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ON NEW POINTS OF THE DISCRETE SPECTRUM UNDER SINGULAR PERTURBATIONS

H. V. TUHAI

Dedicated to Professor V. D. Koshmanenko on his 75 birthday

Abstract. We study the emergence problem of new points in the discrete spectrum under singular perturbations of a positive operator. We start with the sequential approach to construction of additional eigenvalues for perturbed operators, which was produced by V. Koshmanenko on the base of rigged Hilbert spaces methods. Two new observations are established. We show that one can construct a point of the discrete spectrum of any finite multiplicity in a single step. And that the method of rigged Hilbert spaces admits an application to the modified construction of a new point of the discrete spectrum under super-singular perturbations.

1. Introduction

The problem of new points of the discrete spectrum under perturbations of self-adjoint operator has a long history and important applications. In particular, this problem is actual under singular perturbations (see, for example, [1, 7, 11, 12, 18, 23, 24]). The original approach to construction of any number of additional points of the discrete spectrum for singular perturbed operator was developed by V. Koshmanenko in a series of publications [2–10, 16, 21, 22, 25]. Let us shortly recall the main ideas of this approach.

Let $A \geq 1$ denote a self-adjoint operator in a Hilbert space $\mathcal{H}$ with norm $\| \cdot \|$ and inner product $(\cdot,\cdot)$. With A the so-called A–scale of Hilbert spaces is associated,

$$\mathcal{H}_{-k} \sqsupset \mathcal{H}_0 \equiv \mathcal{H} \sqsupset \mathcal{H}_k \equiv \mathcal{H}_k(A), \quad k > 0, \tag{1.1}$$

where $\mathcal{H}_k = \mathrm{Dom} A^{k/2}$ with respect to the positive norm

$$\| \varphi \|_k := \| A^{k/2} f \|, \quad \varphi \in \mathrm{Dom} A^{k/2}$$

and $\mathcal{H}_{-k}$ is the completion of $\mathcal{H}$ in the negative norm

$$\| h \|_{-k} := \| A^{-k/2} h \|, \quad h \in \mathcal{H},$$

(for more details see [2, 13, 14, 16]). The notation $\sqsupset$ in (1.1) stands for a dense and continuous embedding.

In a general approach, the singularly perturbed operator $\tilde{A}$ is defined as a self-adjoint operator uniquely associated with a new rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$ constructed from a given singular perturbation of A. In [15] some parametrization of $\tilde{A}$ is produced in terms of singular quadratic forms γ_S and associated bounded mappings $S : \mathcal{H}_k \to \mathcal{H}_{-k}$ with additional properties.

In accordance with the theory of rigged spaces (see [13, 14]) the operator A can be reconstructed from any pair of spaces $\mathcal{H} \sqsupset \mathcal{H}_k$ with $k > 0$ (or a conjugate couple $\mathcal{H}_{-k} \sqsupset \mathcal{H}$). Since A is positive, one can write $A = \sqrt[k]{D_k}$, where D_k denotes the restriction of the

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unitary mapping $D_{-k,k} : \mathcal{H}_k \to \mathcal{H}_{-k}$ to the set $\mathcal{D}_k \equiv \mathcal{H}_{k/2} = \{\varphi \in \mathcal{H}_k \mid D_{-k,k}\varphi \in \mathcal{H}\}$ (for details see [2,15]). Thus, A is uniquely defined by D_k with each fixed $k > 0$. Therefore one can use the same connection between the singular (or super-singular) positive perturbed operators $\tilde{A}$ and the corresponding associated scale of Hilbert spaces,

$$\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}_0 \equiv \mathcal{H} \sqsupset \tilde{\mathcal{H}}_k, \quad \tilde{\mathcal{H}}_k = \text{Dom}\tilde{A}^{k/2}, \quad k > 0. \tag{1.2}$$

