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ON UNIQUENESS OF FIXED POINTS OF QUADRATIC STOCHASTIC OPERATORS ON A 2D SIMPLEX

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Abstract. The Perron–Frobenius theorem states that a linear stochastic operator associated with a positive square stochastic matrix has a unique fixed point in the simplex and it is strongly ergodic to that fixed point. However, in general, the similar result for quadratic stochastic operators associated with positive cubic stochastic matrices does not hold true. Namely, it may have more than one fixed point in the simplex. Moreover, the uniqueness of fixed points does not imply the strong ergodicity of quadratic stochastic operators. In this paper, for some classes of positive cubic stochastic matrices, we provide a uniqueness criterion for fixed points of quadratic stochastic operators acting on a 2D simplex. Some supporting examples are also presented.

1. Quadratic stochastic operators

Let us first provide some necessary definitions of non-homogeneous Markov chains and quadratic stochastic processes by following the papers [5, 6, 26].

Let $\Omega_{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : \sum\limits_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ \forall\ i = \overline{1,m} \right\}$ be a standard simplex. An element of the simplex Ω_{m-1} is called *a stochastic vector*. A family of square stochastic matrices $\left\{ \mathbb{P}^{[r,t]} = \left(p_{ik}^{[r,t]}\right)_{i,k=1}^{m} : r, t \in \mathbb{N}, t - r \geq 1 \right\}$ is called *a discrete time non-homogeneous Markov chain* if for any natural numbers r, s, t with $r < s < t$ the following condition, known as *the Chapman–Kolmogorov equation*, is satisfied

$$p_{ik}^{[r,t]} = \sum_{j=1}^{m} p_{ij}^{[r,s]} p_{jk}^{[s,t]}, \quad 1 \leq i, k \leq m. \tag{1}$$

A linear operator $\mathcal{L}^{[r,t]} : \Omega_{m-1} \to \Omega_{m-1}$ associated with the square stochastic matrix $\mathbb{P}^{[r,t]} = \left(p_{ik}^{[r,t]}\right)_{i,k=1}^{m}$

$$\left(\mathcal{L}^{[r,t]}(\mathbf{x})\right)_k = \left(\mathbf{x}\mathbb{P}^{[r,t]}\right)_k = \sum_{i=1}^{m} x_i p_{ik}^{[r,t]}, \quad 1 \leq k \leq m, \tag{2}$$

is called *a linear stochastic operator* (*a Markov operator*).

Notice that the Chapman–Kolmogorov equation can be written in the following form:

$$\mathcal{L}^{[r,t]} = \mathcal{L}^{[s,t]} \circ \mathcal{L}^{[r,s]}, \quad r < s < t. \tag{3}$$

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A stochastic vector $\mathbf{x} \in \Omega_{m-1}$ is called *a stationary distribution* of the non-homogeneous Markov chain if one has that $\mathbf{x}\mathbb{P}^{[r,t]} = \mathbf{x}$ for all $r, t \in \mathbb{N}$. It is clear that a set of all stationary distributions of the non-homogeneous Markov chain is nothing but a set of all common fixed points of the linear Markov operators (2) for all $r, t \in \mathbb{N}$.

Let $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ be a cubic matrix and let $\mathbf{p}_{ij\bullet} = (p_{ij1}, p_{ij2}, \cdots, p_{ijm})$ for all $1 \le i, j \le m$. A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *stochastic* if $\mathbf{p}_{ij\bullet}$ is a stochastic vector for all $1 \le i, j \le m$. Without loss of generality, we may assume that $p_{ijk} = p_{jik}$ for any $1 \le i, j, k \le m$.

A family of cubic stochastic matrices $\left\{\mathcal{P}^{[r,t]} = \left(p_{ijk}^{[r,t]}\right)_{i,j,k=1}^m : r, t \in \mathbb{N}, t - r \ge 1\right\}$ with an initial distribution $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$ is called *a discrete time quadratic stochastic process* if for any natural numbers r, s, t with $r < s < t$ either one of the following conditions, the so-called *the nonlinear Chapman–Kolmogorov equations*, is satisfied

(A) $p_{ijk}^{[r,t]} = \sum\limits_{\alpha,\beta=1}^m p_{ij\alpha}^{[r,s]} x_\beta^{(s)} p_{\alpha\beta k}^{[s,t]}, \quad 1 \le i, j, k \le m,$

(B) $p_{ijk}^{[r,t]} = \sum\limits_{\alpha,\beta,\gamma,\delta=1}^m x_\alpha^{(r)} p_{i\alpha\beta}^{[r,s]} x_\gamma^{(r)} p_{j\gamma\delta}^{[r,s]} p_{\beta\delta k}^{[s,t]}, \quad 1 \le i, j, k \le m,$

where $x_k^{(\nu)} = \sum\limits_{i,j=1}^m x_i^{(0)} x_j^{(0)} p_{ijk}^{[0,\nu]}$. We remark that the conditions (A) and (B) are not equivalent to each other. The reader may refer to [5, 26] for an exposition of quadratic stochastic processes.

A quadratic operator $\mathcal{Q}^{[r,t]} : \Omega_{m-1} \to \Omega_{m-1}$ associated with the cubic stochastic matrix $\mathcal{P}^{[r,t]} = \left(p_{ijk}^{[r,t]}\right)_{i,j,k=1}^m$,

$$\left(\mathcal{Q}^{[r,t]}(\mathbf{x})\right)_k = \sum_{i,j=1}^m x_i x_j p_{ijk}^{[r,t]}, \quad 1 \le k \le m, \tag{4}$$

is called *a quadratic stochastic operator* (*a nonlinear Markov operator* [14]).

