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QUASI-INVARIANCE OF COMPLETELY RANDOM MEASURES

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Abstract. Let X be a locally compact Polish space. Let $\mathbb{K}(X)$ denote the space of discrete Radon measures on X. Let μ be a completely random discrete measure on X, i.e., μ is (the distribution of) a completely random measure on X that is concentrated on $\mathbb{K}(X)$. We consider the multiplicative (current) group $C_0(X \to \mathbb{R}_+)$ consisting of functions on X that take values in $\mathbb{R}_+ = (0, \infty)$ and are equal to 1 outside a compact set. Each element $\theta \in C_0(X \to \mathbb{R}_+)$ maps $\mathbb{K}(X)$ onto itself; more precisely, θ sends a discrete Radon measure $\sum_i s_i \delta_{x_i}$ to $\sum_i \theta(s_i) s_i \delta_{x_i}$. Thus, elements of $C_0(X \to \mathbb{R}_+)$ transform the weights of discrete Radon measures. We study conditions under which the measure μ is quasi-invariant under the action of the current group $C_0(X \to \mathbb{R}_+)$ and consider several classes of examples. We further assume that $X = \mathbb{R}^d$ and consider the group of local diffeomorphisms $\mathrm{Diff}_0(X)$. Elements of this group also map $\mathbb{K}(X)$ onto itself. More precisely, a diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ sends a discrete Radon measure $\sum_i s_i \delta_{x_i}$ to $\sum_i s_i \delta_{\varphi(x_i)}$. Thus, diffeomorphisms from $\mathrm{Diff}_0(X)$ transform the atoms of discrete Radon measures. We study quasi-invariance of μ under the action of $\mathrm{Diff}_0(X)$. We finally consider the semidirect product $\mathfrak{G} := \mathrm{Diff}_0(X) \times C_0(X \to \mathbb{R}_+)$ and study conditions of quasi-invariance and partial quasi-invariance of μ under the action of $\mathfrak{G}$.

1. Introduction

Let P be a probability measure on a sample space Ω and let $\mathfrak{G}$ be a group acting on Ω. A fundamental question of the representation theory is whether the probability measure P is quasi-invariant with respect to this action. The latter means that, for each element $g \in \mathfrak{G}$, the pushforward of P under g, denoted by P^g, is equivalent to the measure P, so that the Radon–Nikodym density $\frac{dP^g}{dP}$ exists and is strictly positive P-a.e. If this holds, one can construct a unitary representation of the group $\mathfrak{G}$ in $L^2(\Omega, P)$. To this end, for each $g \in \mathfrak{G}$, one defines a unitary operator U_g in $L^2(\Omega, P)$ by

$$(U_g f)(\omega) = f(g^{-1}\omega)\sqrt{\frac{dP^g}{dP}(\omega)}.$$

Such a representation of $\mathfrak{G}$ is sometimes called quasi-regular.

In the case where the group $\mathfrak{G}$ is big, the problem of quasi-invariance of P with respect to the action of $\mathfrak{G}$ may be very difficult.

Let us consider an important example of such a construction. Let $X = \mathbb{R}^d$ and let dx be the Lebesgue measure on X. Denote by $\Omega = \Gamma(X)$ the space of locally finite subsets of X (configurations). Let $P = \pi_z$ be the Poisson measure on X with intensity measure $z\,dx$, where $z > 0$ is a fixed constant. Let $\mathfrak{G} = \mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X which are equal to the identity outside a compact set. Elements of $\varphi \in \mathrm{Diff}_0(X)$ naturally act on $\Gamma(X)$ by moving each point of the configuration. The measure π_z appears to be quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$. In

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particular, for each $\varphi \in \mathrm{Diff}_0(X)$, the Radon–Nikodym derivative is given by

$$\frac{d\pi_z^\varphi}{d\pi_z}(\gamma) = \prod_{x\in\gamma} J_\varphi(x), \quad \gamma \in \Gamma(X).$$

Here J_φ is the modulus of the determinant of the Jacobian matrix of φ. As a result, we construct a unitary representation of $\mathrm{Diff}_0(X)$ in $L^2(\Gamma(X), \pi_z)$.

The problem of representations of the group of diffeomorphisms of a smooth (noncompact) Riemannian manifold X in the L^2-space with respect to a Poisson measure is a classical one. The fundamental paper [28] by Vershik, Gel'fand, and Graev is a standard reference here.

Let us note that representations of the semidirect product of the additive group $C^\infty(X)$ and $\mathrm{Diff}_0(X)$ in $L^2(\Gamma(X), P)$ are important for nonrelativistic quantum mechanics, see e.g. [10, 11, 12] and the references therein. Here P is a probability measure on the configuration space $\Gamma(X)$, in particular, P can be a Poisson measure.

The representations of the diffeomorphism group $\mathrm{Diff}_0(X)$ in the L^2-space with respect to a Poisson measure naturally led Albeverio, Kondratiev and Röckner [5, 6] to defining elements of differential geometry on the configuration space $\Gamma(X)$ (directional derivative, gradient, tangent space), and developing related analysis on the configuration space equipped with Poisson measure, or more generally, with a Gibbs measure (the Laplace operator, the heat semigroup), and studying the corresponding stochastic processes (Brownian motions) on the configuration space, see also [18, 23, 7, 19]. Laplace operators on the differential forms over the configuration space $\Gamma(X)$ equipped with Poisson measure (and more generally, with a Gibbs measure) were studied by Albeverio, Daletskii and Lytvynov in [1, 2, 3].

Tsilevich, Vershik, and Yor [27] studied quasi-invariance of the gamma measure with respect to the action of the multiplicative group $C_0(X \to \mathbb{R}_+)$. This group consists of functions on X which take values in $\mathbb{R}_+$ and are equal to 1 outside a compact set. The gamma measure is a random measure on X; it belongs to the class of measure-valued Lévy processes. This random measure takes almost surely values in the space $\mathbb{K}(X)$ of discrete Radon measures on X. The latter space consists of Radon measures of the form $\sum_i s_i\delta_{x_i}$, where $s_i > 0$ and δ_{x_i} is the Dirac measure with mass at x_i. Each element $\theta \in C_0(X \to \mathbb{R}_+)$ maps $\mathbb{K}(X)$ onto itself; more precisely, θ sends the discrete Radon measure $\sum_i s_i\delta_{x_i}$ to $\sum_i \theta(s_i)s_i\delta_{x_i}$. The (distribution of) the gamma measure appears to be quasi-invariant under the action of $C_0(X \to \mathbb{R}_+)$.

One can naturally define the semidirect product $\mathfrak{G}$ of the diffeomorphism group $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$. This group consists of all pairs $(\varphi, \theta) \in \mathrm{Diff}_0(X) \times C_0(X \to \mathbb{R}_+)$ and it naturally acts on the space of discrete Radon measures, $\mathbb{K}(X)$: for each $\sum_i s_i\delta_{x_i} \in \mathbb{K}(X)$, its image under the action of (φ, θ) is equal to $\sum_i \theta(\varphi(x_i))s_i\delta_{\varphi(x_i)}$. However, it appears that, if the underlying space X is not compact, the gamma measure is not quasi-invariant with respect to the action of $\mathfrak{G}$. Kondratiev, Lytvynov, and Vershik [21] suggested the notion of partial quasi-invariance and proved that the gamma measure, and more generally, a class of measure-valued Lévy processes, are partially quasi-invariant with respect to the action of $\mathfrak{G}$. The main point of this definition is that, despite absence of quasi-invariant, one can still derive analysis and geometry on space $\mathbb{K}(X)$ equipped with such a measure. One can again construct a gradient, a tangent space, and an associated Laplace operator on $\mathbb{K}(X)$, see [21]. Markov processes on $\mathbb{K}(X)$ which correspond to these Laplace operators are constructed by Conache, Kondratiev, and Lytvynov [8].

Measure-valued Lévy processes form a subclass of completely random measures. A completely random measure [15, 16, 17] is a random measure on X whose values are independent on mutually disjoint sets. We will actually deal with the important class of completely random measures which are discrete Radon measures, i.e., their distribution, μ, is a probability measure on $\mathbb{K}(X)$.

The main problem we solve in this paper is: Under which conditions is a completely random discrete measure μ quasi-invariant, or partially quasi-invariant with respect to the action of $\mathfrak{G}$, the semidirect product of the diffeomorphism group $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$? Our results here extend the related results of [21]. We also refer to the papers [20, 4] which discuss quasi-invariance of a compound Poisson process with respect to the action of the group $\mathfrak{G}$, or its generalization where $\mathbb{R}_+$ is replaced with a Lie group. Also the results on quasi-invariance of the gamma measure with respect to the action of $C_0(X \to \mathbb{R}_+)$ were extended to Poisson processes on $X \times \mathbb{R}_+$ by Lifshifts and Shmileva [22]. (Note that the problem of quasi-invariance of a completely random discrete measure is related to the problem of quasi-invariance of the Poisson process on $X \times \mathbb{R}_+$.)

The paper is organized as follows. In Section 2, we recall the main notions related to completely random measures. We fix a locally compact Polish space X with its Borel σ-algebra $\mathcal{B}(X)$. We denote by $\mathbb{M}(X)$ the set of all Radon measures on X, and the Borel σ-algebra on $\mathbb{M}(X)$ is denoted by $\mathcal{B}(\mathbb{M}(X))$. We define a random measure as a measurable mapping from a probability space that takes values in $\mathbb{M}(X)$. Since we are only interested in the distribution of such a mapping, we agree to call any probability measure μ on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$ a random measure. We define the configuration space $\Gamma(X)$ as a subset of $\mathbb{M}(X)$, and we define a (simple) point process as a random measure which is concentrated on $\Gamma(X)$. We further recall the notion of the Poisson point process π_σ with intensity measure σ. Here σ is a non-atomic Radon measure on $(X, \mathcal{B}(X))$. We discuss the classical result about equivalence of two Poisson point processes, π_ρ and π_λ, [25, 26, 22]. We also discuss the notion and construction of a completely random measure [16]. We finally define a completely random discrete measure as a completely random measure which is concentrated on $\mathbb{K}(X)$.

The results of the paper are in Sections 3–5. Here we study quasi-invariance of completely random discrete measures.

In Section 3, we assume that X is a locally compact Polish space, and we present sufficient conditions for a completely random discrete measure to be quasi-invariant under the action of the group $C_0(X \to \mathbb{R}_+)$ onto $\mathbb{K}(X)$ (transformations of weights of Radon measures).

Note that, for measure-valued Lévy processes, several conditions of their quasi-invariance under the action of the group $C_0(X \to \mathbb{R}_+)$ onto $\mathbb{K}(X)$ were derived in [21]. For a measure-valued Lévy process, its Lévy measure is a measure on $X \times \mathbb{R}_+$ which is a product measure: $dm(x, s) = d\sigma(x)\, d\nu(s)$, where σ is a reference measure on X, while the Lévy process is determined by the measure ν on $\mathbb{R}_+$. However, for a general completely random measure, its Lévy measure m does not have anymore the product structure. This creates technical difficulties when discussing their quasi-invariance. So in Section 3 we overcome these problems and present a number of criteria of the quasi-invariance of general completely random measures.

We also consider three classes of examples of application of these results. We first discuss quasi-invariance of a completely random gamma measure. The latter random measure has the property that its Lévy measure is a measure on $X \times \mathbb{R}_+$ of the form

$$dm(x, s) = \beta(x)\, \frac{e^{-s/\alpha(x)}}{s}\, d\sigma(x)\, ds,$$

where σ is a fixed nonatomic Radon measure on X (typically $d\sigma(x) = dx$ if $X = \mathbb{R}^d$) and $\alpha : X \to \mathbb{R}_+$ and $\beta : X \to [0, \infty)$ are measurable functions satisfying certain conditions.

Next, we consider a class of completely random measures whose Lévy measure is such that, for small values of s,

$$dm(x,s) = \beta(x)(-\log s)^{\alpha(x)}\, d\sigma(x)\, ds,$$

where $\alpha, \beta : X \to \mathbb{R}_+$ are measurable functions satisfying certain conditions.

And finally, we consider a class of completely random measures whose Lévy measure is such that, for small values of s,

$$dm(x,s) = \beta(x) s^{1-\alpha(x)}\, d\sigma(x)\, ds,$$

where $\alpha : X \to (0,1)$ and $\beta : X \to \mathbb{R}_+$ are measurable functions satisfying certain conditions.

In Section 4, we assume that $X = \mathbb{R}^d$ and we present sufficient conditions for a completely random discrete measure to be quasi-invariant under the action of the diffeomorphism group $\mathrm{Diff}_0(X)$ onto $\mathbb{K}(X)$ (transformations of atoms of Radon measures). We also consider applications of these results to the the three classes of examples we mentioned above.

Finally, in Section 5, we discuss quasi-invariance and partial quasi-invariance of a completely random discrete measure under the action of the semidirect product $\mathfrak{G}$ of the groups $C_0(X \to \mathbb{R}_+)$ and $\mathrm{Diff}_0(X)$ onto $\mathbb{K}(X)$ (transformations of both weights and atoms of Radon measures), and we also consider examples.

2. Completely random measures

Let X be a locally compact Polish space, and let $\mathcal{B}(X)$ denote the Borel σ-algebra on X. A measure η on $(X, \mathcal{B}(X))$ is called a *Radon measure* if $\eta(\Lambda) < \infty$ for any compact $\Lambda \subset X$. We denote by $\mathbb{M}(X)$ the set of all Radon measures on X. One defines the *vague topology on* $\mathbb{M}(X)$ as the weakest topology on $\mathbb{M}(X)$ with respect to which any mapping of the following form is continuous:

$$\mathbb{M}(X) \ni \eta \mapsto \int_X f\, d\eta =: \langle f, \eta\rangle \in \mathbb{R}. \tag{1}$$

Here $f \in C_0(X)$, i.e., f is a continuous function $f : X \to \mathbb{R}$ with compact support. We denote by $\mathcal{B}(\mathbb{M}(X))$ the Borel σ-algebra on $\mathbb{M}(X)$.

Remark 1. There is another way of characterization of $\mathcal{B}(\mathbb{M}(X))$. We denote by $\mathcal{B}_0(X)$ the collection of all sets from $\mathcal{B}(X)$ which have compact closure. Then one can show (see e.g. [15]) that $\mathcal{B}(\mathbb{M}(X))$ is the minimal σ-algebra on $\mathbb{M}(X)$ with respect to which every mapping of the following form is measurable:

$$\mathbb{M}(X) \ni \eta \mapsto \eta(\Lambda) = \langle \chi_\Lambda, \eta\rangle \in \mathbb{R},$$

for each $\Lambda \in \mathcal{B}_0(X)$. Here χ_Λ denotes the indicator function of Λ.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable mapping $\xi : \Omega \to \mathbb{M}(X)$ is called a *random measure*. In most cases, we will only be interested in the distribution of a random measure on $\mathbb{M}(X)$. This is why we will often think of a random measure as a probability measure μ on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$. In the latter case, $(\Omega, \mathcal{F}, P) = (\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)), \mu)$ and the mapping ξ is just the identity.