In fact, the developed in [2,3,5,16,21,22] the rigged Hilbert spaces method generalizes the well-known form-sum method, where $\tilde{A}$ is defined as an operator associated with the triplet $\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_1$. Indeed, for consideration of singular perturbations of highest orders one has at first to build one of the spaces $\tilde{\mathcal{H}}_k$ or $\tilde{\mathcal{H}}_{-k},\ k \geq 2$, take an operator of type $\tilde{D}_k$ (see above), and finally to construct its restriction to $\mathcal{H}$. We recall that according to [24], $\tilde{A}$ is a singular perturbation of A in a wide sense, if there exists $k \geq 1$ such that the operators $\tilde{A}^k, A^k$ are coincide on a set $\mathcal{M}_k \subset \tilde{\mathcal{H}}_k \bigcap \mathcal{H}_k$ which is dense in $\mathcal{H}$. Clearly that in such a case the inner products in $\tilde{\mathcal{H}}_k, \mathcal{H}_k$ coincide on this set, $(\varphi, \psi)_k = (\varphi, \psi)^{\sim}_k,\quad \varphi, \psi \in \mathcal{M}_k$. In particular, if $k > 2$, then the set $\mathcal{M}_k$ possibly is dense not only in $\mathcal{H}$ but in $\mathcal{H}_2$ =Dom$A$ too. Therefore the symmetric operator $\dot{A} := A|\mathcal{M}_k$ is essentially self-adjoint and it is impossible to use the standard self-adjoint extensions method. However, one can work in terms of differences between the singular and super-singular perturbations, i.e., between some powers of $\tilde{A}$ and A in terms of the operators S acting from $\mathcal{H}_k$ to $\mathcal{H}_{-k},\ k \geq 2$. In this way the problem of new eigenvalues (points of the discrete spectrum) is viewed as admissible.

The purpose of this article is to demonstrate the rigged spaces method in the problem of emergence of the multiplicative point spectrum and, in addition, to apply it to some special case of super-singular perturbations.

2. The rigged Hilbert spaces method

Here we repeat in more details the rigged Hilbert spaces method presented in [24] for cases $k = 1, 2$.

Consider a part of the A-scale (1.1)

$$\mathcal{H}_{-2} \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H} \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_2 \equiv \mathcal{H}_2(A), \tag{2.1}$$

where $\mathcal{H}_2$ =Dom$A$ with the norm $\|\varphi\|_2 := \|A\varphi\|,\quad \mathcal{H}_1 = \text{Dom}A^{1/2}$ with the norm $\|\varphi\|_1 := \|A^{1/2}\varphi\|$, and $\mathcal{H}_{-2},\ \mathcal{H}_{-1}$ are conjugated spaces. We recall that there exists a one to one correspondence between operators $A = A^* \geq 1$ in $\mathcal{H}$ and the rigged Hilbert spaces of the form (2.1) or the whole scale of Hilbert spaces (1.1) (see Theorem 2.1 in [2]).

By the construction, the linear functional $l_\omega(\varphi) := \langle\varphi, \omega\rangle_{1,-1}$ is defined for each $\omega \in \mathcal{H}_{-1}$ and is continuous on $\mathcal{H}_1\quad (\langle\cdot,\cdot\rangle_{k,-k},\ k > 0$ stands for the dual inner product between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$). Due to the Riesz theorem we have the equality: $l_\omega(\varphi) = (\varphi, \psi)_1$, where $\psi = \psi(\omega) \in \mathcal{H}_1,\quad \|\psi\|_1 = \|\omega\|_{-1}$. Let $D_{-1,1} : \mathcal{H}_1 \ni \psi \to \omega \in \mathcal{H}_{-1}$ denote the canonical unitary isomorphism (see [2,13,14]). And let

$$D_1 := D_{-1,1}|\mathcal{D}_2, \quad \mathcal{D}_1 := \{\varphi \in \mathcal{H}_1 | D_{-1,1}\varphi \in \mathcal{H}\}.$$

Then it is not difficult to understand that $A = D_1$. And $\mathcal{D}_1 = \mathcal{H}_2$ in the norm $\|\varphi\|_2 = \|A\varphi\|$.

Similarity, repeating this way with $k = 2$ we find that $A^2 = D_2$, and therefore $A = \sqrt{D_2}$. Thus, in general case we have $A = (D_k)^{1/k}, k \geq 1$.

If we change the inner product in one of the spaces $\mathcal{H}_k,\ k = 1, 2\ \ (\text{or } \mathcal{H}_{-k})$, i.e., to replace $(\cdot,\cdot)_k$ with $(\cdot,\cdot)^{\sim}_k$ or $((\cdot,\cdot)_{-k}$ on $(\cdot,\cdot)^{\sim}_{-k})$, then using the starting couple $\mathcal{H} \sqsupset \tilde{\mathcal{H}}_k$ (or $\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}$) one can construct a new scale of spaces of the form (1.2)

$$\tilde{\mathcal{H}}_{-2} \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H} \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_2. \tag{2.2}$$

By definition the associated operator $\tilde{A}$ is a singular perturbation with respect to A in a wide sense.