Obviously, we have that $\mathbf{x}^{(\nu)} = \mathcal{Q}^{[0,\nu]}(\mathbf{x}^{(0)})$. Notice that the nonlinear Chapman–Kolmogorov equation can be written in the following form:

$$\mathcal{Q}^{[r,t]}(\mathbf{x}^{(r)}) = \mathcal{Q}^{[s,t]}\left(\mathcal{Q}^{[r,s]}(\mathbf{x}^{(r)})\right), \quad r < s < t. \tag{5}$$

The classical Perron–Frobenius theorem states that if $\mathbb{P} > 0$ then a linear Markov chain has a unique stationary distribution $\mathbf{p} \in \Omega_{m-1}$ and it is *strongly ergodic* (*asymptotically stable*, *regular*) to $\mathbf{p}$, i.e., $\lim\limits_{k\to\infty} \mathcal{L}^{[1,k]}(\mathbf{x}) = \mathbf{p}$ for any $\mathbf{x} \in \Omega_{m-1}$ where $\mathcal{L} : \Omega_{m-1} \to \Omega_{m-1}$, $\mathcal{L}(\mathbf{x}) = \mathbf{x}\mathbb{P}$ is a linear stochastic operator. Unlike the linear case, the structure of a set of all stationary distributions of the higher-order Markov chains (see [1, 21] for definitions) might be as complex as possible (see [16, 24, 25]). In general, an analogue of the Perron–Frobenius theorem for positive higher-order hypermatrices is not true. However, there are some sufficient conditions for the uniqueness of stationary distributions of the higher-order Markov chains and some iterative methods to find the unique stationary distribution of the higher-order Markov chains (see [3, 4, 15, 17, 20]). These sufficient conditions are in the spirit of Banach's contraction principle.

A quadratic stochastic operator has an incredible application in population genetics (see [18]). The quadratic stochastic operator was first introduced in Bernstein's work [2] and considered an important source of analysis to study dynamical properties and modeling in various fields of science such as biology (see [13, 18]), physics (see [27]), game theory (see [7]), control system (see [23]). A fixed point set and an omega limiting set of a special class of quadratic stochastic operators were deeply studied in [9, 10]. Ergodicity and chaotic dynamics of quadratic stochastic operators on the finite dimensional simplex were studied in [8, 22]. In [11, 20], it was given a long self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators. In general, if $\mathcal{P} > 0$ then it is not necessary to be true that $|\mathbf{Fix}(\mathcal{Q})| = 1$. In this paper, we provide a uniqueness criterion for fixed points (stationary distributions) of positive quadratic stochastic operators on a 2D simplex. Moreover, in general, if $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{p}\}$ then it is not necessary to be true that $\lim_{k\to\infty} \mathcal{Q}^{(k)}(\mathbf{x}) = \mathbf{p}$ for any $\mathbf{x} \in \Omega_{m-1}$. It is worth mentioning that there are non-contraction positive quadratic stochastic operators which are still strongly ergodic. We also present some supporting examples.

2. The uniqueness criterion for stationary distributions

Let $\mathcal{Q} : \Omega_2 \to \Omega_2$ be a positive quadratic stochastic operator defined as follows:

$$\mathcal{Q}(\mathbf{x}) = \left(\sum_{i,j=1}^{3} p_{ij} x_i x_j, \sum_{i,j=1}^{3} q_{ij} x_i x_j, \sum_{i,j=1}^{3} r_{ij} x_i x_j \right)^T,$$

where $p_{ij}, q_{ij}, r_{ij} > 0$ and $p_{ij}+q_{ij}+r_{ij} = 1$ with $p_{ij} = p_{ji}$, $q_{ij} = q_{ji}$, $r_{ij} = r_{ji}$, $1 \le i, j \le 3$.

Remark 2.1. *Let $p_1 \neq p_2$, $q_1 \neq q_2$. It is obvious that two quadratic equations $x^2+p_1x+q_1 = 0$ and $x^2+p_2x+q_2 = 0$ have a unique common root if and only if their resultant is equal to zero, i.e.,*

$$(q_2 - q_1)^2 + p_1(q_2 - q_1)(p_1 - p_2) + q_1(p_1 - p_2)^2 = 0.$$

In this case, $x = \dfrac{q_2 - q_1}{p_1 - p_2}$ is the only common root.

We first provide a uniqueness criterion for fixed points (stationary distributions) of positive quadratic stochastic operators on a 2D simplex.

2.1. The uniqueness criterion.

Let us define the following constants:

$$\alpha_{11} = p_{11} + p_{33} - 2p_{13},\ \alpha_{22} = p_{22} + p_{33} - 2p_{23},\ \alpha_{12} = p_{33} + p_{12} - p_{13} - p_{23},$$
$$\beta_{11} = q_{11} + q_{33} - 2q_{13},\ \beta_{22} = q_{22} + q_{33} - 2q_{23},\ \beta_{12} = q_{33} + q_{12} - q_{13} - q_{23},$$
$$\alpha_0 = p_{33},\ \alpha_1 = p_{13} - p_{33},\ \alpha_2 = p_{23} - p_{33},\ \beta_0 = q_{33},\ \beta_1 = q_{13} - q_{33},\ \beta_2 = q_{23} - q_{33},$$
$$\gamma_0 = \beta_0\alpha_{11} - \alpha_0\beta_{11},\quad \gamma_1 = (2\beta_2 - 1)\alpha_{11} - 2\alpha_2\beta_{11},\quad \gamma_2 = \alpha_{11}\beta_{22} - \alpha_{22}\beta_{11},$$
$$\delta_0 = (2\alpha_1 - 1)\beta_{11} - 2\beta_1\alpha_{11},\quad \delta_1 = \alpha_{12}\beta_{11} - \beta_{12}\alpha_{11},\quad \Delta_1 = \gamma_2\delta_0^2 - 2\gamma_1\delta_0\delta_1 + 4\gamma_0\delta_1^2,$$
$$\lambda_0 = \alpha_{11}\gamma_0^2 + (2\alpha_1 - 1)\gamma_0\delta_0 + \alpha_0\delta_0^2,\qquad \lambda_4 = \alpha_{11}\gamma_2^2 + 4\alpha_{12}\gamma_2\delta_1 + 4\alpha_{22}\delta_1^2,$$
$$\lambda_3 = 2\alpha_{11}\gamma_2\gamma_1 + 2\alpha_{12}\gamma_2\delta_0 + 4\alpha_{12}\gamma_1\delta_1 + 4\alpha_1\gamma_2\delta_1 - 2\gamma_2\delta_1 + 4\alpha_{22}\delta_1\delta_0 + 8\alpha_2\delta_1^2,$$
$$\lambda_2 = 2\alpha_{11}\gamma_2\gamma_0 + \alpha_{11}\gamma_1^2 + 2\alpha_{12}\gamma_1\delta_0 + 4\alpha_{12}\gamma_0\delta_1 + 2\alpha_1\gamma_2\delta_0 + 4\alpha_1\gamma_1\delta_1 - \gamma_2\delta_0 - 2\gamma_1\delta_1 + \alpha_{22}\delta_0^2 + 8\alpha_2\delta_1\delta_0 + 4\alpha_0\delta_1^2,$$
$$\lambda_1 = 2\alpha_{11}\gamma_1\gamma_0 + 2\alpha_{12}\gamma_0\delta_0 + 2\alpha_1\gamma_1\delta_0 + 4\alpha_1\gamma_0\delta_1 - \gamma_1\delta_0 - 2\gamma_0\delta_1 + 2\alpha_2\delta_0^2 + 4\alpha_0\delta_1\delta_0.$$