Next, we will discuss a special subset of the set of random measures known as (simple) point processes. The *configuration space over* X is defined by

$$\Gamma(X) := \{\gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here, for a set A, $|A|$ denotes the cardinality of A. Elements γ of $\Gamma(X)$ are called *configurations in* X. One identifies a configuration $\gamma \in \Gamma(X)$ with the measure $\sum_{x\in\gamma} \delta_x$. Here δ_x is the Dirac measure with mass at x. Since a configuration γ contains a finite number of points in each compact set, the measure $\sum_{x\in\gamma} \delta_x$ is Radon. Hence, in the sense of this identification, we get the inclusion $\Gamma(X) \subset \mathbb{M}(X)$.

On $\Gamma(X)$ one defines the *vague topology* as the trace of the vague topology on $\mathbb{M}(X)$. That is, the vague topology on $\Gamma(X)$ is the weakest topology on $\Gamma(X)$ with respect to which every mapping of the following form is continuous:

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle = \sum_{x \in \gamma} f(x) \in \mathbb{R},$$

where $f \in C_0(X)$. One denotes by $\mathcal{B}(\Gamma(X))$ the corresponding Borel σ-algebra on $\Gamma(X)$. One can show that $\Gamma(X) \in \mathcal{B}(\mathbb{M}(X))$ and $\mathcal{B}(\Gamma(X))$ is the trace σ-algebra of $\mathcal{B}(\mathbb{M}(X))$ on $\Gamma(X)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable mapping $\gamma : \Omega \to \Gamma(X)$ is called a *(simple) point process*. In particular, a point process is a random measure. Similarly to the case of random measures, we will often understand by a point process a probability measure μ on $(\Gamma(X), \mathcal{B}(\Gamma(X)))$.

Let σ be a Radon measure on $(X, \mathcal{B}(X))$ and let us assume that σ is nonatomic, i.e., $\sigma(\{x\}) = 0$ for every $x \in X$. A *Poisson point process with intensity measure* σ is defined as the unique probability measure π_σ on $\Gamma(X)$ which has Fourier transform

$$\int_{\Gamma(X)} e^{i\langle f, \gamma \rangle} \, d\pi_\sigma(\gamma) = \exp \left[\int_X (e^{if(x)} - 1) d\sigma(x) \right] \tag{2}$$

for all $f \in C_0(X)$. See e.g.[17] for further details.

Let ρ and λ be non-atomic Radon measures on X. Then we can construct Poisson point processes (or Poisson measures) on $\Gamma(X)$ with intensity ρ and λ, respectively, denoted by π_ρ and π_λ. Now, the following question arises: When are these measures equivalent, i.e. when is π_ρ equivalent to π_λ? The theorem below follows from Skorohod's result [25], from its extension by Takahashi [26] to the case of a rather general underlying space, and from Lifshits and Shmileva's result [22, Theorem 2].

Theorem 2. *Let X be a locally compact Polish space. Let ρ and λ be non-atomic Radon measures on $(X, \mathcal{B}(X))$. The Poisson measures π_ρ and π_λ are equivalent if and only if*

(1) *ρ and λ are equivalent;*
(2) *if density $\phi := \frac{d\rho}{d\lambda}$, then*

$$\int_X \left(\sqrt{\phi} - 1 \right)^2 d\lambda < \infty. \tag{3}$$

In the latter case,

$$\begin{aligned} \frac{d\pi_\rho}{d\pi_\lambda} = \exp \bigg[& (\lambda - \rho)(X \setminus A) + \int_{X \setminus A} \log \phi \, d\lambda \\ & + \int_A (\log \phi - \phi + 1) d\lambda + \int_A \log \phi \, d(\gamma - \lambda) \bigg], \end{aligned} \tag{4}$$

where $A := \left\{ x \in X \mid |1 - \phi(x)| < \frac{1}{2} \right\}$.

Remark 3. As easily seen, if we assume that

$$\phi - 1 \in L^1(X, \lambda), \tag{5}$$

then condition (3) holds as well, i.e., (5) implies (3).

As we see from (4), the density $\frac{d\pi_\rho}{d\pi_\lambda}$ has a rather complicated form. This is why we will not use Theorem 2 in this paper. Instead, we will use the following stronger condition on ϕ to get a much simpler form of $\frac{d\pi_\rho}{d\pi_\lambda}$. The following theorem is taken from Takahashi [26]. (In fact, Theorem 4 is used to prove Theorem 2 in [26]).

Theorem 4. *Let X be a locally compact Polish space. Let ρ and λ be non-atomic Radon measures on $(X, \mathcal{B}(X))$. Assume λ and ρ are equivalent and denote the density $\phi := \frac{d\rho}{d\lambda}$.*

Assume that condition (5) *holds. Then* π_ρ *and* π_λ *are equivalent and*

$$\frac{d\pi_\rho}{d\pi_\lambda}(\gamma) = \exp\left[\langle\gamma, \log\phi\rangle + \int_X (1-\phi)\, d\lambda\right], \tag{6}$$

where $|\log\phi| \in L^1(X, d\gamma)$ *for* π_λ*-a.a.* $\gamma \in \Gamma(X)$.

Remark 5. Note that, in formula (6), $\exp[\langle\gamma, \log\phi\rangle] = \prod_{x\in\gamma}\phi(x)$, where the infinite product converges.

Let us now recall the definition of a completely random measure, given by Kingman [16]. A *completely random measure on* X is defined as a random measure ξ on X such that, for any mutually disjoint sets $A_1, \dots, A_n \in \mathcal{B}_0(X)$ ($n \in \mathbb{N}$, $n \geq 2$), the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent.

The following result is obtained by Kingman [16]. Below we will use the notation $\mathbb{R}_+ := (0, \infty)$.

Theorem 6. (i) *Let* $\xi_d \in \mathbb{M}(X)$ *be a nonatomic Radon measure. Let a set* $\{x_n\}_{n\geq 1} \subset X$ *be at most countable. Let* $(a_n)_{n\geq 1}$ *be a collection of independent, nonnegative-valued random variables such that*

$$\text{for each } A \in \mathcal{B}_0(X){:}\quad \xi_a(A) := \sum_{n=1}^{\infty} a_n \delta_{x_n}(A) < \infty \quad \text{a.s.} \tag{7}$$

Let m *be a measure on* $X \times \mathbb{R}_+$ *such that*

$$m(\{x\} \times \mathbb{R}_+) = 0 \quad \text{for each } x \in X, \tag{8}$$

and

$$\int_{A\times\mathbb{R}_+} \min\{s, 1\}\, dm(x, s) < +\infty \quad \text{for each } A \in \mathcal{B}_0(X) \tag{9}$$

Let N *be a Poisson point process on* $X \times \mathbb{R}_+$ *with intensity measure* m. *Assume that* N *is independent of the random variables* $(a_n)_{n\geq 1}$. *Define a random measure*

$$\xi_r(A) := \int_{A\times\mathbb{R}_+} s\, dN(x, s).$$

Then ξ_d, ξ_a, ξ_r *are independent, completely random measures on* X. *Furthermore,* $\xi = \xi_d + \xi_a + \xi_r$ *is a also a completely random measure on* X.

(ii) *Let* ξ *be a completely random measure on* X. *Then there exist independent, completely random measures* ξ_d, ξ_a, ξ_r *as in part* (i) *such that* $\xi = \xi_d + \xi_a + \xi_r$.

Remark 7. (9) is equivalent to

$$\int_{A\times(0,1]} s\, dm(x, s) < \infty, \tag{10}$$

$$\int_{A\times[1,+\infty)} dm(x, s) < \infty \tag{11}$$

for each $A \in \mathcal{B}_0(X)$.

Remark 8. In fact, Kingman [16] (see also [17]) does not assume that a random measure takes values in the space of Radon measures. He allows a random measure to take values in the space of *all* measures on $(X, \mathcal{B}(X))$ and assumes that, for each $A \in \mathcal{B}(X)$, $\xi(A)$ is a random variable (i.e., a measurable mapping.) In that case, one does not need condition (7) to hold. However, Daley and Vere-Jones [9, Theorem 6.3.VIII] do assume that a random measure takes values in the space of Radon measures, but they do not assume (7). It is clear that, without this condition, a measure ξ_a may not be a Radon measure (even possibly a.s.) So, Theorem 6 is a refinement of [9, Theorem 6.3.VIII].

In this paper, we will only use part (i) of Theorem 6. For the reader's convenience and for our references below, we will now present the proof of part (i) and we will also discuss in detail the construction of the completely random measure ξ_r, cf. [14], Section 3 in [13], and subsection 2.2 in [21].

Proof of part (i) of Theorem 6. Since the measure ξ_d is deterministic, it is trivially a completely random measure.

Next, we need to prove that $\xi_a(A)$ is a completely random measure. By the definition of $\xi_a(A)$, for each $A \in \mathcal{B}_0(X)$, we have that

$$\xi_a(A) := \sum_{k=1}^{\infty} a_k \delta_{x_k}(A) = \sum_{k\geq 1,\ x_k\in A} a_k.$$

If sets $A_1, \dots, A_n \in \mathcal{B}_0(X)$ are disjoint, then the random variables a_k appearing in each sum $\xi_a(A_i) = \sum_{k\geq 1,\ x_k\in A_i} a_k$ are different, so $\xi_a(A_1), \dots, \xi_a(A_n)$ are independent random variables. Furthermore, (7) ensures that $\xi_a(A)$ is a Radon measure a.s. Thus, ξ_a is a completely random measure.

Now, we need to prove that ξ_r a completely random measure. Consider the product space $\hat{X} := X \times \mathbb{R}_+$ where $\mathbb{R}_+ := (0, +\infty)$. We need to make $\mathbb{R}_+$ a locally compact Polish space. Consider the bijective mapping $\mathbb{R} \ni x \mapsto e^x \in \mathbb{R}_+$. Its inverse mapping is the logarithm function $\ln(x)$. For $s_1, s_2 \in \mathbb{R}_+$, we then take the distance between them in $\mathbb{R}_+$ as the usual distance in $\mathbb{R}$ between $\ln(s_1)$ and $\ln(s_2)$. Thus,

$$\operatorname{dist}(s_1, s_2) = |\ln s_1 - \ln s_2| = \left|\ln\left(\frac{s_1}{s_2}\right)\right|.$$

Equipped with this metric, $\mathbb{R}_+$ is a locally compact Polish space. Taking the product of X and $\mathbb{R}_+$, we obtain a locally compact Polish space $\hat{X}$. The Borel σ-algebra on $\hat{X}$ is denoted by $\mathcal{B}(\hat{X})$.

Next, on the space $\hat{X}$ we want to construct a Poisson point process with intensity measure m. To this end, we should prove that m is a Radon measure on $\hat{X}$. It suffices to prove that, for each $A \in \mathcal{B}_0(X)$ and each closed interval $[a, b] \subset \mathbb{R}_+$, $m(A \times [a, b]) < \infty$. In fact, we will prove that, for each $A \in \mathcal{B}_0(X)$ and $\epsilon > 0$,

$$m(A \times [\epsilon, \infty)) < \infty. \tag{12}$$

By (10), for each $0 < \epsilon \leq 1$, we have that

$$\int_{A\times[\epsilon,1]} dm(x, s) \leq \int_{A\times[\epsilon,1]} \frac{s}{\epsilon}\, dm(x, s) \leq \frac{1}{\epsilon}\int_{A\times(0,1]} s\, dm(s) < +\infty. \tag{13}$$

Hence, by (11), this implies (12).

By (8), the Radon measure m is nonatomic. Hence, we can construct π_m, the Poisson measure on $(\Gamma(\hat{X}), \mathcal{B}(\Gamma(\hat{X})))$ with intensity measure m.

Let $\Gamma_p(\hat{X})$ denote the set of all pinpointing configurations in $\hat{X}$:

$$\Gamma_p(\hat{X}) := \big\{\gamma \in \Gamma(\hat{X}) \mid \text{if } (x_1, s_1), (x_2, s_2) \in \gamma, (x_1, s_1) \neq (x_2, s_2),\ \text{then } x_1 \neq x_2\big\}.$$

It is known that

$$\Gamma_p(\hat{X}) \in \mathcal{B}(\Gamma(\hat{X})), \tag{14}$$

see [14].

By (8) and the explicit construction of Poisson measure in a finite volume (see e.g. [17]), we conclude that

$$\pi_m(\Gamma_p(\hat{X})) = 1, \tag{15}$$

i.e., the Poisson measure π_m is concentrated on the set of pinpointing configurations.

Now for each $\gamma \in \Gamma_p(\hat{X})$ and $A \in \mathcal{B}_0(X)$, we define a local mass by

$$\mathfrak{M}_A(\gamma) := \sum_{(x,s)\in\gamma} \chi_A(x)s = \int_{\hat{X}} \chi_A(x)s\, d\gamma(x,s) \in [0,+\infty].$$

We then define the set of pinpointing configurations with finite local mass by

$$\Gamma_{pf}(\hat{X}) := \big\{\gamma \in \Gamma(\hat{X}) \mid \mathfrak{M}_A(\gamma) < \infty \text{ for each } A \in \mathcal{B}_0(X)\big\}.$$

Lemma 9. *We have* $\pi_m(\Gamma_{pf}(\hat{X})) = 1$.

Proof. Let $A \in \mathcal{B}_0(X)$. By condition (10) and the Mecke identity (e.g. [24]),

$$\int_{\Gamma(\hat{X})} \sum_{(x,s)\in\gamma} \chi_A(x)\chi_{(0,1]}(s)s\, d\pi_m(\gamma) = \int_{A\times(0,1]} s\, dm(x,s) < +\infty.$$

Hence,

$$\sum_{(x,s)\in\gamma} \chi_A(x)\chi_{(0,1]}(s)s < +\infty \quad \text{for } \pi_m\text{-a.a. } \gamma \in \Gamma_p(X). \tag{16}$$

By condition (11) and construction of the Poisson measure,

$$|\gamma \cap (A \times (1,+\infty))| < \infty \quad \text{for } \pi_m\text{-a.a. } \gamma \in \Gamma_p(X).$$

This implies

$$\sum_{(x,s)\in\gamma} \chi_A(x)\chi_{(1,+\infty)}(s)s < +\infty \quad \text{for } \pi_m\text{-a.a. } \gamma \in \Gamma_p(X). \tag{17}$$

Note that X can be represented as a countable union of compact sets. Hence, the lemma follows. □

Next, we define on X the set of discrete Radon measures

$$\mathbb{K}(X) := \left\{\eta = \sum_i s_i\delta_{x_i} \in \mathbb{M}(X) \mid s_i > 0, x_i \in X\right\}.$$

Here, δ_{x_i} is the Dirac measure with mass at x_i, the atoms x_i are assumed to be distinct and their total number is at most countable. By convention, the cone $\mathbb{K}(X)$ contains the null mass $\eta = 0$, which is represented by the sum over the empty set of indices i. We denote $\tau(\eta) := \{x_i\}$, i.e., the set on which the measure η is concentrated. For $\eta \in \mathbb{K}(X)$ and $x \in \tau(\eta)$, we denote by s_x the mass of η at point x, i.e., $s_x := \eta(\{x\})$. Thus, each $\eta \in \mathbb{K}(X)$ can be written in the form $\eta = \sum_{x\in\tau(\eta)} s_x\delta_x$.

