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SCHRÖDINGER OPERATORS WITH MEASURE-VALUED POTENTIALS: SEMIBOUNDEDNESS AND SPECTRUM

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To V. D. Koshmanenko on the occasion of his 75th birthday

Abstract. We study 1-D Schrödinger operators in the Hilbert space $L^2(\mathbb{R})$ a with real-valued Radon measure $q'(x)$, $q \in \mathrm{BV}_{loc}(\mathbb{R})$ as potentials. New sufficient conditions for minimal operators to be bounded below and selfadjoint are found. For such operators, a criterion for discreteness of the spectrum is proved, which generalizes Molchanov's, Brinck's, and the Albeverio–Kostenko–Malamud criteria. The quadratic forms corresponding to the investigated operators are described.

1. Introduction and main results

We consider the 1-D Schrödinger operator

$$\mathrm{S}(q)u \equiv \mathrm{S}u := -u'' + q'(x)u, \tag{1.1}$$

in the complex Hilbert space $L^2(\mathbb{R})$. The potential of (1.1) is the generalized derivative $q'(x)$ of a certain real-valued function $q \in L^2_{loc}(\mathbb{R})$. Following [13], we define $\mathrm{S}(q)$ as a quasi-differential operator,

$$l_q[u] := -(u' - qu)' - q(u' - qu) - q^2u,$$
$$\mathrm{Dom}(l_q) := \{u : \mathbb{R} \to \mathbb{C} \mid u, u' - qu \in \mathrm{AC}_{loc}(\mathbb{R})\}.$$

The quasi-differential expression $l_q[u]$ is equal to $-u''+q'(x)u$ in the sense of distributions,

$$\langle l_q[u], \varphi\rangle = \langle -u'' + q'(x)u, \varphi\rangle \quad \text{for every} \quad \varphi \in C^\infty_{comp}(\mathbb{R}).$$

Hereafter $u^{[1]} := u' - qu$ denotes the quasi-derivative. Then the operators (1.1) are defined as

$$\mathrm{S}(q)u := l_q[u],$$
$$\mathrm{Dom}(\mathrm{S}(q)) := \left\{u \in L^2(\mathbb{R}) \,\middle|\, u, u' - qu \in \mathrm{AC}_{loc}(\mathbb{R}),\ l_q[u] \in L^2(\mathbb{R})\right\},$$

and

$$\dot{\mathrm{S}}_0(q)u := l_q[u], \quad \mathrm{Dom}(\dot{\mathrm{S}}_0(q)) := \{u \in \mathrm{Dom}(\mathrm{S}(q)) \mid \mathrm{supp}\, u \Subset \mathbb{R}\}.$$

As usual, the operators $\mathrm{S}(q)$ and $\dot{\mathrm{S}}_0(q)$ are called maximal and preminimal, respectively. Under these assumptions the operator $\dot{\mathrm{S}}_0(q)$ is symmetric and closable, its closure being denoted by $\mathrm{S}_0(q)$ (see Proposition in Appendix).

Necessary and sufficient conditions for the operators $\mathrm{S}_0(q)$ to be bounded below and to have discrete spectrum are found in [11]. However, they are not constructive. Nonetheless, in physical applications the most interesting situation is where the potentials $q'(x)$ in (1.1) are real-valued Radon measures on a locally compact space $\mathbb{R}$, i. e. $q \in \mathrm{BV}_{loc}(\mathbb{R})$ (see, for instance, references in [2, 1, 8]). This situation is investigated in this paper. The

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case where the Radon measure is absolutely continuous, i. e. $q' \in L^1_{loc}(\mathbb{R})$, was studied in [3, 12]. The approach applied in [3] may be generalized onto arbitrary Radon measures on $\mathbb{R}$.

Let us suppose that there exists a finite number $C > 0$ such that for all intervals J of the real axis $\mathbb{R}$ with length ≤ 1 we have

$$\text{(Br)} \qquad \int_J d\,q(x) \ge -C.$$

Without loss of generality we may assume that in the Brinck condition (Br) $C \ge 2$ and we assume this in what follows.

Theorem A. *Under the condition* (Br) *the operator* $\mathrm{S}_0(q)$ *is bounded below, selfadjoint and* $\mathrm{S}_0(q) = \mathrm{S}(q)$.

The following theorem gives necessary and sufficient conditions for the spectra of the minimal operators to be discrete.

Theorem B. *Let the potential* $q'(x)$ *satisfy the condition* (Br). *Then spectrum of the operator* $\mathrm{S}_0(q)$ *is discrete if and only if the Molchanov condition is satisfied,*

$$\lim_{|a|\to\infty} \int_a^{a+h} d\,q(x) = +\infty,$$

for all $h > 0$.

The following theorem gives a description of the quadratic forms generated by the Schrödinger operators. We use notations and definitions from [7].

Theorem C. *Let the potential* $q'(x)$ *satisfy the condition* (Br). *Then following statements are fulfilled.*

(I) *The sesquilinear form*

$$\dot{t}_{\dot{\mathrm{S}}_0(q)}[u,v] \equiv \dot{t}[u,v] := \Big(\dot{\mathrm{S}}_0(q)u, v\Big)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u'\overline{v'}d\,x + \int_{\mathbb{R}} u\overline{v}d\,q(x),$$
$$\mathrm{Dom}(\dot{t}_{\dot{\mathrm{S}}_0(q)}) := \mathrm{Dom}(\dot{\mathrm{S}}_0(q)),$$

is densely defined, symmetric, and bounded from below,

$$\Big(\dot{\mathrm{S}}_0(q)u, u\Big)_{L^2(\mathbb{R})} \ge -2C^2\|u\|^2_{L^2(\mathbb{R})}.$$

The form $\dot{t}_{\dot{\mathrm{S}}_0(q)}$ *is closable.*

(II) *"Potential energy"*

$$\mathrm{Q}(u) := \lim_{M,N\to\infty} \int_{-M}^{N} |u(x)|^2 d\,q(x)$$

exists and is finite for all $u \in \mathrm{Dom}(\mathrm{S}(q))$, *moreover,*

$$\mathrm{Dom}(\mathrm{S}(q)) \subset H^1(\mathbb{R}).$$

(III) *The closure* t *of the sesquilinear form* $\dot{t}$, $t := (\dot{t})^{\sim}$, *may be represented as*

$$t[u,v] = \int_{\mathbb{R}} u'\overline{v'}d\,x + \lim_{M,N\to\infty} \int_{-M}^{N} u\overline{v}d\,q(x),$$
$$\mathrm{Dom}(t) = \left\{ u \in H^1(\mathbb{R}) \;\middle|\; \exists \lim_{M,N\to\infty} \int_{-M}^{N} |u(x)|^2 d\,q(x) \in \mathbb{R} \right\}.$$

The sesquilinear form t *is densely defined, closed, symmetric, and bounded from below.*

2. Proof of Theorem A

We begin with formulating two necessary lemmas.

Lemma 2.1 (T. Ganelius [4])**.** *Let $f \geq 0$ and g be functions of bounded variation on a compact interval J. Then*

$$\int_J f\,d\,g \leq \left(\inf_J f + \operatorname*{var}_J f\right) \sup_{K\subset J} \int_K d\,g,$$

where K is a compact subinterval of J.

Lemma 2.1 is crucial in our proof of the fact that the preminimal operator $\dot{\mathrm{S}}_0(q)$ is bounded below under the condition (Br).