Theorem 2.2. *Let $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. The positive quadratic stochastic operator $\mathcal{Q} : \Omega_2 \to \Omega_2$ has a unique fixed point (a stationary distribution) if and only if the quartic equation*

$$\lambda_4 a^4 + \lambda_3 a^3 + \lambda_2 a^2 + \lambda_1 a + \lambda_0 = 0$$

has a unique real root $a_0 \in (0, 1) \setminus \{-\frac{\delta_0}{2\delta_1}\}$ which satisfies $0 < A_0 < 1$ and $0 < B_0 < 1$, where

$$A_0 = \frac{\gamma_2 a_0^2 + \gamma_1 a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0},$$
$$B_0 = \frac{(\gamma_2 + 2\delta_1)a_0^2 + (\gamma_1 + \delta_0)a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0}.$$

Moreover, in this case, the only fixed point (a stationary distribution) is $(A_0, a_0, 1-B_0)^T$.

Proof. In order to find all fixed points of the positive quadratic operator $\mathcal{Q} : \Omega_2 \to \Omega_2$, we have to solve the following system of equations:

$$\begin{cases} x_1 = p_{11}x_1^2 + p_{22}x_2^2 + p_{33}x_3^2 + 2p_{12}x_1x_2 + 2p_{13}x_1x_3 + 2p_{23}x_2x_3, \\ x_2 = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + 2q_{12}x_1x_2 + 2q_{13}x_1x_3 + 2q_{23}x_2x_3, \\ x_3 = r_{11}x_1^2 + r_{22}x_2^2 + r_{33}x_3^2 + 2r_{12}x_1x_2 + 2r_{13}x_1x_3 + 2r_{23}x_2x_3. \end{cases} \tag{6}$$

Since $x_3 = 1 - x_1 - x_2$, it is enough to find all solutions (x_1, x_2) of the first two equations of the system (6) which satisfy the following conditions $x_1, x_2 > 0$ and $0 < x_1 + x_2 < 1$.

By plugging $x_3 = 1 - x_1 - x_2$ into the first and second equations of the system (6), we can get the following system of equations with respect to (x_1, x_2):

$$\begin{cases} \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + 2\alpha_{12}x_1x_2 + (2\alpha_1 - 1)x_1 + 2\alpha_2x_2 + \alpha_0 = 0, \\ \beta_{11}x_1^2 + \beta_{22}x_2^2 + 2\beta_{12}x_1x_2 + 2\beta_1x_1 + (2\beta_2 - 1)x_2 + \beta_0 = 0. \end{cases}.$$

Let $x_1 = x$ be a variable and $x_2 = a$ be a parameter. Since $\alpha_{11}\beta_{11} \neq 0$, the last system of equations takes the following form:

$$\begin{cases} x^2 + \dfrac{2\alpha_{12}a + 2\alpha_1 - 1}{\alpha_{11}}x + \dfrac{\alpha_{22}a^2 + 2\alpha_2 a + \alpha_0}{\alpha_{11}} = 0, \\ x^2 + \dfrac{2\beta_{12}a + 2\beta_1}{\beta_{11}}x + \dfrac{\beta_{22}a^2 + (2\beta_2 - 1)a + \beta_0}{\beta_{11}} = 0. \end{cases}$$

Let

$$A_1 = \frac{2\alpha_{12}a + 2\alpha_1 - 1}{\alpha_{11}}, \quad B_1 = \frac{\alpha_{22}a^2 + 2\alpha_2 a + \alpha_0}{\alpha_{11}},$$
$$A_2 = \frac{2\beta_{12}a + 2\beta_1}{\beta_{11}}, \quad B_2 = \frac{\beta_{22}a^2 + (2\beta_2 - 1)a + \beta_0}{\beta_{11}}.$$

We then have the following two quadratic equations:

$$x^2 + A_1x + B_1 = 0, \quad x^2 + A_2x + B_2 = 0.$$

Since $\Delta_1 \neq 0$, we have that $A_1 \neq A_2$ and $B_1 \neq B_2$. This means that for any $\mathbf{x}, \mathbf{y} \in \mathbf{Fix}(\mathcal{Q})$ one has that $x_2 \neq y_2$. Therefore, the system of equations (6) has a solution (x_1, x_2), $x_1, x_2 > 0$ with $0 < x_1 + x_2 < 1$ if and only if the last two quadratic equations should have a unique common root in $(0, 1)$ for $a \in (0, 1)$. Due to Remark 2.1, the last two quadratic equations have a unique common root if and only if

$$(B_2 - B_1)^2 + A_1(B_2 - B_1)(A_1 - A_2) + B_1(A_1 - A_2)^2 = 0$$

and the unique fixed point is $x = \dfrac{B_2 - B_1}{A_1 - A_2}$. It is clear that

$$A_1 - A_2 = \frac{1}{\alpha_{11}\beta_{11}}(2\delta_1 a + \delta_0), \quad B_2 - B_1 = \frac{1}{\alpha_{11}\beta_{11}}(\gamma_2 a^2 + \gamma_1 a + \gamma_0).$$