Note that the closure of $\mathbb{K}(X)$ in the vague topology coincides with $\mathbb{M}(X)$. As shown in [14], $\mathbb{K}(X) \in \mathcal{B}(\mathbb{M}(X))$. We denote by $\mathcal{B}(\mathbb{K}(X))$ the trace σ-algebra of $\mathcal{B}(\mathbb{M}(X))$ on $\mathbb{K}(X)$.

Let us now construct a bijective mapping

$$\mathcal{R} : \Gamma_{pf}(\hat{X}) \to \mathbb{K}(X) \tag{18}$$

as follows: For each $\gamma = \{(x_i, s_i)\} \in \Gamma_{pf}(\hat{X})$, we set

$$\mathcal{R}\gamma := \sum_i s_i\delta_{x_i} \in \mathbb{K}(X). \tag{19}$$

By [14, Theorem 6.2], we have

$$\mathcal{B}(\mathbb{K}(X)) = \{\mathcal{R}A \mid A \in \mathcal{B}(\Gamma_{pf}(\hat{X}))\}. \tag{20}$$

Hence, both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are measurable mappings.

Let ξ_r be the pushforward of π_m under $\mathcal{R}:\Gamma_{pf}(\hat{X})\mapsto\mathbb{K}(X)$. If $A_1,\dots,A_n\in\mathcal{B}_0(X)$ are mutually disjoint, then $\gamma(B_1),\dots,\gamma(B_n)$ are independent random variables under π_m if $B_1\subset A_1\times\mathbb{R}_+,\dots,B_n\subset A_n\times\mathbb{R}_+$. Therefore, the random variables

$$\int_{\hat{X}}\chi_{A_1}(x)s\,d\gamma(x,s),\dots,\int_{\hat{X}}\chi_{A_n}(x)s\,d\gamma(x,s)$$

are independent under π_m. This implies that $\eta(A_1),\dots,\eta(A_n)$ are independent under ξ_r. Thus, ξ_r is a completely random measure.

Trivially, the sum $\xi_d+\xi_a+\xi_r$ is a completely random measure as well. Thus, part (i) Theorem 6 is proven. □

The following result is immediate now.

Corollary 10. *Let m be a measure on $X\times\mathbb{R}_+$ which satisfies* (8) *and* (9)*. Then there exists a completely random measure μ_m such that $\mu_m(\mathbb{K}(X))=1$ and which has Fourier transform*

$$\int_{\mathbb{K}(X)}e^{i\langle f,\eta\rangle}d\mu_m(\eta)=\exp\left[\int_X\int_{\mathbb{R}_+}\left(e^{isf(x)}-1\right)dm(x,s)\right],\quad f\in C_0(X). \tag{21}$$

The measure m will be called the Lévy measure of the completely random measure μ_m.

Remark 11. It is easy to see that (21) remains true if $f\in B_0(X)$, i.e., $f:X\to\mathbb{R}$ is a measurable bounded function with compact support. In particular, for any $A\in\mathcal{B}_0(X)$ and $t\in\mathbb{R}$, we may take $f(x)=t\chi_A(x)$. Then by (21)

$$\int_{\mathbb{K}(X)}e^{it\eta(A)}d\mu_m(\eta)=\exp\left[\int_A\int_{\mathbb{R}_+}\left(e^{ist}-1\right)dm(x,s)\right].$$

In particular, if m is product measure:

$$dm(x,s)=d\sigma(x)d\lambda(s),$$

then

$$\int_{\mathbb{K}(X)}e^{it\eta(A)}d\mu_m(\eta)=\exp\left[\sigma(A)\int_{\mathbb{R}_+}\left(e^{ist}-1\right)d\lambda(s)\right].$$

Thus, in this case the distribution of the random variable $\eta(A)$ only depends on $\sigma(A)$. This is why in such a case, one calls μ_m a *measure-valued Lévy processes.*

The corollary below follows immediately from Theorem 6 and its proof.

Corollary 12. *Let ξ be a completely random measure on X. Then there exist a deterministic, nonatomic Radon measure ξ_d and completely random measure ξ', taking values a.s. in the space $\mathbb{K}(X)$ of discrete Radon measures on X, such that $\xi=\xi_d+\xi'$.*

A completely random measure on X which takes a.s. values in $\mathbb{K}(X)$ is called a *completely random discrete measure.* In particular, the measure ξ_r from Theorem 6 is a *completely random discrete measure without fixed atoms.* Below we will only be interested in such completely random measures.

3. Quasi-invariance of completely random measures with respect to transformations of weights

In this section, we will consider the current group which transforms the weights. Let σ be a fixed Radon non-atomic measure on $(X,\mathcal{B}(X))$.

3.1. **General theory.** We define

$$C_0(X \to \mathbb{R}_+) := \{\theta : X \to \mathbb{R}_+ \mid \theta \text{ is continuous and } \\ \theta = 1 \text{ outside a compact set in } X\}.$$

$C_0(X \to \mathbb{R}_+)$ is a (commutative) group under the usual point-wise multiplication of functions. In particular, the identity element in this group is the function which is identically equal to 1 on X. We call $C_0(X \to \mathbb{R}_+)$ a *current group.*

We define the action of the group $C_0(X \to \mathbb{R}_+)$ on $\mathbb{M}(X)$ (the set of Radon measures) by

$$\mathbb{M}(X) \ni \eta \mapsto \theta\eta \in \mathbb{M}(X) \quad \text{for each } \theta \in C_0(X \to \mathbb{R}_+).$$

Here $\theta\eta$ denotes the measure on X which has density θ with respect to the measure η.

Assume μ_m is a completely random measure on X which has Fourier transform (21). We are interested whether μ_m is quasi-invariant with respect to the action of the group $C_0(X \to \mathbb{R}_+)$ on $\mathbb{M}(X)$.

Let us assume that

$$dm(x, s) = \frac{l(x, s)}{s}\, d\sigma(x)\, ds, \tag{22}$$

where

$$\text{for each } x \in X, \quad \text{either } l(x, s) > 0 \text{ for all } s \in \mathbb{R}_+ \quad \text{or} \quad l(x, s) = 0 \text{ for all } s \in \mathbb{R}_+. \tag{23}$$

Below, for a set $Y \in \mathcal{B}(X)$, we denote by $\mathcal{B}(Y)$ the trace σ-algebra of $\mathcal{B}(X)$ on Y, i.e., the collection of all $A \in \mathcal{B}(X)$ satisfying $A \subset Y$. We will also denote by $\mathcal{B}_0(Y)$ the collection of all $A \in \mathcal{B}(X)$ which satisfy $A \subset Y$.

Let

$$Y := \{x \in X \mid l(x, \cdot) > 0\}. \tag{24}$$

Then, under (22) and (23), condition (9) becomes

$$\int_{A\times\mathbb{R}_+} l(x, s) \min\{s^{-1}, 1\}\, d\sigma(x)\, ds < +\infty \quad \text{for all } A \in \mathcal{B}_0(Y). \tag{25}$$

Note also that that condition (8) is now satisfied.

The following theorem and Corollary 15 below are the main result of this section. They extend Theorem 4 and Corollary 5 in [21], proved for measure-valued Lévy processes.

Theorem 13. *Assume* (22), (23) *and* (25) *hold. Assume that, for each $n \in \mathbb{N}$, there exists $\epsilon > 0$ such that, for each $A \in \mathcal{B}_0(Y)$,*

$$\int_A \left[\sup_{r\in\left[\frac{1}{n}, n\right]} \int_{(0,\epsilon)} \frac{|l(x, rs) - l(x, s)|}{s}\, ds\right] d\sigma(x) < \infty. \tag{26}$$

Then the measure μ_m is quasi-invariant with respect to all transformations from the group of currents, $C_0(X \to \mathbb{R}_+)$, i.e., each $\theta \in C_0(X \to \mathbb{R}_+)$ maps $\mathbb{K}(X)$ into itself, and μ_m^θ is equivalent to μ_m. Furthermore, the corresponding density is given by

$$\frac{d\mu_m^\theta}{d\mu_m}(\eta) = \exp\left[\int_Y \log\left(\frac{l(x, \theta^{-1}(x)s_x)}{l(x, s_x)}\right) s_x^{-1} d\eta(x) \right. \\ \left. + \int_Y \int_{\mathbb{R}_+} \frac{\big(l(x, s) - l(x, \theta^{-1}(x)s)\big)}{s}\, ds\, d\sigma(x)\right]. \tag{27}$$

In (27)*, the function appearing under the sign of integral with respect to measure η belongs to $L^1(Y, \eta)$ for μ_m-a.a. $\eta \in \mathbb{K}(X)$.*

Proof. We divide the proof of this theorem into several steps.

Step 1. Let us first prove that, for each $\theta \in C_0(X \to \mathbb{R}_+)$,

$$\begin{aligned}&\int_Y \int_{\mathbb{R}_+} \frac{\left|l(x,s) - l(x,\theta^{-1}(x)s)\right|}{s}\, ds\; d\sigma(x)\\ &\qquad = \int_X \int_{\mathbb{R}_+} \frac{\left|l(x,s) - l(x,\theta^{-1}(x)s)\right|}{s}\, ds\, d\sigma(x) < \infty. \qquad (28)\end{aligned}$$

The function θ is continuous and takes values in $\mathbb{R}_+$. By the definition of $C_0(X \to \mathbb{R}_+)$, there exists a compact set $C \subset X$ such that $\theta(x) = 1$ for al $x \notin C$. The function θ is continuous on the compact set C. Hence θ attains its infimum and supremum on C. Thus,

$$\inf_{x\in C} \theta(x) > 0, \quad \sup_{x\in C} \theta(x) < +\infty.$$

But this implies that, for all $y \in X$,

$$0 < \inf_{x\in X} \theta(x) \le \theta(y) \le \sup_{x\in X} \theta(x) < \infty.$$

Hence, there exists $n \in \mathbb{N}$ such that, for all $x \in X$,

$$\frac{1}{n} \le \theta(x) \le n.$$

So, fix this $n \in \mathbb{N}$, and choose the corresponding $\epsilon > 0$ as in the formulation of the theorem. Denote $A = C \cap Y$, $A \in \mathcal{B}_0(Y)$. We have

$$\begin{aligned}\int_{A\times\mathbb{R}_+} \frac{\left|l(x,\theta^{-1}(x)s) - l(x,s)\right|}{s}\, d\sigma(x)\, ds &= \int_{A\times(0,\epsilon)} \frac{|l(x,\theta^{-1}(x)s) - l(x,s)|}{s}\, d\sigma(x)\, ds\\ &\quad + \int_{A\times[\epsilon,+\infty)} \frac{|l(x,\theta^{-1}(x)s) - l(x,s)|}{s}\, d\sigma(x)\, ds.\end{aligned}$$

To prove the finiteness of the first integral, we have, for a fixed $x \in A$,

$$\int_{(0,\epsilon)} \frac{|l(x,\theta^{-1}(x)s) - l(x,s)|}{s} ds \le \sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l(x,rs) - l(x,s)|}{s} ds.$$

Hence, by (26),

$$\begin{aligned}&\int_{A\times(0,\epsilon)} \frac{|l(x,\theta^{-1}(x)s) - l(x,s)|}{s}\, d\sigma(x)\, ds\\ &\qquad \le \int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l(x,rs) - l(x,s)|}{s} ds\right] d\sigma(x) < +\infty.\end{aligned}$$

For the second integral, we have

$$\begin{aligned}&\int_{A\times[\epsilon,+\infty)} \frac{|l(x,\theta^{-1}(x)s) - l(x,s)|}{s}\, ds\, d\sigma(x)\\ &\quad \le \frac{1}{\epsilon} \int_{A\times[\epsilon,+\infty)} |l(x,\theta^{-1}(x)s) - l(x,s)|\, d\sigma(x)\, ds\\ &\quad \le \frac{1}{\epsilon} \left[\int_{A\times[\epsilon,+\infty)} l(x,\theta^{-1}(x)s)\, d\sigma(x)\, ds + \int_{A\times[\epsilon,+\infty)} l(x,s)\, d\sigma(x)\, ds\right]. \qquad (29)\end{aligned}$$

By (25) the second integral in (29) is finite. Let us consider the first integral

$$\int_{A\times[\epsilon,+\infty)} l(x,\theta^{-1}(x)s)\, d\sigma(x)\, ds.$$

Let G denote the image of $A \times [\epsilon, +\infty)$ under the mapping $(x, s) \to (x, \theta^{-1}(x)s)$. Then, as $\frac{1}{n} \leq \theta(x) \leq n$, we obtain from (25):

$$\begin{aligned}
&\int_{A\times[\epsilon,+\infty)} l(x, \theta^{-1}(x)s)\, d\sigma(x)\, ds \\
&\quad = \int_G l(x, s)\theta(x)\ d\sigma(x)\ ds \\
&\quad \leq n \int_G l(x, s)\ d\sigma(x)\ ds \\
&\quad \leq n \int_{A\times\left[\frac{\epsilon}{n},+\infty\right)} l(x, s) ds < \infty.
\end{aligned}$$

Thus,

$$\int_A \int_{\mathbb{R}_+} \frac{|l(x, s) - l(x, \theta^{-1}(x)s)|}{s}\ ds\ d\sigma(x) < \infty. \tag{30}$$

If $x \notin A$, then either $\theta(x) = 1$ or $l(x, s) = 0$ for all $s \in \mathbb{R}_+$. Hence

$$l(x, s) - l(x, \theta^{-1}(x)s) = 0.$$

Therefore the integral in (30) is equal to

$$\int_X \int_{\mathbb{R}_+} \frac{|l(x, s) - l(x, \theta^{-1}(x)s)|}{s}\ ds\ d\sigma(x).$$

Thus (28) holds.

Step 2. We will now bring the problem of equivalence of the measures μ_m and μ_m^θ to the configuration space $\Gamma_{pf}(\hat{X})$.