The following lemma plays a technical role.

Lemma 2.2 (I. Brinck [3])**.** *Let J be a compact interval of length l. Then for all $x \in J$ and $f \in H^1(J)$ we have*

$$\frac{1}{2}l^{-1}\|f\|^2_{L^2(J)} - \frac{1}{2}l\|f'\|^2_{L^2(J)} \leq |f(x)|^2 \leq 2t^{-1}\|f\|^2_{L^2(J)} + t\|f'\|^2_{L^2(J)}, \quad 0 < t \leq l,$$

and

$$\inf_{x\in J} |f(x)|^2 \leq l^{-1}\|f\|^2_{L^2(J)}.$$

Lemma 2.3. *Let $q'(x)$ satisfy the condition* (Br)*. If I is a finite interval of length l and if $f \in H^1(I)$, then*

$$\int_I |f(x)|^2 d\,q(x) \geq -C\left(2(hl/n)^{-1}\|f\|^2_{L^2(I)} + (hl/n)\|f'\|^2_{L^2(I)}\right), \tag{2.1}$$

where n is an integer such that $n-1 < l \leq n$, and h is an arbitrary number from $(0,1]$.

Proof. There is no loss of generality in supposing that $I = (0,l)$.

We first suppose $l = 1$ and apply Lemma 2.1. Thus

$$-\int_I |f(x)|^2 d\,q(x) \leq -\left(\inf_I |f|^2 + \operatorname*{var}_J |f|^2\right) \sup_{K\subset I} \int_K d\,q(x).$$

Due to (Br) the factor $-\sup_{K\subset I}\int_K d\,q(x)$ is majorized by C, and from Lemma 2.2 we get

$$\inf_{x\in I} |f(x)|^2 \leq \|f\|^2_{L^2(I)} \leq h^{-1}\|f\|^2_{L^2(I)}, \quad h \in (0,1].$$

We now write $f(x) = f_1(x) + if_2(x)$, where f_1 and f_2 are real-valued functions. Due to Cauchy's inequality we get

$$\operatorname*{var}_I |f(x)|^2 = \int_I \left|\frac{d}{dx}|f(x)|^2\right| dx = \int_I |f_1 f_1' + f_2 f_2'|\,dx \leq 2\|f\|_{L^2(I)}\|f'\|_{L^2(I)},$$

and, hence,

$$\begin{aligned} -\int_I |f(x)|^2 d\,q(x) &\leq Ch^{-1}\left(\|f\|^2_{L^2(I)} + 2h\|f\|_{L^2(I)}\|f'\|_{L^2(I)}\right) \\ &\leq Ch^{-1}\left(2\|f\|^2_{L^2(I)} + h^2\|f'\|^2_{L^2(I)}\right), \end{aligned}$$

which proves the lemma for $l = 1$.

To prove the lemma for arbitrary l we put $Q(x) = q(ln^{-1}x)$. Then

$$\begin{aligned} \int_0^l |f(x)|^2 d\,q(x) &= \int_0^n |f(ln^{-1}x)|^2 d\,q(ln^{-1}x) = \int_0^n |f(ln^{-1}x)|^2 d\,Q(x) \\ &= \sum_{k=1}^n \int_{k-1}^k |f(ln^{-1}x)|^2 d\,Q(x). \end{aligned}$$

Note that the function Q satisfies condition (Br) with the same constant C for all intervals of length $\leq n/l$ and, hence, for all intervals of length ≤ 1. Therefore the assumption of lemma for intervals of unit length implies

$$\int_{k-1}^{k} |f(ln^{-1}x)|^2 d\,Q(x) \geq -C\left(2h^{-1}\int_{k-1}^{k} |f(ln^{-1}x)|^2 d\,x + h\int_{k-1}^{k} \frac{d}{dx}|f(ln^{-1}x)|^2 d\,x\right),$$

and hence, summing over k, we get

$$\begin{aligned}\int_{0}^{n} |f(ln^{-1}x)|^2 d\,Q(x) &\geq -C\left(2h^{-1}\int_{0}^{n} |f(ln^{-1}x)|^2 d\,x + h\int_{0}^{n} \frac{d}{dx}|f(ln^{-1}x)|^2 d\,x\right)\\ &= -C\left(2h^{-1}l^{-1}n\int_{0}^{l} |f(x)|^2 d\,x + hln^{-1}\int_{0}^{l} |f'(x)|^2 d\,x\right),\end{aligned}$$

which proves the lemma. □

Corollary 2.3.1. *If the length of an interval I does not exceed 1, then*

$$\int_I |u'(x)|^2 d\,x + 2C^2\int_I |u(x)|^2 d\,x + \int_I |u(x)|^2 d\,q(x) \geq 0$$

for any $u \in H^1(I)$.

Proof. Due to the choice of n in Lemma 2.3, we get $n/lC < (l+1)/lC$. Since we assume that $C \geq 2$, we may conclude that $n/lC < 1$ if $l \geq 1$. Thus, we may put $h = n/lC$ in (2.1), which yields the corollary. □

Corollary 2.3.2. *Let the condition* (Br) *be satisfied. Then*

$$\int_{\mathbb{R}} |u(x)|^2 d\,q(x) \geq -C\left(2h^{-1}\|u\|^2_{L^2(\mathbb{R})} + h\|u'\|^2_{L^2(\mathbb{R})}\right) \tag{2.2}$$

for all $u \in H^1_{comp}(\mathbb{R})$ and $h \in (0,1]$.

Proof. We divide the real axis into a sum of disjoint intervals of unit length. Then (2.1) holds on each of these intervals and the summation gives (2.2). □

Remark. If the support of u is not compact, Corollary 2.3.2 obviously still holds if

$$\lim_{M,N\to\infty}\int_{-M}^{N} |u(x)|^2 d\,q(x)$$

exists as improper Riemann–Stieltjes integral. Then the integral in (2.2) must, of course, be interpreted accordingly.

Lemma 2.3 allows us to prove that the preminimal operator is bounded from below.

Theorem 2.4. *Let the potential $q'(x)$ satisfy the condition* (Br)*. Then the preminimal operator $\dot{\mathrm{S}}_0(q)$ is bounded from below and the following estimate holds:*

$$(\dot{\mathrm{S}}_0(q)u, u) \geq -2C^2\|u\|^2_{L^2(\mathbb{R})}, \quad u \in \mathrm{Dom}(\dot{S}_0(q)).$$

Proof. For arbitrary $u \in \mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ there is a positive integer N such that $\operatorname{supp} u \subseteq [-N, N]$ (recall that $\mathrm{Dom}(\dot{\mathrm{S}}_0(q)) \subset H^2_{comp}(\mathbb{R})$, see property 6^0 of Proposition in Appendix). Therefore,

$$\begin{aligned}(\dot{\mathrm{S}}_0(q)u, u)_{L^2(\mathbb{R})} &= (l_q[u], u)_{L^2(\mathbb{R})} = \|u'\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |u(x)|^2 d\,q(x)\\ &= \|u'\|^2_{L^2(\mathbb{R})} + \sum_{n=-N}^{N}\int_{[n,n+1)} |u(x)|^2 d\,q(x).\end{aligned} \tag{2.3}$$