After simple algebra, we get the following quartic equation:

$$\lambda_4 a^4 + \lambda_3 a^3 + \lambda_2 a^2 + \lambda_1 a + \lambda_0 = 0. \tag{7}$$

Hence, the number of solutions (x_1, x_2), $x_1, x_2 > 0$ with $0 < x_1 + x_2 < 1$ of the system of equations (6) is the same as the number of positive roots in the interval $(0, 1)$ of the quartic equation (7). For every positive root $a \in (0, 1)$ of the quartic equation (7), the unique common root of two quadratic equations is $x = \dfrac{\gamma_2 a^2 + \gamma_1 a + \gamma_0}{2\delta_1 a + \delta_0}$. Consequently, the positive quadratic stochastic operator has a unique fixed point if and only if the quartic equation (7) must have a unique real root $a_0 \in (0,1)$ which satisfies

$$0 < A_0 = \frac{\gamma_2 a_0^2 + \gamma_1 a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0} < 1,$$
$$0 < B_0 = A_0 + a_0 = \frac{(\gamma_2 + 2\delta_1)a_0^2 + (\gamma_1 + \delta_0)a_0 + \gamma_0}{2\delta_1 a_0 + \delta_0} < 1.$$

In this case, the only fixed point is $(A_0, a_0, 1 - B_0)^T$. This completes the proof. □

Remark 2.3. *Theorem 2.2 provides a uniqueness criterion for the fixed point of the positive quadratic stochastic operators in the case $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. It is worth mentioning that there are also some positive quadratic stochastic operators having three fixed points in the case $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. Similarly, the positive quadratic stochastic operator may have one or three fixed points regardless of the condition $\alpha_{11}\beta_{11}\Delta_1 = 0$. Some supporting examples are provided in the next section.*

We now present an application of Theorem 2.2.

2.2. **Applications.**

Example 2.4 (Lyubich's Example)**.** Y. I. Lyubich, without proof (see [18], p. 296), had provided an example for a positive quadratic stochastic operator $\mathcal{V}_\varepsilon : \Omega_2 \to \Omega_2$ having three fixed points

$$\mathcal{V}_\varepsilon : \begin{cases} (\mathcal{V}_\varepsilon(\mathbf{x}))_1 = (1-4\varepsilon)x_1^2 + 2\varepsilon x_2^2 + 10\varepsilon x_3^2 + 4\varepsilon x_1x_2 + (1+4\varepsilon)x_1x_3 + 8\varepsilon x_2x_3 \\ (\mathcal{V}_\varepsilon(\mathbf{x}))_2 = 2\varepsilon x_1^2 + (1-3\varepsilon)x_2^2 + \varepsilon x_3^2 + (\frac{1}{2} + 2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + (1-12\varepsilon)x_2x_3 \\ (\mathcal{V}_\varepsilon(\mathbf{x}))_3 = 2\varepsilon x_1^2 + \varepsilon x_2^2 + (1-11\varepsilon)x_3^2 + (\frac{3}{2} - 6\varepsilon)x_1x_2 + (1-6\varepsilon)x_1x_3 + (1+4\varepsilon)x_2x_3 \end{cases},$$

where $0 < \varepsilon < \frac{9-5\sqrt{2}}{124}$. However, it turns out that Lyubich's example has a unique fixed point. We want to show that $\mathcal{V}_\varepsilon : \Omega_2 \to \Omega_2$ has the unique fixed point

$$\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)}, a_0, \frac{1-2a_0}{2(1+a_0)}\right)^T \in \Omega_2$$

for any $0 < \varepsilon < \frac{1}{12}$ where $a_0 \in (0, \frac{1}{2})$ is the unique root of the quartic equation

$$(2-12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0.$$

It is clear that $\mathcal{V}_\varepsilon : \Omega_2 \to \Omega_2$ is a positive quadratic stochastic operator whenever $0 < \varepsilon < \frac{1}{12}$ in which $\frac{9-5\sqrt{2}}{124} < \frac{1}{12}$. Let us calculate the following constants for $\mathcal{V}_\varepsilon$:

$$\alpha_{11} = 2\varepsilon,\ \alpha_{22} = 4\varepsilon,\ \alpha_{12} = 6\varepsilon - 0.5,\ \alpha_1 = 0.5 - 8\varepsilon,\ \alpha_2 = -6\varepsilon,\ \alpha_0 = 10\varepsilon$$
$$\beta_{11} = \varepsilon,\ \beta_{22} = 10\varepsilon,\ \beta_{12} = -0.25 + 7\varepsilon,\ \beta_1 = 0,\ \beta_2 = 0.5 - 7\varepsilon,\ \beta_0 = \varepsilon,$$
$$\gamma_2 = 16\varepsilon^2,\ \gamma_1 = -16\varepsilon^2,\ \gamma_0 = -8\varepsilon^2,\ \delta_1 = -8\varepsilon^2,\ \delta_0 = -16\varepsilon^2,\ \Delta_1 = 6144\varepsilon^6,$$
$$\lambda_4 = 256\varepsilon^4 - 1536\varepsilon^5,\ \lambda_3 = 2048\varepsilon^5,\ \lambda_2 = 2048\varepsilon^5 - 384\varepsilon^4,$$
$$\lambda_1 = -2048\varepsilon^5 - 128\varepsilon^4,\ \lambda_0 = 640\varepsilon^5.$$

