{f(x) \in L_2([1,\infty), dx) \mid xf(x) \in L_2([1,\infty), dx)\}.$$

In such a case, $\mathcal{H}_+ = \mathfrak{D}(A^{1/2})$ is the space with the scalar product

$$(\psi,\varphi)_+ = (A^{1/2}\psi, A^{1/2}\varphi)_0 = \int_1^\infty \psi(x)\varphi(x)x\,dx,$$

i.e. $\mathcal{H}_+ = L_2([1,\infty), xdx)$. The space $\mathcal{H}_-$ has a scalar product,

$$(\omega,\phi)_- = \int_1^\infty \omega(x)\phi(x)\frac{1}{x}\,dx,$$

i.e., $\mathcal{H}_- = L_2([1,\infty), \frac{1}{x}dx)$. Hence we the have rigged Hilbert spaces

$$L_2([1,\infty)\frac{1}{x}, dx) \supset L_2([1,\infty), dx) \supset L_2([1,\infty), xdx).$$

Let us choose $\omega = \frac{1}{\sqrt{x}}$, and consider the functional $\mathcal{I}_\omega(\varphi)$ on $\mathcal{H}_+$, i.e., $\varphi \in \mathcal{H}_+$, because $\omega \in L_2([1,\infty)\frac{1}{x}, dx)$, since $\int_1^\infty \frac{1}{\sqrt{x}}\frac{1}{\sqrt{x}}\frac{1}{x}\,dx = 1 < \infty$ and $\omega \notin L_2([1,\infty), dx)$, since $\int_1^\infty \frac{1}{\sqrt{x}}\frac{1}{\sqrt{x}}\,dx = \infty$.

But, for $\varphi = \frac{1}{x}$ we have $\frac{1}{x} \in L_2([1,\infty)\,dx)$, since $\int\limits_1^\infty \frac{1}{x^2}\,dx = 1 < \infty$ and what is unexpectedly $\int\limits_1^\infty \frac{1}{\sqrt{x}}\frac{1}{x}\,dx = 2 < \infty$. Hence, this is a reason to put

$$\mathcal{I}_{\omega^{ex}} = \langle \omega^{ex}, \varphi\rangle = \int\limits_1^\infty \omega(x), \varphi(x)\,dx.$$

But it is not only one possible way to extend the functional $\mathcal{I}_\omega$. Hence, for $\varphi = \frac{1}{x}$ we can put

$$\mathcal{I}_{\omega^{ex}}(\varphi) = \begin{cases} \int\limits_1^\infty \omega(x)\varphi(x)\,dx = 1, & \text{naturally,} \\ c_\varphi \in \mathbb{R}, & \text{in general.} \end{cases} \tag{9}$$

Remark 1. The considered in the article functionals are useful in corresponding parts of singular self-adjoint [4, 5, 13] and non-self-adjoint [9] perturbation theory. We illustrate this below.

Remark 2. For example, the functional $\mathcal{I}_{\omega^{ex}} = \langle \omega^{ex}, \varphi\rangle$, where $\varphi \in \mathcal{H}_+ \cup \Phi$, can be extended in general to $\Phi \subset \mathcal{H}_{-\alpha}$, for an arbitrary $\alpha > 1$. Each of the considered above functionals can be extended in such a way.

2. Some remarks on singular non-symmetric perturbations of $\mathcal{H}_{-1}$-class

Let us consider a linear operator V acting from $\mathcal{H}_{+1}$ into $\mathcal{H}_{-1}$. Let it have the form $V = V^{\omega_1,\omega_2} = \langle\cdot,\omega_1\rangle\omega_2$, $\omega_1,\omega_2 \in \mathcal{H}_{-1}$. If we define by $\mathbf{A}$ an extension by linearity of the operator A that is a bounded operator acting from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$, then $\mathbf{A}+V$ is also a linear operator from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$. For the operator in the scale we know the concept of an adjoint operator $(\mathbf{A}+V)^*$, which acts also from $\mathcal{H}_{+1}$ into $\mathcal{H}_{-1}$ [6].

For a formal expression $A + \langle\cdot,\omega_1\rangle\omega_2$ we can give the meaning of an operator in $\mathcal{H}$ [10, 12]. We take $\mathbf{A} + \langle\cdot,\omega_1\rangle\omega_2$ and restrict it to $\mathcal{H}$ and denote it by A^{ω_1,ω_2},

$$A^{\omega_1,\omega_2} := (\mathbf{A} + \langle\cdot,\omega_1\rangle\omega_2)\restriction_{\mathcal{H}}. \tag{10}$$

Sometimes we will also be using the notation of the operator A instead of $\mathbf{A}$, if it will not lead to any obvious contradiction.

The restriction process is not always convenient, hence we used the following definition of singularly non-symmetrically perturbed operator [8, 9].

Definition 1. Let $A > 1$ be a positive self-adjoint operator defined in a separable Hilbert space $\mathcal{H}$. For $\omega_1,\omega_2 \in \mathcal{H}_{-1}\setminus\mathcal{H}$, $\omega_1 \neq \omega_2$, we put $\eta_i = \mathbf{A}^{-1}\omega_i$, $i = 1,2$.

The operator A^{ω_1,ω_2} is called singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-1}$-class with respect to A, if

$$\begin{aligned} \mathfrak{D}(A^{\omega_1,\omega_2}) = \Big\{\psi = \varphi - b\eta_2 \mid \varphi \in \mathfrak{D}(A), \\ b = b(\varphi) = \frac{(A\varphi,\eta_1)}{1 + (A^{1/2}\eta_2, A^{1/2}\eta_1)}\Big\} \end{aligned} \tag{11}$$

in the case $(A^{1/2}\eta_2, A^{1/2}\eta_1) \neq -1$; and

$$\mathfrak{D}(A^{\omega_1,\omega_2}) = \mathfrak{D}_{\mathcal{H}_1} \dot{+} \{c\eta_2\}, \quad \mathfrak{D}_{\mathcal{H}_1} = \{\varphi \in \mathfrak{D}(A) \mid (A\varphi,\eta_1) = 0\}, \tag{12}$$

($c \in \mathbb{C}$) in the case $(A^{1/2}\eta_2, A^{1/2}\eta_1) = -1$, (and we denote $A^{\omega_1,\omega_2} \in \mathcal{P}(A)$).

The action is given by the rule $A^{\omega_1,\omega_2}\psi = A\varphi$.