To estimate the terms $\int_{[n,n+1)} |u(x)|^2 d\,q(x)$ we apply Lemma 2.3 with $l = n = 1$ and $h = C^{-1}$ (recall that $C \geq 2$) and get

$$\int_{[n,n+1)} |u(x)|^2 d\,q(x) \geq -2C^2 \|u\|^2_{L^2([n,n+1))} - \|u'\|^2_{L^2([n,n+1))}. \tag{2.4}$$

Substituting the estimate (2.4) into (2.3) we receive the estimate we require,

$$(\dot{\mathrm{S}}_0(q)u, u)_{L^2(\mathbb{R})} \geq \|u'\|^2_{L^2(\mathbb{R})} + \sum_{n=-N}^{N} \Big(-2C^2 \|u\|^2_{L^2([n,n+1))} - \|u'\|^2_{L^2([n,n+1))} \Big) = -2C^2 \|u\|^2_{L^2(\mathbb{R})}.$$

Theorem is proved. □

If the preminimal operator $\dot{\mathrm{S}}_0(q)$ is bounded from below, the minimal operator $\mathrm{S}_0(q)$ is selfadjoint and coincides with the maximal operator $\mathrm{S}(q)$ (see [1, Remark III.2] and [10, Corollary 2]). Therefore Theorem 2.4 implies Theorem A.

Theorem A is proved.

3. Auxiliary results

We shall make use of a set of functions $\varphi(x)$ with compact supports and uniformly bounded derivatives. We define φ as follows:

(3.1)
- (i) $\varphi(x) = \varphi(x, r, R) = \begin{cases} 1 & \text{for } \quad -r \leq x \leq R, \\ 0 & \text{for } \quad x < -r - 1 \text{ and } x > R + 1. \end{cases}$
- (ii) For every x the function $\varphi(x)$ is increasing in r and R.
- (iii) The derivatives $\varphi'(x)$ and $\varphi''(x)$ are continuous and uniformly bounded in x, r and R.

It follows from this definition that $0 \leq \varphi \leq 1$ and that $\varphi \to 1$ as $\min(r, R) \to \infty$.

Lemma 3.1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a bounded, twice continuously differentiable function with bounded first and second derivatives. If*

$$\int_J \omega(x) d\,q(x) \geq -C \tag{3.2}$$

for all intervals J of length ≤ 1, then

$$\int_{\mathbb{R}} \omega^2 |u'|^2 d\,x < \infty$$

for all $u \in \mathrm{Dom}(\mathrm{S}(q))$.

Proof. Let φ be one of the functions introduced above and put $\psi = \varphi^2\omega^2$. If u is any function in $\mathrm{Dom}(\mathrm{S}(q))$ we get, integrating by parts,

$$\int_{\mathbb{R}} \psi l_q[u]\overline{u} d\,x = \int_{\mathbb{R}} \psi' u' \overline{u} d\,x + \int_{\mathbb{R}} \psi |u'|^2 d\,x + \int_{\mathbb{R}} \psi |u|^2 d\,q. \tag{3.3}$$

Now, let u be a real-values function in $\mathrm{Dom}(\mathrm{S}(q))$. Then the first integral in the right-hand side can be integrated by parts, yielding

$$\int_{\mathbb{R}} \psi l_q[u]\overline{u} d\,x = -\frac{1}{2} \int_{\mathbb{R}} \psi'' |u|^2 d\,x + \int_{\mathbb{R}} \psi |u'|^2 d\,x + \int_{\mathbb{R}} \psi |u|^2 d\,q. \tag{3.4}$$

The functions ψ and ψ'' tend boundedly to ω^2 and $(\omega^2)''$, respectively, as $\varphi \to 1$, that is as $\min(r, R) \to \infty$, and, since $|\omega^2 l_q[u]\overline{u}|$ and $(\omega^2)''|u|^2$ are both integrable, the first two integrals in (3.4) tend to the finite limits $\int_{\mathbb{R}} \omega^2 l_q[u]\overline{u} d\,x$ and $\int_{\mathbb{R}} (\omega^2)''|u|^2 d\,x$, respectively as $\varphi \to 1$. Since the convergence of ψ is also monotone, we conclude that $\int_{\mathbb{R}} \psi |u'|^2 d\,x$

must tend to $\int_{\mathbb{R}} \omega^2|u'|^2 d\,x$ although this limit may not be finite, and therefore $\int_{\mathbb{R}} \psi|u|^2 d\,q$ must also have limit (possibly $-\infty$).

We put $d\,W(x) = \omega(x) d\,q(x)$. It follows from (3.2) that W satisfies a condition of the type (Br). Therefore, we apply Lemma 2.1 (as in the proof of Lemma 2.3) to obtain

$$-\int_{\mathbb{R}} \psi|u|^2 d\,q = -\int_{\mathbb{R}} \omega(x)\varphi^2(x)|u(x)|^2 d\,W(x) \le C\left(2\int_{\mathbb{R}} \varphi^2\omega|u|^2 d\,x + \mathrm{var}\varphi^2\omega|u|^2\right).$$

But $\mathrm{var}\varphi^2\omega|u|^2$ is bounded by

$$\int_{\mathbb{R}} \varphi^2|\omega'||u|^2 d\,x + 2\int_{\mathbb{R}} \varphi\omega|\varphi'||u|^2 d\,x + 2\int_{\mathbb{R}} \omega\varphi^2|uu'|d\,x,$$

which in turn is majorized by

$$M\|u\|^2 + 2\|u\|\|\varphi\omega u'\|,$$

where the coefficient M depends only on the bounds for ω, ω', and φ'. Hence, it follows from (3.4) that

$$\|\varphi\omega u'\|^2 \le O(1) + 2\|u\|\|\varphi\omega u'\|.$$

Thus, $\|\varphi\omega u'\|^2 = \int_{\mathbb{R}} \varphi^2\omega^2|u'|^2 d\,x = \int_{\mathbb{R}} \psi|u'|^2 d\,x$ must be bounded. Therefore,

$$\int_{\mathbb{R}} \omega^2|u'|^2 d\,x < \infty, \tag{3.5}$$

and the lemma is proved for every real $u \in \mathrm{Dom}(\mathrm{S}(q))$.

Since every u in $\mathrm{Dom}(\mathrm{S}(q))$ can be written as $u_1 + iu_2$, where u_1 and u_2 are real and from $\mathrm{Dom}(\mathrm{S}(q))$, the proof for real u shows that $\int_{\mathbb{R}} \omega^2|u_1'|^2 d\,x < \infty$ and $\int_{\mathbb{R}} \omega^2|u_2'|^2 d\,x < \infty$. Hence, $\int_{\mathbb{R}} \omega^2|u'|^2 d\,x < \infty$ for all $u \in \mathrm{Dom}(\mathrm{S}(q))$. The proof of the lemma is complete. □

We observe that $\int_{\mathbb{R}} \psi|u|^2 d\,q$ has a finite limit for all u in $\mathrm{Dom}(\mathrm{S}(q))$, and that $|u'\overline{u}|$ is integrable. Hence $\int_{\mathbb{R}} \psi' u'\overline{u} d\,x$ in (3.3) tends to $\int_{\mathbb{R}} (\omega^2)' u'\overline{u} d\,x$ for all $u \in \mathrm{Dom}(\mathrm{S}(q))$.

We obtain the following useful result from Lemma 3.1 with $\omega(x) \equiv 1$.