Recall that the measure μ_m was constructed as the pushforward of the Poisson measure π_m under the bijective mapping $\mathcal{R}$, see (18) and (19). Consider the inverse mapping

$$\mathcal{R}^{-1} : \mathbb{K}(X) \to \Gamma_{pf}(\hat{X}),$$

with

$$\mathcal{R}^{-1}\left(\sum_i s_i \delta_{x_i}\right) = \{(x_i, s_i)\}.$$

As we already know $\mathcal{R}^{-1}$ is measurable. Denote by π_m^θ the pushforward of μ_m^θ under $\mathcal{R}^{-1}$. Note that

$$\begin{aligned}
&\mathcal{R}^{-1}\theta\mathcal{R} : \Gamma_{pf}(\hat{X}) \to \Gamma_{pf}(\hat{X}), \quad \text{and} \\
&\gamma = \{(x_i, s_i)\} \to \{(x_i, \theta(x_i)s_i)\}.
\end{aligned} \tag{31}$$

Hence, π_m^θ is the pushforward of the measure π_m under the transformation (31). Thus, for each $f \in C_0(X \times \mathbb{R}_+)$ and $\gamma = \{(x_i, s_i)\} \in \Gamma_{pf}(\hat{X})$,

$$\langle f, \mathcal{R}^{-1}\theta\mathcal{R}\gamma\rangle = \sum_i f(x_i, \theta(x_i)s_i) = \langle f^\theta, \gamma\rangle,$$

where $f^\theta : X \times \mathbb{R}_+ \to \mathbb{R}$ and $f^\theta(x, s) = f(x, \theta(x)s)$. Hence, the Fourier transform of π_m^θ is

$$\begin{aligned}
\int_{\Gamma_{pf}(\hat{X})} e^{i\langle f,\gamma\rangle} d\pi_m^\theta(\gamma) &= \int_{\Gamma_{pf}(\hat{X})} e^{\langle f^\theta,\gamma\rangle} d\pi_m(\gamma) \\
&= \exp\left[\int_X \int_{\mathbb{R}_+} (e^{if(x,\theta(x)s)} - 1)\frac{l(x, s)}{s}\ ds\ d\sigma(x)\right] \\
&= \exp\left[\int_X \int_{\mathbb{R}_+} (e^{if(x,s)} - 1)\frac{l(x, \theta^{-1}(x)s)}{s}\ d\sigma(x)\ ds\right].
\end{aligned}$$

Hence, π_m^θ is the Poisson measure on $\Gamma_{pf}(\hat{X})$ with intensity measure

$$dm^\theta(x,s) := \frac{l(x,\theta^{-1}(x)s)}{s}\, d\sigma(x)\, ds.$$

Thus, to prove that the measures μ_m and μ_m^θ are equivalent, it is sufficient to prove that the measures π_m and π_m^θ are equivalent.

Step 3. By using Theorem 4, we will now show that the measures π_m and π_m^θ are equivalent.

By (23), both measures m and m^θ are concentrated on $Y\times\mathbb{R}_+$, are equivalent and

$$\frac{dm^\theta}{dm}(x,s) = \frac{l(x,\theta^{-1}(x)s)s}{s\; l(x,s)}\,\chi_Y(x) = \frac{l(x,\theta^{-1}(x)s)}{l(x,s)}\,\chi_Y(x).$$

We have by (28),

$$\begin{aligned}\int_{Y\times\mathbb{R}_+}&\left|\frac{l(x,\theta^{-1}(x)s)}{l(x,s)}-1\right| dm(x,s)\\ &= \int_{Y\times\mathbb{R}_+}\frac{|l(x,\theta^{-1}(x)s)-l(x,s)|}{s}\, d\sigma(x)\, ds<\infty.\end{aligned}$$

Hence, by Theorem 4, the measures π_m and π_m^θ are equivalent, hence so are μ_m and μ_m^θ.

Also by Theorem 4, for $\gamma=\{(x_i,s_i)\}\in\Gamma_{pf}(\hat{X})$,

$$\begin{aligned}\frac{d\pi_m^\theta}{d\pi_m}(\gamma) &= \exp\left[\left\langle \log\left(\frac{dm^\theta}{dm}\right)\chi_{\hat{Y}},\gamma\right\rangle + \int_{\hat{Y}}\left(1-\frac{dm^\theta}{dm}\right)dm\right]\\ &= \exp\left[\sum_i \log\left(\frac{l(x_i,\theta^{-1}(x_i)s_i)}{l(x_i,s_i)}\right)\chi_{\hat{Y}}(x_i,s_i)\right.\\ &\qquad\left.+\int_{\hat{Y}}\left(1-\frac{l(x,\theta^{-1}(x)s)}{l(x,s)}\right)\frac{l(x,s)}{s}\, d\sigma(x)\, ds\right]\\ &= \exp\left[\sum_i \log\left(\frac{l(x_i,\theta^{-1}(x_i)s_i)}{l(x_i,s_i)}\right)\frac{s_i}{s_i}\,\chi_{\hat{Y}}(x_i,s_i)\right.\\ &\qquad\left.+\int_{\hat{Y}}(l(x,s)-l(x,\theta^{-1}(x)s)\,\frac{1}{s}\, d\sigma(x)\, ds\right],\end{aligned}$$

where $\hat{Y}=Y\times\mathbb{R}_+$. From here formula (27) follows. □

Corollary 14. *Assume that the condition of Theorem 13 hold. For each $\theta\in C_0(X\to\mathbb{R}_+)$, we define a unitary operator $\mathscr{U}_\theta$ in $L^2(\mathbb{K}(X)\to\mathbb{C},\mu_m)$ by*

$$(\mathscr{U}_\theta f)(\eta) = f(\theta^{-1}\eta)\sqrt{\frac{d\mu_m^\theta}{d\mu_m}(\eta)},$$

where the Radon–Nikodym density $\frac{d\mu_m^\theta}{d\mu_m}$ is given by (27). Then the operators $\mathscr{U}_\theta$, $\theta\in C_0(X\to\mathbb{R}_+)$, form a unitary representation of the current group $C_0(X\to\mathbb{R}_+)$.

Corollary 15. *Assume (22)–(25) are satisfied. Assume that, for some $\epsilon>0$,*

$$l(x,s)=l_1(x,s)+l_2(x,s)\quad \text{for } x\in Y,\ s\in(0,\epsilon),$$

where Y is defined by (24). Here, for each fixed $x\in Y$, the function $l_1(x,s)$ is differentiable in s on $(0,\epsilon)$, and for each $n\in\mathbb{N}$ and $A\in\mathcal{B}_0(Y)$,

$$\int_A\int_{(0,\frac{\epsilon}{n})}\sup_{u\in[\frac{s}{n},sn]}\left|\frac{\partial}{\partial u}l_1(x,u)\right| ds\, d\sigma(x)<\infty \tag{32}$$

and

$$\int_A \int_{(0,\epsilon)} \frac{l_2(x,s)}{s}\, ds\, d\sigma(x) < \infty. \tag{33}$$

Then condition (26) *is satisfied, and so the conclusion of Theorem 13 holds.*

Proof. Using that $l(x,s) = l_1(x,s) + l_2(x,s)$, we get

$$\begin{aligned} &\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l(x,rs)-l(x,s)|}{s}\, ds \\ &\le \sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l_1(x,rs)-l_1(x,s)|}{s}\, ds + \sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l_2(x,rs)-l_2(x,s)|}{s}\, ds. \end{aligned}$$

Hence, it suffices to prove that (26) holds for both $l(x,s) = l_1(x,s)$ and for $l(x,s) = l_2(x,s)$. By Taylor's formula,

$$|l_1(x,rs) - l_1(x,s)| = \left| \frac{\partial}{\partial u} l_1(x,u)\Big|_{u=u_o} \right| |rs - s|,$$

where u_0 is a point between rs and s, that is for $r<1$, $u_0 \in (rs,s)$ and $r>1$, $u_0 \in (s,rs)$. Therefore, for $r \in \left[\frac{1}{n}, n\right]$, we have $u_0 \in \left[\frac{s}{n}, sn\right]$. Hence, for $r \in \left[\frac{1}{n}, n\right]$,

$$|l_1(x,rs) - l_1(x,s)| \le \sup_{u\in\left[\frac{s}{n},sn\right]} \left| \frac{\partial}{\partial u} l_1(x,u) \right| ns.$$

This implies, by (32),

$$\begin{aligned} &\int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \frac{|l_1(x,rs)-l_1(s,s)|}{s}\, ds \right] d\sigma(x) \\ &\qquad \le \int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon)} \sup_{u\in\left[\frac{s}{n},n\right]} \left| \frac{\partial}{\partial u} l_1(x,u) \right| n\, ds \right] d\sigma(x) \\ &\qquad\qquad = n \int_A \int_{(0,\epsilon)} \sup_{u\in\left[\frac{s}{n},sn\right]} \left| \frac{\partial}{\partial u} l_1(x,u) \right| ds\, d\sigma(x) < \infty, \end{aligned}$$

where $A \in \mathcal{B}_0(Y)$. Thus, the statement is proven for l_1.

Now, let us prove the statement for l_2. For $r \in \left[\frac{1}{n}, n\right]$, and $A \in \mathcal{B}_0(Y)$,

$$\begin{aligned} &\int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon/n)} \frac{|l_2(x,rs)-l_2(x,s)|}{s}\, ds \right] d\sigma(x) \\ &\le \int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon/n)} \frac{l_2(x,rs)}{s}\, ds \right] d\sigma(x) + \int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon/n)} \frac{l_2(x,s)}{s}\, ds \right] d\sigma(x) \\ &= \int_A \left[\sup_{r\in\left[\frac{1}{n},n\right]} \int_{(0,\epsilon/n)} \frac{l_2(x,s)}{s}\, ds \right] d\sigma(x) + \int_A \int_{(0,\epsilon/n)} \frac{l_2(x,s)}{s}\, ds\, d\sigma(x) \\ &\le 2 \int_A \int_{(0,\epsilon)} \frac{l_2(x,s)}{s}\, ds\, d\sigma(x) < \infty \end{aligned}$$

by (33). □

3.2. **Examples.** We will now consider examples of completely random measures which satisfy the assumptions of Corollary 15.

3.2.1. *Completely random gamma measures.* Let us fix two parameters $\alpha > 0$ and $\beta > 0$. We first consider the function

$$l(x,s) = l(s) = \beta e^{-\frac{s}{\alpha}},$$

so that

$$dm(x,s) = \beta \frac{e^{-\frac{s}{\alpha}}}{s}\, d\sigma(x)\, ds. \tag{34}$$

Note that

$$dm(x,s) = d\sigma(x)\, d\lambda(s),$$

where

$$d\lambda(s) = \beta \frac{e^{-\frac{s}{\alpha}}}{s}\, ds.$$

Following [27], we will call the measure μ_m the *gamma measure*, or the *measure-valued gamma process with parameters* α *and* β.

Proposition 16. *The Laplace transform of the measure* μ_m *with* m *given by* (34) *is*

$$\int_{\mathbb{K}(X)} \exp[-\langle \eta, f\rangle] d\mu_m(\eta) = \exp\left[-\beta \int_X \log(1+\alpha f(x))\, d\sigma(x)\right], \tag{35}$$

where $f : X \to \mathbb{R}$ *is a bounded measurable function with compact support which satisfies* $f(x) > -\frac{1}{\alpha}$ *for all* $x \in X$.

This result is known, see [27], but we will now give a complete proof of it, since we will later on need it.

Proof. We start with the following known result.

Lemma 17. *For* $u > -1$,

$$\int_0^\infty \frac{e^{-us} - 1}{s}\, e^{-s}\, ds = -\log(1+u). \tag{36}$$

By Lemma 17, for $u > \frac{1}{\alpha}$,

$$\begin{aligned} \int_0^\infty \frac{e^{-us} - 1}{s}\, e^{-\frac{s}{\alpha}}\, ds &= \int_0^\infty \frac{e^{-s\alpha u} - 1}{s}\, e^{-s}\, ds \\ &= -\log\left(1 + \alpha u\right). \end{aligned} \tag{37}$$

Using the construction of the measure μ_m and the Laplace transform of the Poisson measure, we have

$$\int_{\mathbb{K}(X)} \exp\left[-\langle \eta, f\rangle\right] d\mu_m(\eta) = \exp\left[\int_X \int_{\mathbb{R}_+} (e^{-f(x)s} - 1) e^{-\frac{s}{\alpha}} \frac{\beta}{s}\, ds\, d\sigma(x)\right]. \tag{38}$$

By (37), for each $x \in X$,

$$\int_{\mathbb{R}_+} \left(e^{-f(x)s} - 1\right) e^{-\frac{s}{\alpha}} \frac{1}{s}\, ds = -\log\left(1 + \alpha f(x)\right).$$

Now, substituting the above result into the right hand side of equation (38), we get (35). □

Let $\Delta \in \mathcal{B}_0(X)$. By (35), for each $t > -\frac{1}{\alpha}$,

$$\begin{aligned} \int_{\mathbb{K}(X)} \exp[-t\eta(\Delta)] &= \exp\left[-\beta \int_X \log(1 + \alpha t \chi_\Delta(x))\, d\sigma(x)\right] \\ &= (1+\alpha t)^{-\beta\, \mathrm{vol}(\Delta)}. \end{aligned} \tag{39}$$

Let us recall that the gamma distribution on $\mathbb{R}$ with parameters α and θ is defined by

$$\frac{u^{\theta-1}}{\alpha^\theta\Gamma(\theta)}\, e^{-\frac{u}{\alpha}}\, \chi_{(0,\infty)}(u)\, du.$$

The Laplace transform of the gamma distribution is given by

$$\int_{\mathbb{R}} e^{-tu}\, \frac{u^{\theta-1}}{\alpha^\theta\Gamma(\theta)}\, e^{-\frac{u}{\alpha}}\, \chi_{(0,\infty)}(u)\, du = (1+\alpha t)^\theta, \quad t > -\frac{1}{\alpha}.$$

Hence, under μ_m, the random variable $\eta(\Delta)$ has gamma distribution with parameters α and $\theta = \beta\ \mathrm{vol}(\Delta)$.

Now, we will produce a generalization by making the parameters α and β to be positive functions on X. Thus, let us consider measurable functions

$$\alpha : X \to \mathbb{R}_+, \quad \beta : X \to [0,\infty).$$

We define

$$l(x,s) = \beta(x)\, e^{-\frac{s}{\alpha(x)}},$$

so that

$$dm(x,s) = \frac{\beta(x)e^{-\frac{s}{\alpha(x)}}}{s}\, d\sigma(x)\, ds. \tag{40}$$

We denote by $L^1_{\mathrm{loc}}(X,\sigma)$ the space of all measurable functions $f: X\to\mathbb{R}$ such that, for each $A\in\mathcal{B}_0(X)$, $f\chi_A \in L^1(X,\sigma)$, i.e., $\int_A |f(x)|\, d\sigma(x) < \infty$.

Lemma 18. *Assume that the function $\alpha\beta$ belongs to $L^1_{\mathrm{loc}}(X,\sigma)$. Then the measure m given by* (40) *satisfies* (25).

Proof. For each $A\in\mathcal{B}_0(X)$, we have

$$\begin{aligned}\int_{A\times\mathbb{R}_+} l(x,s)\min\{s^{-1},1\}\, d\sigma(x)\, ds &\le \int_{A\times\mathbb{R}_+} l(x,s)\, d\sigma(x)\, ds\\ &= \int_A\int_0^\infty \beta(x)e^{-\frac{s}{\alpha(x)}}\, ds\, d\sigma(x)\\ &= \int_A \beta(x)\left(\int_0^\infty e^{-\frac{s}{\alpha(x)}}\, ds\right) d\sigma(x)\\ &= \int_A \alpha(x)\beta(x)\, d\sigma(x) < \infty.\end{aligned}$$

□

Proposition 19. *The Laplace transform of the measure μ_m with m given by* (40) *is*

$$\int_{\mathbb{K}(X)} \exp[-\langle\eta,f\rangle] d\mu_m(\eta) = \exp\left[-\int_X (1+\alpha(x)f(x))\beta(x)d\sigma(x)\right],$$

where $f: X\to\mathbb{R}$ is a bounded, measurable function with compact support which satisfies $\alpha(x)f(x) > -1$ for all $x\in X$.