The operator A is called (initial) non-perturbed, and $V = \langle \cdot, \omega_1 \rangle \omega_2$ is called the perturbation (of the $\mathcal{H}_{-1}$-class). Hence, A^{ω_1,ω_2} is naturally to call as perturbed operator.

The presented definition generalizes non-local interactions [5, 14, 15] with a self-adjoint operator to the case where the perturbed operator is non-self-adjoint.

The fact that the space $\mathcal{H}$ is separable is not obligatory in the Definition 1. The positivity of the operator A or its semiboundedness is also not obligatory. In such a case it is need to require for (11) the following: the positive space $\mathcal{H}_{+1}$ with the norm $\|\varphi\|_{+1} = \|(|A|^{1/2} + I)\varphi\|$, $\varphi \in \mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$ is a completion $\mathcal{H}$ with respect to the norm $\|f\|_{-1} = \|(|A|^{-1/2} + I)f\|$, $f \in \mathcal{H}$; $\mathcal{H}_{+2}$ with the norm $\|\varphi\|_{+2} = \|(|A| + I)\varphi\|$, $\varphi \in \mathcal{H}_{+2}$ and $\mathcal{H}_{-2}$ is a completion of $\mathcal{H}$ with respect to the norm $\|f\|_{-2} = \|(|A| + I)f\|$, $f \in \mathcal{H}$.

In particular, in what follows we consider for convenience that the operator V is of the form $V = \alpha\langle \cdot, \omega_1 \rangle \omega_2$ with the constant $\alpha \in \mathbb{C}, 0 < |\alpha| < \infty$. Such a form does not differ from the previous consideration, since we can every time write

$$V = \alpha\langle \cdot, \omega_1 \rangle \omega_2 = \langle \cdot, \omega_1 \rangle \alpha\omega_2 = \langle \cdot, \bar{\alpha}\omega_1 \rangle \omega_2, \tag{13}$$

but it is convenient for applications. In particular, $A^{\bar{\alpha}\omega_1,\omega_2} = A^{\omega_1,\alpha\omega_2}$.

Therefore, Definition 1 can be generalized.

Definition 2. Let A be a self-adjoint operator defined in a separable Hilbert space $\mathcal{H}$. For $\omega_1, \omega_2 \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $\omega_1 \neq \omega_2$, we put $\eta_i(z) = (\mathbf{A} - z)^{-1}\omega_i$, $i = 1, 2$, $z \in \rho(\mathbf{A})$, where $\rho(\cdot)$ is a resolvent set of the corresponding operator.

The operator $A^{\omega_1,\alpha\omega_2}$ is called singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-1}$-class with respect to A, if

$$\begin{aligned}\mathfrak{D}(A^{\omega_1,\alpha\omega_2}) = \Big\{ &\psi = \varphi - b_z\eta_2(z) \mid \varphi \in \mathfrak{D}(A), \\ &b_z = \frac{((A-z)\varphi, \eta_1(\bar{z}))}{1/\alpha + \langle (A-z)\eta_2(z), \eta_1(\bar{z})\rangle} \Big\}\end{aligned} \tag{14}$$

in the case $((A - z)\eta_2(z), \eta_1(\bar{z})) \neq -1/\alpha$ for a fixed z; and

$$\mathfrak{D}(A^{\omega_1,\alpha\omega_2}) = \mathfrak{D}_{\mathcal{H}_1} \dot{+} \{c\eta_2(z)\}, \quad \mathfrak{D}_{\mathcal{H}_1}(z) = \{\varphi \in \mathfrak{D}(A) \mid ((A-z)\varphi, \eta_1(\bar{z}) = 0\} \tag{15}$$

($c \in \mathbb{C}$) in the case $((A - z)\eta_2(z), \eta_1(\bar{z})) = -1/\alpha$, (and we denote $A^{\omega_1,\alpha\omega_2} \in \mathcal{P}(A)$).

The action is given by the rule $(A^{\omega_1,\omega_2} - z)\psi = (A - z)\varphi$.

But this Definition has also its own drawbacks. The written form of domains (14) and (15) depend on z. In spite of such a drawback, it is useful for the next example with the Schrödinger operator.

Proposition 2. *Definition 1 is equivalent to Definition 2, if we consider positive (semi-bounded with the corresponding shift) self-adjoint operator A.*

Proof. In one way the proof is trivial. It is enough to put $z = 0$.

In general the proof is very complicated. In this short article we present only a sketch of the proof. For operators from Definitions 1 an 2 we write the corresponding resolvents (see the Theorem 2 below), and comparing the obtained resolvents we conclude equivalence of the operators. By the writing the resolvent corresponding to the operator from Definition 2, we prove independence of (14) and (15) on z. □

Schrödinger operator with non-local interactions. Let us consider the operator $A = -\frac{d^2}{dx^2}$ with the domain $\mathfrak{D}(A) = W_2^2(\mathbb{R})$ in the space $L_2(\mathbb{R}^1, dx)$. The operator $\mathbf{A}$ is defined by the extension of A to $W_2^1(\mathbb{R}^1)$ in the sense of generalized functions, i.e., as an operator acting from $W_2^1(\mathbb{R}^1)$ into $W_2^{-1}(\mathbb{R}^1)$.

Proposition 3. *The operator*

$$-\Delta^{\delta_{x_1},\alpha\delta_{x_2}} = -\frac{d^2}{dx^2} + \alpha\langle \cdot, \delta_{x_1}\rangle \delta_{x_2}, \quad \alpha \in \mathbb{C}, \tag{16}$$

where δ_{x_1} and δ_{x_2} are δ-functions of Dirac concentrated at the points $x_1,\ x_2,\ x_1 \neq x_2$, $x_1,\ x_2 \in \mathbb{R}^1$, has the domain

$$\begin{aligned}(17)\quad \mathfrak{D}(-\Delta^{\delta_{x_1},\alpha\delta_{x_2}}) = \big\{\psi \in W_2^1(\mathbb{R}^1) \cap W_2^2(\mathbb{R}^1 \setminus \{x_1\}) \mid \psi(x_1+) = \psi(x_1-),\\ \psi'(x_2+) - \psi'(x_2-) = \alpha\psi(x_1)\}\end{aligned}$$

and it acts as follows:

$$(18)\qquad -\Delta^{\delta_{x_1},\alpha\delta_{x_2}}\psi = -\varphi''.$$

Proof. Firstly we consider the case

$$\langle (A-z)\eta_2(z), \eta_1(\bar{z})\rangle \neq -1/\alpha.$$

From the Definition 2, in accordance with expression (14), the vector ψ has the form

$$\psi = \varphi - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|x-x_2|},$$

where $z = k^2$ and $k = \sqrt{\pm i}$, $\operatorname{Im} k > 0$, $\eta_j(z) = \frac{i}{2k}e^{ik|x-x_j|}$, $j = 1, 2$ are taken from [2]. In particular, in the sense of generalized functions, we have

$$\langle (A-z)\varphi, \eta_1(\bar{z})\rangle = \langle \varphi, (A-\bar{z})\eta_1(\bar{z})\rangle = \langle \varphi, \delta_{x_1}\rangle = \varphi(x_1)$$

and

$$\langle \eta_2(z), (A-\bar{z})\eta_1(\bar{z})\rangle = \langle \eta_2(z), \delta_{x_1}\rangle = \frac{i}{2k}e^{ik|x_1-x_2|}.$$

The first equality in (17), namely $\psi(x_1+) = \psi(x_1-)$, is fulfilled,

$$\begin{aligned}\varphi(x_1+) - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|(x_1+)-x_2|}\\ = \varphi(x_1-) - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|(x_1-)-x_2|}.\end{aligned}$$

Let us show the second equality in (17). Since

$$\psi\prime(x) = \varphi\prime(x) - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|x-x_2|}ik\theta(x-x_2),$$

where $\theta(x-x_2)$ is the Heaviside function with the jump at the point x_2, we have

$$\begin{aligned}&\psi\prime(x_2+) - \psi\prime(x_2-)\\ &= \left\{\varphi\prime - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|x-x_2|}ik\theta(x-x_2)\right\}_{x=x_2+}\\ (19)\qquad &- \left\{\varphi\prime - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|x-x_2|}ik\theta(x-x_2)\right\}_{x=x_2-}\\ &= -\frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}ik(1-(-1)) = \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}},\end{aligned}$$

where it is taken into account that $\varphi'(x_2+) = \varphi'(x_2-)$, since $\varphi(x) \in W_2^2(\mathbb{R}^1)$.

The left part of the second equality in (17) has the form

$$\begin{aligned}&\varphi(x_1) - \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\ \frac{i}{2k}e^{ik|x_1-x_2|}\\ (20)\qquad &= \frac{\varphi(x_1)\frac{1}{\alpha} + \varphi(x_1)\frac{i}{2k}e^{ik|x_1-x_2|} - \varphi(x_1)e^{ik|x_1-x_2|}}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}\\ &= \frac{1}{\alpha}\ \frac{\varphi(x_1)}{\frac{1}{\alpha} + \frac{i}{2k}e^{ik|x_1-x_2|}}.\end{aligned}$$

Comparing the ends of the expressions in (19) and (20), we obtain

$$\psi'(x_2+) - \psi'(x_2-) = \alpha\psi(x_1).$$

Hence, we proved the case $-\frac{1}{\alpha} \neq \frac{i}{2k}e^{ik|x_1-x_2|}$.

Let us consider the case (15), namely,

$$-1/\alpha = \langle (A-z)\eta_2(z), \eta_1(\bar{z})\rangle = \frac{i}{2k}e^{ik|x_1-x_2|}. \tag{21}$$

In such a case,

$$\begin{aligned}\mathfrak{D}_{\mathcal{H}_1}(z) &= \left\{\varphi \in W_2^2(\mathbb{R}) \mid \langle (A-z)\varphi, \eta_1(\bar{z})\rangle = 0\right\} \\ &= \left\{\varphi \in W_2^2(\mathbb{R}) \mid \varphi(x_1) = 0\right\}.\end{aligned}$$

For $h(x) \in \mathfrak{D}_{\mathcal{H}_1}(z)$, we have $h(x_1+) = h(x_1-) = h(x_1) = 0$.

For $\eta_2(z) = \frac{i}{2k}e^{ik|x-x_2|}$ we also have

$$e^{ik|(x_1+)-x_2|} = e^{ik|(x_1-)-x_2|}.$$

For $h(x) \in \mathfrak{D}_{\mathcal{H}_1}(z)$, we have $h'(x_2+) = h'(x_2-)$ and $h(x_1) = 0$, i.e., the second equality in (17) holds true.

Let us write the second equality in (17) for $\eta_2(z) = \frac{i}{2k}e^{ik|x-x_2|}$. The left-hand side has the form

$$\begin{aligned}\psi\prime(x_2+) - \psi\prime(x_2-) = \frac{i}{2k}&e^{ik|x-x_2|}ik\theta(x-x_2)|_{x=x_2+} \\ &- \frac{i}{2k}e^{ik|x-x_2|}ik\theta(x-x_2)|_{x=x_2-} = -\frac{1}{2}(1-(-1)).\end{aligned} \tag{22}$$

The right-hand side of the second identity in (17) has the form

$$\alpha\eta_2(z)|_{x=x_1} = \alpha\frac{i}{2k}e^{ik|x-x_2|}|_{x=x_1} = \alpha\frac{i}{2k}e^{ik|x_1-x_2|}. \tag{23}$$

Indeed, (22) is equal to (23), if we take into account (21). Hence, (17) is also true for vectors from (15).

The action (18) is obvious. □

Let us denote the resolvent $R_z = (A-z)^{-1}$, $z \in \rho(A)$ of the operator A and find a general form for the resolvent $\tilde{R}_z = (\tilde{A}-z)^{-1}$, $z \in \rho(\tilde{A})$ of perturbed operator $\tilde{A}$.