Corollary 3.1.1. *Let the condition* (Br) *be satisfied. Then*

$$\mathrm{Dom}(\mathrm{S}(q)) \subset H^1(\mathbb{R}).$$

We see from (3.3), (3.4), and (3.5) with $\omega(x) \equiv 1$ that $\|u'\|^2$ is finite and that

$$\lim_{\varphi\to 1} \int_{\mathbb{R}} \varphi^2|u|^2 d\,q(x) \quad \text{exists}$$

and also that

$$(\mathrm{S}(q)u, u)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} l_q[u]\overline{u} d\,x = \int_{\mathbb{R}} |u|^2 d\,x + \lim_{\varphi\to 1} \int_{\mathbb{R}} \varphi^2|u|^2 d\,q(x).$$

This enables us to prove that the "potential energy"

$$\mathrm{Q}(u) = \lim_{M,N\to\infty} \int_{-M}^{N} |u(x)|^2 d\,q(x) \tag{3.6}$$

exists and is finite for every $u \in \mathrm{Dom}(\mathrm{S}(q))$ as improper Riemann–Stieltjes integral.

Let $\varphi_1 = \varphi^2(x, r, R)$ and $\varphi_2 = \varphi^2(x, r-1, R-1)$, with φ being defined by (3.1). Then obviously

$$\int_{-r}^{R} |u|^2 d\,q = \int_{\mathbb{R}} \varphi_1|u|^2 d\,q - \int_{-r-1}^{-r} \varphi_1|u|^2 d\,q - \int_{R}^{R+1} \varphi_1|u|^2 d\,q$$

and

$$\int_{-r}^{R} |u|^2 d\,q = \int_{\mathbb{R}} \varphi_2|u|^2 d\,q + \int_{-r}^{-r+1} (1-\varphi_2)|u|^2 d\,q + \int_{R-1}^{R} (1-\varphi_2)|u|^2 d\,q.$$

In these two identities, each of the four integrals over intervals of unit length can be one-sidedly estimated in terms of the norms of u and u' over the interval by Lemma 2.3. Since u and u' are both from $L^2(\mathbb{R})$, those norms vanish with increasing r and R. Thus

$$\int_{\mathbb{R}} \varphi_2 |u|^2 d\,q - o(1) \le \int_{-r}^{R} |u|^2 d\,q \le \int_{\mathbb{R}} \varphi_1 |u|^2 d\,q + o(1),$$

and, hence,

$$\int_{-r}^{R} |u|^2 d\,q \to \lim_{\varphi\to 1} \int_{\mathbb{R}} \varphi |u|^2 d\,q$$

as $\min(r,R)\to\infty$. Thus, the limit in (3.6) exists. It also follows that

$$\mathrm{Q}(u) = \int_{\mathbb{R}} |u|^2 d\,q = (\mathrm{S}(q)u,u)_{L^2(\mathbb{R})} - (u',u')_{L^2(\mathbb{R})} \tag{3.7}$$

for all $u \in \mathrm{Dom}(\mathrm{S}(q))$, which is equivalent to

$$(\mathrm{S}(q)u,u)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u'|^2 d\,x + \int_{\mathbb{R}} |u|^2 d\,q(x).$$

We have just proved the first half of the following.

Theorem 3.2. *If $q'(x)$ satisfies* (Br)*, then the potential energy* $\mathrm{Q}(u)$ *defined by* (3.6) *exists and is finite for any $u \in \mathrm{Dom}(\mathrm{S}(q))$ as improper Riemann–Stieltjes integral. Moreover, for any $h \in (0, C^{-1}]$ and every $u \in \mathrm{Dom}(\mathrm{S}(q))$, we have*

$$(1-Ch)(u',u')_{L^2(\mathbb{R})} \le 2Ch^{-1}(u,u)_{L^2(\mathbb{R})} + (\mathrm{S}(q)u,u)_{L^2(\mathbb{R})} \tag{3.8}$$

and

$$(1-Ch)\mathrm{Q}(u) \ge -2Ch^{-1}(u,u)_{L^2(\mathbb{R})} - Ch\,(\mathrm{S}(q)u,u)_{L^2(\mathbb{R})}\,. \tag{3.9}$$

Proof. For all $h \le C^{-1}$ (<1) and every $u \in \mathrm{Dom}(\mathrm{S}(q))$ we have

$$\mathrm{Q}(u) = \int_{\mathbb{R}} |u|^2 d\,q(x) \ge -2Ch^{-1}\|u\|^2_{L^2(\mathbb{R})} - Ch\|u'\|^2_{L^2(\mathbb{R})}$$

due to Corollary 2.3.2 and the remark to this Corollary. Then (3.8) and (3.9) follow from (3.7). □

4. Proof of Theorem B

Let us first prove some preliminary results.

If $q'(x)$ satisfies an upper estimate of a type corresponding to (Br), that is,

$$\int_J d\,q(x) \le C_1 \tag{4.1}$$

for all intervals J of length ≤ 1, then $-q'(x)$ satisfies (Br) with C replaced by C_1. Hence, Lemma 2.3 and Corollary 2.3.2 give upper bounds for $\int |u|^2 d\,q$. For convenience we state them in a separate statement.

Proposition 4.1. *Let $q'(x)$ satisfy* (4.1)*. If I is any finite interval of length l and $f \in H_2^1(I)$, then*

$$\int_I |u(x)|^2 d\,q(x) \le C_1 \left\{ 2(hl/n)^{-1} \parallel u \parallel^2_{L^2(I)} + (hl/n) \parallel f' \parallel^2_{L^2(I)} \right\},$$

where n is the integer determined by $n-1 < l \le n$ and h is any number in the interval $0 < h \le 1$.

If u belongs to $H^1(\mathbb{R})$ and has compact support, then also

$$\int_{\mathbb{R}} |u(x)|^2 d\,q(x) \le C_1 \left\{ 2h^{-1} \parallel u \parallel^2_{L^2(\mathbb{R})} + h \parallel u' \parallel^2_{L^2(\mathbb{R})} \right\}$$

for any positive $h \le 1$.

Lemma 4.2. *Assume that* I *is an interval of length* ≤ 1, $q'(x)$ *satisfies* (Br) *and*

$$\int_I |h(x)|^2 d\,q(x) \le C_1$$

for some function $h \in H^1(I)$ *such that* $0 < m \le |h(x)| \le M$ *for all* $x \in I$. *Then*

$$\int_I d\,q(x) \le C_0, \tag{4.2}$$

where C_0 *depends only on* C, C_1, m, M, *and* $\|h'\|^2_{L^2(I)}$.

Proof. We apply Lemma 2.1 with $f = |h|^{-2}$ and $d\,g = |h|^2 d\,q$ to obtain

$$\int_I d\,q(x) = \int_I |h|^{-2}|h|^2 d\,q(x) \le \left(\inf_I |h|^{-2} + \operatorname*{var}_I |h|^{-2}\right) \sup_{J\subset I} \int_J |h|^2 d\,q(x), \tag{4.3}$$

and we shall exhibit a bound for each of the factors in the right-hand side.