3. The singular perturbations with the multiple point spectrum

We are able to formulate our main result on the multiple point spectrum as follows.

Theorem 3.1. *Let A be a positive self-adjoint operator in Hilbert space $\mathcal{H}$, any real $E \in \mathbb{R}$, and orthogonal vectors $\psi_i \in \mathcal{H}_{+1}$, $i = 1, ..., n \geq 2$ which satisfy the condition: span$\{\psi_i\} \cap \mathcal{D}(A) = 0$. Let $\tilde{A} = A_n$ be a rank n singular perturbation of the operator A, i.e., A_n solves the multiple eigenvalues problem (for its existence see [4, 9, 16, 21])*

$$A_n \psi_i = E\psi_i, \quad i = 1, \dots, n. \tag{3.1}$$

We assert that A_n admits a one step construction as a generalized operator sum,

$$A_n = A \tilde{+} T_n,$$

where

$$T_n = -\sum_{i,j=1}^{n} \beta_{ij}^{(n)} \langle \cdot, (\mathbf{A} - E)\psi_i \rangle (\mathbf{A} - E)\psi_j, \quad \mathbf{A} \equiv \mathbf{A}_0 = A^{\rm cl} : \mathcal{H}_{+1} \to \mathcal{H}_{-1}. \tag{3.2}$$

Here

$$\beta_{ij}^{(n)} = (-1)^{i+j} \frac{|M_n^{ji}|}{|M_n|}, \tag{3.3}$$

where

$$M_n = (c_{ij})_{i,j=1}^{n}, \quad c_{ij} = \langle \psi_i, (\mathbf{A} - E)\psi_j \rangle, \quad i, j = 1, \dots, n, \tag{3.4}$$

and $|M_n^{i,j}|$ denotes the minor of the matrix M_n which is produced by omitting the ith row and the jth column.

Proof. It is known [4, 9, 16, 21] that one can construct the operator T_n consistently in n steps using the rank one singular perturbations $\alpha_i \langle \cdot, \omega_i \rangle \omega_i$ at each step, where $\omega_i = (\mathbf{A}_{i-1} - E)\psi_i, \alpha_i = -1/\langle \psi_i, \omega_i \rangle, i = 1, \dots, n$. Here $\mathbf{A}_i = \mathbf{A}_{i-1} + \alpha_i \langle \cdot, \omega_i \rangle \omega_i = \mathbf{A}_0 + T_i : \mathcal{H}_{+1} \to \mathcal{H}_{-1}$, with $\mathbf{A} \equiv \mathbf{A}_0$, $T_0 = 0$. Thus

$$T_n = \sum_{i=1}^{n} \alpha_i \langle \cdot, \omega_i \rangle \omega_i. \tag{3.5}$$

We will prove (3.2) using the method of mathematical induction. So at the the first step we have

$$A_1 = A \widetilde{+} T_1$$

with

$$T_1 = \alpha_1 \langle \cdot, \omega_1 \rangle \omega_1, \quad \text{where} \quad \omega_1 = (\mathbf{A} - E)\psi_1, \alpha_1 = -\frac{1}{\langle \psi_1, \omega_1 \rangle} = \beta_{11}^{(1)},$$

where we introduced the notations $\beta_{11}^{(1)} := -1/c_{11}$, $c_{11} := \langle \psi_1, \omega_1 \rangle$. At the second step we obtain

$$A_2 = A \widetilde{+} T_2, \tag{3.6}$$

where

$$T_2 = \alpha_1 \langle \cdot, \omega_1 \rangle \omega_1 + \alpha_2 \langle \cdot, \omega_2 \rangle \omega_2 \tag{3.7}$$

with

$$\omega_2 = (\mathbf{A}_1 - E)\psi_2 = (\mathbf{A} - E)\psi_2 + \alpha_1 \langle \psi_2, \omega_1 \rangle \omega_1 = (\mathbf{A} - E)\psi_2 - \frac{c_{21}}{c_{11}} \omega_1, \tag{3.8}$$