Theorem 2. *Let $A > 1$ be a positive self-adjoint operator defined in the separable Hilbert space $\mathcal{H}$ and, $\tilde{A}$ be an operator singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-1}$-class with respect to A defined in Definition 1.*

For the resolvents $R_z = (A-z)^{-1}$ and $\tilde{R}_z = (\tilde{A}-z)^{-1}$, the M. Krein type formula

$$\tilde{R}_z = R_z + b_z(\cdot, \eta_1(\bar{z}))\eta_2(z), \quad z, \xi \in \rho(A) \cap \rho(\tilde{A}), \tag{24}$$

holds true with the vector-valued functions

$$\eta_1(z) = (A-\xi)(A-z)^{-1}\eta_1(\xi), \quad \eta_2(z) = (A-\xi)(A-z)^{-1}\eta_2(\xi), \tag{25}$$

where $\eta_1(z), \eta_2(z) \in \mathcal{H}_{+1}$ and with the scalar-valued function

$$b_z^{-1} - b_\xi^{-1} = (\xi - z)(\eta_1(\xi), \eta_2(\bar{z})). \tag{26}$$

The vectors $\eta_1(z), \eta_2(z)$ are connected with the value b_z and ω_1, ω_2 by the relations

$$\begin{aligned}&\eta_1(z) = R_z\omega_1, \quad \eta_2(z) = R_z\omega_2, \\ &-b_z^{-1} = \alpha^{-1} + \langle (A-z)^{-1}\omega_2, \omega_1\rangle,\end{aligned} \tag{27}$$

where $0 < |\alpha| < \infty$.

For the main idea of the proof of Theorem 2, see in [8, 9].

In general, case $\alpha = 0$ can also be taken into consideration by putting $b_z \equiv 0$ and understanding as $\tilde{R}_z \equiv R_z$. We can also put $|\alpha| = \infty$, then in the last expression in (27) we do not have the first term, i.e., α^{-1}.

For the operator in Definition 2, we can formulate a theorem similar to Theorem 2 but it needs new independent proof.

Let us define an adjoint operator $(A^{\omega_1,\omega_2})^*$ for a given operator A^{ω_1,ω_2}. We will use Theorem 2 and, in particular, the next general but obvious proposition. For this reason, let us introduce the bounded linear operator

$$\tilde{T} := T + b_0(\cdot, \eta_1)\eta_2,$$

where T is a bounded self-adjoint operator defined in the space $\mathcal{H}$ and $b_0 \in \mathbb{C}$, $\eta_1, \eta_2 \in \mathcal{H}_{+1} \subset \mathcal{H}$.

Proposition 4. *For an arbitrary bounded self-adjoint operator T defined everywhere in the space $\mathcal{H}$ and for arbitrary vectors $\eta_1, \eta_2 \in \mathcal{H}_{+1} \subset \mathcal{H}$ and a number $b_0 \in \mathbb{C}$, the operator*

$$(\tilde{T})^* = T + \bar{b_0}(\cdot, \eta_2)\eta_1 \tag{28}$$

is an adjoint operator to $\tilde{T}$.

Proof. For all vectors $f, g \in \mathcal{H}$,

$$(\tilde{T}f, g) = ([Tf + b_0(f, \eta_1)\eta_2], g) = (Tf, g) + b_0(f, \eta_1)(\eta_2, g). \tag{29}$$

On the another hand, we have

$$\begin{aligned} (f, (\tilde{T})^* g) &= (f, [Tg + \bar{b}_0(g, \eta_2)\eta_1]) \\ &= (f, Tg) + (f, \bar{b}_0(g, \eta_2)\eta_1) \\ &= (Tf, g) + b_0\overline{(g, \eta_2)}(f, \eta_1) \\ &= (Tf, g) + b_0(\eta_2, g)(f, \eta_1). \end{aligned} \tag{30}$$

Comparing (29) and (30) we verify (28). □

Using Proposition 4, we can define $(A^{\omega_1,\omega_2})^*$. Let A be a positive self-adjoint operator in the separable Hilbert space $\mathcal{H}$. For $\omega_1, \omega_2 \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $\omega_1 \neq \omega_2$ we put $\eta_i = \mathbf{A}\omega_i$, $i = 1, 2$. The operator A^{ω_2,ω_1} in accordance with the Definition 1 is an operator that is singular non-symmetric rank one perturbed $\mathcal{H}_{-1}$-class with respect to A, if

$$\mathfrak{D}(A^{\omega_2,\omega_1}) = \Big\{\psi = \varphi - b\eta_1 \mid \varphi \in \mathfrak{D}(A), \quad b = b(\varphi) = \frac{(A\varphi, \eta_2)}{1 + (A^{1/2}\eta_1, A^{1/2}\eta_2)}\Big\}$$

in the case $(A^{1/2}\eta_1, A^{1/2}\eta_2) \neq -1$; and

$$\mathfrak{D}(A^{\omega_2,\omega_1}) = \mathfrak{D}^*_{\mathcal{H}_1} \dot{+} \{c\eta_1\}, \quad \mathfrak{D}^*_{\mathcal{H}_1} = \{\varphi \in \mathfrak{D}(A) \mid (A\varphi, \eta_2) = 0\}$$

in the case $(A^{1/2}\eta_1, A^{1/2}\eta_2) = -1$.

The action is the same as in Definition 1,

$$A^{\omega_2,\omega_1}\psi = A\varphi.$$

Analogously to Definition 2, we can define $((A^{\omega_1,\alpha\omega_2})^* - \bar{z})$, $z \in \rho(A)$, which we need for the next consideration. Namely,

$$\mathfrak{D}((A^{\omega_1,\alpha\omega_2})^*) = \Big\{\psi = \varphi - b_z\eta_1(\bar{z}) \mid \varphi \in \mathfrak{D}(A), \quad b_z = \frac{((A - \bar{z})\varphi, \eta_2(z))}{1/\bar{\alpha} + \langle (A - \bar{z})\eta_1(\bar{z}), \eta_2(z)\rangle}\Big\} \tag{31}$$

in the case $((A-\bar z)\eta_1(\bar z),\eta_2(z))\neq -1/\bar\alpha$ for fixed z, $\mathrm{Im}(z)\neq 0$; and

(32) $$\mathfrak{D}((A^{\omega_1,\alpha\omega_2})^*)=\mathfrak{D}^*_{\mathcal{H}_1}\dot+\{c\eta_1(\bar z)\},\quad \mathfrak{D}^*_{\mathcal{H}_1}(z)=\{\varphi\in\mathfrak{D}(A)\mid((A-\bar z)\varphi,\eta_2(z))=0\}$$

in the case $((A-\bar z)\eta_1(\bar z),\eta_2(z))=-1/\bar\alpha$.