For any $J \subset I$ the set $I \setminus J$ consists of at most two intervals K and L, of length k and l, respectively. From Lemma 2.3 with $h = 1$ we find

$$\int_K |h|^2 d\,q(x) \ge -C\left(2k^{-1}\|h\|^2_{L^2(K)} + k\|h'\|^2_{L^2(K)}\right).$$

Since $\|h\|^2_{L^2(K)} \le kM^2$ and $k \le 1$, this yields

$$\int_K |h|^2 d\,q(x) \ge -C\left(2M^2 + \|h'\|^2_{L^2(K)}\right).$$

Because a similar estimate holds for the interval L, we have

$$\int_{I\setminus J} |h|^2 d\,q(x) \ge -C\left(4M^2 + \|h'\|^2_{L^2(I)}\right).$$

Hence,

$$\int_J |h|^2 d\,q(x) = \int_I |h|^2 d\,q(x) - \int_{I\setminus J} |h|^2 d\,q(x) \le C_1 + C\left(4M^2 + \|h'\|^2_{L^2(I)}\right). \tag{4.4}$$

Thus, there exists a bound of the required type for the second factor in (4.2).

On the other hand,

$$\inf_J \|h\|^2 \le m^{-2}, \tag{4.5}$$

and

$$\begin{aligned} \operatorname*{var}_I |h|^{-2} &= \int_I \left|\frac{d}{dx}|h(x)|^{-2}\right| d\,x = \int_I 2|h|^{-4}\left|Re(h\overline{h'})\right| d\,x \\ &\le 2m^{-4}\|h\|_{L^2(I)}\|h'\|_{L^2(I)} \le 2m^{-4}M\|h'\|_{L^2(I)}. \end{aligned} \tag{4.6}$$

So, in virtue of (4.3), (4.4), (4.5), and (4.6), we get a desired bound for $\int_I d\,q(x)$. □

Now we are ready to prove Theorem B.

Let us consider the operator

$$\mathrm{B} = \mathrm{S}(q) + (2C^2 + 1)\mathrm{I},$$

where I is the identity operator with the domain $\mathrm{Dom}(\mathrm{S}(q))$. Let us recall that $\mathrm{S}_0(q) = \mathrm{S}(q)$. It is obvious that the operator $\mathrm{S}(q)$ has discrete spectrum if and only if the operator B has. We get

$$(\mathrm{B}u, u)_{L^2(\mathbb{R})} \ge (u, u)_{L^2(\mathbb{R})}. \tag{4.7}$$

Then due to Rellich Theorem the operator B has discrete spectrum if and only if the set

$$\mathcal{M} = \{u \in \mathrm{Dom}(\mathrm{S}(q)) | (\mathrm{B}u, u)_{L^2(\mathbb{R})} \leq 1\}$$

is precompact (i.e., every infinite sequence contains a Cauchy-sequence).

The norms of elements of $\mathcal{M}$ are uniformly bounded according to (4.7). Hence, choosing h appropriately in (3.9), we see that $\|u'\|^2_{L^2(\mathbb{R})}$ is also uniformly bounded with respect to $u \in \mathcal{M}$. Thus, $\mathcal{M}$ is an equicontinuous family of functions $u \in L^2(\mathbb{R})$, i.e.,

$$\|u(x+h) - u(x)\|^2_{L^2(\mathbb{R})}$$

vanishes uniformly as $h \to 0$. A compactness theorem of M. Riesz can now be applied: The set $\mathcal{M}$ is precompact if and only if

$$\lim_{n\to\infty} \left(\sup_{u\in\mathcal{M}} \int_{x>n} |u|^2 d\,x \right) = 0. \tag{4.8}$$

We shall now prove that the condition

$$\lim_{|a|\to\infty} \int_a^{a+h} d\,q(x) = +\infty \quad \text{for all} \quad h > 0 \tag{4.9}$$

is sufficient for the discreteness of the spectrum. To this end we suppose that (4.8) is not fulfilled. This means that we assume the existence of a sequence of functions $u_n \in \mathcal{M}$ for which

$$\int_{|x|>n} |u_n|^2 d\,x \geq \eta^{-1} > 0 \tag{4.10}$$

for some η independent of n. Now

$$(\mathrm{B}u_n, u_n)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u_n'|^2 d\,x + (2C^2+1) \int_{\mathbb{R}} |u_n|^2 d\,x + \int_{\mathbb{R}} |u_n|^2 d\,q(x) \leq 1,$$

according to (3.7), and if $n \geq 1$, then

$$\int_{-n}^{n} |u_n'|^2 d\,x + (2C^2+1) \int_{-n}^{n} |u_n|^2 d\,x + \int_{-n}^{n} |u_n|^2 d\,q(x) \geq 0$$

due to Corollary 2.3.1. Therefore, in view of (4.10),

$$\int_{\mathbb{R}} |u_n'|^2 d\,x + (2C^2+1) \int_{\mathbb{R}} |u_n|^2 d\,x + \int_{\mathbb{R}} |u_n|^2 d\,q(x) \leq 1 \leq \eta \int_{x>n} |u_n|^2 d\,x.$$

We split the set $(-\infty, -n) \cup (n, \infty)$ into a sum of disjoint intervals J_k of equal length $l \leq 1$. (This number l shall be the same for all n. It will be clear below how l is most suitable chosen, depending on the numbers C and η only.) Then

$$\sum_k \left[\int_{J_k} |u_n'|^2 d\,x + (2C^2+1) \int_{J_k} |u_n|^2 d\,x + \int_{J_k} |u_n|^2 d\,q(x) \right] \leq \eta \sum_k \int_{J_k} |u_n|^2 d\,x. \tag{4.11}$$

Hence, there exists at least one interval $I_n = I$ among J_k such that

$$\int_I |u_n'|^2 d\,x + (2C^2+1) \int_I |u_n|^2 d\,x + \int_I |u_n|^2 d\,q(x) \leq \eta \int_I |u_n|^2 d\,x. \tag{4.12}$$

Lemma 2.3 and (4.12) yield

$$(1 - Cl)\|u_n'\|^2_{L^2(I)} + (2C^2 + 1 - 2Cl^{-1})\|u_n\|^2_{L^2(I)} \leq \eta \|u_n\|^2_{L^2(I)}.$$

Let v_n be a multiple of u_n such that $\|v_n\|^2_{L^2(I)} = l$, and let $l < 1/C$. Then

$$\|v_n'\|^2_{L^2(I)} \leq (1 - Cl)^{-1}(\eta + 2Cl^{-1} - 2C^2 - 1)l,$$

which yields

$$l\|v_n'\|^2_{L^2(I)} \leq l(1 - Cl)^{-1}(\eta l + 2C - l(2C^2 + 1)).$$

Since the expression on the right vanishes as $l \to 0$, there exists a number $l_0(\eta, C)$ depending only on η and C such that $l \le l_0$ implies $l\|v_n'\|^2_{L^2(I)} \le \frac{1}{2}$. Letting the intervals in (4.11) have precisely the length l_0 we conclude from the Lemma 2.2 that

$$1/4 \le |v_n(x)|^2 \le 9/4 \tag{4.13}$$

for all x in I. Finally, we conclude from (4.12), which also holds for v_n by homogeneity, that

$$\int_I |v_n(x)|^2 d\,q(x) \le \|v_n\|^2_{L^2(I)}(\eta - 2C^2 - 1) - \|v_n'\|^2_{L^2(I)} \le l_0(\eta - 2C^2 - 1) = K. \tag{4.14}$$

In view of (4.13) and (4.14), the assumptions of Lemma 4.2 are satisfied. Hence,

$$\int_I d\,q(x) \le C_0,$$

where C_0 depends only on $\|v_n'\|^2_{L^2(I)}$, C, and K, i. e. on η and C only.