Obviously, $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. Hence, we get the following quartic equation:

$$(2-12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon = 0. \tag{8}$$

Let $f(a) = (2-12\varepsilon)a^4 + 16\varepsilon a^3 + (16\varepsilon - 3)a^2 - (16\varepsilon + 1)a + 5\varepsilon$. Since $0 < \varepsilon < \frac{1}{12}$, it is easy to check that $f(0) = 5\varepsilon > 0,\ f(1) = 9\varepsilon - 2 < 0,\ f(2) = 18 - 27\varepsilon > 0$. This means that the quartic equation has at least two positive roots. On the other hand, due to Descartes's theorem, the number of positive roots is less or equal to, the number of sign differences between consecutive nonzero coefficients $2-12\varepsilon,\ 16\varepsilon,\ 16\varepsilon-3,\ -(16\varepsilon+1),\ 5\varepsilon$ of the quartic equation, which is, two. Therefore, the quartic equation has exactly two positive roots in which one of them belongs to $(0,1)$ and another one belongs to $(1,2)$.

Hence, for any $0<\varepsilon<\frac{1}{12}$, there exists a unique positive root a_0 of the quartic equation which belongs to the interval $(0,1)$. Since $f(0)=5\varepsilon>0,\ f\left(\frac{1}{2}\right)=\frac{18\varepsilon-9}{8}<0$, one has that $a_0\in(0,\frac{1}{2})$. Consequently, for any $0<\varepsilon<\frac{1}{12}$, the quadratic stochastic operator $\mathcal{V}_\varepsilon:\Omega_2\to\Omega_2$ given by (8) has a unique fixed point $\left(\frac{1+2a_0(1-a_0)}{2(1+a_0)},\ a_0,\ \frac{1-2a_0}{2(1+a_0)}\right)^T\in\Omega_2$ where a_0 is a unique root in the interval $(0,\frac{1}{2})$ of the quartic equation (8).

Example 2.5. We define a positive quadratic stochastic operator $\mathcal{R}_\varepsilon:\Omega_2\to\Omega_2$ for any $0<\varepsilon<\frac{1}{1000}$ as

$$\mathcal{R}_\varepsilon:\begin{cases}(\mathcal{R}_\varepsilon(\mathbf{x}))_1=(0.9-\varepsilon)x_1^2+\varepsilon x_2^2+0.1x_3^2+2(1-2\varepsilon)x_1x_2+2\varepsilon x_1x_3+2\varepsilon x_2x_3\\(\mathcal{R}_\varepsilon(\mathbf{x}))_2=0.1x_1^2+(0.9-\varepsilon)x_2^2+\varepsilon x_3^2+2\varepsilon x_1x_2+2\varepsilon x_1x_3+2(1-2\varepsilon)x_2x_3\\(\mathcal{R}_\varepsilon(\mathbf{x}))_3=\varepsilon x_1^2+0.1x_2^2+(0.9-\varepsilon)x_3^2+2\varepsilon x_1x_2+2(1-2\varepsilon)x_1x_3+2\varepsilon x_2x_3\end{cases}.$$

We would like to show that $\mathbf{Fix}(\mathcal{R}_\varepsilon)=\left\{\mathbf{c}=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^T\right\}$. Let us calculate the following constants for $\mathcal{R}_\varepsilon$:

$$\alpha_{11}=1-3\varepsilon,\ \alpha_{22}=0.1-\varepsilon,\ \alpha_{12}=1.1-4\varepsilon,\ \alpha_1=\varepsilon-0.1,\alpha_2=\varepsilon-0.1,\ \alpha_0=0.1,$$
$$\beta_{11}=0.1-\varepsilon,\ \beta_{22}=-1.1+4\varepsilon,\ \beta_{12}=-1+3\varepsilon,\ \beta_1=0,\ \beta_2=1-3\varepsilon,\ \beta_0=\varepsilon,$$
$$\gamma_2=-13\varepsilon^2+7.5\varepsilon-1.11,\ \gamma_1=20\varepsilon^2-9.4\varepsilon+1.02,\ \gamma_0=-3\varepsilon^2+1.1\varepsilon-0.01,$$
$$\delta_1=13\varepsilon^2-7.5\varepsilon+1.11,\ \delta_0=-2\varepsilon^2+1.4\varepsilon-0.12,$$
$$\Delta_1=-1040\varepsilon^6+1369.6\varepsilon^5-715.84\varepsilon^4+176.616\varepsilon^3-16.2708\varepsilon^2-0.94212\varepsilon+0.20646,$$

$$\lambda_4=1521\varepsilon^5-2262\varepsilon^4+1350.99\varepsilon^3-405.18\varepsilon^2+61.0389\varepsilon-3.6963,$$
$$\lambda_3=-2028\varepsilon^5+3016\varepsilon^4-1801.32\varepsilon^3+540.24\varepsilon^2-81.3852\varepsilon+4.9284,$$
$$\lambda_2=390\varepsilon^5-846.4\varepsilon^4+609.42\varepsilon^3-196.548\varepsilon^2+28.9314\varepsilon-1.5318,$$
$$\lambda_1=84\varepsilon^5+3.2\varepsilon^4-54.68\varepsilon^3+23.544\varepsilon^2-3.2268\varepsilon+0.0996,$$
$$\lambda_0=-15\varepsilon^5+9.2\varepsilon^4+0.55\varepsilon^3-1.016\varepsilon^2+0.1217\varepsilon+0.0001.$$

Obviously, $\alpha_{11}\beta_{11}\Delta_1\neq0$ for any $0<\varepsilon<\frac{1}{1000}$. Hence, we get the following quartic equation:

$$\lambda_4a^4+\lambda_3a^3+\lambda_2a^2+\lambda_1a+\lambda_0=0. \tag{9}$$

Let $g(a)=\lambda_4a^4+\lambda_3a^3+\lambda_2a^2+\lambda_1a+\lambda_0$. It is easy to check that

$$g(-1)<0,\quad g(0)>0,\quad g(0.2)<0,\quad g(0.5)>0,\quad g(1)<0,\quad \forall\ 0<\varepsilon<\frac{1}{1000}.$$