The action is defined by

(33) $$((A^{\omega_1,\alpha\omega_2})^*-\bar z)\psi=(A-\bar z)\varphi.$$

The operator adjoint to the Schrödinger operator with non-local interactions. The operator

$$-\Delta^{\alpha\delta_{x_2},\delta_{x_1}}=-\frac{d^2}{dx^2}+\bar\alpha\langle\cdot,\delta_{x_2}\rangle\delta_{x_1},\quad \alpha\in\mathbb{C},$$

is adjoint to the operator (16) and has domain

(34) $$\mathfrak{D}(-\Delta^{\alpha\delta_{x_2},\delta_{x_1}})=\big\{\psi\in W_2^1(\mathbb{R}^1)\cap W_2^2(\mathbb{R}^1\setminus\{x_2\})\mid\psi(x_2+)=\psi(x_2-),\\ \psi'(x_1+)-\psi'(x_1-)=\bar\alpha\psi(x_2)\}$$

and acts as follows: $-\Delta^{\alpha\delta_{x_2},\delta_{x_1}}\psi=-\varphi''$, where φ, ψ are from (34).

Proof. Firstly we consider the case

$$((A-\bar z)\eta_1(\bar z),\eta_2(z))\neq -1/\alpha.$$

In accordance with (31) for $\psi\in\mathfrak{D}(\Delta^{\alpha\delta_{x_2},\delta_{x_1}})$

$$\psi=\varphi-\frac{\varphi(x_2)}{\frac{1}{\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|x-x_1|},$$

where $z=k^2$, $k=\sqrt{\pm i}$, $\mathrm{Im}k>0$, $\eta_j(\bar z)=\frac{i}{2\bar k}e^{i\bar k|x-x_j|}$, $j=1,2$.

In particular, in the sense of generalized functions we have

$$((A-\bar z)\varphi,\eta_2(z))=(\varphi,(A-z)\eta_2(z))=\langle\varphi,\delta_{x_2}\rangle=\varphi(x_2)$$

and

$$\langle(A-\bar z)\eta_1(\bar z),\eta_2(z)\rangle=\langle\eta_1(\bar z),\delta_{x_2}\rangle=\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}.$$

The first equality in (34), namely $\psi(x_2+)=\psi(x_2-)$, is true,

$$\psi(x_2+)-\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|(x_2+)-x_1|}\\ =\psi(x_2-)-\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|(x_2-)-x_1|}.$$

Let us show the second equality in (34). Since

$$\psi'=\varphi'-\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|x-x_1|}i\bar k\theta(x-x_1),$$

where $\theta(x-x_1)$ is the Heaviside function with the jump at the point x_2, we have

(35) $$\begin{aligned}&\psi'(x_1+)-\psi'(x_1-)\\ &\quad=\left\{\varphi'-\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|x-x_1|}i\bar k\theta(x-x_1)\right\}|_{x=x_1+}\\ &\quad-\left\{\varphi'-\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}e^{i\bar k|x-x_1|}i\bar k\theta(x-x_1)\right\}|_{x=x_1-}\\ &\quad=\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}}\;\frac{i}{2\bar k}i\bar k(1-(-1))=\frac{\varphi(x_2)}{\frac{1}{\bar\alpha}+\frac{i}{2\bar k}e^{i\bar k|x_2-x_1|}},\end{aligned}$$

where we use that $\varphi'(x_2+) = \varphi'(x_2-)$, since $\varphi(x) \in W_2^2(\mathbb{R})$.

The left-hand side of the second equality in (34) has the form

$$\begin{aligned}
&\varphi(x_2) - \frac{\varphi(x_2)}{\frac{1}{\bar\alpha} + \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}} \, \frac{i}{2\bar k} e^{i\bar k|x-x_1|} \\
&= \frac{\varphi(x_2) + \varphi(x_2)\frac{i}{2\bar k} e^{i\bar k|x_2-x_1|} - \varphi(x_2)\frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}}{\frac{1}{\bar\alpha} + \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}} \\
&= \frac{1}{\bar\alpha} \frac{\varphi(x_2)}{\frac{1}{\bar\alpha} + \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}}.
\end{aligned} \tag{36}$$

Comparing the expressions (35) and (36), we obtain

$$\psi'(x_1+) - \psi'(x_1-) = \bar\alpha \psi(x_2).$$

Hence we proved the case $-\frac{1}{\alpha} \neq \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}$.

Let us consider the case (32), namely

$$-\frac{1}{\alpha} = ((A-\bar z)\eta_1(\bar z), \eta_2(z)) = \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}. \tag{37}$$

In such a case,

$$\begin{aligned}
\mathfrak{D}^*_{\mathcal{H}_1}(z) &= \left\{\varphi \in W_2^2(\mathbb{R}) \mid ((A-\bar z)\varphi, \eta_2(z)) = 0\right\} \\
&= \left\{\varphi \in W_2^2(\mathbb{R}) \mid \varphi(x_2) = 0\right\}.
\end{aligned}$$

For $h(x) \in \mathfrak{D}^*_{\mathcal{H}_1}(z)$ we have $h(x_1+) = h(x_1-) = h(x_1) = 0$.

For $\eta_1(\bar z) = \frac{i}{2\bar k} e^{i\bar k|x-x_1|}$ we also have

$$e^{i\bar k|(x_2+)-x_1|} = e^{i\bar k|(x_2-)-x_1|}.$$

For $h(x) \in \mathfrak{D}^*_{\mathcal{H}_1}(z)$ we also have $h'(x_2+) = h'(x_2-)$ and $h(x_2) = 0$, i.e., the second equality in (34) holds true.

Let us consider the second equality in (34) for $\eta_2(\bar z) = \frac{i}{2\bar k} e^{i\bar k|x-x_1|}$. The left-hand side has the form

$$\begin{aligned}
\psi'(x_2+) - \psi'(x_2-) &= \frac{i}{2\bar k} e^{i\bar k|x-x_1|} i\bar k \theta(x-x_1)|_{x=x_1+} \\
&\quad - \frac{i}{2\bar k} e^{i\bar k|x-x_1|} i\bar k \theta(x-x_1)|_{x=x_1-} = -\frac{1}{2}(1-(-1)) = -1.
\end{aligned} \tag{38}$$

The right-hand side of the equality (34) is

$$\bar\alpha \eta_1(z)|_{x=x_2} = \bar\alpha \frac{i}{2\bar k} e^{i\bar k|x-x_1|}|_{x=x_2} = \bar\alpha \frac{i}{2\bar k} e^{i\bar k|x_2-x_1|}. \tag{39}$$