where $c_{21} = \langle \psi_2, \omega_1 \rangle$ and by the construction,

$$\alpha_2^{-1} = -\langle \psi_2, \omega_2 \rangle. \tag{3.9}$$

Using (3.8), (3.4), we get

$$\begin{aligned}
\alpha_2^{-1} &= -\langle \psi_2, (\mathbf{A}-E)\psi_2\rangle + \frac{1}{\langle \psi_1, (\mathbf{A}-E)\psi_1\rangle} |\langle \psi_2, (\mathbf{A}-E)\psi_1\rangle|^2 \alpha_2^{-1} \\
&= -\left(c_{22} - \frac{c_{12}c_{21}}{c_{11}} \right) = -\frac{|M_2|}{|M_1|} = \beta_{22}^{(2)}.
\end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) in (3.7) we have

$$\begin{aligned}
T_2 = &-\frac{1}{c_{11}} \langle \cdot, (\mathbf{A}-E)\psi_1\rangle \\
&+ \frac{c_{11}}{|M_2|} \langle \cdot, (\mathbf{A}-E)\psi_2 - \frac{c_{21}}{c_{11}}(\mathbf{A}-E)\psi_1\rangle \left((\mathbf{A}-E)\psi_2 - \frac{c_{21}}{c_{11}}(\mathbf{A}-E)\psi_1 \right).
\end{aligned}$$

After simple transformations we obtain

$$\begin{aligned}
T_2 = &-\frac{1}{c_{11}} \left(1 + \frac{c_{12}c_{21}}{|M_2|} \right) \langle \cdot, (\mathbf{A}-E)\psi_1\rangle (\mathbf{A}-E)\psi_1 \\
&+ \frac{c_{21}}{|M_2|} \langle \cdot, (\mathbf{A}-E)\psi_2\rangle (\mathbf{A}-E)\psi_1 \\
&+ \frac{c_{12}}{|M_2|} \langle \cdot, (\mathbf{A}-E)\psi_1\rangle\rangle (\mathbf{A}-E)\psi_2 - \frac{c_{22}}{|M_2|} \langle \cdot, (\mathbf{A}-E)\psi_2\rangle (\mathbf{A}-E)\psi_2.
\end{aligned}$$

Thus

$$T_2 = \sum_{i,j=1}^{2} \beta_{ij}^{(2)} \langle \cdot, (\mathbf{A}-E)\psi_i\rangle (\mathbf{A}-E)\psi_j,$$

where

$$\beta_{ij}^{(2)} = (-1)^{i+j} \frac{|M_2^{ji}|}{|M_2|}, \quad i,j = 1,2.$$

Assume that (3.2) is true for $k = n-1$,

$$T_{n-1} = -\sum_{i,j=1}^{n-1} (-1)^{i+j} \frac{|M_{n-1}^{ji}|}{|M_{n-1}|} \langle \cdot, (\mathbf{A}-E)\psi_i\rangle (\mathbf{A}-E)\psi_j. \tag{3.11}$$

Let us prove a similar formula for $k = n$. By construction,

$$T_n = T_{n-1} + \alpha_n \langle \cdot, \omega_n \rangle \omega_n, \tag{3.12}$$

where

$$\omega_n = (\mathbf{A}_{n-1} - E)\psi_n = (\mathbf{A}-E)\psi_n + T_{n-1}\psi_n, \quad \alpha_n^{-1} = -\langle \psi_n, \omega_n \rangle. \tag{3.13}$$

We will find α_n using (3.11),

$$\begin{aligned}
\alpha_n^{-1} &= -\langle \psi_n, (\mathbf{A}-E)\psi_n\rangle + \langle \psi_n, T_{n-1}\psi_n\rangle \\
&= -c_{nn} + \sum_{i,j=1}^{n-1} (-1)^{i+j} \frac{|M_{n-1}^{ji}|}{|M_{n-1}|} \overline{\langle \psi_n, (\mathbf{A}-E)\psi_i\rangle} \langle \psi_n, (\mathbf{A}-E)\psi_j\rangle \\
&= -c_{nn} + \sum_{i,j=1}^{n-1} (-1)^{i+j} \frac{|M_{n-1}^{ji}|}{|M_{n-1}|} c_{in} c_{nj} \\
&= -\frac{1}{|M_{n-1}|} \left(c_{nn}|M_{n-1}| - \sum_{i,j=1}^{n-1} (-1)^{i+j} |M_{n-1}^{ji}| c_{in} c_{nj} \right) = -\frac{|M_n|}{|M_{n-1}|}.
\end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.12) we obtain

$$T_n = T_{n-1} - \frac{|M_{n-1}|}{|M_n|}\langle \cdot, (\mathbf{A}-E)\psi_n + T_{n-1}\psi_n\rangle \left((\mathbf{A}-E)\psi_n + T_{n-1}\psi_n\right). \tag{3.15}$$