Proof. Analogously to (38), we have

$$\int_{K(X)} \exp[-\langle\eta,f\rangle] d\mu_m(\eta) = \exp\left[\int_X\int_{\mathbb{R}_+}\left(e^{-f(x)s}-1\right) e^{-\frac{s}{\alpha(x)}}\, \frac{\beta(x)}{s}\, ds\, d\sigma(x)\right].$$

By (37),

$$\int_{\mathbb{R}_+}\left(e^{-f(x)s}-1\right) e^{-\frac{s}{\alpha(x)}}\, \frac{\beta(x)}{s}\, ds = -\beta(x)\log(1+\alpha(x)f(x)),$$

which implies the proposition. □

Lemma 20. *Assume that the functions $\alpha\beta$ and β belong to $L^1_{\mathrm{loc}}(X)$. Then the measure m satisfies the conditions of Corollary 15.*

Proof. Fix any $\epsilon > 0$. In the notations of Corollary 15, we set $l_1 = l$ and $l_2 = 0$. The function $l(x, s)$ is evidently differentiable in the s variable. Thus, we only have to check that, for each $n \in \mathbb{N}$ and $A \in \mathcal{B}_0(X)$, $A \subset Y = \{y \in X \mid \beta(y) = 0\}$,

$$\int_A \int_0^{\frac{\epsilon}{n}} \sup_{u \in \left[\frac{s}{n}, sn\right]} \left| \frac{\partial}{\partial u} l(x, u) \right| ds \, d\sigma(x) < \infty.$$

We have

$$\left| \frac{\partial}{\partial u} l(x, u) \right| = \frac{\beta(x)}{\alpha(x)} e^{-\frac{u}{\alpha(x)}},$$

thus

$$\sup_{u \in \left[\frac{s}{n}, sn\right]} \left| \frac{\partial}{\partial u} l(x, u) \right| = \frac{\beta(x)}{\alpha(x)} e^{-\frac{s}{\alpha(x) n}}.$$

We have

$$\begin{aligned}
&\int_A \int_0^{\frac{\epsilon}{n}} \frac{\beta(x)}{\alpha(x)} e^{-\frac{s}{\alpha(x) n}} ds \, d\sigma(x) \\
&\quad = \int_A \frac{\beta(x)}{\alpha(x)} (-n) \alpha(x) \left[e^{-\frac{\epsilon}{\alpha(x) n^2}} - 1 \right] d\sigma(x) \\
&\quad \le n \int_A \beta(x) d\sigma(x) < \infty.
\end{aligned}$$

Therefore, (32) holds. $\square$

Remark 21. Obviously, the conditions of Lemma 20 are satisfied when, for example, the function β is locally integrable, while the function α is locally bounded.

Theorem 22. *Let the measure m be given by* (40) *and assume that the conditions of Lemma 20 are satisfied. Then, for $\theta \in C_0(X \to \mathbb{R}_+)$, the corresponding Radon–Nikodym derivative of the measure μ_m, $\frac{d\mu_m^\theta}{d\mu_m}$, is given by*

$$\frac{d\mu_m^\theta}{d\mu_m}(\eta) = \exp\left[\int_Y (1 - \theta^{-1}(x)) \frac{1}{\alpha(x)} d\eta(x) - \int_Y \log(\theta(x)) \beta(x) \, d\sigma(x) \right].$$

Proof. We have

$$\begin{aligned}
&\int_Y \log\left(\frac{l(x, \theta^{-1}(x) s_x)}{l(x, s_x)} \right) s_x^{-1} d\eta(x) \\
&\quad = \int_Y \left[\log\left(l(x, \theta^{-1}(x) s_x) \right) - \log(l(x, s_x)) \right] s_x^{-1} d\eta(x) \\
&\quad = \int_Y \left[\log\left(\beta(x) e^{\frac{-s_x \theta^{-1}(x)}{\alpha(x)}} \right) - \log\left(\beta(x) e^{\frac{-s_x}{\alpha(x)}} \right) \right] s_x^{-1} d\eta(x) \\
&\quad = \int_Y \left[-s_x \theta^{-1}(x) + s_x \right] \frac{1}{\alpha(x)} s_x^{-1} d\eta(x) \\
&\quad = \int_Y (1 - \theta^{-1}(x)) \frac{1}{\alpha(x)} d\eta(x),
\end{aligned}$$

and by (37)

$$\begin{aligned}
&\int_Y \int_{\mathbb{R}_+} \frac{(l(x,s) - l(x,\theta^{-1}(x)s)}{s}\, ds\, d\sigma(x)\\
&= \int_Y \int_{\mathbb{R}_+} \frac{\beta(x)e^{-\frac{s}{\alpha(x)}} - \beta(x)e^{-\frac{s\theta^{-1}(x)}{\alpha(x)}}}{s}\, ds\, dx\\
&= \int_Y \beta(x)\left[-\int_{\mathbb{R}_+}\left(\frac{e^{\frac{s}{\alpha(x)}(1-\theta^{-1}(x))} - 1}{s}\right) e^{-\frac{s}{\alpha(x)}}\, ds\right] d\sigma(x)\\
&= \int_Y \beta(x)\log\left(1 - \alpha(x)\frac{1-\theta^{-1}(x)}{\alpha(x)}\right) d\sigma(x)\\
&= \int_Y \beta(x)\log(\theta^{-1}(x))\, d\sigma(x)\\
&= -\int_Y \beta(x)\log(\theta(x))\, d\sigma(x).
\end{aligned}$$

Thus, we have

$$\int_Y \int_{\mathbb{R}_+} \frac{(l(x,s) - l(x,\theta^{-1}(x)s)}{s}\, ds\, d\sigma(x) = -\int_Y \beta(x)\log(\theta(x)) d\sigma(x).$$

Now, by Theorem 13, the statement follows. □

Remark 23. Note that, for any $A \in \mathcal{B}_0(X)$ such that $\sigma(A) > 0$ and $\beta(x) > 0$ for all $x \in A$, we have $m(A \times \mathbb{R}_+) = \infty$.

3.2.2. *Completely random measures with a Lévy measure of logarithmic type near zero.* Let us consider another example of a quasi-invariant measure. Let $Y \in \mathcal{B}(X)$. Consider measurable functions $\alpha : Y \to \mathbb{R}_+$ and $\beta : Y \to \mathbb{R}_+$. Let $\epsilon \in (0, e^{-1})$ and we define, for $(x,s) \in X \times \mathbb{R}_+$,

$$l(x,s) = \begin{cases} \beta(x)(-\log s)^{-\alpha(x)}, & x \in Y,\ s \in (0,\epsilon),\\ g(x,s), & x \in Y,\ s \in [\epsilon,\infty),\\ 0, & x \notin Y,\ s \in \mathbb{R}_+, \end{cases} \tag{41}$$

so that on $Y \times (0,\epsilon)$

$$dm(x,s) = \beta(x)\,\frac{(-\log s)^{-\alpha(x)}}{s}\, d\sigma(x)\, ds, \tag{42}$$

and on $Y \times [\epsilon,\infty)$

$$dm(x,s) = \frac{g(x,s)}{s}\, d\sigma(x)\, ds. \tag{43}$$

Here we assume that the function $g(x,s)$ is strictly positive and satisfies

$$\int_A \int_\epsilon^\infty \frac{g(x,s)}{s}\, ds\, d\sigma(x) < \infty \tag{44}$$

for all $A \in \mathcal{B}_0(Y)$.

Lemma 24. *Let $\beta \in L^1_{\mathrm{loc}}(Y,\sigma)$. Then the measure m with the function $l(x,s)$ given by* (41) *satisfies* (25).

Proof. By (44), we only need to check that, for any $A \in \mathcal{B}_0(Y)$

$$\int_{A\times(0,\epsilon]} l(x,s)\, d\sigma(x)\, ds = \int_{A\times(0,\epsilon]} \beta(x)(-\log s)^{-\alpha(x)}\, d\sigma(x)\, ds < +\infty.$$

But, for all $s \in (0, e^{-1}]$, $-\log s \geq 1$, and since $\alpha(x) > 0$, $(-\log s)^{-\alpha(x)} \leq 1$. Hence, the statement trivially follows. □

Proposition 25. *Let $\beta \in L^1_{\mathrm{loc}}(Y, \sigma)$. Then the measure m with the function $l(x,s)$ given by* (41) *satisfies the conditions of Corollary 15.*

Proof. Let us set $l_1(x,s) = l(x,s)$ and $l_2(x,s) = 0$. It suffices to show that, for each $n \in \mathbb{N}$ and $A \in \mathcal{B}_0(Y)$,

$$\int_A \int_0^{\frac{\epsilon}{n}} \sup_{u \in [\frac{s}{n}, sn]} \left| \frac{\partial}{\partial u} l(x,u) \right| d\sigma(x)\, ds < \infty.$$

We have

$$\frac{\partial}{\partial u} l(x,u) = \frac{\beta(x)\alpha(x)(-\log u)^{-\alpha(x)-1}}{u}.$$

Hence, for each $s \in (0, \frac{\epsilon}{n})$,

$$\begin{aligned}
\sup_{u \in [\frac{s}{n}, sn]} \left| \frac{\partial}{\partial u} l(x,u) \right| &= \beta(x)\alpha(x) \sup_{u \in [\frac{s}{n}, sn]} \frac{(-\log u)^{-\alpha(x)-1}}{u} \\
&\leq \beta(x)\alpha(x) \left(\sup_{u \in [\frac{s}{n}, sn]} \frac{1}{u} \right) \left(\sup_{u \in [\frac{s}{n}, sn]} (-\log u)^{-\alpha(x)-1} \right) \\
&= \beta(x)\alpha(x) \frac{n}{s(-\log(sn))^{\alpha(x)+1}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_A d\sigma(x) \int_0^{\frac{\epsilon}{n}} ds\, \beta(x)\alpha(x) \frac{n}{s} \frac{1}{(-\log(sn))^{\alpha(x)+1}} \\
&\quad = n \int_A d\sigma(x) \beta(x)\alpha(x) \int_{-\infty}^{\log \epsilon} ds\, \frac{1}{(-s)^{\alpha(x)+1}} \\
&\quad = n \int_A d\sigma(x) \frac{\beta(x)}{(-\log \epsilon)^{\alpha(x)}} \\
&\quad \leq \int_A \beta(x)\, d\sigma(x) < \infty.
\end{aligned}$$

Therefore, the conditions of Corollary 15 are satisfied. □

We finish this part with the following observation, which we will use later on.

Proposition 26. *(i) Assume that $\alpha(x) > 1$ for all $x \in X$ and*

$$\int_A \frac{\beta(x)}{\alpha(x) - 1}\, d\sigma(x) < \infty,$$

for each $A \in \mathcal{B}_0(Y)$. Then

$$m(A \times \mathbb{R}_+) = \int_A d\sigma(x) \int_{\mathbb{R}_+} \frac{l(x,s)}{s} ds < \infty.$$

(ii) Assume that $\alpha(x) \leq 1$ for all $x \in X$. Then, for each $A \in \mathcal{B}_0(Y)$ with $\sigma(A) > 0$, we have

$$m(A \times \mathbb{R}_+) = \int_A d\sigma(x) \int_{\mathbb{R}_+} \frac{l(x,s)}{s} ds = +\infty.$$

Proof. For each $A \in \mathcal{B}_0(Y)$, we have

$$\int_A d\sigma(x) \int_0^{+\infty} ds\, \frac{l(x,s)}{s} ds = \int_A d\sigma(x) \int_0^{\epsilon} ds\, \frac{l(x,s)}{s} + \int_A d\sigma(x) \int_{\epsilon}^{+\infty} \frac{g(x,s)}{s}.$$

By (44) the second integral on the right hand side is finite. Hence, we need to calculate the first integral on the right hand side.

(i) We have

$$\begin{aligned}\int_A d\sigma(x) \int_0^{\epsilon} ds\, \frac{l(x,s)}{s} &= \int_A d\sigma(x) \int_0^{\epsilon} \frac{ds}{s} \frac{\beta(x)}{(-\log s)^{\alpha(x)}} \\ &= \int_A \frac{1}{\alpha(x)-1} (-\log \epsilon)^{-\alpha(x)} d\sigma(x) \\ &\le \int_A \frac{\beta(x)}{\alpha(x)-1}\, d\sigma(x) < +\infty.\end{aligned}$$

(ii) We have

$$\int_A d\sigma(x) \int_0^{\epsilon} ds\, \frac{l(x,s)}{s} = \int_A d\sigma(x)\, \beta(x) \int_{-\log\epsilon}^{+\infty} ds \frac{1}{s^{\alpha(x)}} = +\infty.$$

□

3.2.3. *Completely random measures with a Lévy measure of power type near zero.* Let $Y \in \mathcal{B}(X)$. Let functions $\alpha : Y \to (0,1)$ and $\beta : Y \to \mathbb{R}_+$ be measurable. Let $\epsilon \in (0,1)$. We define for $(x,s) \in X \times \mathbb{R}_+$

$$l(x,s) = \begin{cases} \beta(x) s^{1-\alpha(x)}, & x \in Y,\ s \in (0,\epsilon), \\ g(x,s), & x \in Y,\ s \in [\epsilon, \infty), \\ 0, & x \notin Y. \end{cases} \tag{45}$$

Thus, on $Y \times (0,\epsilon)$,

$$dm(x,s) = \frac{\beta(x)}{s^{\alpha(x)}}\, d\sigma(x)\, ds, \tag{46}$$

and on $Y \times [\epsilon, \infty)$

$$dm(x,s) = \frac{g(x,s)}{s}\, d\sigma(x)\, ds. \tag{47}$$

Lemma 27. *Let the measure m have the function $l(x,s)$ defined by* (45). *Let $\beta \in L^1_{\mathrm{loc}}(Y,\sigma)$. Then m satisfies* (25).