This means that the quartic equation given by (9) has three positive roots $a_1<a_2<a_3$ in which $a_1\in(0,0.2)$, $a_2\in(0.2,0.5)$, and $a_3\in(0.5,1)$. Moreover, it is easy to check that $a_2=\frac{1}{3}$ is a root of the quartic equation (9) which belongs to $(0.2,0.5)$. Among these three roots, only for $a_2=\frac{1}{3}$ we have that

$$0<A=\frac{\left(-1.11+7.5\varepsilon-13\varepsilon^2\right)a^2+\left(1.02-9.4\varepsilon+20\varepsilon^2\right)a-0.01+1.1\varepsilon-3\varepsilon^2}{2\left(1.11-7.5\varepsilon+13\varepsilon^2\right)a-2\varepsilon^2+1.4\varepsilon-0.12}<1,$$

$$0<B=\frac{\left(1.11-7.5\varepsilon+13\varepsilon^2\right)a^2+\left(0.90-8\varepsilon+18\varepsilon^2\right)a-3\varepsilon^2+1.1\varepsilon-0.01}{2\ \left(1.11-7.5\varepsilon+13\varepsilon^2\right)a-2\varepsilon^2+1.4\varepsilon-0.12}<1.$$

Hence, in this case, we obtain that $A_0=1-B_0=\frac{1}{3}$ and $\mathbf{Fix}(\mathcal{R}_\varepsilon)=\left\{\mathbf{c}=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^T\right\}$ for any $0<\varepsilon<\frac{1}{1000}$.

We now provide some examples for positive quadratic stochastic operators which may have one or three fixed points regardless of the condition $\alpha_{11}\beta_{11}\Delta_1=0$.

Example 2.6. *We pick up three points* $\boldsymbol{X} = (0.1, 0.3, 0.6)^T$, $\boldsymbol{Y} = (0.2, 0.3, 0.5)^T$ *and* $\boldsymbol{Z} = (0.7, 0.2, 0.1)^T$ *from the simplex* Ω_2. *We define a positive quadratic stochastic operator* $\mathcal{Q} : \Omega_2 \to \Omega_2$ *as follows:*

$$\mathcal{Q}: \begin{cases} (\mathcal{Q}(\mathbf{x}))_1 = \frac{376708}{440200}x_1^2 + \frac{2952}{440200}x_2^2 + \frac{292}{440200}x_3^2 + \frac{17}{10}x_1x_2 + \frac{3}{5}x_1x_3 + \frac{1}{50}x_2x_3, \\ (\mathcal{Q}(\mathbf{x}))_2 = \frac{300}{2201}x_1^2 + \frac{3}{5}x_2^2 + \frac{2}{5}x_3^2 + \frac{1467902}{6096770}x_1x_2 + \frac{2942114}{3048385}x_1x_3 + \frac{240522}{1219354}x_2x_3, \\ (\mathcal{Q}(\mathbf{x}))_3 = \frac{873}{110050}x_1^2 + \frac{21641}{55025}x_2^2 + \frac{65957}{110050}x_3^2 + \frac{722258}{12193540}x_1x_2 + \frac{530250}{1219354}x_1x_3 + \frac{108689946}{60967700}x_2x_3. \end{cases}$$

A straightforward calculation shows that $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are fixed points of the positive quadratic stochastic operator* $\mathcal{Q} : \Omega_2 \to \Omega_2$ *for which* $\Delta_1 = 0$. *Therefore, we obtain that* $|\boldsymbol{Fix}(\mathcal{Q})| = 3$ *in which* $\alpha_{11}\beta_{11}\Delta_1 = 0$.

Example 2.7. *We define a positive quadratic stochastic operator* $\mathcal{Q} : \Omega_2 \to \Omega_2$ *as follows:*

$$\mathcal{Q}: \begin{cases} (\mathcal{Q}(\mathbf{x}))_1 = 0.15x_1^2 + 0.17x_2^2 + 0.17x_3^2 + 0.32x_1x_2 + 0.32x_1x_3 + 0.30x_2x_3, \\ (\mathcal{Q}(\mathbf{x}))_2 = 0.68x_1^2 + 0.67x_2^2 + 0.66x_3^2 + 1.32x_1x_2 + 1.34x_1x_3 + 1.32x_2x_3, \\ (\mathcal{Q}(\mathbf{x}))_3 = 0.17x_1^2 + 0.16x_2^2 + 0.17x_3^2 + 0.36x_1x_2 + 0.34x_1x_3 + 0.38x_2x_3. \end{cases}$$

A straightforward calculation shows that the positive quadratic stochastic operator $\mathcal{Q} : \Omega_2 \to \Omega_2$ *is a contraction for which* $\alpha_{11} = \beta_{11} = 0$. *Therefore, we obtain that* $|\boldsymbol{Fix}(\mathcal{Q})| = 1$ *in which* $\alpha_{11}\beta_{11}\Delta_1 = 0$.

3. Positivity $\not\Rightarrow$ Uniqueness of fixed points $\not\Rightarrow$ Strong ergodicity $\not\Rightarrow$ Contraction

Let $\mathcal{Q} : \Omega_2 \to \Omega_2$ be a positive quadratic stochastic operator defined as follows:

$$\mathcal{Q}(\mathbf{x}) = \left(\sum_{i,j=1}^{3} p_{ij}x_ix_j, \sum_{i,j=1}^{3} q_{ij}x_ix_j, \sum_{i,j=1}^{3} r_{ij}x_ix_j \right)^T,$$

where $p_{ij}, q_{ij}, r_{ij} > 0$ and $p_{ij} + q_{ij} + r_{ij} = 1$ with $p_{ij} = p_{ji}$, $q_{ij} = q_{ji}$, and $r_{ij} = r_{ji}$, $1 \le i, j \le 3$.