Consider the expression $T_{n-1}\psi_n$

$$\begin{aligned} T_{n-1}\psi_n &= \sum_{i,j=1}^{n-1}(-1)^{i+j}\frac{|M_{n-1}^{ji}|}{|M_{n-1}|}\langle \psi_n, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_j \\ &= \frac{1}{|M_{n-1}|}\sum_{j=1}^{n-1}(-1)^{j+n-1}\sum_{i=1}^{n-1}(-1)^{i+n-1}|M_{n-1}^{ji}|(\mathbf{A}-E)\psi_j \\ &= \frac{1}{|M_{n-1}|}\sum_{j=1}^{n-1}(-1)^{j+n-1}|M_n^{jn}|(\mathbf{A}-E)\psi_j. \end{aligned} \tag{3.16}$$

Now (3.15) has the form

$$\begin{aligned} T_n = T_{n-1} &- \frac{|M_{n-1}|}{|M_n|}\langle \cdot, (\mathbf{A}-E)\psi_n\rangle(\mathbf{A}-E)\psi_n - \frac{|M_{n-1}|}{|M_n|}\langle \cdot, (\mathbf{A}-E)\psi_n\rangle T_{n-1}\psi_n \\ &- \frac{|M_{n-1}|}{|M_n|}\langle \cdot, T_{n-1}\psi_n\rangle(\mathbf{A}-E)\psi_n - \frac{|M_{n-1}|}{|M_n|}\langle \cdot, T_{n-1}\psi_n\rangle T_{n-1}\psi_n. \end{aligned}$$

Using (3.11), (3.4) and (3.16) we get

$$\begin{aligned} T_n = &-\frac{|M_{n-1}|}{|M_n|}\langle \cdot, (\mathbf{A}-E)\psi_n\rangle(\mathbf{A}-E)\psi_n \\ &- \sum_{i,j=1}^{n-1}(-1)^{i+j}\frac{|M_{n-1}^{ji}|}{|M_{n-1}|}\langle \cdot, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_j \\ &+ \frac{1}{|M_n|}\sum_{j=1}^{n-1}(-1)^{j+n}|M_n^{jn}|\langle \cdot, (\mathbf{A}-E)\psi_n\rangle(\mathbf{A}-E)\psi_j \\ &+ \frac{1}{|M_n|}\sum_{i=1}^{n-1}(-1)^{i+n}|M_n^{ni}|\langle \cdot, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_n \\ &- \frac{1}{|M_n||M_{n-1}|}\sum_{i=1}^{n-1}(-1)^{i+n}|M_n^{ni}|\langle \cdot, (\mathbf{A}-E)\psi_i\rangle\sum_{j=1}^{n-1}(-1)^{j+n}|M_n^{jn}|(\mathbf{A}-E)\psi_j. \end{aligned}$$

Consider the second and the last terms,

$$\begin{aligned} &-\sum_{i,j=1}^{n-1}(-1)^{i+j}\frac{|M_{n-1}^{ji}|}{|M_{n-1}|}\langle \cdot, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_j \\ &- \frac{1}{|M_n||M_{n-1}|}\sum_{i=1}^{n-1}(-1)^{i+n}|M_n^{ni}|\langle \cdot, (\mathbf{A}-E)\psi_i\rangle\sum_{j=1}^{n-1}(-1)^{j+n}|M_n^{jn}|(\mathbf{A}-E)\psi_j \\ &= -\frac{1}{|M_n||M_{n-1}|}\sum_{i,j=1}^{n-1}(-1)^{i+j}\left(|M_n||M_{n-1}^{ji}| + |M_n^{ni}||M_n^{jn}|\right) \\ &\times \langle \cdot, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_j \\ &= -\frac{1}{|M_n|}\sum_{i,j=1}^{n-1}(-1)^{i+j}|M_n^{ji}|\langle \cdot, (\mathbf{A}-E)\psi_i\rangle(\mathbf{A}-E)\psi_j, \end{aligned}$$

where we used the Sylvester's determinant identity [17]

$$|M_n||M_{n-1}^{ji}| = |M_n^{nn}||M_n^{ji}| - |M_n^{ni}||M_n^{jn}|, \quad |M_n^{nn}| = |M_{n-1}|.$$