Therefore, if $\mathcal{M}$ is not precompact, we can find a sequence of intervals I_n of equal length l_0 and with I_n outside the interval $|x| \le n$ such that $\int_{I_n} d\,q(x)$ is uniformly bounded. Then (4.9) cannot be true; hence, $\mathcal{M}$ must be precompact if (4.9) holds. This proves the sufficiency assertion of Theorem B.

It remains to prove that condition (4.9) for the discreteness of the spectrum is necessary. To do this let us consider the operator $\mathrm{B}^{1/2}$ instead of B. The operator $\mathrm{B}^{1/2}$ has discrete spectrum if and only if the operator B has. Then Rellich Theorem for $\mathrm{B}^{1/2}$ reads as follows: spectrum of $\mathrm{B}^{1/2}$ is discrete if and only if the set

$$\mathcal{M}' = \left\{ u \in \mathrm{Dom}(\mathrm{B}^{1/2}) \,\middle|\, \|\mathrm{B}^{1/2}u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})} \le 1 \right\}$$

is precompact. The operator $\mathrm{B}^{1/2}$ is more convenient than the operator B for proving the necessity because

$$C^\infty_{comp}(\mathbb{R}) \subset \mathrm{Dom}(\mathrm{B}^{1/2}).$$

Let notice that $\mathrm{Dom}(\mathrm{B}^{1/2})$ coincides with the domain of the closure of the quadratic form $\dot{t}_{\dot{\mathrm{S}}_0(q)}$ generated by the preminimal operator $\dot{\mathrm{S}}_0(q)$.

Now, suppose that condition (4.8) is not satisfied. This is equivalent to the existence of a sequence $\{\Delta\}_1^\infty$ of disjoint intervals of equal length $\kappa > 0$ such that

$$\int_{\Delta_\nu} d\,q(x) \le C_1 \tag{4.15}$$

for all ν. Obviously there is no loss of generality to suppose that $\kappa \le 1$, for otherwise we can find a sequence of intervals contained in Δ_ν of length ≤ 1 for which (4.15) holds.

We observe that (4.15) implies the existence of an upper bound for the corresponding integral over any sub-interval J contained in Δ_ν, because

$$\int_J d\,q(x) = \int_{\Delta_\nu} d\,q(x) - \int_{\Delta_\nu - J} d\,q(x) \le C_1 + 2C = K$$

in view of (Br).

Let $\varphi_1 \not\equiv 0$ be a twice continuously differentiable function with support contained in Δ_1 and let φ_ν be the translate of φ_1 to the interval Δ_ν. Applying Proposition 4.1 we then get

$$\begin{aligned} &\int_{\mathbb{R}} |\varphi_\nu'|^2 d\,x + (2C^2+1)\int_{\mathbb{R}} |\varphi_\nu|^2 d\,x + \int_{\mathbb{R}} |\varphi_\nu|^2 d\,q(x) \\ &\le (1+K\kappa)\|\varphi_\nu'\|^2_{L^2(\mathbb{R})} + (2C^2 + 1 + 2K\kappa^{-1})\|\varphi_\nu\|^2_{L^2(\mathbb{R})} \\ &= (1+K\kappa)\|\varphi_1'\|^2_{L^2(\mathbb{R})} + (2C^2 + 1 + 2K\kappa^{-1})\|\varphi_1\|^2_{L^2(\mathbb{R})} \end{aligned} \tag{4.16}$$

for all ν. Since the functions φ_ν have disjoint supports, it follows that

$$\|\varphi_j - \varphi_k\|^2_{L^2(\mathbb{R})} = 2\|\varphi_1\|^2_{L^2(\mathbb{R})} > 0 \quad \text{when} \quad j \neq k.$$

Hence a set containing all the functions φ_ν cannot be precompact.

Further, supposing that φ_1 is normed so that the right hand side of (4.16) does not exceed, say, $\frac{1}{2}$, and using the fact that $\mathrm{B}^{1/2} \geq \mathrm{I}$, we conclude that the set $\mathcal{M}'$ contains the sequence $\{\varphi_\nu\}_1^\infty$. Therefore $\mathcal{M}'$ is not precompact, and hence the spectrum of $\mathrm{S}(q)$ cannot be discrete. Thus, assumption (4.15) must be false if $\mathrm{S}(q)$ has discrete spectrum. Consequently, (4.9) is a necessary condition.

The proof of Theorem B is thereby complete.

5. Proof of Theorem C

Theorem 2.4 and Theorem 3.2 together with Corollary 3.1.1 prove assertions (I) and (II) of Theorem C respectively.

Let us prove assertion (III) of Theorem C.

We shall deal with the domain of $\mathrm{B}^{1/2}$ instead of the domain of the sesquilinear form $t[u, v]$ (which is a closure of the form $\dot{t}_{\dot{\mathrm{S}}_0(q)}[u, v]$ generated by the preminimal operator $\dot{\mathrm{S}}_0(q)$). Recall that

$$\mathrm{B} = \mathrm{S}(q) + (2C^2 + 1)\mathrm{I} \quad \text{and} \quad \mathrm{Dom}(\mathrm{B}) = \mathrm{Dom}(\mathrm{S}(q)).$$

The operator B is selfadjoint and $\mathrm{B} \geq \mathrm{I}$. It is well known that $\mathrm{Dom}(\mathrm{B}^{1/2})$ coincides with $\mathrm{Dom}(t)$.

For arbitrary $f, g \in H^1_{comp}(\mathbb{R})$ we define a new inner product

$$\langle f, g\rangle := \int_{\mathbb{R}} f'\overline{g'}dx + \int_{\mathbb{R}} f\overline{g}dp(x), \tag{5.1}$$

where $p(x) := q(x) + (2C^2 + 1)x$. Then in view of Corollary 2.3.2 we conclude that

$$\begin{aligned}\langle f, f\rangle = \int_{\mathbb{R}} |f'|^2_{L^2(\mathbb{R})}dx + \int_{\mathbb{R}} |f'|^2_{L^2(\mathbb{R})}dp(x) \\ \geq (1 - Ch)\|f'\|^2_{L^2(\mathbb{R})} + (2C^2 + 1 - 2Ch^{-1})\|f\|^2_{L^2(\mathbb{R})}\end{aligned} \tag{5.2}$$

for all positive $h \leq 1$. Therefore with a proper choice of h we get

$$\langle f, f\rangle \geq C_1(\|f'\|^2_{L^2(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}) \tag{5.3}$$

for some positive constant C_1. Closing $H^1_{comp}(\mathbb{R})$ in the norm (5.2) we get a Hilbert space $\mathcal{R}$.

Lemma 5.1. *The embedding $\mathcal{R} \subset H^1(\mathbb{R})$ holds true and the inner product in $\mathcal{R}$ is given by*

$$\langle f, h\rangle = \int_{\mathbb{R}} f'\overline{h'}dx + \int_{\mathbb{R}} f\overline{h}dp \tag{5.4}$$

for any $h \in H^1_{comp}(\mathbb{R})$ and $f \in \mathcal{R}$.