Proposition 3.1 ([18, 25])**.** *If* $\mathcal{P} > 0$ *then one has that* $|\mathbf{Fix}(\mathcal{Q}) \cap \Omega_2| = 1$ or 3.

3.1. Positivity $\not\Rightarrow$ The uniqueness of fixed points.

Now, we are aiming to provide an example for positive quadratic stochastic operators having three fixed points in the simplex.

Example 3.2 ([24, 25])**.** *We pick up three points* $\boldsymbol{A} = (0.1, 0.2, 0.7)^T$, $\boldsymbol{B} = (0.4, 0.3, 0.3)^T$ *and* $\boldsymbol{C} = (0.59, 0.31, 0.1)^T$ *from the simplex* Ω_2. *We define a positive quadratic stochastic operator* $\mathcal{Q}_0 : \Omega_2 \to \Omega_2$ *as follows:*

$$\mathcal{Q}_0: \begin{cases} (\mathcal{Q}_0(\mathbf{x}))_1 = \frac{232873}{319300}x_1^2 + \frac{4717}{10300}x_2^2 + \frac{207}{63860}x_3^2 + \frac{7}{5}x_1x_2 + \frac{3}{5}x_1x_3 + \frac{1}{50}x_2x_3, \\ (\mathcal{Q}_0(\mathbf{x}))_2 = \frac{27}{100}x_1^2 + \frac{1}{2}x_2^2 + \frac{3}{20}x_3^2 + \frac{470171}{814300}x_1x_2 + \frac{378421}{407150}x_1x_3 + \frac{158157}{814300}x_2x_3, \\ (\mathcal{Q}_0(\mathbf{x}))_3 = \frac{54}{79825}x_1^2 + \frac{433}{10300}x_2^2 + \frac{27037}{31930}x_3^2 + \frac{18409}{814300}x_1x_2 + \frac{191589}{407150}x_1x_3 + \frac{1454157}{814300}x_2x_3. \end{cases}$$

A straightforward calculation shows that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *are fixed points of the positive quadratic stochastic operator* $\mathcal{Q}_0 : \Omega_2 \to \Omega_2$. *We can define another positive quadratic stochastic operator* $\mathcal{Q}_1 : \Omega_2 \to \Omega_2$ *as follows:*

$$\mathcal{Q}_1: \begin{cases} (\mathcal{Q}_1(\mathbf{x}))_1 = \frac{17322871}{22351000}x_1^2 + \frac{990257}{2163000}x_2^2 + \frac{1559}{13410600}x_3^2 + \frac{13}{10}x_1x_2 + \frac{16}{25}x_1x_3 + \frac{11}{500}x_2x_3, \\ (\mathcal{Q}_1(\mathbf{x}))_2 = \frac{224}{1000}x_1^2 + \frac{488}{1000}x_2^2 + \frac{125}{1000}x_3^2 + \frac{703327}{1017875}x_1x_2 + \frac{19461451}{24429000}x_1x_3 + \frac{8271787}{24429000}x_2x_3, \\ (\mathcal{Q}_1(\mathbf{x}))_3 = \frac{4301}{4470200}x_1^2 + \frac{117199}{2163000}x_2^2 + \frac{2933179}{3352650}x_3^2 + \frac{18371}{2035750}x_1x_2 + \frac{13761989}{24429000}x_1x_3 + \frac{1601951}{977160}x_2x_3. \end{cases}$$

A straightforward calculation shows that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *are also fixed points of the positive quadratic stochastic operator* $\mathcal{Q}_1 : \Omega_2 \to \Omega_2$. *Now, we can define a family of positive quadratic stochastic operators* $\mathcal{Q}_\varepsilon : \Omega_2 \to \Omega_2$ *as* $\mathcal{Q}_\varepsilon(\mathbf{x}) = (1 - \varepsilon)\mathcal{Q}_0(\mathbf{x}) + \varepsilon\mathcal{Q}_1(\mathbf{x})$ *for any*

$x \in \Omega_2$ and $0 \le \varepsilon \le 1$. It is clear that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are also fixed points of the family of positive quadratic stochastic operators $\mathcal{Q}_\varepsilon : \Omega_2 \to \Omega_2$.

Remark 3.3. *It is easy to check in Example 3.2 that $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. This shows that one can have $|\mathbf{Fix}(\mathcal{Q})| = 3$ in the case $\alpha_{11}\beta_{11}\Delta_1 \neq 0$.*

3.2. **Uniqueness of fixed points $\not\Rightarrow$ Strong ergodicity.** Now, we shall provide examples for positive quadratic stochastic operators having a unique fixed point in which they are not strongly ergodic.

Example 3.4. *We again consider a positive quadratic stochastic operator $\mathcal{R}_\varepsilon : \Omega_2 \to \Omega_2$,*

$$\mathcal{R}_\varepsilon : \begin{cases} (\mathcal{R}_\varepsilon(\mathbf{x}))_1 = (0.9-\varepsilon)x_1^2 + \varepsilon x_2^2 + 0.1x_3^2 + 2(1-2\varepsilon)x_1x_2 + 2\varepsilon x_1x_3 + 2\varepsilon x_2x_3, \\ (\mathcal{R}_\varepsilon(\mathbf{x}))_2 = 0.1x_1^2 + (0.9-\varepsilon)x_2^2 + \varepsilon x_3^2 + 2\varepsilon x_1x_2 + 2\varepsilon x_1x_3 + 2(1-2\varepsilon)x_2x_3, \\ (\mathcal{R}_\varepsilon(\mathbf{x}))_3 = \varepsilon x_1^2 + 0.1x_2^2 + (0.9-\varepsilon)x_3^2 + 2\varepsilon x_1x_2 + 2(1-2\varepsilon)x_1x_3 + 2\varepsilon x_2x_3. \end{cases}$$