Finally, by (3.3) we obtain

$$T_n = -\sum_{i,j=1}^{n} \beta_{ij}^{(n)} \langle \cdot, (\mathbf{A}-E)\psi_i \rangle (\mathbf{A}-E)\psi_j,$$

which proves the theorem. □

4. The case of a rank-one super-singular perturbation

Here we recall, following [15,24], a constructive way for consideration of super singular perturbed operators $\tilde{A}$ from the class $\mathcal{P}_k^n(A)$ with $k>2$. More precisely we will explain this approach for the case of rank-one super-singular perturbations.

We start with a perturbation of A in $\mathcal{H}$ which is given by the quadratic form

$$\gamma_\omega[\varphi] = \langle \varphi, \omega \rangle_{k,-k} \langle \omega, \varphi \rangle_{-k,k}, \quad \omega \in \mathcal{H}_{-k} \backslash \mathcal{H}_{\frac{k}{2}}, \quad k>2.$$

The associated with γ_ω operator acts in the rigged space as follows:

$$T_\omega \quad : \quad \mathcal{H}_k \ni \varphi \to \langle \varphi, \omega \rangle_{k,-k} \omega \in \mathcal{H}_{-k}.$$

It belongs to the $\mathcal{S}_{-k}$-class since the set

$$\mathrm{Ker} S_\omega = \mathrm{Ker}\gamma_\omega = \{\varphi \in \mathcal{H}_k | \langle \varphi, \omega \rangle_{k,-k} = 0\}$$

is dense in $\mathcal{H}_{-\frac{k}{2}}$ due to $\omega \notin \mathcal{H}_{-\frac{k}{2}}$. So, for sufficiently large k the set $\mathrm{Ker}S_\omega$ is dense in $\mathcal{H}_2$ =Dom$A$ and one can not apply any usual way for construction of the perturbed operator $\tilde{A}$. Another way is to consider γ_ω or S_ω as perturbations for $A^{\frac{k}{2}} \; : \mathcal{H}_k \to \; \mathcal{H}$. In particular, one can define $\tilde{A}$ by using the inverse operator $(A^{-\frac{k}{2}} + B_\omega)^{\frac{2}{k}}$, where B_ω acts in $\mathcal{H}$ as the rank one operator

$$B_\omega = \beta_\omega(\cdot, \eta_0)\eta_0, \quad \eta_0 := \mathbf{A}^{-\frac{\mathbf{k}}{\mathbf{2}}}\omega,$$

with an appropriate constant β_ω. That is, we have to take $\beta_\omega = 1 - c_\omega$, where the constant c_ω should satisfy the inequality $\|\eta_0\|_{-1}^2 \le c_\omega < 1$, which guaranties $\tilde{A} \ge 1$ (see [2]).

If we need to obtain a new point of the discrete spectrum then we take $\omega = c_\omega(A^{\frac{k}{2}}\psi - \lambda\psi)$ with some $\psi \in \mathcal{H} \backslash \mathcal{H}_{\frac{k}{2}}, \;\; \|\psi\| = 1, \;\; \lambda \in \mathbb{R}$. Then, according to the above, the Krein's resolvent formula has the form

$$(\tilde{A}^{\frac{k}{2}} - z)^{-1} = (A^{\frac{k}{2}} - z)^{-1} + B_\omega(z), \quad B_\omega(z) = \beta_{\omega,z}(\cdot, \eta_{\bar{z}})\eta_z$$

with

$$\eta_z = (A^{\frac{k}{2}} - \lambda)(A^{\frac{k}{2}} - z)^{-1}\psi, \quad \beta_{\omega,z} = \frac{1}{(\lambda - z)(\psi, \eta_{\bar{z}})}.$$

By construction, the operator $\tilde{A}^{\frac{k}{2}}$ solves the eigenvalue problem: $\tilde{A}^{\frac{k}{2}}\psi = \lambda\psi$. Therefore the operator $\tilde{A}$ solves the eigenvalue problem too, $\tilde{A}\psi = \lambda^{\frac{2}{k}}\psi$.