Proof. For each $A \in \mathcal{B}_0(Y)$,

$$\int_A \int_0^{\epsilon} \beta(x) s^{1-\alpha(x)}\, ds\, d\sigma(x) \le \int_A \beta(x)\, d\sigma(x) < \infty.$$

By (44), the statement follows. □

Proposition 28. *Let the measure m have the function $l(x,s)$ given by* (45). *Let $\beta \in L^1_{\mathrm{loc}}(Y,\sigma)$. Then the measure m satisfies the conditions of Corollary 15.*

Proof. We set $l_1(x,s) = l(x,s)$ and $l_2(x,s) = 0$. Then

$$\frac{\partial}{\partial u} l(x,u) = \frac{\beta(x)(1-\alpha(x))}{u^{\alpha(x)}}.$$

Hence,

$$\sup_{u \in \left[\frac{s}{n}, sn\right]} \left| \frac{\partial}{\partial u} l(x,u) \right| = \frac{\beta(x)(1-\alpha(x)) n^{\alpha(x)}}{s^{\alpha(x)}} \le \frac{\beta(x)(1-\alpha(x)) n}{s^{\alpha(x)}}.$$

Hence, for each $A \in \mathcal{B}_0(Y)$,

$$\begin{aligned}
&\int_A \int_0^{\frac{\epsilon}{n}} \sup_{u \in \left[\frac{s}{n}, sn\right]} \left| \frac{\partial}{\partial u} l(x,u) \right| d\sigma(x)\, ds \\
&\quad \le \int_A \int_0^{\frac{\epsilon}{n}} \frac{\beta(x)(1-\alpha(x))n}{s^{\alpha(x)}}\, d\sigma(x)\, ds \\
&\quad = \int_A \beta(x) n \left(\frac{\epsilon}{n}\right)^{-\alpha(x)+1} d\sigma(x) \\
&\quad \le n \int_A \beta(x)\, d\sigma(x) < \infty.
\end{aligned}$$

Therefore, the conditions of Corollary 15 are satisfied. □

Proposition 29. *Let the conditions of Proposition 28 be satisfied.*

(i) Assume additionally that $\frac{\beta}{1-\alpha} \in L^1_{\mathrm{loc}}(X, \sigma)$. *Then, for each* $A \in \mathcal{B}_0(Y)$, *we have* $m(A \times \mathbb{R}_+) < \infty$.

(ii) Assume that $A \in \mathcal{B}_0(Y)$ *and*

$$\int_A \frac{\beta(x)}{1-\alpha(x)}\, d\sigma(x) = \infty.$$

Then $m(A \times \mathbb{R}_+) = \infty$.

Proof. Analogously to the proof of Proposition 26, we only need to consider the integral

$$\int_A d\sigma(x) \int_0^\epsilon ds\, \frac{l(x,s)}{s} = \int_A \frac{\beta(x)}{1-\alpha(x)}\, \epsilon^{1-\alpha(x)}\, d\sigma(x).$$

Noting that

$$\epsilon \le \epsilon^{1-\alpha(x)} \le 1,$$

we easily conclude the statement. □

4. Quasi-invariance of completely random measures with respect to transformations of atoms

From now on, we will assume that $X = \mathbb{R}^d$ and σ is the Lebesgue measure dx. (More generally, we could assume that X is a smooth Riemannian manifold and σ is a volume measure on it.)

In this section, we will consider the transformations of the atoms of completely random measures by the action of the group of diffeomorphisms which are identical outside a compact set.

4.1. **General theory.** A *diffeormorphism of* $X = \mathbb{R}^d$ is a bijective mapping $\varphi : X \to X$ such that both φ and φ^{-1} are infinitely differentiable. We say that a diffeomorphism φ has *compact support* if there exists a compact set $\Lambda \subset X$ such that $\varphi(x) = x$ for all $x \in \Lambda^c$. We denote by $\mathrm{Diff}_0(X)$ the set of all diffeomorphisms of X which have compact support.

It is clear that for any $\varphi, \psi \in \mathrm{Diff}_0(X)$, their composition $\varphi \circ \psi$ again belongs to $\mathrm{Diff}_0(X)$. So we define a group product on $\mathrm{Diff}_0(X)$ as the composition of two diffeomorphisms. The neutral element of this group is the identity mapping e. Note that the product in this group is non-commutative.

The group $\mathrm{Diff}_0(X)$ naturally acts on X: for each $\varphi \in \mathrm{Diff}_0(X)$, $\varphi(x)$ is the action of φ on $x \in X$. Furthermore, the group $\mathrm{Diff}_0(X)$ naturally acts on $\mathbb{M}(X)$, the space of Radon measures on X. For each $\varphi \in \mathrm{Diff}_0(X)$ and $\eta \in \mathbb{M}(\mathbb{X})$, the action of φ on η is defined by $\varphi^*\eta$, the pushforward of η under φ;

$$\varphi^*\eta(\Delta) = \eta(\varphi^{-1}\Delta), \quad \Delta \in \mathcal{B}(X).$$

Clearly $\varphi^*\eta \in \mathbb{M}(X)$.

Let $\eta \in \mathbb{K}(X)$,

$$\eta = \sum_i s_i \delta_{x_i}. \tag{48}$$

Then, for $\varphi \in \mathrm{Diff}_0(X)$

$$\varphi^*\eta = \sum_i s_i \delta_{\varphi(x_i)}. \tag{49}$$

In particular, $\varphi^*\eta \in \mathbb{K}(X)$, that is the group $\mathrm{Diff}_0(X)$ acts on $\mathbb{K}(X)$.

Note that each $\varphi \in \mathrm{Diff}_0(X)$ transforms the atoms of a discrete measure, leaving the weights without changes.

If μ is a probability measure on $\mathbb{K}(X)$, there is a natural question whether μ is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$. If this is indeed the case, one gets a quasi-regular representation of $\mathrm{Diff}_0(X)$ in $L^2(\mathbb{K}(X), \mu)$.

Theorem 30. *Let m be a measure on $X \times \mathbb{R}_+$ which satisfies* (8), (9). *Let μ_m be the corresponding completely random measure, see Corollary 10. For each $\varphi \in \mathrm{Diff}_0(X)$, we extend the action of φ to $X \times \mathbb{R}_+$ by setting*

$$X \times \mathbb{R}_+ \ni (x, s) \mapsto (\varphi(x), s) \in X \times \mathbb{R}_+, \tag{50}$$

which is a smooth diffeomorphism of $X \times \mathbb{R}_+$. Let $m_\varphi := \varphi^* m$ be the pushforward of the measure m under (50). *Then μ_m is quasi-invariant with respect to the action of* $\mathrm{Diff}_0(X)$ *if and only if, for each $\varphi \in \mathrm{Diff}_0(X)$,*

- *m and m_φ are equivalent;*
- $\displaystyle\int_{\hat{X}} \left(\sqrt{\frac{dm_\varphi}{dm}} - 1\right)^2 dm < \infty.$

Proof. In view of (48), (49) and the construction of the measure μ_m, μ_m is quasi-invariant with respect to $\mathrm{Diff}_0(X)$ if and only if the Poisson measure π_m is quasi-invariant under the following action of $\mathrm{Diff}_0(X)$ onto $\Gamma(\hat{X})$:

$$\gamma = \{(x_i, s_i)\} \mapsto \varphi\gamma := \{(\varphi(x_i), s_i)\}, \tag{51}$$

where $\varphi \in \mathrm{Diff}_0(X)$. Note that, for each $\gamma \in \Gamma(\hat{X})$, $\varphi\gamma$ indeed belongs to $\Gamma(\hat{X})$.

Let $\varphi^*\pi_m$ be the pushforward of π_m under (51). We claim that

$$\varphi^*\pi_m = \pi_{\varphi^* m} = \pi_{m_\varphi},$$

i.e., the Poisson measure on $\Gamma(\hat{X})$ with intensity measure m_φ. Indeed, for each $f \in C_0(\hat{X})$, we have

$$\begin{aligned}
\int_{\Gamma_{pf}(\hat{X})} e^{i\langle f,\gamma\rangle} d(\varphi^*\pi_m)(\gamma) &= \int_{\Gamma_{pf}(\hat{X})} e^{i\sum_{(x,s)\in\gamma} f(x,s)} d(\varphi^*\pi_m)(\gamma) \\
&= \int_{\Gamma_{pf}(\hat{X})} e^{i\sum_{(x,s)\in\gamma} f(\varphi(x),s)} d\pi_m(\gamma) \\
&= \exp\left[\int_{\hat{X}} \left(e^{if(\varphi(x),s)} - 1\right) dm(x,s)\right] \\
&= \exp\left[\int_{\hat{X}} \left(e^{if(x,s)} - 1\right) dm_\varphi(x,s)\right] \\
&= \int_{\Gamma_{pf}(\hat{X})} e^{i\langle f,\gamma\rangle} d\pi_{m_\varphi}(\gamma).
\end{aligned}$$

Now the statement of the theorem immediately follows from the Theorem 2. □

Corollary 31. *Let the measure m on $\hat{X}$ be of the form (22), let $l(x,s) > 0$ for all $x \in X$ and $s \in \mathbb{R}_+$, and let (25) be satisfied for all $A \in \mathcal{B}_0(X)$. Let μ_m be the corresponding completely random measure. Then μ_m is quasi-invariant with respect to the action of* $\mathrm{Diff}_0(X)$ *if and only if, for each $\varphi \in \mathrm{Diff}_0(X)$,*

$$\begin{aligned}(52)\qquad &\int_{\hat{X}} \left(\sqrt{\frac{l(\varphi^{-1}(x),s)}{l(x,s)}\, J_\varphi(x)} - 1 \right)^2 \frac{l(x,s)}{s}\, dx\, ds \\ &\qquad = \int_{\hat{X}} \left(\sqrt{l(\varphi^{-1}(x),s) J_\varphi(x)} - \sqrt{l(x,s)} \right)^2 \frac{1}{s}\, dx\, ds < +\infty,\end{aligned}$$

where $J_\varphi(x)$ is the modulus of the determinant of the Jacobian matrix of φ.

Proof. By the definition of m_φ, for each $\Delta \in \mathcal{B}(\hat{X})$,

$$\begin{aligned} m_\varphi(\Delta) &= \int_X \chi_\Delta(\varphi(x),s)\, dm(x,s) \\ &= \int_X \chi_\Delta(\varphi(x),s)\, \frac{l(x,s)}{s}\, dx\, ds \\ &= \int_X \chi_\Delta(\varphi(x),s)\, \frac{l(\varphi^{-1}(\varphi(x))),s)}{s}\, dx\, ds \\ &= \int_X \chi_\Delta(x,s)\, \frac{l(\varphi^{-1}(x),s)}{s}\, J_\varphi(x)\, dx\, ds \\ &= \int_\Delta \frac{l(\varphi^{-1}(x),s)}{l(x,s)}\, J_\varphi(x)\, dm(x,s). \end{aligned}$$

Therefore, we have the Radon–Nikodym derivative

$$(53)\qquad \frac{dm_\varphi}{dm}(x,s) = \frac{l(\varphi^{-1}(x)s)}{l(x,s)}\, J_\varphi(x).$$

Therefore, the second condition in Theorem 30 becomes (52). □

The following result was shown in [21].

Corollary 32. *Let m be a measure on $X \times \mathbb{R}_+$ of the form*

$$dm(x,s) = dx\, d\lambda(s),$$

where λ is a measure on $\mathbb{R}_+$. Further assume that

$$\int_{\mathbb{R}_+} \min\{1,s\}\, d\lambda(s) < \infty.$$

Then μ_m is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ *if and only if* $\lambda(\mathbb{R}_+) < \infty$.

Proof. Note that (8) and (9) are satisfied. In this case,

$$\frac{dm_\varphi}{dm}(x,s) = J_\varphi(x).$$

Hence,

$$(54)\qquad \int_{\hat{X}} \left(\sqrt{\frac{dm_\varphi}{dm}} - 1 \right)^2 dm = \int_X \left(\sqrt{J_\varphi(x)} - 1 \right)^2 dx\, \lambda(\mathbb{R}_+).$$

Since the function $\left(\sqrt{J_\varphi(x)} - 1 \right)^2$ is smooth and has compact support in X, we have

$$\int_X \left(\sqrt{J_\varphi(x)} - 1 \right)^2 < \infty.$$

Hence (54) is finite, if and only if, $\lambda(\mathbb{R}_+) < \infty$. □

The following result generalizes Corollary 32.

Corollary 33. *Let m be a measure on $X \times \mathbb{R}_+$ which satisfies* (8). *Assume that, for each $\varphi \in \mathrm{Diff}_0(X)$, the measures m and m_φ are equivalent. Further assume that*

$$m(\Lambda \times \mathbb{R}_+) < \infty, \quad \Lambda \in \mathcal{B}_0(X). \tag{55}$$

Then μ_m is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ and for each $\varphi \in \mathrm{Diff}_0(X)$, the corresponding Radon–Nikodym density is given by

$$\frac{d\mu_m^\varphi}{d\mu_m}(\eta) = \prod_{x \in \tau(\eta)} \frac{dm_\varphi}{dm}(x, s_x). \tag{56}$$

Proof. Note that (55) implies (9). According to Theorem 4, to prove quasi-invariance, it suffices to prove that, for each $\varphi \in \mathrm{Diff}_0(X)$,

$$\int_{\hat{X}} \left| \frac{dm_\varphi}{dm} - 1 \right| dm < \infty.$$

Choose $\Lambda \in \mathcal{B}_0(X)$ such that $\varphi(x) = x$ for all $x \in \Lambda^c$. Then

$$\frac{dm_\varphi}{dm}(x, s) = 1, \quad \text{for all } (x, s) \in \Lambda^c \times \mathbb{R}_+.$$

Hence

$$\begin{aligned} \int_{\hat{X}} \left| \frac{dm_\varphi}{dm} - 1 \right| dm &= \int_{\Lambda \times \mathbb{R}_+} \left| \frac{dm_\varphi}{dm} - 1 \right| dm \\ &\le \int_{\Lambda \times \mathbb{R}_+} \left(\frac{dm_\varphi}{dm} + 1 \right) dm \\ &= m_\varphi(\Lambda \times \mathbb{R}_+) + m(\Lambda \times \mathbb{R}_+) \\ &= 2m(\Lambda \times \mathbb{R}_+) < \infty. \end{aligned}$$

Here id denotes the identity map.