Indeed, (38) is equal to (39), if we take into account (37). Hence, (34) for (32) is also true. The action $-\Delta^{\alpha\delta_{x_1},\delta_{x_2}}\psi_2 = -\varphi''$ follows from (33). □

3. Some remarks about singular non-symmetric perturbations of $\mathcal{H}_{-2}$-class

Let us consider an extension of the operator A to the space $\mathcal{H}_{-2}$ as a bounded operator acting from $\mathcal{H}_0$ into $\mathcal{H}_{-2}$, or as an unbounded operator with the domain $\mathcal{H}_0$ in $\mathcal{H}_{-2}$. We denote such an extension in this section also by $\mathbf{A}$. In the scale

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \supset \mathcal{H}_{+1} \supset \mathcal{H}_{+2},$$

we consider the linear operator V from $\mathcal{H}_{+2}$ into $\mathcal{H}_{-2}$. Let it be of the form $V = V^{\omega_1,\omega_2} = \langle\cdot,\omega_1\rangle\omega_2$, $\omega_1,\omega_2 \in \mathcal{H}_{-2}$. Since the operator $\mathbf{A}$ is bounded as acting from $\mathcal{H}_0$ into $\mathcal{H}_{-2}$, we see that $\mathbf{A}+V$ is also a bonded linear operator from $\mathcal{H}_0$ into $\mathcal{H}_{-2}$. The concept of an adjoint operator $(\mathbf{A}+V)^*$ is valid and it acts from $\mathcal{H}_0$ into $\mathcal{H}_{-2}$ [6].

For a formal expression $A+\langle\cdot,\omega_1\rangle\omega_2$ we can give a sense of an operator in $\mathcal{H}$ [10]. For this reason we restrict $\mathbf{A}+\langle\cdot,\omega_1\rangle\omega_2$ to $\mathcal{H}$ and denote it also by A^{ω_1,ω_2},

$$A^{\omega_1,\omega_2} := (\mathbf{A}+\langle\cdot,\omega_1\rangle\omega_2)\restriction_{\mathcal{H}} .$$

If this will not lead to a confusion, we will use the notation A, instead of $\mathbf{A}$.

The next definition for the perturbed operator A^{ω_1,ω_2} of $\mathcal{H}_{-2}$-class is presented in [9].

Definition 3. Let $A>1$ be a positive self-adjoint operator defined in a separable Hilbert space $\mathcal{H}$. For $\omega_1,\omega_2\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$, $\omega_1\neq\omega_2$, either $\omega_1\in\mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$, or $\omega_2\in\mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$, we put $\eta_i=\mathbf{A}^{-1}\omega_i$, $i=1,2$.

The operator A^{ω_1,ω_2} is called an operator singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-2}$-class with respect to A, iff

$$\mathfrak{D}(A^{\omega_1,\omega_2})=\Big\{\psi=\varphi-b\eta_2 \mid \varphi\in\mathfrak{D}(A),\quad b=b(\varphi)=\frac{(A\varphi,\eta_1)}{1+\tau+(A(A^2+1)^{-1}\eta_2,\eta_1)}\Big\} \tag{40}$$

in the case $1+\tau+(A(A^2+1)^{-1}\eta_2,\eta_1)\neq 0$, where $\tau\in\mathbb{C}$ is an arbitrary parameter; and

$$\mathfrak{D}(A^{\omega_1,\omega_2})=\mathfrak{D}_{\mathcal{H}_2}\dot{+}\{c\eta_2\},\quad \mathfrak{D}_{\mathcal{H}_2}=\{\varphi\in\mathfrak{D}(A)\mid (A\varphi,\eta_1)=0\}, \tag{41}$$

($c\in\mathbb{C}$) in the case $1+\tau+(A(A^2+1)^{-1}\eta_2,\eta_1)=0$, (and it is denoted by $A^{\omega_1,\omega_2}\in\mathcal{P}_\tau(A)$).

The action is given by $A^{\omega_1,\omega_2}\psi=A\varphi$.

The operator A is also called (initial) non-perturbed, and $V=\langle\cdot,\omega_1\rangle\omega_2$ is called a perturbation (of $\mathcal{H}_{-2}$-class). Hence, A^{ω_1,ω_2} is naturally called a perturbed operator of $\mathcal{H}_{-2}$-class, or an operator that needs an additional parameterization.

The proposed definition generalizes the case of non-local interactions (of $\mathcal{H}_{-2}$-class) [14, 15] for self-adjoint operators to the case of non-symmetric perturbations.

Next we consider V in the form $V=\alpha\langle\cdot,\omega_1\rangle\omega_2$, with a constant $\alpha\in\mathbb{C}$, $0<|\alpha|<\infty$, that has no difference from the previous one due to (13). In particular, we have $A^{\bar{\alpha}\omega_1,\omega_2}=A^{\omega_1,\alpha\omega_2}$ too. Therefore, the Definition 3 has the following form.

Definition 4. Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. For $\omega_1,\omega_2\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$, $\omega_1\neq\omega_2$, either $\omega_1\in\mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$, or $\omega_2\in\mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$, we put $\eta_k(z)=(\mathbf{A}-z)^{-1}\omega_k$, $k=1,2$, $z\in\rho(\mathbf{A})$.

The operator $A^{\omega_1,\alpha\omega_2}$ is called an operator singularly non-symmetrically rank-one perturbed of $\mathcal{H}_{-2}$-class with respect to the A iff

$$\mathfrak{D}(A^{\omega_1,\alpha\omega_2})=\Big\{\psi=\varphi-b_z\eta_2(z)\mid\varphi\in\mathfrak{D}(A),\quad b_z=b_z(\varphi)=\frac{((A-z)\varphi,\eta_1(\bar z))}{1/\alpha+\tau+((A-z)(1+zA)(A^2+1)^{-1}\eta_2(z),\eta_1(\bar z))}\Big\} \tag{42}$$

in the case $((A-z)(1+zA)(A^2+1)^{-1}\eta_2(z),\eta_1(\bar z))+1/\alpha+\tau\neq 0$ for a fixed z, where $\tau\in\mathbb{C}$ is a parameter; and

$$\mathfrak{D}(A^{\omega_1,\alpha\omega_2})=\mathfrak{D}_{\mathcal{H}_2}\dot{+}\{c\eta_2(z)\},\quad \mathfrak{D}_{\mathcal{H}_2}=\{\varphi\in\mathfrak{D}(A)\mid\langle(A-z)\varphi,\eta_1(\bar z)\rangle=0\} \tag{43}$$

($c\in\mathbb{C}$), in the case $((A-z)(1+zA)(A^2+1)^{-1}\eta_2(z),\eta_1(\bar z))+1/\alpha+\tau=0$, (and it is also denoted by $A^{\omega_1,\alpha\omega_2}\in\mathcal{P}_\tau(A)$).