We have already showed that $\mathbf{Fix}(\mathcal{R}_\varepsilon) = \left\{\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T\right\}$ for any $0 < \varepsilon < \frac{1}{1000}$. Let us study the local behavior of the unique fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$ of the quadratic stochastic operator $\mathcal{R}_\varepsilon$. To do so, we have to calculate absolute values of eigenvalues of the Jacobian matrix $J(\mathcal{R}_\varepsilon)$ at the fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$ under the constrain $x_1 + x_2 + x_3 = 1$. The eigenvalues are roots of the following quadratic equation (see [18]*, pp. 292–293)*

$$\left|\begin{matrix} \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_1}{\partial x_1} - \lambda & \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_1}{\partial x_2} & \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_1}{\partial x_3} \\ \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_2}{\partial x_1} & \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_2}{\partial x_2} - \lambda & \frac{\partial(\mathcal{R}_\varepsilon(\mathbf{x}))_2}{\partial x_3} \\ 1 & 1 & 1 \end{matrix}\right|_{\mathbf{x}=\mathbf{c}} = 0\,,$$

which is

$$\lambda^2 + (4\varepsilon - 1.8)\,\lambda + \frac{16}{3}\varepsilon^2 - 4.8\varepsilon + 1.08 = 0.$$

The last quadratic equation has two complex roots in which

$$|\lambda_1| = |\lambda_2| = \sqrt{1.08 - 4.8\varepsilon + \frac{16}{3}\varepsilon^2} > 1, \quad \forall\ 0 < \varepsilon < \frac{1}{1000}.$$

Consequently, the fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$ is repelling. This means that the trajectory of the quadratic stochastic operator $\mathcal{R}_\varepsilon$ starting from any initial point does not converge to the unique fixed point $\mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$, i.e., it is not strongly ergodic.

3.3. **Strong ergodicity $\not\Rightarrow$ Contraction.** Now, we shall provide examples for the non-contraction positive quadratic stochastic operators which are strongly ergodic.

Example 3.5 ([23]). *Let $\mathbf{a}_1 = (0.1, 0.3, 0.6)^T$, $\mathbf{a}_2 = (0.7, 0.1, 0.2)^T$, $\mathbf{a}_3 = (0.2, 0.6, 0.2)^T$. Let us consider the following positive quadratic stochastic operator $\mathcal{Q}_{\mathbf{a_1a_2a_3}} : \Omega_2 \to \Omega_2$,*

$$\mathcal{Q}_{\mathbf{a_1a_2a_3}}(\mathbf{x}) = \mathbf{a}_1x_1^2 + \mathbf{a}_2x_2^2 + \mathbf{a}_3x_3^2 + 2\mathbf{a}_3x_1x_2 + 2\mathbf{a}_2x_1x_3 + 2\mathbf{a}_1x_2x_3.$$

One can see that

$$\mathbb{P}_1 = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_2 \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} \mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_1 \end{pmatrix}, \qquad \mathbb{P}_3 = \begin{pmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{pmatrix}$$

are doubly stochastic matrices. Therefore, $\mathcal{Q}_{\mathbf{a_1a_2a_3}} : \Omega_2 \to \Omega_2$ is strongly ergodic, i.e., $\lim\limits_{k\to\infty} \mathcal{Q}^{(k)}_{\mathbf{a_1a_2a_3}}(\mathbf{x}) = \mathbf{c}$ for any $\mathbf{x} \in \Omega_2$ where $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. However, $\mathcal{Q}_{\mathbf{a_1a_2a_3}} : \Omega_2 \to \Omega_2$ is not contraction. Indeed, if $\mathbf{x}_0 = (0.85, 0.1, 0.05)^T$ and $\mathbf{y}_0 = (0.9, 0.1, 0)^T$ then $\|\mathcal{Q}_{\mathbf{a_1a_2a_3}}(\mathbf{x}_0) - \mathcal{Q}_{\mathbf{a_1a_2a_3}}(\mathbf{y}_0)\|_1 = 0.1005 > 0.1 = \|\mathbf{x}_0 - \mathbf{y}_0\|_1$.

Remark 3.6. *It is well-known that if* $\mathbb{P} > 0$ *then the corresponding positive linear stochastic operator* $\mathcal{L} : \Omega_n \to \Omega_n$, $\mathcal{L}(\mathbf{x}) = \mathbb{P}\mathbf{x}$ *is strongly ergodic if and only if it is a contraction. Example 3.5 shows that unlike the linear case, the class of positive strongly ergodic quadratic stochastic operators does not coincide with the class of contraction quadratic stochastic operators. This is an unexpected situation. However, there are a lot of non-negative linear (quadratic) stochastic operators which are strongly ergodic but not contractions. For example, let us consider the following linear stochastic operator* $\mathcal{L} : \Omega_3 \to \Omega_3$, $\mathcal{L}(\mathbf{x}) = \mathbb{P}\mathbf{x}$, *where* $\mathbb{P} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ *is a column-stochastic matrix. It is clear* $\boldsymbol{Fix}(\mathcal{L}) = \{\mathbf{e}_1\}$. *Since* $\mathbb{P}^n = \begin{pmatrix} 1 & 1 - \frac{1}{2^n} & 1 - a_n - \frac{1}{2^n} \\ 0 & \frac{1}{2^n} & a_n \\ 0 & 0 & \frac{1}{2^n} \end{pmatrix}$ *where* $a_{n+1} = \frac{1}{2}(\frac{1}{2^n} + a_n)$ *with* $a_1 = \frac{1}{2}$, *the linear stochastic operator* $\mathcal{L} : \Omega_3 \to \Omega_3$ *is strongly ergodic. However, it is not a contraction because of* $\|\mathcal{L}(\mathbf{e}_1) - \mathcal{L}(\mathbf{e}_3)\|_1 = \|\mathbf{e}_1 - \mathbf{e}_3\|_1 = 2$. *In this sense, it was naturally expected to have a lot of non-positive quadratic stochastic operators which are strongly ergodic but not contractions (for example, see* [12, 19]*).*

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