5. On eigenvalue problem for $\mathcal{H}_{-4}$ perturbations

In this section we are interesting in new eigenvalues and eigenvectors for a rank-one super-singular perturbed operators [15, 24]. We will again use the method developed in [21,24]. Especially we will show that one can construct a rank-one singular perturbations of a strictly positive self-adjoint operator A and its square power A^2 in such a way that both of them solve the eigenvalue problem with the same eigenvector but different eigenvalues: λ for $\widetilde{A}$ and λ^2 for $\widetilde{A^2}$.

Let $A \geqslant c > 1$ be a positive unbounded self-adjoint operator with $\mathrm{dom} A \equiv \mathcal{D}(A)$ in a Hilbert space $\mathcal{H}$.

Let us recall that in accordance with [24] a self-adjoint operator $\widetilde{A} \neq A$ is said to be (pure) singularly perturbed with respect to A (write $\tilde{A} \in \mathcal{P}_s(A)$), if the set

$$\mathcal{D} = \left\{ \varphi \in \mathcal{D}(A) \cap \mathcal{D}(\tilde{A}) : A\varphi = \tilde{A}\varphi \right\}$$

is dense in $\mathcal{H}$.

Let us introduce from A an extended rigged space [13], which is a part of the A-scale of Hilbert spaces,

$$\mathcal{H}_{-4} \supset \mathcal{H}_{-2} \supset \mathcal{H} \supset \mathcal{H}_{+2} \supset \mathcal{H}_{+4}, \tag{5.1}$$

where $\mathcal{H}_{+2} = \mathcal{D}(A),\ \mathcal{H}_{+4} = \mathcal{D}(A^2)$ with graph-norms A and A^2, respectively.

By definition (see. [15]) each rank-one $\mathcal{H}_{-4}$-super-singular perturbation of the operator A is given by a vector $\omega \in \mathcal{H}_{-4} \setminus \mathcal{H}_{-2}$. In terminology [15], such kind of the perturbation is super-singular since the set $\Phi_0 = \{\varphi \in \mathcal{H}_{+4} : (\omega, \varphi)_{-4,4} = 0\}$ is a domain of essential self-adjoint for A. However for A^2 this vector ω produces a rank-one $\mathcal{H}_{-2}$-singular perturbation, which can be considered using the well-known methods. Namely, using the Krein resolvent formula, the corresponding constructions developed in [2, 4, 24, 26]) we can define a singularly perturbed operator $\widetilde{A^2}$,

$$(\widetilde{A^2})^{-1} = (A^2)^{-1} + b_0\,(\cdot, \eta)\,\eta, \tag{5.2}$$

where $\eta = (A^2)^{-1}\omega \in \mathcal{H}$, $b_0 \in \mathbb{R}$ is a fixed positive number.

According to [4, 16, 21, 24] the operator $\widetilde{A^2}$ solves the eigenvalue problem, $\widetilde{A^2}\varphi = \lambda\varphi$ with $\lambda > 0$ and φ, which are defined by ω and b_0 which satisfy the equation

$$b_0\,(\varphi, \eta)\,\eta = \left(\lambda^{-1} - (A^2)^{-1}\right)\varphi\left(A^2 - \lambda\right)\lambda^{-1}(A^2)^{-1}\varphi,$$

where

$$\lambda b_0\,(\varphi, \eta)\,A^2(A^2 - \lambda)^{-1}\eta = \varphi,$$

and

$$\lambda b_0\,(\varphi, \eta)\left(A^2(A^2 - \lambda)^{-1}\eta, \eta\right) = (\varphi, \eta)\,.$$

In particular, due to (5.2), it follows that

$$\left(\widetilde{A^2}\right)^{-1}\varphi = (A^2)^{-1}\varphi + b_0\,(\varphi, \eta)\,\eta = \frac{1}{\lambda}\varphi. \tag{5.3}$$

Thus, one can solve the inverse problem putting $\varphi = A^2(A^2 - \lambda)^{-1}\eta$, and $\eta = (A^2 - \lambda)^{-1}A^{-2}\varphi$ with

$$b_0 = \frac{1}{\lambda\,(A^2(A^2 - \lambda)^{-1}\eta, \eta)}.$$

Now we will construct an operator $\widetilde{A}$ associated with $\widetilde{A^2}$, which solves the eigenvalue problem $\widetilde{A}\varphi = \sqrt{\lambda}\varphi$ and can be considered as a rank-one $\mathcal{H}_{-2}$-singular perturbation for A.