The action is given by the rule $(A^{\omega_1,\alpha\omega_2}-z)\psi=(A-z)\varphi$.

Definition 3 has also a limitation, i.e., it is formulated for a positive (semibounded) operator. Definition 4 removes this flaw but the domains in (42) and (43) depend on z.

The next theorem (as Theorem 2) gives a description of non-symmetrically singular $\mathcal{H}_{-2}$-class perturbations of self-adjoint operators in terms of their resolvents.

Theorem 3. *Let $A > 1$ be a positive self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and $\tilde{A}$ be an operator singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-2}$-class with respect to A defined in the Definition 4.*

The resolvents $R_z = (A - z)^{-1}$ and $\tilde{R}_z = (\tilde{A} - z)^{-1}$ satisfy a M. Krein type formula,

$$\tilde{R}_z = R_z + b_z(\cdot, \eta_1(\bar{z}))\eta_2(z), \quad z, \xi \in \rho(A) \cap \rho(\tilde{A}), \tag{44}$$

with the vector-valued functions

$$\eta_1(z) = (A - \xi)(A - z)^{-1}\eta_1(\xi), \quad \eta_2(z) = (A - \xi)(A - z)^{-1}\eta_2(\xi), \tag{45}$$

where $\eta_1(z), \eta_2(z) \in \mathcal{H}$ and

$$-b_z^{-1} = \alpha^{-1} + \tau + ((A - z)(1 + zA)(A^2 + 1)^{-1}\eta_2(z), \eta_1(z)), \tag{46}$$

where $\alpha \in \mathbb{C}$, $0 < |\alpha| < \infty$ and $\forall \tau \in \mathbb{C}$; the scalar-valued function for which satisfies the equality

$$b_z^{-1} - b_\xi^{-1} = (\xi - z)(\eta_2(\xi), \eta_1(\bar{z})). \tag{47}$$

The vectors $\eta_1(z), \eta_2(z)$ are connected with ω_1, ω_2 by the relations

$$\eta_1(z) = R_z\omega_1, \;\; \eta_2(z) = R_z\omega_2, \tag{48}$$

In general, the case $\alpha = 0$ we can also be considered by putting $b_z \equiv 0$ and supposing $\tilde{R}_z \equiv R_z$. We can also put $|\alpha| = \infty$. In such a case, the first term, i.e., α^{-1}, in the equality (46) must be absent.

The proof of Theorem 3 (as a part) contains the proof the fact that the domains in (42) and (43) are in fact independent on z.

Analogously we can also define $(A^{\omega_1,\omega_2})^*$ of $\mathcal{H}_{-2}$-class. Let $A > 1$ be a positive self-adjoint operator in a separable Hilbert space $\mathcal{H}$. For $\omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}_0$, $\omega_1 \neq \omega_2$, either $\omega_1 \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, or $\omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, we put $\eta_i = \mathbf{A}\omega_i$, $i = 1, 2$. The operator A^{ω_2,ω_1}, due to Definition 3, is an operator singularly non-symmetrically rank one perturbed of $\mathcal{H}_{-2}$-class with respect A iff

$$\begin{aligned} \mathfrak{D}(A^{\omega_2,\omega_1}) = \Big\{ \psi = \varphi - b\eta_1 \mid \varphi \in \mathfrak{D}(A), \\ b = b(\varphi) = \frac{(A\varphi, \eta_2)}{1 + \tau + (A(A^2 + 1)^{-1}\eta_1, \eta_2)} \Big\} \end{aligned} \tag{49}$$

in the case $1 + \tau + (A(A^2 + 1)^{-1}\eta_1, \eta_2) \neq 0$, where $\tau \in \mathbb{C}$ is a parameter; and

$$\mathfrak{D}(A^{\omega_2,\omega_1}) = \mathfrak{D}^*_{\mathcal{H}_2} \dot{+} \{c\eta_1\}, \quad \mathfrak{D}^*_{\mathcal{H}_2} = \{\varphi \in \mathfrak{D}(A) \mid (A\varphi, \eta_2) = 0\} \tag{50}$$

in the case $1 + \tau + (A(A^2 + 1)^{-1}\eta_1, \eta_2) = 0$.

The action is given by $A^{\omega_2,\omega_1}\psi = A\varphi$, where φ, ψ form (49) and (50).

Proposition 5. *The operator A^{ω_1,ω_2}, defined in Definition 3, is adjoint to the operator A^{ω_2,ω_1} defined also using Definition 3 by expressions (49) and (50).*

Proof. The case $1 + \tau + (A(A^2 + 1)^{-1}\eta_1, \eta_2) \neq 0$ is fulfilled.

Indeed, for A^{ω_1,ω_2}, we have

$$(A^{\omega_1,\omega_2})^{-1} = A^{-1} - \frac{1}{1 + \tau + (A(A^2 + 1)^{-1}\eta_2, \eta_1)}(\cdot, \eta_1)\eta_2 \tag{51}$$

with a parameter $\tau \in \mathbb{C}$.

And for A^{ω_2,ω_1} we have

$$(A^{\omega_2,\omega_1})^{-1} = A^{-1} - \frac{1}{1 + \bar{\tau} + (A(A^2 + 1)^{-1}\eta_1, \eta_2)}(\cdot, \eta_2)\eta_1, \tag{52}$$

where we put the parameter $\bar{\tau}$.