Proof. The first assertion of the lemma follows immediately from (5.3). To prove the second one, let f be defined by a sequence $\{f_\nu\}_1^\infty$ of elements from $H^1_{comp}(\mathbb{R})$. Then

$$\langle f_\nu, h\rangle = \int_{\mathbb{R}} f'_\nu\overline{h'}dx + \int_{\mathbb{R}} f_\nu\overline{h}dp \tag{5.5}$$

by definition. But f'_ν and f_ν converge in $L_2(\mathbb{R})$ to f' and f respectively. Hence, f_ν converges uniformly to f on the support of h. Thus, the integral in (5.5) tends to the integral in (5.4), which proves the lemma. □

Lemma 5.2. *The domain $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ is dense in $\mathcal{R}$.*

Proof. Suppose that $\langle f, u\rangle = 0$ for every $u \in \mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ and some $f \in \mathcal{R}$. Integrating by parts we obtain

$$0 = \langle f, u\rangle = \int_{\mathbb{R}} f'\overline{u'}dx + \int_{\mathbb{R}} f\overline{u}dp = -\int_{\mathbb{R}} f\overline{u''}dx + \int_{\mathbb{R}} f\overline{u}dp = (f, \mathrm{B}u)_{L^2(\mathbb{R})},$$

according to lemma 5.1.

But $\mathrm{B}(\mathrm{Dom}(\dot{\mathrm{S}}_0(q)))$ is dense in $L_2(\mathbb{R})$, hence $f = 0$, which proves the lemma. □

Theorem 5.3. *The domain of the operator* $\mathrm{B}^{1/2}$ *coincides with* $\mathcal{R}$.

Proof. We first note that $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ is dense in $\mathrm{Dom}(\mathrm{S}(q))$ in the graph norm, because S is the closure of its restriction to $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$. Using well-known functional calculus for operators, we then conclude that $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ is also dense in the domain of $\mathrm{B}^{1/2}$ in the corresponding graph norm. Since

$$\left(\mathrm{B}^{1/2}u, \mathrm{B}^{1/2}u\right)_{L^2(\mathbb{R})} = (\mathrm{B}u, u)_{L^2(\mathbb{R})} = \langle u, u\rangle$$

for all $u \in \mathrm{Dom}(\dot{\mathrm{S}}_0(q))$, then the domain of $\mathrm{B}^{1/2}$ is obtained by closing $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ with respect to the norm in $\mathcal{R}$. Thus

$$\mathrm{Dom}(\mathrm{B}^{1/2}) \subset \mathcal{R}.$$

But Lemma 5.2 shows that $\mathrm{Dom}(\mathrm{B}^{1/2})$ cannot be a proper subset of $\mathcal{R}$, for then some $f \in \mathcal{R}\setminus\{0\}$ would be orthogonal to all $h \in \mathrm{Dom}(\mathrm{B}^{1/2})$ and hence to all $u \in \mathrm{Dom}(\dot{\mathrm{S}}_0(q))$, which is possible only for $f = 0$. Thus

$$\mathrm{Dom}(\mathrm{B}^{1/2}) = \mathcal{R}$$

and the theorem is proved. □

Remark that we have not given any explicit form for the inner product $\langle f, g\rangle$ of arbitrary elements in $\mathcal{R}$. It may be of interest to note, however, that an integral expression corresponding to (5.1) does give the inner product $\langle f, g\rangle$ for arbitrary $f, g \in \mathcal{R}$.

Lemma 5.4. *The inner product in* $\mathcal{R}$ *is given by*

$$\langle f, g\rangle = \lim_{M,N\to\infty}\left(\int_{-M}^{N} f'\overline{g'}dx + \int_{-M}^{N} f\overline{g}dp\right).$$

Proof. It is sufficient to prove that for every $f \in \mathcal{R}$

$$\langle f, f\rangle = \lim_{M,N\to\infty}\left(\int_{-M}^{N} |f'|^2dx + \int_{-M}^{N} |f|^2dp\right), \tag{5.6}$$

because then

$$\begin{aligned}4\langle f, g\rangle &= \langle f+g, f+g\rangle - \langle f-g, f-g\rangle + i\left[\langle f+ig, f+ig\rangle - \langle f-ig, f-ig\rangle\right]\\ &= \lim_{M,N\to\infty} 4\left(\int_{-M}^{N} f'\overline{g'}dx + \int_{-M}^{N} f\overline{g}dp\right).\end{aligned}$$

We define

$$\langle f, f\rangle_n = \int_{n-1}^{n} |f'|^2dx + \int_{n-1}^{n} |f|^2dp$$

for any $f \in \mathcal{R}$ and infer from the Corollary 2.3.1 to Lemma 2.3 that the number $\langle f, f\rangle_n$ is non-negative for all n. We proceed to prove that the series

$$P(f) = \sum_{n=-\infty}^{\infty} \langle f, f\rangle_n \tag{5.7}$$

converges to $\langle f, f\rangle$.

For any $h \in H^1_{comp}(\mathbb{R})$ the series in (5.7) is finite and $P(h) = \langle h, h\rangle$. Now, let f be an arbitrary element in $\mathcal{R}$, defined by a Cauchy-sequence $\{f_\nu\}_1^\infty$ of elements in $H^1_{comp}(\mathbb{R})$. Then, as we have seen, f'_ν converges in $L_2(\mathbb{R})$ to f' and f_ν converges uniformly to f on compacts. Thus the individual terms $\langle f_\nu, f_\nu\rangle_n$ converge to $\langle f, f\rangle_n$ for every n. But $\langle f_\nu, f_\nu\rangle$ converges to $\langle f, f\rangle$ and hence Fatou's lemma shows that

$$P(f) = \sum_{n=-\infty}^{\infty} \langle f, f\rangle_n = \sum_{n=-\infty}^{\infty} \lim_{\nu\to\infty} \langle f_\nu, f_\nu\rangle_n \le \lim_{\nu\to\infty} \sum_{n=-\infty}^{\infty} \langle f_\nu, f_\nu\rangle_n$$
$$= \lim_{\nu\to\infty} P(f_\nu) = \lim_{\nu\to\infty} \langle f_\nu, f_\nu\rangle = \langle f, f\rangle.$$

Thus, the series $P(f)$ converges, because its terms are non-negative, and $P(f) \le \langle f, f\rangle$.

To obtain the converse inequality, we define $\langle f, h\rangle_n$ for $f \in \mathcal{R}$ and $h \in H^1_{comp}(\mathbb{R})$ by

$$\langle f, h\rangle_n = \int_{n-1}^{n} f'\overline{h'}d\,x + \int_{n-1}^{n} f\overline{h}d\,p.$$

Lemma 5.1 shows that

$$\langle f, h\rangle = \sum_{n=-\infty}^{\infty} \langle f, h\rangle_n,$$

the series in fact being finite. Since $\langle f, f\rangle_n$ is positive definite we get by Schwarz' inequality

$$|\langle f, h\rangle_n|^2 \le \langle f, f\rangle_n \langle h, h\rangle_n.$$

Hence

$$|\langle f, h\rangle|^2 = \left|\sum_{n=-\infty}^{\infty} \langle f, h\rangle_n\right|^2 \le \sum_{n=-\infty}^{\infty} \langle f, f\rangle_n \sum_{n=-\infty}^{\infty} \langle h, h\rangle_n = P(f)\langle h, h\rangle.$$

This proves $P(f) \ge \langle f, f\rangle$, as $H^1_{comp}(\mathbb{R})$ is dense in $\mathcal{R}$. Therefore, $P(f) = \langle f, f\rangle$ in view of the inequality obtained above.