Put

$$\mu = (A - \sqrt{\lambda})A^{-1}\varphi, \quad p_0 = \frac{1}{\sqrt{\lambda}\left(A(A - \sqrt{\lambda})^{-1}\mu, \mu\right)}$$

and define

$$\widetilde{A}^{-1} = A^{-1} + p_0\,(\cdot, \mu)\,\mu.$$

Using η and λ we rewrite the last formula as

$$\widetilde{A}^{-1} = A^{-1} + \frac{\left(\cdot, A(A + \sqrt{\lambda})^{-1}\eta\right)}{\sqrt{\lambda}\left(A^2(A^2 - \lambda)^{-1})^{-1}\eta, A(A + \sqrt{\lambda})^{-1}\eta\right)} A(A + \sqrt{\lambda})^{-1}\eta. \tag{5.4}$$

Now it is easy to check that our operator solves the above mentioned eigenvalue problem. Thus the following statement is true.

Theorem 5.1. *Let in a separable Hilbert space $\mathcal{H}$ for a positive self-adjoint operator $A = A^* \geqslant c > 1$, a rank-one super-singular perturbation in a form of vector $\omega \in \mathcal{H}_{-4} \setminus \mathcal{H}_{-2}$ be given. And let the operator $\widetilde{A^2}$ be singularly perturbed with respect to A^2. Assume it is defined by the Krein formula on ω and some real b_0. Assume also that $\widetilde{A^2}$ solves the eigenvalue problem, $\widetilde{A^2}\varphi = \lambda\varphi$. Then the associated operator $\widetilde{A} \in \mathcal{P}_s^1(A)$ defined by formula (5.4) solves the eigenvalue problem, $\widetilde{A}\varphi = \sqrt{\lambda}\varphi$.*

We note that under the above construction, $\sqrt{\widetilde{A^2}}\varphi = \widetilde{A}\varphi$, however, in general, $\sqrt{\widetilde{A^2}} \neq \widetilde{A}$.

Example. Let A is the multiplication by x in $L_2 = L_2([1,\infty), dx)$. Then the part of scale (5.1) has the form

$$L_2([1,\infty), x^{-4}dx) \supset L_2([1,\infty), x^{-2}dx) \supset L_2([1,\infty), dx) \\ \supset L_2([1,\infty), x^2dx) \supset L_2([1,\infty), x^4dx).$$

Take $\omega = x,\ \lambda = 0,25$. Then, due to $A^2 f(x) = x^2 f(x)$ we have

$$\widetilde{A^2}^{-1} = x^{-2} + b_0\left(\cdot, x^{-1}\right)x^{-1},$$

where

$$b_0 = 4\left(x^2\left(x^2 - 0,25\right)^{-1} x^{-1}, x^{-1}\right)^{-1}.$$

Now

$$\varphi = x^2\left(x^2 - 0,25\right)^{-1} x^{-1} = \frac{4x}{4x^2 - 1}.$$

It is easy to calculate that

$$\int_1^\infty \frac{x^2}{x^2\left(x^2 - 0,25\right)}dx = 2\int_1^\infty \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)dx = \ln \frac{2x-1}{2x+1}\bigg|_1^\infty = \ln 3.$$

Therefore $b_0 = 4/\ln 3$. Let us denote

$$\mu = (x - 0,5)\,x^{-1}\varphi = x\,(x + 0,5)^{-1}\,x^{-1} = \frac{2}{2x+1}$$

and

$$p_0 = \frac{1}{\sqrt{\lambda}(A(A - \sqrt{\lambda})^{-1}\mu, \mu)}.$$

Since

$$\int_1^\infty \frac{4xdx}{(2x-1)(2x+1)^2} = \left(\frac{1}{4}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{1}{2}\frac{1}{2x+1}\right)\bigg|_1^\infty = \frac{1}{6} + \frac{1}{4}\ln 3,$$

we find that

$$\widetilde{A}^{-1} = A^{-1} + \frac{12}{2 + 3\ln 3}\left(\cdot, \frac{2}{2x+1}\right)\frac{2}{2x+1}.$$

Now it is easy to check that

$$\widetilde{A^2}\varphi = 0,25\varphi, \quad \widetilde{A}\varphi = 0,5\varphi.$$

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National Aviation University, prosp. Kosmonavta Komarova 1, Kyiv, 03058, Ukraine
E-mail address: ttugay@ukr.net

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