Comparing (51) and (52) we get $((A^{\omega_1,\omega_2})^{-1})^* = (A^{\omega_2,\omega_1})^{-1}$, since

$$\overline{(A(A^2+1)^{-1}\eta_2,\eta_1)} = \overline{(\eta_2, A(A^2+1)^{-1}\eta_1)} = (A(A^2+1)^{-1}\eta_1,\eta_2).$$

Let us also show the case

$$1+\tau+(A(A^2+1)^{-1}\eta_2,\eta_1) = 0. \tag{53}$$

Let $\psi_2 = \varphi_0 \dot{+} c_2\eta_2 \in \mathfrak{D}(A^{\omega_1,\omega_2})$, defined in (41), and $\psi_2^* = \varphi_0^* \dot{+} c_1\eta_1 \in \mathfrak{D}(A^{\omega_2,\omega_1})$ defined in (50). Then

$$(A^{\omega_1,\omega_2}\psi_2,\psi_2^*) = (A\varphi_0, \psi_0^* \dot{+} c_1\eta_1) = (A\varphi_0,\varphi_0^*) + c_1(A\varphi_0,\eta_1),$$

$$(A^{\omega_2,\omega_1}\psi_2^*,\psi_2) = (A\varphi_0^*, \varphi_0 \dot{+} c_2\eta_2) = (A\varphi_0^*,\varphi_0^*) + c_2(A\varphi_0^*,\eta_2).$$

From the last two expression we have

$$(A^{\omega_1,\omega_2}\psi_2,\psi_2^*) = \overline{(A^{\omega_2,\omega_1}\psi_2^*,\psi_2)} = (\psi_2, A^{\omega_2,\omega_1}\psi_2^*),$$

because of

$$(A\varphi_0,\varphi_2^*) = \overline{(A\varphi_0^*,\varphi_0)} = (\varphi_0, A\varphi_0^*)$$

and $(A\varphi_0,\eta_1) = 0$ for ψ defined in (29) and $(A\varphi_0,\eta_2) = 0$ for ψ^* defined in (50).

The case where, for example, ψ_1 is defined in (40) and ψ_2 is from (50) (or conversely, ψ_1 from (41) and ψ_2 from (49)) is impossible due to (53)). □

By analogy with Definition 4 we define $((A^{\omega_1,\alpha\omega_2})^* - \bar{z})$. Namely,

$$\mathfrak{D}((A^{\omega_1,\alpha\omega_2})^*) = \Big\{\psi = \varphi - b_z\eta_1(\bar{z}) \mid \varphi\in\mathfrak{D}(A), \quad b_z = \frac{((A-\bar{z})\varphi,\eta_2(z))}{1/\bar{\alpha}+\bar{\tau}+((A-\bar{z})(1+\bar{z}A)(A^2+1)^{-1}\eta_1(\bar{z}),\eta_2(z))}\Big\} \tag{54}$$

in the case $1/\bar{\alpha}+\bar{\tau}+((A-\bar{z})(1+\bar{z}A)(A^2+1)^{-1}\eta_1(\bar{z}),\eta_2(z)) \neq 0$ for a fixed z; and

$$\mathfrak{D}((A^{\omega_1,\alpha\omega_2})^*) = \mathfrak{D}^*_{\mathcal{H}_2}\dot{+}\{c\eta_1(\bar{z})\}, \quad \mathfrak{D}^*_{\mathcal{H}_2}(z) = \{\varphi\in\mathfrak{D}(A) \mid ((A-\bar{z})\varphi,\eta_2(z)) = 0\} \tag{55}$$

in the case $1/\bar{\alpha}+\bar{\tau}+((A-\bar{z})(1+\bar{z}A)(A^2+1)^{-1}\eta_1(\bar{z}),\eta_2(z)) = 0$.

The action is given by $((A^{\omega_1,\alpha\omega_2})^* - \bar{z})\psi = (A-\bar{z})\varphi$, where φ, ψ from (54) and (55).

4. A spacial case of singular non-symmetric perturbations of $\mathcal{H}_{-2}$-class

Let us compare the coefficient b in (11) and in (40). If $\omega_1,\omega_2\in\mathcal{H}_{-1}$, i.e., $\eta_1,\eta_2\in\mathcal{H}_{+1}$, then $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ is a well defined expression, and we have an $\mathcal{H}_{-1}$-class perturbation.

If $\omega_1,\omega_2\in\mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$, i.e., $\eta_1,\eta_2\in\mathcal{H}_0\setminus\mathcal{H}_{+1}$, then $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ is not well defined. In such a case, we consider $(A(A^2+1)^{-1}\eta_2,\eta_1)$ instead of $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ and we have an $\mathcal{H}_{-2}$-class perturbation.

But in the important Example 1 we meet a situation in which $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ at first glance exists and in spite of $\eta_1,\eta_2\notin\mathcal{H}_0$. It was possible because the scalar product $(\cdot,\cdot) = \int_1^\infty \cdot,\cdot\, dx$ has a "natural" extension (9), i.e., we meet the existence of the corresponding integral.

At the end of the article we propose some typical (but not general) variant which allows us to consider b in (11) instead of in (40), but with $\eta_1,\eta_2\notin\mathcal{H}_0$.

In the spirit of the first section, we extend the scalar product $(\cdot,\cdot)$ to the case (f,g), when $f\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$, $g\in\mathcal{H}_0\setminus\mathcal{H}_{+1}$, $(f\neq g)$. For example, if we can decompose the vectors $f = f_1+f_2$ and $g = g_1+g_2$ such that $\operatorname{spsupp}(f_i)\subseteq F_i\subset\mathbb{R}$, $\operatorname{spsupp}(g_i)\subseteq G_i\subset\mathbb{R}$, $i=1,2$; $F_1\cap F_2=\emptyset$, $G_1\cap G_2=\emptyset$, $F_2=G_2$ and $f_1\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$, $g_1\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$, but $f_2,g_2\in\mathcal{H}_0$.

In such a case it would be natural to put $(f_1,g_1) =: 0$ and (f,g) understood as

$$(f,g) = (f_1+f_2, g_1+g_2) = (f_1,g_1)+(f_1,g_2)+(f_2,g_1)+(f_2,g_2) = (f_2,g_2).$$

In the last considerations by spsupp$(\cdot)$ we denote the spectral support (of a vector) with respect to the corresponding operator A. Let us remark that each extension of a functional depends on some self-adjoint operator A.

Example 1 show that the described above version of an extension of a scalar product is not necessarily typical and unique. Due to this remark we can put $(f_1, g_1) =: c$, with an arbitrary constant $c \in \mathbb{R} \setminus \{0, \infty\}$ without loss of correctness.

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National Technical University of Ukraine (KPI), 37 Peremogy Av., Kyiv, 03056, Ukraine
E-mail address: dudkin@imath.kiev.ua

National Technical University of Ukraine (KPI), 37 Peremogy Av., Kyiv, 03056, Ukraine
E-mail address: tanyavdovenko@meta.ua

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