Formula (56) will follow from formula (6) (see also Remark 5) if we show

$$\int_{\hat{X}} \left(1 - \frac{dm_\varphi}{dm} \right) dm = 0.$$

Choose again $\Lambda \in \mathcal{B}_0(X)$ such that φ is equal to the identity on Λ^c. Then, for any $(x, s) \in \Lambda^c \times \mathbb{R}_+$, we have

$$\frac{dm_\varphi}{dm}(x, s) = 1.$$

Hence

$$\int_{\hat{X}} \left(1 - \frac{dm_\varphi}{dm} \right) dm = \int_{\Lambda \times \mathbb{R}_+} \left(1 - \frac{dm_\varphi}{dm} \right) dm) = 0.$$

□

Corollary 34. *Assume that the measure m satisfies* (22) *with $l(x, s) > 0$ for all $(x, s) \in X \times \mathbb{R}_+$ and assume that* (55) *holds. Then μ_m is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ and for each $\varphi \in \mathrm{Diff}_0(X)$ we have*

$$\frac{d\mu_m^\varphi}{d\mu_m}(\eta) = \prod_{x \in \tau(\eta)} \frac{l(\varphi^{-1}(x), s_x)}{l(x, s_x)} J_\varphi(x). \tag{57}$$

Proof. Corollary 34 follows from Corollary 33 and (53). □

Corollary 35. *Let the assumptions of Corollary 31 be satisfied. Assume that there exists an open set $\Lambda \subset X$, $\Lambda \neq \emptyset$, such that, for all $x \in \Lambda$,*

$$\int_{\mathbb{R}_+} \frac{l(x,s)}{s}\, ds = \infty. \tag{58}$$

Assume that, for each $x \in \Lambda$, the limit $\lim_{s\to 0} l(x,s) =: l(x,0)$ exists, $l(x,0) \neq 0$, and the function $\Lambda \ni x \mapsto l(x,0)$ is continuous. Then the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

Proof. Without loss of generality, we may assume that the set Λ is bounded. Assume that μ_m is quasi-invariant with respect to $\mathrm{Diff}_0(X)$. Then by Corollary 31, for each diffeomorphism φ with support in Λ, we have

$$\int_\Lambda \int_{\mathbb{R}_+} \left(\sqrt{\frac{l(\varphi^{-1}(x),s)}{l(x,s)} J_\varphi(x)} - 1 \right)^2 \frac{l(x,s)}{s}\, ds\, dx < \infty.$$

Hence, for a.a. $x \in \Lambda$,

$$\int_{\mathbb{R}_+} \left(\sqrt{\frac{l(\varphi^{-1}(x),s)}{l(x,s)} J_\varphi(x)} - 1 \right)^2 \frac{l(x,s)}{s}\, ds < \infty. \tag{59}$$

Note that, for each $x \in \Lambda$,

$$\lim_{s\to 0} \left(\sqrt{\frac{l(\varphi^{-1}(x),s)}{l(x,s)} J_\varphi(x)} - 1 \right)^2 = \left(\sqrt{\frac{l(\varphi^{-1}(x),0)}{l(x,0)} J_\varphi(x)} - 1 \right)^2. \tag{60}$$

By (58), (59) and (60), for a.a. $x \in \Lambda$,

$$\frac{l(\varphi^{-1}(x),0)}{l(x,0)} J_\varphi(x) = 1,$$

or equivalently, for a.a. $x \in \Lambda$,

$$l(\varphi^{-1}(x),0) = \frac{l(x,0)}{J_\varphi(x)}. \tag{61}$$

By the continuity of the function $l(\cdot,0)$, we get that equality (61) holds, in fact, for all $x \in \Lambda$ and all diffeomorphisms $\varphi \in \mathrm{Diff}_0(X)$ with support in Λ.

But equality (61) is impossible. Just choose any $x, y \in \Lambda$ and any diffeomorphisms $\varphi, \psi \in \mathrm{Diff}_0(X)$ with support in Λ such that, for some $x \in \Lambda$, $\varphi^{-1}(x) = \psi^{-1}(x) = y$ and $J_\varphi(x) \neq J_\psi(x)$. Then

$$l(y,0) = \frac{l(x,0)}{J_\varphi(x)} \neq \frac{l(x,0)}{J_\psi(x)} = l(y,0),$$

which is a contradiction. □

Corollary 36. *Let the assumptions of Corollary 31 be satisfied. Assume that there exists an open set $\Lambda \subset X$, $\Lambda \neq \emptyset$, such that, for all $x \in \Lambda$,*

$$\int_{\mathbb{R}_+} \frac{l(x,s)}{s}\, ds = \infty.$$

Assume that there exists a diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ such that, for all $x \in \Lambda$, we have

$$\lim_{s\to 0} \frac{l(\varphi^{-1}(x),s)}{l(x,s)} J_\varphi(x) \neq 1.$$

Then the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

Proof. It immediately follows from the assumptions of the corollary that, for this diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$, we get

$$\int_{\hat{X}} \left(\sqrt{\frac{l(\varphi^{-1}(x), s)}{l(x,s)}\, J_\varphi(x)} - 1 \right)^2 \frac{l(x,s)}{s}\, dx\, ds = \infty.$$

Hence, the condition of Corollary 35 is not satisfied and the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$. □

4.2. **Examples.**

4.2.1. *Completely random gamma measures.* Just as in subsec. 3.2.1, consider the measure m with

$$l(x,s) = \beta(x) e^{-\frac{s}{\alpha(x)}},$$

where $\alpha, \beta : X \to \mathbb{R}_+$. Assume that the function β is continuous and $\alpha \in L^1_{\mathrm{loc}}(X)$. This, in particular implies that $\alpha\beta \in L^1_{\mathrm{loc}}(X)$, hence the condition of Lemma 18 is satisfied.

Condition (58) is evidently satisfied for each $x \in X$. We also evidently have

$$l(x,0) = \lim_{s\to 0} \beta(x) e^{-\frac{s}{\alpha(x)}} = \beta(x).$$

Hence, the conditions of Corollary 35 are satisfied and the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

4.2.2. *Completely random measures with a Lévy measure of logarithmic type near zero.* We consider two cases.

Case 1. Let $l(x,s)$ be given by formula (41) with $Y = X$ and α, β being continuous functions. Since $\beta \in L^1_{\mathrm{loc}}(X, dx)$, the condition of Lemma 24 is satisfied.

Let us also assume that $\alpha(x) \le 1$. By Proposition 26, we then get that equation (58) holds for all $x \in X$.

Let us assume that the function α is not constant. We get

$$\lim_{s\to 0} \frac{l(\varphi^{-1}(x), s)}{l(x,s)} J_\varphi(x) = J_\varphi(x)\, \frac{\beta(\varphi^{-1}(x))}{\beta(x)} \lim_{s\to 0} (-\log s)^{\alpha(x)-\alpha(\varphi^{-1}(x))}. \tag{62}$$

Choose an open set $\Lambda \subset X$ and a diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ so that, for all $x \in \Lambda$,

$$\alpha(x) > \alpha(\varphi^{-1}(x)).$$

Hence,

$$\lim_{s\to 0} \frac{l(\varphi^{-1}(x), s)}{l(x,s)} J_\varphi(x) = +\infty.$$

Hence, the condition of Corollary 36 is satisfied and the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

If the function α is constant, then evidently formula (62) becomes

$$\lim_{s\to 0} \frac{l(\varphi^{-1}(x), s)}{l(x,s)} J_\varphi(x) = J_\varphi(x)\, \frac{\beta(\varphi^{-1}(x))}{\beta(x)}.$$

By Corollary 36, we will conclude that the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ if we show that there exist $\varphi \in \mathrm{Diff}_0(X)$ and an open non-empty set Λ such that, for all $x \in \Lambda$,

$$J_\varphi(x) - \frac{\beta(x)}{\beta(\varphi^{-1}(x))} \ne 0. \tag{63}$$

Since J_φ and $\frac{\beta}{\beta\circ\varphi^{-1}}$ are continuous functions, this will follow from the statement that there exist $\varphi \in \mathrm{Diff}_0(X)$ and $x \in X$ such that (63) holds. But for this, we can easily construct a diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ such that $\varphi(x) = x$ but $J_\varphi(x) \ne 1$.

Case 2. Let $l(x,s)$ be given by formula (41) with $Y = X$. Assume that the conditions of Proposition 26, (i) are satisfied. In particular $\alpha(x) > 1$ for all $x \in X$. Then, by Proposition 26, (i) and Corollary 33, μ_m is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

4.2.3. *Completely random measures with a Lévy measure of power type near zero.* Let $l(x,s)$ be as in subsec. 3.2.3 with $Y = X$. If $\beta \in L^1_{\mathrm{loc}}(X, dx)$ and $\frac{\beta}{1-\alpha} \in L^1_{\mathrm{loc}}(X, dx)$. Then, by Proposition 29 and Corollary 33, μ_m is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$.

5. Quasi-invariance and partial quasi-invariance with respect to the semidirect product

In this section, we will study quasi-invariance of μ_m with respect to the semidirect product of the groups $C_0(X \to \mathbb{R}_+)$ and $\mathrm{Diff}_0(X)$.

5.1. **Quasi-invariance with respect to the semidirect product.** We recall that an *automorphism* α of a group $(G, \cdot)$ is a bijective mapping $\alpha : G \to G$ such that, for any $g_1, g_2 \in G$, we have $\alpha(g_1 \cdot g_2) = \alpha(g_1) \cdot \alpha(g_2)$.

Following [21], we define the semidirect product of $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$. The group $\mathrm{Diff}_0(X)$ acts on $C_0(X \to \mathbb{R}_+)$ by automorphisms. More precisely, for each $\varphi \in \mathrm{Diff}_0(X)$, we may define an automorphism of $C_0(X \to \mathbb{R}_+$ by

$$C_0(X \to \mathbb{R}_+) \ni \theta \mapsto \alpha(\varphi)\theta = \theta \circ \varphi^{-1} \in C_0(X \to \mathbb{R}_+).$$

Let $\mathfrak{G}$ be the Cartesian product of $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$:

$$\mathfrak{G} = \mathrm{Diff}_0(X) \times C_0(X \to \mathbb{R}_+).$$

We define a group multiplication on $\mathfrak{G}$ as follows: for any $g_1 = (\varphi_1, \theta_1)$, $g_2 = (\varphi_2, \theta_2) \in \mathfrak{G}$, we set

$$g_1 g_2 = (\varphi_1 \circ \varphi_2, \theta_1(\theta_2 \circ \varphi_1^{-1})).$$

Then $\mathfrak{G}$ becomes a group. One denotes this group by

$$\mathfrak{G} = \mathrm{Diff}_0(X) \leftthreetimes C_0(X \to \mathbb{R}_+)$$

and one calls $\mathfrak{G}$ the *semidirect product of* $\mathrm{Diff}_0(X)$ *and* $C_0(X \to \mathbb{R}_+)$ *with respect to* α.

The group $\mathfrak{G}$ naturally acts on $\mathbb{M}(X)$, the space of Radon measures on X: for any $g = (\varphi, \theta) \in \mathfrak{G}$ and any $\eta \in \mathbb{M}(X)$, we define the Radon measure $g\eta$ by

$$d(g\eta)(x) := \theta(x) d(\varphi^* \eta)(x). \tag{64}$$

Here $\varphi^*\eta$ is the push-forward of η under φ. Note that when $g = (\varphi, \theta)$ acts on η, we first act on η by φ, i.e., we take $\varphi^*\eta$, and then we act by θ, i.e., we multiply the measure $\varphi^*\eta$ by θ. Note that each $g \in \mathfrak{G}$ maps $\mathbb{K}(X)$ into $\mathbb{K}(X)$.

Proposition 37. *Let μ be a measure on $\mathbb{K}(X)$ (or $\mathbb{M}(X)$). The measure μ is quasi-invariant with respect to $\mathfrak{G}$ if and only if μ is quasi-invariant with respect to the action of both groups* $\mathrm{Diff}_0(X)$ *and* $C_0(X \to \mathbb{R}_+)$*. In the latter case, we have, for each $g = (\varphi, \theta) \in \mathfrak{G}$,*

$$\frac{d\mu^\theta}{d\mu}(\eta) = \frac{d\mu^\varphi}{d\mu}(\theta^{-1}\eta) \frac{d\mu^\theta}{d\mu}(\eta). \tag{65}$$

Proof. If μ is quasi-invariant with respect to $\mathfrak{G}$, then automatically it is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$, since $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$ are subgroups of $\mathfrak{G}$. So assume that μ is quasi-invariant with respect to $\mathrm{Diff}_0(X)$ and $C_0(X \to \mathbb{R}_+)$ and let us prove that μ is quasi-invariant with respect to $\mathfrak{G}$.

Let $F : \mathbb{M}(X) \to [0, +\infty]$ be a measurable function. Let $g = (\varphi, \theta) \in \mathfrak{G}$. We have, by (64),

$$\begin{aligned}\int_{\mathbb{M}(X)} F(\eta) d\mu^g(\eta) &= \int_{\mathbb{M}(X)} F(g\eta) d\mu(\eta)\\ &= \int_{\mathbb{M}(X)} F(\theta(\varphi^*\eta)) d\mu(\eta)\\ &= \int_{\mathbb{M}(X)} F(\theta\eta) d\mu^\varphi(\eta). \end{aligned} \tag{66}$$

Since μ is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$, we continue (66) as follows:

$$\begin{aligned}&= \int_{\mathbb{M}(X)} F(\theta\eta) \frac{d\mu^\varphi}{d\mu}(\eta)\, d\mu(\eta)\\ &= \int_{\mathbb{M}(X)} F(\eta) \frac{d\mu^\varphi}{d\mu}(\theta^{-1}\eta) d\mu^\theta(\eta).\end{aligned}$$

Since μ is quasi-invariant with respect to $C_0(X \to \mathbb{R}_+)$, we continue:

$$= \int_{\mathbb{M}(X)} F(\eta) \frac{d\mu^\varphi}{d\mu}(\theta^{-1}\eta) \frac{d\mu^\theta}{d\mu}(\eta) d\mu(\eta).$$

The functions $\frac{d\mu^\theta}{d\mu}$ and $\frac{d\mu^\varphi}{d\mu}$ are strictly positive on $\mathbb{M}(X)$ μ-almost everywhere. Let

$$\begin{aligned}A &:= \left\{ \eta \in \mathbb{M}(X) \mid \frac{d\mu^\varphi}{d\mu}(\theta^{-1}\eta) = 0 \right\},\\ B &:= \left\{ \eta \in \mathbb{M}(X) \mid \frac{d\mu^\varphi}{d\mu}(\eta) = 0 \right\}.\end{aligned}$$

As we already said $\mu(B) = 0$. But $A = \theta B$. Hence $\mu(A) = 0$ because of quasi-invariance with respect to $C_0(X \to \mathbb{R}_+)$. Thus

$$\frac{d\mu^\varphi}{d\mu}(\theta^{-1}\eta) \frac{d\mu^\theta}{d\mu}(\eta) > 0 \quad \mu\text{-a.e.}$$

Hence, the probability measures μ^g and μ are equivalent and (65) holds. □

Theorem 38. *Let m satisfy (22) with $l(x, s) > 0$ for all $(x, s) \in \hat{X}$, (25), and (55). Then μ_m is quasi-invariant with respect to $g = (\varphi, \theta) \in \mathfrak{G}$ and the corresponding Radon-Nikodym derivative is given by*

$$\begin{aligned}\frac{d\mu_m^g}{d\mu_m}(\eta) &= \left(\prod_{x \in \tau(\eta)} \frac{l(\varphi^{-1}(x), \theta^{-1}(x) s_x)}{l(x, \theta^{-1}(x) s_x)} J_\varphi(x) \right)\\ &\times \exp\left[\int_X \log\left(\frac{l(x, \theta^{-1}(x) s_x)}{l(x, s_x)} \right) s_x^{-1} d\eta(x) + \int_X \int_{\mathbb{R}_+} \frac{\big(l(x, s) - l(x, \theta^{-1}(x) s)\big)}{s}\, ds\, dx \right].\end{aligned}$$

Proof. By Corollary 34, μ_m is quasi-invariant with respect to $\mathrm{Diff}_0(X)$ and

$$\frac{d\mu_m^\varphi}{d\mu_m}(\eta) = \prod_{x \in \tau(\eta)} \frac{l(\varphi^{-1}(x), s_x)}{l(x, s_x)} J_\varphi(x). \tag{67}$$

In Corollary 15, we set $l_1(x, s) = 0$ and $l_2(x, s) = l(x, s)$. Then (32) and (33) are satisfied and by Theorem 13 and Corollary 15, μ_m is quasi-invariant with respect to $C_0(X \to \mathbb{R}_+)$

and

$$\frac{d\mu_m^\theta}{d\mu_m}(\eta) = \exp\left[\int_X \log\left(\frac{l(x,\theta^{-1}(x)s_x)}{l(x,s_x)}\right) s_x^{-1}d\eta(x) + \int_X\int_{\mathbb{R}_+} \frac{\big(l(x,s)-l(x,\theta^{-1}(x)s)\big)}{s}\,ds\,dx\right]. \tag{68}$$

Now the statement of the theorem follows from Proposition 37, (67) and (68). □

Example 39. Let $l(x,s)$ be given by formula (41) with $Y = X$. Assume that the conditions of Proposition 26, (i) are satisfied. In particular $\alpha(x) > 1$ for all $x \in X$. Then by Proposition 25 and subsec. 4.2.2, (ii), the measure μ_m is quasi-invariant with respect to the action of $\mathfrak{G}$.