We have thus proved that the integral in (5.6) converges to $\langle f, f\rangle$ when $\mathbb{Z} \ni M, N \to \infty$. But f and f' are both in $L^2(\mathbb{R})$; therefore we can apply Lemma 2.3 to arbitrary M and N (as in the proof of Theorem 3.2) to obtain

$$\int_{-[M]-1}^{[N]+1} |f'|^2 d\,x + \int_{-[M]-1}^{[N]+1} |f|^2 d\,p + o(1) \ge \int_{-M}^{N} |f'|^2 d\,x + \int_{-M}^{N} |f|^2 d\,p$$
$$\ge \int_{-[M]}^{[N]} |f'|^2 d\,x + \int_{-[M]}^{[N]} |f|^2 d\,p - o(1),$$

with $[N]$ denoting the greatest integer $\le N$. But we have proved that the expressions on the left and on the right both tend to $\langle f, f\rangle$. Hence the lemma is proved. □

Theorem 5.5. *The equality*

$$\mathrm{Dom}(\mathrm{B}^{1/2}) = \left\{ u \in H^1(\mathbb{R}) \,\middle|\, \exists \int_{\mathbb{R}} |u|^2 d\,q \in \mathbb{R} \right\},$$

holds, where the integral $\int_{\mathbb{R}} u\overline{v} d\,q(x)$ is considered as improper Riemann–Stieltjes integral.

Proof. We have just shown that the limit in (5.6) exists and is finite for all $f \in \mathcal{R}$. Since f and f' are in $L^2(\mathbb{R})$, then the potential energy exists and is finite.

Conversely, if f satisfies the conditions of the theorem, the formula

$$F(g) = \lim_{M,N\to\infty} \left(\int_{-M}^{N} g'\overline{f'}d\,x + \int_{-M}^{N} g\overline{f}d\,p \right)$$

defines a continuous functional realized by some element $h \in \mathcal{R}$, and it is not difficult to prove that the function $f - h$ must then be an L^2-solution to the equation $Bu = 0$. Since B is positive this implies $f = h$, hence $f \in \mathcal{R}$ and the theorem is proved. □

6. Some remarks

Standard arguments show that the minimal operator $\mathrm{S}_0(q)$ is bounded from below in the Hilbert space $L^2(\mathbb{R})$ if and only if minimal operators $\mathrm{S}_0^{\pm}(q)$, generated by the differential expression $\mathrm{S}(q)$ in Hilbert spaces $L^2(\mathbb{R}_{\pm})$ correspondingly are bounded from below. Herein the discreteness of the spectrum of operator $\mathrm{S}_0(q)$ is equivalent to the discreteness of the both spectra of the operators $\mathrm{S}_D^{\pm}(q)$ that correspond to the selfadjoint extensions of operators $\mathrm{S}_0^{\pm}(q)$ with homogeneous Dirichlet condition at the end of the semi-axis $\mathbb{R}_{\pm}$. Therefore Theorems A and B (reformulated accordingly) also hold for the Schrödinger operators on the semi-axis, which were studied in [1]. These theorems generalize the results [1, Lemma III.1] and [1, Theorem IV.1].

The following example illustrates the difference between our results and the former ones.

Example. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary strictly increasing unbounded sequence of positive numbers such that $x_{n+1} - x_n \to 0$ as $n \to \infty$. Choose $\rho > 0$ and $\{\alpha_{2n-1}\}_{n=1}^{\infty} \subset \mathbb{R}_+$ arbitrarily. Consider the potential of the form

$$q'(x) = \sum_{n=1}^{\infty} (\rho + \alpha_{2n-1})\delta(x - x_{2n-1}) - \sum_{n=1}^{\infty} \rho\delta(x - x_{2n}).$$

Simple verification shows that the Radon measure $q'(x)$ does not satisfy conditions (A) and (B) from paper [2] and conditions of Theorem IV.1 from [1]. However, $q'(x)$ satisfies condition (Br). Therefore, operator $\mathrm{S}_D^{+}(q)$ is bounded from below and self-adjoint. Due to Theorem B its spectrum is discrete if and only if

$$\sum_{x_{2n-1} \in \Delta} \alpha_{2n-1} \to +\infty,$$

where the interval $\Delta \subset \mathbb{R}_+$ moves to $+\infty$ preserving its length.

Appendix

Let us formulate some known statements about the operators $\dot{\mathrm{S}}_0(q)$, $\mathrm{S}_0(q)$ and $\mathrm{S}(q)$, which are used in the paper. Their proofs may be found in [9, 10, 6, 5].

Proposition. *The operators* $\dot{\mathrm{S}}_0(q)$, $\mathrm{S}_0(q)$, *and* $\mathrm{S}(q)$ *have the following properties:*

- 1^0. *The domain* $\mathrm{Dom}(\dot{\mathrm{S}}_0(q))$ *of the preminimal operator* $\dot{\mathrm{S}}_0(q)$ *is dense in the Hilbert space* $L^2(\mathbb{R})$.
- 2^0. *The operator* $\dot{\mathrm{S}}_0(q)$ *is symmetric and therefore it is closable.*
- 3^0. *Let* $\mathrm{S}_0(q) := \left(\dot{\mathrm{S}}_0(q)\right)^{\sim}$. *Then*

$$\left(\dot{\mathrm{S}}_0(q)\right)^* = \mathrm{S}(q) \quad \text{and} \quad \dot{\mathrm{S}}_0(q) \subset \mathrm{S}_0(q) \subset \mathrm{S}(q).$$

- 4^0. *The minimal operator* $\mathrm{S}_0(q)$ *is a densely defined, closed, and symmetric operator with the deficiency index* (d, d), *where* $0 \leq d \leq 2$. *The operators* $\mathrm{S}_0(q)$ *and* $\mathrm{S}(q)$ *are mutually adjoint, i. e.*

$$\mathrm{S}_0^*(q) = \mathrm{S}(q) \quad \text{and} \quad \mathrm{S}^*(q) = \mathrm{S}_0(q).$$

5^0. *The domain* $\mathrm{Dom}(\mathrm{S}_0(q))$ *of the minimal operator* $\mathrm{S}_0(q)$ *has the form*

$$\mathrm{Dom}(\mathrm{S}_0(q)) = \{u \in \mathrm{Dom}(\mathrm{S}(q))\,|\,[u,v]_{+\infty} - [u,v]_{-\infty} = 0 \quad \forall v \in \mathrm{Dom}(\mathrm{S}(q))\},$$

where $[u,v] \equiv [u,v](x) := u(x)\overline{v^{[1]}(x)} - u^{[1]}(x)\overline{v(x)}$.

6^0. *The domains of the operators* $\dot{\mathrm{S}}_0(q)$, $\mathrm{S}_0(q)$ *and* $\mathrm{S}(q)$ *satisfy the embeddings*

$$\mathrm{Dom}(\dot{\mathrm{S}}_0(q)) \subset H^1_{comp}(\mathbb{R}),$$
$$\mathrm{Dom}(\mathrm{S}_0(q)) \subset H^1_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}),$$
$$\mathrm{Dom}(\mathrm{S}(q)) \subset H^1_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}).$$

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