Example 40. Let l be as in subsec. 4.2.3. Then it follows from Proposition 28 subsec. 4.2.3 that he measure μ_m is quasi-invariant with respect to the action of $\mathfrak{G}$.

5.2. **Partial quasi-invariance with respect to the semidirect product.** The following definition is taken from [21].

Let $(\Omega,\mathcal{F},P)$ be a probability space, and let G be a group which acts on Ω. We say that *the probability measure P is partially quasi-invariant with respect to transformations $g \in G$* if there exists a filtration $(\mathcal{F}_n)_{n=1}^\infty$ such that

- $\mathcal{F}$ is the minimal σ-algebra on Ω which contains all $\mathcal{F}_n$, $n \in \mathbb{N}$;
- For each $g \in G$ and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that g maps $\mathcal{F}_n$ into $\mathcal{F}_k$;
- For any $n \in \mathbb{N}$ and $g \in G$, there exists a measurable function $R_g^{(n)} : \Omega \to [0,+\infty]$ such that, for each $F : \Omega \to [0,\infty]$ which is $\mathcal{F}_n$-measurable,
$$\int_\Omega F(\omega)dP^g(\omega) = \int_\Omega F(\omega)R_g^{(n)}(\omega)dP(\omega).$$
Here P^g is the push-forward of P under g.

Remark 41. If P is quasi-invariant with respect to the action of G, then it is partially quasi-invariant. In this case, just choose $\mathcal{F}_n = \mathcal{F}$ and $R_g^{(n)} = \frac{dP^g}{dP}$.

Theorem 42. *Assume that the conditions of Theorem 13 are satisfied. Assume that there exists $\Lambda \in \mathcal{B}_0(X)$ such that $m(\Lambda\times\mathbb{R}_+) = +\infty$. Then the measure μ_m is partially quasi-invariance with respect to the action of the group $\mathfrak{G}$.*

Proof. The Borel σ-algebra $\mathcal{B}(\Gamma_{pf}(\hat{X}))$ may be identified as the minimal σ-algebra on $\Gamma_{pf}(\hat{X})$ with respect to which any mapping of the following form is measurable:

$$\Gamma_{pf}(\hat{X}) \ni \gamma \mapsto |\gamma\cap\Lambda|, \quad \Lambda \in \mathcal{B}_0(\hat{X}), \tag{69}$$

see e.g. Section 1.1, in particular Lemma 1.4 in [15]. For each $n \in \mathbb{N}$, we denote by $\mathcal{B}_n(\Gamma_{pf}(\hat{X}))$ the minimal σ-algebra on $\Gamma_{pf}(\hat{X})$ with respect to which each mapping of the form (69) is measurable with $\Lambda \subset [\frac{1}{n},\infty)\times X$. Obviously $(\mathcal{B}_n(\Gamma_{pf}(\hat{X})))_{n=1}^\infty$ is a filtration and $\mathcal{B}(\Gamma_{pf}(\hat{X}))$ is the minimal σ-algebra on $\Gamma_{pf}(\hat{X})$ which contains all $\mathcal{B}_n(\Gamma_{pf}(\hat{X}))$.

Recall (20). Let $\mathcal{B}_n(\mathbb{K}(X))$ denote the image of $\mathcal{B}_n(\Gamma_{pf}(\hat{X}))$ under the mapping $\mathcal{R}$. Therefore, $(\mathcal{B}_n(\mathbb{K}(X)))_{n=1}^\infty$ is a filtration and $\mathcal{B}(\mathbb{K}(X))$ is the minimal σ-algebra on $\mathbb{K}(X)$ which contains all $\mathcal{B}_n(\mathbb{K}(X))$.

The following lemma follows immediately from the definition of $\mathcal{B}_n(\mathbb{K}(X))$.

Lemma 43. *A function F is $\mathcal{B}_n(\mathbb{K}(X))$-measurable if and only if F is $\mathcal{B}(\mathbb{K}(X))$-measurable and for each $\eta = \sum_i s_i\delta_{x_i} \in \mathbb{K}(X)$*

$$F(\eta) = F\left(\sum_{i:\, s_i\ge\frac{1}{n}} s_i\delta_{x_i}\right). \tag{70}$$

Lemma 44. *Let $g = (\varphi, \theta) \in \mathfrak{G}$. Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}$ be such that*

$$\frac{1}{k} \le \frac{1}{n} \inf_{x \in X} \theta(x). \tag{71}$$

Then g maps $\mathcal{B}_n(\mathbb{K}(X))$ into $\mathcal{B}_k(\mathbb{K}(X))$.

Proof. Let $F : \mathbb{K}(X) \to [0, +\infty)$ be a $\mathcal{B}_n(\mathbb{K}(X))$-measurable function. Thus, by Lemma 43, formula (70) holds. We note that the inverse element of $g = (\varphi, \theta)$ in the algebra $\mathfrak{G}$ is $g^{-1} = (\varphi^{-1}, \theta^{-1} \circ \varphi)$. Let us the consider the function

$$\mathbb{K}(X) \ni \eta \mapsto F(g^{-1}\eta) \in [0, +\infty). \tag{72}$$

This function is evidently $\mathcal{B}(\mathbb{K}(X))$-measurable. Then, by Lemma 43 and (71), for $\eta = \sum_i s_i \delta_{x_i} \in \mathbb{K}(X)$,

$$\begin{aligned}
F(g^{-1}\eta) &= F\left(\sum_i \theta^{-1}(\varphi(\varphi^{-1}(x_i))) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(\sum_i \theta^{-1}(x_i) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(\sum_{i:\, \theta^{-1}(x_i) s_i \ge \frac{1}{n}} \theta^{-1}(x_i) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(\sum_{i:\, s_i \ge \frac{1}{n}\, \theta(x_i)} \theta^{-1}(x_i) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(\sum_{i:\, s_i \ge \frac{1}{n}\, \inf_{x\in X} \theta(x)} \theta^{-1}(x_i) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(\sum_{i:\, s_i \ge \frac{1}{k}} \theta^{-1}(x_i) s_i \delta_{\varphi^{-1}(x_i)}\right) \\
&= F\left(g^{-1}\Big(\sum_{i:\, s_i \ge \frac{1}{k}} s_i \delta_{x_i}\Big)\right).
\end{aligned}$$

Hence, by Lemma 43, the function $F(g^{-1}\cdot)$ is $\mathcal{B}_k(\mathbb{K}(X))$-measurable.

Let $A \in \mathcal{B}_n(\mathbb{K}(X))$ and let $F = \chi_A$. Thus, F is a $\mathcal{B}_n(\mathbb{K}(X))$-measurable function. Therefore, $F(g^{-1}\cdot) = \chi_A(g^{-1}\cdot)$ is a $\mathcal{B}_k(\mathbb{K}(X))$-measurable function. But

$$\chi_A(g^{-1}\eta) = \chi_{gA}(\eta),$$

which implies $gA \in \mathcal{B}_k(\mathbb{K}(X))$. □

Next, let $F : \mathbb{K}(X) \to [0, +\infty]$ be measurable with respect to $\mathcal{B}_n(\mathbb{K}(X))$. Let $g = (\varphi, \theta) \in \mathscr{G}$. Then

$$\begin{aligned}
\int_{\mathbb{K}(X)} F(\eta)\, d\mu_m^g(\eta) &= \int_{\mathbb{K}(X)} F(g\eta)\, d\mu_m(\eta) \\
&= \int_{\mathbb{K}(X)} F(\theta(\varphi^*\eta))\, d\mu_m(\eta) \\
&= \int_{\mathbb{K}(X)} F(\theta\eta)\, d\mu_m^{\varphi}(\eta). \qquad (73)
\end{aligned}$$

Let $k \in \mathbb{N}$ be chosen so that

$$\frac{1}{k} \le \frac{1}{n} \inf_{x \in X} \theta^{-1}(x). \tag{74}$$

It follows from the proof of this lemma that the function $\eta \mapsto F(\theta\eta)$ is $\mathcal{B}_k(\mathbb{K}(X))$-measurable.

By the construction of the σ-algebra $\mathcal{B}_k(\Gamma_{pf}(\hat{X}))$, this σ-algebra can be identified with the σ-algebra $\mathcal{B}\left(\Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)\right)$. More precisely, each set

$$A \in \mathcal{B}\left(\Gamma_{pf}\left(X \times [\frac{1}{k}, +\infty)\right)\right)$$

is identified with the set

$$\left\{\gamma \in \Gamma_{pf}(\hat{X} \mid \gamma \cap \left(X \times [\frac{1}{k}, +\infty)\right) \in A\right\}.$$

Under this identification, the restriction of the Poisson measure π_k on $\Gamma_{pf}(\hat{X})$ to the σ-algebra $\mathcal{B}_k(\Gamma_{pf}(\hat{X})) = \mathcal{B}(\Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)$ coincides with the Poisson measure $\pi_{m(k)}$, where $m(k)$ is the restriction of the measure m to $X \times \left[\frac{1}{k}, +\infty\right)$.

Note that, for each $\gamma \in \Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)$ and each $\Lambda \in \mathcal{B}_0(X)$,

$$\gamma \cap \left(\Lambda \times \left[\frac{1}{k}, +\infty\right)\right)$$

is a finite set. Hence, for each $\varphi \in \mathrm{Diff}_0(X)$, the mapping

$$\gamma = \sum_i \delta_{(x_i, s_i)} \mapsto \varphi\gamma = \sum_i \delta_{(\varphi(x_i), s_i)} \tag{75}$$

maps $\Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)$ into $\Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)$. We denote by $\mu^{\varphi}_{m(k)}$ the pushforward of the Poisson measure $\pi_{m(k)}$ under the transformation (75). Thus, we get

$$\begin{aligned}
\int_{\mathbb{K}(X)} F(\theta\eta)\, d\mu^{\varphi}_m(\eta) &= \int_{\Gamma_{pf}\left(X \times \left[\frac{1}{k}, +\infty\right)\right)} F(\theta(\mathcal{R}\gamma))\, d\pi^{\varphi}_{m(k)}(\gamma)\\
&= \int_{\Gamma_{pf}(\hat{X})} F(\theta(\mathcal{R}\gamma))\, d\pi^{\varphi}_{m(k)}(\gamma)\\
&= \int_{\mathbb{K}(X)} F(\theta\eta)\, d\mu^{\varphi}_{m(k)}(\gamma).
\end{aligned} \tag{76}$$

By (25), for each $\Lambda \in \mathcal{B}_0(X)$,

$$m\left(\Lambda \times \left[\frac{1}{k}, +\infty\right)\right) < \infty.$$

Hence, by (73), (76), Theorem 13 and Corollary 34,

$$\begin{aligned}
&\int_{\mathbb{K}(X)} F(\eta) d\mu^{g}_m(\eta) = \int_{\mathbb{K}(X)} F(\theta\eta)\, d\mu^{\varphi}_{m(k)}(\gamma)\\
&= \int_{\mathbb{K}(X)} F(\theta\eta) \prod_{x \in \tau(\eta):\, s_x \ge \frac{1}{k}} \frac{l(\varphi^{-1}(x), s_x)}{l(x, s_x)} J_\varphi(x)\, d\mu_{m(k)}(\eta)\\
&= \int_{\mathbb{K}(X)} F(\theta\eta) \prod_{x \in \tau(\eta):\, s_x \ge \frac{1}{k}} \frac{l(\varphi^{-1}(x), s_x)}{l(x, s_x)} J_\varphi(x)\, d\mu_{m}(\eta)\\
&= \int_{\mathbb{K}(X)} F(\theta\eta) \prod_{x \in \tau(\eta):\, \theta(x) s_x \ge \frac{\theta(x)}{k}} \frac{l(\varphi^{-1}(x), \theta^{-1}(x)\theta(x) s_x)}{l(x, \theta^{-1}(x)\theta(x) s_x)} J_\varphi(x)\, d\mu_{m}(\eta)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{K}(X)} F(\eta) \left(\prod_{x\in\tau(\eta):\, s_x \geq \frac{\theta(x)}{k}} \frac{l(\varphi^{-1}(x), \theta^{-1}(x)s_x)}{l(x, \theta^{-1}(x)s_x)} \right) \\
&\quad \times \exp\left[\int_X \log\left(\frac{l(x, \theta^{-1}(x)s_x)}{l(x, s_x)} \right) s_x^{-1} d\eta(x) \right. \\
&\quad \left. + \int_X \int_{\mathbb{R}_+} \frac{\big(l(x,s) - l(x, \theta^{-1}(x)s)\big)}{s}\, ds\, dx \right] d\mu_m(\eta) \\
&= \int_{\mathbb{K}(X)} F(\eta) R_g^{(n)}(\eta)\, d\mu_m(\eta),
\end{aligned}
$$

where

$$
\begin{aligned}
R_g^{(n)}(\eta) = &\left(\prod_{x\in\tau(\eta):\, s_x \geq \frac{\theta(x)}{k}} \frac{l(\varphi^{-1}(x), \theta^{-1}(x)s_x)}{l(x, \theta^{-1}(x)s_x)} \right) \\
&\times \exp\left[\int_X \log\left(\frac{l(x, \theta^{-1}(x)s_x)}{l(x, s_x)} \right) s_x^{-1} d\eta(x) \right. \\
&\left. + \int_X \int_{\mathbb{R}_+} \frac{\big(l(x,s) - l(x, \theta^{-1}(x)s)\big)}{s}\, ds\, dx \right].
\end{aligned}
$$

(Recall that k depends on n through (74).) □

Example 45. Let m be as in subsec. 3.2.1 and let the functions $\alpha\beta$ and β belong to $L^1_{\mathrm{loc}}(X)$. Further assume that $\beta(x) > 0$. Then by Theorem 22, Remark 23, and Theorem 42, the measure μ_m is partially quasi-invariant with respect to the action of the group $\mathfrak{G}$. By subsec. 4.2.1, the measure μ_m is not quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$, hence it is not quasi-invariant with respect to the action of $\mathfrak{G}$.

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